Asymptotic expansion of the invariant measure for ballistic random walk in the low disorder regime

David Campos[*,†,??] **and Alejandro F. Ramírez**[*,??]

David Campos
Escuela de Matemática
Universidad de Costa Rica
e-mail: josedavid.campos@ucr.ac.cr

Alejandro F. Ramírez
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
e-mail: aramirez@mat.uc.cl

Abstract: We consider a random walk in random environment in the low disorder regime on $\mathbb{Z}^d$. That is, the probability that the random walk jumps from a site x to a nearest neighboring site $x+e$ is given by $p(e)+\epsilon\xi(x,e)$, where $p(e)$ is deterministic, $\{\{\xi(x,e) : |e|_1 = 1\} : x \in \mathbb{Z}^d\}$ are i.i.d. and $\epsilon > 0$ is a parameter which is eventually chosen small enough. We establish an asymptotic expansion in ϵ for the invariant measure of the environmental process whenever a ballisticity condition is satisfied. As an application of our expansion, we derive a numerical expression up to first order in ϵ for the invariant measure of random perturbations of the simple symmetric random walk in dimensions $d = 2$.

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1. Introduction

We derive an asymptotic expansion for the invariant measure of the environmental process of random walks moving on $\mathbb{Z}^d$ in the low disorder regime within the spirit of previous expansions of Sabot [12] for the velocity. Our result is one of the few instances where explicit quantitative information about the invariant measure of the environmental process is given for random walks in random environments in dimensions $d \geq 2$ with nonvanishing velocity.

For $x \in \mathbb{R}^d$ we denote by $|x|_1$ and $|x|_2$ its l^1 and l^2 norms respectively. Let $V := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ and $\mathcal{P} := \{p_e : e \in V\}$ where $p_e \geq 0$ and $\sum_{e\in V} p_e = 1$. We define $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ endowed with its Borel σ-algebra and denote any $\omega =$

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$\{\omega(x) : x \in \mathbb{Z}^d\} \in \Omega$ where for each $x \in \mathbb{Z}^d$ we let $\omega(x) = \{\omega(x,e) : e \in V\} \in \mathcal{P}$, an *environment*. We now define the *random walk in the environment* ω as the Markov chain $\{X_n : n \geq 0\}$ with state space $\mathbb{Z}^d$ defined by the transition probabilities

$$P(X_{n+1} = y + e | X_n = y) = \omega(y,e),$$

for all $e \in V$ and $y \in \mathbb{Z}^d$. For each $x \in \mathbb{Z}^d$, we denote by $P_{x,\omega}$ its law if it starts from x. Throughout we will assume that the space of environments Ω is endowed with a probability measure $\mathbb{P}$. We will call $P_{x,\omega}$ the *quenched law* of the random walk, while $P_x := \int P_{x,\omega} d\mathbb{P}$ the *averaged* or *annealed* law of the random walk. We will suppose that $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. under $\mathbb{P}$. The law $\mathbb{P}$ is said to be uniformly elliptic if there exists a $\kappa > 0$, which we will call the ellipticity constant, such that for all $x \in \mathbb{Z}^d$ and $e \in V$,

$$\mathbb{P}(\omega(x,e) \geq \kappa) = 1.$$

Define $\mathcal{P}_0 := \{p \in \mathcal{P} : \min_{e \in V} p(e) > 0\}$. Consider a transition kernel $p_0 = \{p_0(e) : e \in V\} \in \mathcal{P}_0$. For our main result, we will consider laws $\mathbb{P}$ which are perturbations of a simple random walk which jumps according to the transition kernel p_0: for each $\epsilon > 0$ we define

$$\Omega_{p_0,\epsilon} := \left\{\omega \in \Omega : |\omega(x,e) - p_0(e)| \leq \epsilon \text{ for all } x \in \mathbb{Z}^d, e \in V\right\}. \tag{1.1}$$

