THE BLOW-UP PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION WITH A POTENTIAL

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Abstract. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$. We consider the problem $u_t = \Delta u + V(x) u^p$ in $\Omega \times (0, T)$, with Dirichlet boundary conditions $u = 0$ on $\partial \Omega \times (0, T)$ and initial datum $u(x, 0) = M u_0(x)$ where $M \geq 0$, $u_0$ is positive and compatible with the boundary condition. We give estimates for the blow up time of solutions for large values of $M$. As a consequence of these estimates we find that, for $M$ large, the blow up set concentrates near the points where $u_0^{p-1}V$ attains its maximum.

1. INTRODUCTION

In this paper we study the blow-up phenomena for the following semilinear parabolic problem with a potential

\begin{equation}
\begin{aligned}
&u_t = \Delta u + V(x) u^p \quad \text{in } \Omega \times (0, T),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&u(x, 0) = M u_0(x) \quad \text{in } \Omega.
\end{aligned}
\end{equation}

First, let us state our basic assumptions. They are: $\Omega$ is a bounded, convex, smooth domain in $\mathbb{R}^N$ and the exponent $p$ is subcritical, that is, $1 < p < (N + 2)/(N - 2)$. The potential $V$ is Lipschitz continuous and there exists a constant $c > 0$ such that $V(x) \geq c$ for all $x \in \Omega$. As for the initial condition we assume that $M \geq 0$ and that $u_0$ is a smooth positive function compatible with the boundary condition. Moreover, we impose that

\begin{equation}
\begin{aligned}
&M \Delta u_0 + \frac{\min_{x \in \Omega} V(x)}{2} M^p u_0^p \geq 0.
\end{aligned}
\end{equation}

Key words and phrases. Blow-up, semilinear parabolic equations.

Supported by Universidad de Buenos Aires under grant TX048, by ANPCyT PICT No. 03-00000-00137 and CONICET (Argentina) and by Fondecyt 1030798 and Fondecyt Coop. Int. 7050118 (Chile).

2000 Mathematics Subject Classification 35K57, 35B40.
We note that (7) holds for $M$ large if $\Delta u_0$ is nonnegative in a neighborhood of the set where $u_0$ vanishes.

It is known that, and we will prove it later for the sake of completeness, once $u_0$ is fixed the solution to (7) blows up in finite time for any $M$ sufficiently large. By this we understand that there exists a time $T = T(M)$ such that $u$ is defined in $\Omega \times [0, T)$ and

$$\lim_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty.$$  

The study of the blow-up phenomena for parabolic equations and systems has attracted considerable attention in recent years, see for example [7], [8], [9], [10], [11], [12], [13], [14] and the corresponding references. A good review in the topic can be found in [7]. When a large or small diffusion is considered, see [7], [8].

Important issues in a blow-up problem are to obtain estimates for the blow-up time, $T(M)$, and determine the spatial structure of the set where the solution becomes unbounded, that is, the blow-up set. More precisely, the blow-up set of a solution $u$ that blows up at time $T$ is defined as

$$B(u) = \{ x / \text{there exist } x_n \to x, t_n / T, \text{ with } u(x_n, t_n) \to \infty \}.$$  

The problem of estimating the blow-up time and the description and location of the blow-up set has proved to be a subtle problem and has been addressed by several authors. See for example [7], [8] and the corresponding bibliographies.

Our interest here is the description of the asymptotic behavior of the blow-up time, $T(M)$, and of the blow-up set, $B(u)$, as $M \to \infty$. It turns out that their asymptotics depend on a combination of the shape of both the initial condition, $u_0$, and the potential $V$. Roughly speaking one expects that if $u_0 \equiv 1$ then the blow-up set should concentrate near the points where $V$ attains its maximum. On the other hand if $V \equiv 1$ the blow-up set should be near the points where $u_0$ attains its maximum. Here we show that the quantity that plays a major role is $(\max_x u_0^{p-1}(x)V(x))^{-1}$.