Let us note that for $\epsilon > \min_{e \in V} p_0(e)$, each probability measure concentrated on $\Omega_{p_0,\epsilon}$ is uniformly elliptic with ellipticity constant

$$\kappa = \min_{e \in V} p_0(e) - \epsilon. \tag{1.2}$$

Throughout this article, κ will be given by (1.2). Also, recall the definition of the local drift for $x \in \mathbb{Z}^d$ as

$$d(x,\omega) := \sum_{e \in V} \omega(x,e)e.$$

For $\omega \in \Omega$, define the canonical shifts $\{\theta_x : x \in \mathbb{Z}^d\}$ as $\theta_x \omega(y) := \omega(x+y)$ for all $y \in \mathbb{Z}^d$. Finally, define the *environmental process* $\{\bar{\omega}_n : n \geq 0\}$ starting from $\bar{\omega}_0 = \omega$ as

$$\bar{\omega}_n := \theta_{X_n} \omega.$$

The transition kernel of this process is defined as the map R from the set of functions $f : \Omega \to \mathbb{R}$ to itself given by

$$Rf(\omega) := \sum_{e \in V} \omega(0,e) f(\theta_e \omega). \tag{1.3}$$

To state the main result of this article, let us define for each $\omega \in \Omega$, $x \in \mathbb{Z}^d$ and $e \in V$,

$$\xi(x,e) := \frac{1}{\epsilon}\left(\omega(x,e) - p_0(e)\right), \tag{1.4}$$

so that

$$\omega(x,e) = p_0(e) + \epsilon\xi(x,e), \tag{1.5}$$

and

$$\bar{\xi}(x,e) := \xi(x,e) - \mathbb{E}[\xi(x,e)],$$

where the notation $\mathbb{E}$ denotes taking expectation with respect to the measure $\mathbb{P}$. Define also

$$p_\epsilon(e) := p_0(e) + \epsilon\mathbb{E}[\xi(0,e)]. \tag{1.6}$$

Furthermore, define for $n \geq 0$ and $x, y \in \mathbb{Z}^d$, $p_n(x,y)$ as the probability that a random walk with transition kernel $p \in \mathcal{P}$ jumps from x at time 0 to site y at time n, and the function

$$J_{p^*}(x) := \lim_{n\to\infty} \sum_{k=0}^{n} \left(p_k(0,-x) - p_k(0,0)\right), \tag{1.7}$$

where here for $p \in \mathcal{P}$, the subscript p^* in the above expression is the transition kernel $p^* \in \mathcal{P}$ defined by

$$p^*(e) := p(-e) \quad \text{for} \quad e \in V. \tag{1.8}$$

Note that for each p which defines a transient random walk, the above expression can be written as a difference of a Green function evaluated at different points. On the other hand, in the two-dimensional recurrent case, (1.7) is equal to the negative of the potential kernel of a random walk with transition kernel p^*.

In the main result of this article, we establish an asymptotic expansion for the invariant measure of random walks in environments whose law is supported in $\Omega_{p_0,\epsilon}$ for a given $p_0 \in \mathcal{P}_0$ and ϵ small enough. To formulate it, we will assume the following condition on the local drift. Given $p_0 \in \mathcal{P}_0$, and $\epsilon > 0$ we will say that a probability measure $\mathbb{P}$ defined on Ω satisfies the local drift condition **(LD)** with bound ϵ if $\mathbb{P}(\Omega_{p_0,\epsilon}) = 1$ and

$$\mathbb{E}[d(0,\omega)] \cdot e_1 \geq \epsilon. \tag{1.9}$$

Whenever the local drift condition **(LD)** is satisfied, the random walk satisfies Kalikow's condition (see [16] or [12] for its definition), and hence by Theorem 3.1 of Sznitman and Zerner [16], the environmental process has a marginal law at fixed time which converges in distribution to an invariant measure. We will call this invariant measure, the *limiting invariant measure* of the environmental process. Furthermore, given a measure μ defined on Ω and a subset $B \subset \mathbb{Z}^d$, we will call the marginal law of μ in $\mathcal{P}^B$ *the restriction of* μ *to* B.

Theorem 1. *Let $\eta > 0$ and B be a finite subset of $\mathbb{Z}^d$. Then, there is an $\epsilon_0 > 0$ such that whenever $\epsilon \leq \epsilon_0$, $p_0 \in \mathcal{P}_0$, and $\mathbb{P}$ satisfies the local drift condition* **(LD)** *[c.f. (1.9)], the limiting invariant measure $\mathbb{Q}$ has a restriction $\mathbb{Q}_B$ to B which is absolutely continuous with respect to the restriction $\mathbb{P}_B$ to B of $\mathbb{P}$, with a Radon-Nikodym derivative admitting $\mathbb{P}$-a.s. the expansion*

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \epsilon \sum_{z \in B} \sum_{e \in V} \bar{\xi}(z,e) J_{p_\epsilon^*}(e+z) + O\left(\epsilon^{2-\eta}\right), \tag{1.10}$$

where $\left|O\left(\epsilon^{2-\eta}\right)\right| \leq c_1 \epsilon^{2-\eta}$, for some constant $c_1 = c_1(\eta, \kappa, d, B)$ depending only on η, κ, d and B.