Theorem 1.1. There exists $\bar{M} > 0$ such that if $M \geq \bar{M}$ the solution of (7) blows up in a finite time that we denote by $T(M)$. Moreover, let

$$A = A(u_0, V) := \frac{1}{(\max_x u_0^{p-1}(x)V(x))}.$$
then there exist two positive constants $C_1$, $C_2$, such that, for $M$ large enough,

\begin{equation}
- \frac{C_1}{M^{\frac{1}{p-1}}} \leq T(M)M^{p-1} - \frac{A}{p-1} \leq \frac{C_2}{M^{\frac{1}{p-1}}},
\end{equation}

and the blow-up set verifies,

\begin{equation}
u_0^{p-1}(a)V(a) \geq \frac{1}{A} - \frac{C}{M^\gamma}, \quad \text{for all } a \in B(u),
\end{equation}

where $\gamma = \min(\frac{p-1}{4}, \frac{1}{3})$.

Note that this result implies that

$$
\lim_{M \to \infty} T(M)M^{p-1} = \frac{A}{p-1}.
$$

Moreover, it provides precise lower and upper bounds on the difference $T(M)M^{p-1} - \frac{A}{p-1}$.

We also observe that (3.3) shows that the set of blow-up points concentrates for large $M$ near the set where $u_0^{p-1}V$ attains its maximum.

If in addition the potential $V$ and the initial datum $u_0$ are such that $u_0^{p-1}V$ has a unique non degenerate maximum at a point $\tilde{a}$, then there exist constants $c > 0$ and $d > 0$ such that

$$
u_0^{p-1}(\tilde{a})V(\tilde{a}) - \nu_0^{p-1}(x)V(x) \geq c|\tilde{a} - x|^2 \quad \text{for all } x \in B(\tilde{a}, d).
$$

Therefore, according to our result, if $M$ is large enough one has

$$
|\tilde{a} - a| \leq \frac{C}{M^\gamma} \quad \text{for any } a \in B(u),
$$

with $\gamma = \min(\frac{p-1}{4}, \frac{1}{3})$.

Throughout the paper we will denote by $C$ a constant that does not depends on the relevant parameters involved but may change at each step.

2. Proof of Theorem 3.3.

We begin with a lemma that provides us with an upper estimate of the blow-up time. This upper estimate gives the upper bound for $T(M)M^{p-1}$ in (3.3) and will be crucial in the rest of the proof of Theorem 3.3.

**Lemma 2.1.** There exist a constant $C > 0$ and $M_0 > 0$ such that for every $M \geq M_0$, the solution of (3.3) blows up in a finite time that verifies

\begin{equation}
T(M) \leq \frac{A}{M^{p-1}(p-1)} + \frac{C}{M^{\frac{1}{p-1}}M^{p-1}}.
\end{equation}
**Proof:** Let $\bar{a} \in \Omega$ be such that
\[
u_0^{p-1}(\bar{a})V(\bar{a}) = \max_x u_0^{p-1}(x)V(x),
\]
$L$ the constant of Lipschitz continuity of $V$, and $K$ an upper bound for the first derivatives of $u_0$ and $L$.

In order to get the upper estimate let $M$ be fixed and $\varepsilon = \varepsilon(M) > 0$ to be defined latter, small enough so all functions involved are well defined. Pick
\[
\delta = \frac{\varepsilon}{2K},
\]
then
\[
V(x) \geq V(\bar{a}) - \frac{\varepsilon}{2} \quad \text{and} \quad u_0(x) \geq u_0(\bar{a}) - \varepsilon \quad \text{for all} \ x \in B(\bar{a}, \delta).
\]

Let $w$ be the solution of
\[
\begin{align*}
w_t &= \Delta w + \left( V(\bar{a}) - \frac{\varepsilon}{2} \right) w^p \quad \text{in} \ B(\bar{a}, \delta) \times (0, T_w),
\quad 
w &= 0 \quad \text{on} \ \partial B(\bar{a}, \delta) \times (0, T_w),
\quad 
w(x, 0) = M(u_0(\bar{a}) - \varepsilon), \quad \text{in} \ B(\bar{a}, \delta)
\end{align*}
\]
and $T_w$ its corresponding blow up time. A comparison argument shows that $u \geq w$ in $B(\bar{a}, \delta) \times (0, T)$ and hence
\[
T \leq T_w.
\]