Expanding $J_{p_\epsilon^*}$ it is possible rewrite in dimensions $d \geq 2$, the expansion (1.10).

Corollary 2. *Let $\eta > 0$ and B be a finite subset of $\mathbb{Z}^d$. Then, there is an $\epsilon_0 > 0$ such that whenever $\epsilon \leq \epsilon_0$, $p_0 \in \mathcal{P}_0$, and $\mathbb{P}$ satisfies the local drift condition* **(LD)** *[c.f. (1.9)], the limiting invariant measure $\mathbb{Q}$ has a restriction $\mathbb{Q}_B$ to B which is absolutely continuous with respect to the restriction $\mathbb{P}_B$ to B of $\mathbb{P}$, with a Radon-Nikodym derivative admitting $\mathbb{P}$-a.s. the expansion*

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \epsilon \sum_{z \in B} \sum_{e \in V} \bar{\xi}(z,e) J_{p_0^*}(e+z) + O\left(\epsilon^{2-\eta}\right), \tag{1.11}$$

where $\left|O\left(\epsilon^{2-\eta}\right)\right| \leq c_2 \epsilon^{2-\eta}$, for some constant $c_2 = c_2(\eta, \kappa, d, B)$ depending only on η, κ, d and B.

From the point of view of its explicitness, a startling consequence of Corollary 2 is stated in Corollary 3 of section 3, where due to the fact that the potential kernel of a simple symmetric random walk in dimension $d = 2$ can be recursively computed, we can obtain a numerical expression up to first order for the limiting invariant measure. Furthermore, the Radon-Nikodym derivative (1.10) plays an important role in local limit theorems (see for example Theorem 1.11 of [2] valid for $d \geq 4$).

On the other hand, by the fact that the marginal law of the environmental process converges to the limiting invariant measure, and the fact that

$$X_n - \sum_{i=0}^{n-1} d(0, \bar{\omega}_i) \qquad n \geq 0,$$

is a P_0-martingale, we can recover through Theorem 1 Sabot's expansion for the velocity [12], under the local drift condition **(LD)**,

$$v = \int d(0,\omega)d\mathbb{Q} = d_0 + \epsilon d_1 + \epsilon^2 d_2^\epsilon + O(\epsilon^{3-\eta}), \tag{1.12}$$

where $d_0 := \sum_{e\in V} ep_0(e)$, $d_1 := \sum_{e\in V} e\mathbb{E}[\xi(0,e)]$ and

$$d_2^\epsilon := \sum_{e\in V}\sum_{e'\in V} C_{e,e'} J_{p_\epsilon^*}(e),$$

where $C_{e,e'} := \mathrm{Cov}(\xi(0,e),\xi(0,e'))$.

The absolute continuity of the invariant measure $\mathbb{Q}$ of Theorem 1 with respect to the law of the environment restricted to finite sets follows from the proof of Theorem 3.1 of Sznitman and Zerner in [16]. In dimensions $d \geq 4$, since Kalikow's condition is satisfied, by a result of Berger, Cohen and Rosenthal [2] (see also a previous result of Bolthausen and Sznitman [4] valid at low disorder), we also know that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, and in dimensions $d \geq 2$, by [11], we know that it is absolutely continuous with respect to $\mathbb{P}$ in every forward half space perpendicular to e_1.

Random perturbations of random walks have been already considered (see [8] for perturbations leading to the Einstein relation and [6, 15, 5, 1] for perturbations of the simple symmetric random walk). In particular, in [15] it is proven that under a condition weaker than **(LD)**, the random walk is ballistic even though it might not satisfy Kalikow's condition, and the expansion presented in [12] and here breaks down. On the other hand, we would like to emphasize that Theorem 1 is one of the first results for the model of random walks in random environment in the ballistic regime, and hence non-reversible, giving explicit quantitative information about the invariant measure of the environmental process.

The proof of Theorem 1 is based on adequate expansions of the Green function of the random walk within the spirit of [12]. Nevertheless, a key ingredient that we have to incorporate here is to obtain an expression for the limiting invariant measure in terms of accumulation points of a Cesàro type average performed at a stopping time with a geometric distribution. This is the content of Proposition 5 in section 4. Furthermore, it is necessary to obtain careful expansions of Green functions of random walks perturbed at multiple points.