Our task now is to estimate $T_w$ for large values of $M$. To this end, let $\lambda_1(\delta)$ be the first eigenvalue of $-\Delta$ in $B(\bar{a}, \delta)$ and let $\varphi_1$ be the corresponding positive eigenfunction normalized so that
\[
\int_{B(\bar{a}, \delta)} \varphi_1(x) \, dx = 1.
\]
That is,
\[
\begin{align*}
-\Delta \varphi_1 &= \lambda_1(\delta) \varphi_1, \quad \text{in} \ B(\bar{a}, \delta),
\quad 
\varphi_1 &= 0 \quad \text{on} \ \partial B(\bar{a}, \delta).
\end{align*}
\]
Now, set
\[
\Phi(t) = \int_{B(\bar{a}, \delta)} w(x, t)\varphi_1(x) \, dx.
\]
Then $\Phi(t)$ satisfies $\Phi(0) = M(u_0(\bar{a}) - \varepsilon)$ and
\[
\Phi'(t) = \int_{B(\bar{a},\delta)} w_t(x, t)\varphi_1(x) \, dx
\]
\[
= \int_{B(\bar{a},\delta)} \left( \Delta w(x, t)\varphi_1(x) + \left( V(x_1) - \frac{\varepsilon}{2} \right) w^p(x, t)\varphi_1(x) \right) \, dx
\]
\[
\geq -\lambda_1(\delta) \int_{B(\bar{a},\delta)} w(x, t)\varphi_1(x) \, dx
\]
\[
+ \left( V(\bar{a}) - \frac{\varepsilon}{2} \right) \left( \int_{B(\bar{a},\delta)} w(x, t)\varphi_1(x) \, dx \right)^p
\]
\[
= -\lambda_1(\delta) \Phi(t) + \left( V(\bar{a}) - \frac{\varepsilon}{2} \right) \Phi(t)^p.
\]

Let us recall that there exists a constant $D$, depending on the dimension only, such that the eigenvalues of the laplacian scale according to the rule $\lambda_1(\delta) = D\delta^{-2}$.

Now, we choose $\varepsilon$ such that
\[
\lambda_1(\delta) = D\delta^{-2} = D \left( \frac{\varepsilon}{2K} \right)^{-2} = \frac{\varepsilon}{2} (M(u_0(\bar{a}) - \varepsilon))^{p-1}.
\]

So, $\varepsilon$ is of order
\[
\varepsilon \sim \frac{C}{M^{\frac{p-1}{p}}}.
\]

Choose $M_0$ such that for $M \geq M_0$ the resulting $\varepsilon$ is small enough.

Then for any $M \geq M_0$ we have that
\[
\Phi'(t) \geq (V(\bar{a}) - \varepsilon)\Phi(t)^p,
\]
for all $t \geq 0$ for which $\Phi$ is defined.

Since $\Phi(0) = M(u_0(\bar{a}) - \varepsilon)$ and $T_w$ is less or equal than the blow up time of $\Phi$ integrating (??) it follows that
\[
T_w \leq \frac{1}{M^{p-1}(p-1)(V(\bar{a}) - \varepsilon)(u_0(\bar{a}) - \varepsilon)^{p-1}}
\]
\[
\leq \frac{1}{M^{p-1}(p-1)V(\bar{a})u_0(\bar{a})^{p-1}} \frac{C}{M^{\frac{p-1}{p}} M^{p-1}},
\]
for all $M \geq M_0$.

Now we prove a lemma that provides us with an upper bound for the blow up rate. We observe that this is the only place where we use hypothesis (??).

**Lemma 2.2.** Assume (??). Then there exists a constant $C$ independent of $M$ such that
\[
u(x, t) \leq C(T - t)^{-\frac{1}{p-1}}.
\]

Proof: Let \( m = \min_{x \in \Omega} V \). Following ideas of \cite{4}, set
\[
v = u_t - \frac{m}{2} u^p.
\]
Then \( v \) verifies
\[
v_t - \Delta v - V(x) pu^{p-1} v = \frac{m}{2} p(p - 1) u^{p-2} |\nabla u|^2 \geq 0 \quad \text{in } \Omega \times (0, T),
\]
\[
v = 0 \quad \text{on } \partial \Omega \times (0, T),
\]
\[
v(x, 0) = M \Delta u_0 + \left( V(x) - \frac{m}{2} \right) M^p u_0^p \geq 0 \quad \text{in } \Omega.
\]
Therefore \( v \geq 0 \) and hence \( u_t \geq \frac{m}{2} u^p \).

Integrating this inequality from 0 to \( T \) we get
\[
u(x, t) \leq \frac{2^{\frac{1}{p-1}}}{(m(p - 1)(T - t))^{\frac{1}{p-1}}} \equiv C(T - t)^{-\frac{1}{p-1}},
\]
as we wanted to prove. \( \square \)

We are now in a position to prove Theorem \( 2.3 \).

Proof of Theorem \( 2.3 \): The idea of the proof is to combine the estimate of the blow-up time proved in Lemma \( 2.2 \) with local energy estimates near a blow-up point \( a \), like the ones considered in \cite{4} and \cite{5}, to obtain an inequality that forces \( u_{b}^{p-1}(a)V(a) \) to be close to \( \max_x u_{b}^{p-1}V \).

Let us now proceed with the proof of the estimates on the blow-up set. We fix for the moment \( M \) large enough such that \( u \) blows up in finite time \( T = T(M) \) and let \( a = a(M) \) be a blow up point. As in \cite{5}, for this fixed \( a \) we define
\[
w(y, s) = (T - t)^{\frac{1}{p-1}} u(a + y(T - t)^{\frac{1}{2}}, t)|_{t= T(1 - e^{-s})}.
\]
Then \( w \) satisfies
\[
w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p - 1} w + V(a + yT e^{-\frac{s}{2}}) w^p,
\]
in \( \cup_{s \in (0, \infty)} \Omega(s) \times \{ s \} \) where \( \Omega(s) = \{ y : a + yT e^{-\frac{s}{2}} \in \Omega \} \) with \( w(y, 0) = T^{\frac{1}{p-1}} u_0(a + y T^{\frac{1}{2}}) \). The above equation can rewritten as
\[
w_s = \frac{1}{\rho} \nabla (\rho \nabla w) - \frac{1}{p - 1} w + V(a + yT e^{-\frac{s}{2}}) w^p
\]
where \( \rho(y) = \exp\left(-\frac{|y|^2}{4}\right) \).

Consider the energy associated with the "frozen" potential
\[
V \equiv V(a),
\]
that is
\[ E(w) = \int_{\Omega(s)} \left( \frac{1}{2} |\nabla w|^2 + \frac{1}{2(p-1)} w^2 - \frac{1}{p+1} V(a) w^{p+1} \right) \rho(y) \, dy. \]

Then, using the fact that \( \Omega \) is convex, we get
\[ \frac{dE}{ds} \leq - \int_{\Omega(s)} (w_s)^2 \rho(y) \, dy + \int_{\Omega(s)} (V(a + ye^{-s^2}) - V(a)) w^p w_s \rho(y) \, dy. \]

Since \( V(x) \) is Lipschitz and \( w \) is bounded due to Lemma ??, then there exists a constant \( C \) depending only on \( N, p \) and \( V \), recall that the constant in Lemma ?? does not depend on \( M \), such that
\[ \frac{dE}{ds} \leq - \int (w_s)^2 \rho(y) \, dy + Ce^{-s^2 T} \left( \int (w_s)^2 \rho(y) \, dy \right)^{1/2}. \]

Maximizing the right hand side of the above expression with respect to \( \int (w_s)^2 \rho(y) \, dy \) we obtain
\[ \frac{dE}{ds} \leq Ce^{-s^2 T^2} \]
and integrating is \( s \) we get
\[ (2.4) \quad E(w) \leq E(w_0) + CT^2. \]

Since \( w \) is bounded and satisfies (??), following the arguments given in [?] and [?], one can prove that \( w \) converges as \( s \to \infty \) to a non trivial bounded stationary solution of the limit equation
\[ (2.5) \quad 0 = \Delta z - \frac{1}{2} y \cdot \nabla z - \frac{1}{p-1} z + V(a) z^p \]
in the whole \( \mathbb{R}^N \).

Again by the results of [?] and [?], since \( p \) is subcritical, \( 1 < p < (N+2)/(N-2) \), the only non trivial bounded positive solution of (??) with \( V(a) = 1 \) is the constant \( (p-1)^{-\frac{1}{p-1}} \). A scaling argument gives that the only non trivial bounded positive solution of (??) is the constant \( k = k(a) \) given by
\[ k(a) = \frac{1}{(V(a)(p-1))^{\frac{1}{p-1}}}. \]

Therefore, we conclude that
\[ \lim_{s \to \infty} w = k(a) \]
if \( a \) is a blow-up point. Also by the results of [?], [?] we have
\[ (2.6) \quad E(w(\cdot, s)) \to E(k(a)) \quad \text{as } s \to \infty, \]
where

\[ E(k(a)) = \int \left( \frac{1}{2(p-1)}(k(a))^2 - \frac{1}{p+1}V(a)(k(a))^{p+1} \right) \rho(y) \, dy \]

\[ = (k(a))^2 \left( \frac{1}{2(p-1)} - \frac{1}{(p+1)(p-1)} \right) \int \rho(y) \, dy. \]

By (??) and (??) we obtain that, if \( a \) is a blow-up point, then

\[ E(k(a)) \leq E(w_0) + CT^2. \]

where \( w_0(y) = w(y, 0) = T^{\frac{1}{p-1}} Mu_0(a + yT^\frac{1}{2}) \).

As \( u_0 \) is smooth, \( y \rho(y) \) integrable, and \( T^{\frac{1}{p-1}} M \) is bounded by Lemma ??, there are constants \( C \) independent of \( a \) such that for \( M \geq M_0 \)

\[ E(w(\cdot, 0)) = \int_{\Omega(0)} \left( \frac{1}{2} |\nabla w_0(y)|^2 + \frac{1}{2(p-1)} w_0^2(y) \right) \rho(y) \, dy \]

\[ \leq \int_{\Omega(0)} \left( \frac{1}{2} (T^{\frac{1}{p-1}} M)^2 T |\nabla u_0(a)|^2 \right) \rho(y) \, dy \]

\[ + \int_{\Omega(0)} \left( \frac{1}{2(p-1)} (T^{\frac{1}{p-1}} Mu_0(a))^2 \right) \rho(y) \, dy \]

\[ - \int_{\Omega(0)} \left( \frac{1}{p+1} V(a)(T^{\frac{1}{p-1}} Mu_0(a))^{p+1} \right) \rho(y) \, dy \]

\[ + CT^\frac{3}{2} + CT^\frac{1}{2}. \]

Therefore, since \( |\nabla u_0| \) is bounded,

\[ E(w(\cdot, 0)) \leq \int_{\Omega(0)} \left( \frac{1}{2(p-1)} (T^{\frac{1}{p-1}} Mu_0(a))^2 \right) \rho(y) \, dy \]

\[ - \int_{\Omega(0)} \left( \frac{1}{p+1} V(a)(T^{\frac{1}{p-1}} Mu_0(a))^{p+1} \right) \rho(y) \, dy \]

\[ + CT^\frac{3}{2} + CT^\frac{1}{2}. \]

Or, since \( T \leq 1 \) for \( M \) large

\[ E(w(\cdot, 0)) \leq E(T^{\frac{1}{p-1}} Mu_0(a)) + CT^\frac{1}{2}. \]

Hence we arrive to the following bound for \( E(k(a)) \)

\[ (2.7) \quad E(k(a)) \leq E(w(\cdot, 0)) + CT^2 \leq E(T^{\frac{1}{p-1}} Mu_0(a)) + CT^\frac{1}{2}. \]

Observe that if \( b \) is a constant then the energy can be written as

\[ E(b) = \Gamma F(b), \]
where $\Gamma$ is the constant

$$\Gamma = \int \rho(y) \, dy$$

and $F$ is the function

$$F(z) = \left( \frac{1}{2(p-1)}z^2 - \frac{1}{p+1}V(a)z^{p+1} \right).$$

As $F$ attains a unique maximum at $k(a)$ and $F''(k(a)) = -1$ there are $\alpha$ and $\beta$ such that if $|z - k(a)| \leq \alpha$ then

$$F''(z) \leq -\frac{1}{2},$$

and if $|F(z) - F(k(a))| \leq \beta$ then

$$|z - k(a)| \leq \alpha.$$

From (??) we obtain

$$F(k(a)) \leq F(T^{\frac{1}{p+1}}M u_0(a)) + CT^{\frac{1}{2}}.$$

If $M_1$ is such that $C(T(M_1))^{\frac{1}{2}} = \beta$ then for $M \geq \max(M_0, M_1)$

$$\beta \geq CT^{\frac{1}{2}} \geq F(k(a)) - F(T^{\frac{1}{p+1}}M u_0(a)).$$

Hence by the properties of $F$,

$$|k(a) - T^{\frac{1}{p+1}}M u_0(a)| \leq \alpha.$$

Therefore

$$CT^{\frac{1}{2}} \geq F(k(a)) - F(T^{\frac{1}{p+1}}M u_0(a)) \geq \frac{1}{4}(T^{\frac{1}{p+1}}M u_0(a) - k(a))^2.$$

So, using Lemma ??,

$$k(a) - CT^{\frac{1}{2}} \leq T^{\frac{1}{p+1}}M u_0(a)$$

(2.8)

$$\leq \frac{u_0(a)}{(p-1)^{\frac{1}{p+1}}V^{\frac{1}{p+1}}(\bar{a}) u_0(\bar{a})} + \frac{C u_0(a)}{M^{\frac{1}{2}}},$$

where

$$\theta(a) = \left( \frac{u_0(a)V(a)^{\frac{1}{p+1}}}{u_0(\bar{a})V(\bar{a})^{\frac{1}{p+1}}} \right)$$

and $\bar{a}$ is such that

$$u_0^{p-1}(\bar{a})V(\bar{a}) = \max_x u_0^{p-1}(x)V(x).$$
Recall that
\[ T \leq \frac{C}{M^{p-1}}. \]

Therefore, we get
\[ k(a)(1 - \theta(a)) \leq \frac{C u_0(a)}{M^{\frac{1}{4}}} + \frac{C}{M^{\frac{p-1}{4}}} \leq \frac{C}{M^\gamma}, \]
with \( \gamma = \min\left(\frac{p-1}{4}, \frac{1}{3}\right) \).

As \( V \) is bounded we have that \( k(a) \) is bounded from below, hence
\[ (1 - \theta(a)) \leq \frac{C}{M^\gamma}, \]
that is,
\[ \theta(a) \geq 1 - \frac{C}{M^\gamma} \]
and we finally obtain
\[ u_0(a)V(a)^{\frac{1}{p-1}} \geq u_0(\bar{a})V(\bar{a})^{\frac{1}{p-1}} - \frac{C}{M^\gamma}. \]

This proves (??).

To obtain the lower estimate for the blow-up time observe that from (??) and the fact that \( V(a) \geq c > 0 \) we get
\[
\begin{align*}
\frac{1}{(V(a)(p-1))^{\frac{1}{p-1}}} - CT^\frac{1}{2} & \leq T^{\frac{1}{p-1}} M u_0(a). \\
\frac{1}{u_0(a)(V(a)(p-1))^{\frac{1}{p-1}}} - CT^\frac{1}{2} & \leq T^{\frac{1}{p-1}} M.
\end{align*}
\]

By (??) and \( u_0^{p-1}(\bar{a})V(\bar{a}) = \max_x u_0^{p-1}(x)V(x) \) we get
\[
\frac{1}{u_0(\bar{a})(V(\bar{a})(p-1))^{\frac{1}{p-1}}} - CT^\frac{1}{2} \leq T^{\frac{1}{p-1}} M
\]
and using
\[ T \leq \frac{C}{M^{p-1}}. \]
we obtain
\[
\frac{1}{u_0(\bar{a})(V(\bar{a})(p - 1))^{\frac{1}{p - 1}}} - \frac{C}{M^{\frac{1}{p - 1}}} \leq T^{\frac{1}{p - 1}}M
\]
as we wanted to prove.

References


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