In the next section of this article, we will derive a more explicit version of Corollary 2 for the case of perturbations of the simple symmetric random walk in dimension $d = 2$. Then, in section 3, we will give a heuristic explanation of the expansions (1.10) and (1.11) of Theorem 1 and Corollary 2. In section 4, we will derive an expression for the limiting invariant measure in terms of Cesàro averages of the marginal laws of the environmental process up to a stopping time with a geometric distribution. In section 5, we will show how to perturb at a finite number of sites the Green function of the random walk. The results of section 5 will be used to expand a typical term of the expression giving the limiting measure in Proposition 8 of section 6. Using this expansion, Theorem 1 is proved in section 7. Finally, Corollary 2 will be proved in section 8.

2. Random perturbations of the simple symmetric random walk in $d = 2$

Here we will derive explicit numerical expressions for some marginal laws of the invariant measure which will be a consequence of Theorem 1 and Corollary 2, for the case in which the perturbations are done on a simple symmetric random walk, so that $p_0(e) = \frac{1}{2d}$ for all $e \in V$. To simplify notation, we will drop the subindex from J_{p_0} writing instead just $J := J_{p_0}$ for this choice of p_0.

Firstly, let us note that when $d = 2$, the explicit values

$$J(z) = \begin{cases} 0 & \text{for z} = (0,0) \\ -1 & \text{for z} = (0,\pm 1), (\pm 1, 0) \\ -\frac{4}{\pi} & \text{for z} = (1,\pm 1), (\pm 1, 1) \\ \frac{8}{\pi} - 4 & \text{for z} = (0,\pm 2), (\pm 2, 0) \end{cases}$$

can be recursively derived (see McCrea and Whipple [10] or Spitzer [13]). We hence obtain the following corollary from Corollary 2. Here we define $z_0 := (0,0)$ and $z_1 := (0,1)$.

Corollary 3. *Let p_0 be the jump probabilities of a simple symmetric random walk, $\eta > 0$ and $d = 2$. Then, there is an $\epsilon_0 > 0$ such that whenever $\epsilon \le \epsilon_0$, $p_0 \in \mathcal{P}_0$, and $\mathbb{P}$ satisfies the local drift condition* **(LD)** *[c.f. (1.9)], the Radon-Nikodym derivative of the restriction $\mathbb{Q}_{z_0,z_1}$ to $\{z_0, z_1\}$ of the limiting invariant measure $\mathbb{Q}$ with respect to the restriction $\mathbb{P}_{z_0,z_1}$ of $\mathbb{P}$ to $\{z_0, z_1\}$, admits $\mathbb{P}$-a.s. the following expansion*

$$\frac{d\mathbb{Q}_{z_0,z_1}}{d\mathbb{P}_{z_0,z_1}} = 1 - \frac{4}{\pi}\left(\bar{\xi}(z_1,e_1) + \bar{\xi}(z_1,-e_1)\right)\epsilon + \left(\frac{8}{\pi} - 4\right)\bar{\xi}(z_1,e_2)\epsilon + O\left(\epsilon^{2-\eta}\right).$$

In particular, we have that $\mathbb{P}$-a.s.

$$\frac{d\mathbb{Q}_{z_i}}{d\mathbb{P}_{z_i}} = \begin{cases} 1 + O\big(\epsilon^{2-\eta}\big) & \text{if } i = 0 \\ 1 - \left(\frac{4}{\pi}\left(\bar{\xi}(z_1,e_1) + \bar{\xi}(z_1,-e_1)\right) + \left(\frac{8}{\pi} - 4\right)\bar{\xi}(z_1,e_2)\right)\epsilon + O\big(\epsilon^{2-\eta}\big) & \text{if } i = 1. \end{cases}$$

Here, $\left|O\left(\epsilon^{2-\eta}\right)\right| \le c_2'\epsilon^{2-\eta}$, for some constant $c_2' = c_2'(\eta)$ depending only on η.

Similar estimates can be obtained for the marginal law of the limiting invariant measure $\mathbb{Q}$ restricted to other finite subsets of $\mathbb{Z}^2$ using the recursive method presented in [10] (see also [13]) to compute J.

3. Formal derivation of the invariant measure perturbative expansion

Here we will show how one can formally derive the expansion (1.10) of Theorem 1. Given any $p \in \mathcal{P}$, defining a nonvanishing drift $\sum_{e\in V} ep(e) \neq 0$, we define for each $x, y \in \mathbb{Z}^d$ the Green function $g^p(x,y)$ as the expectation of the number of visits to site y of the random walk starting from site x.

Consider a perturbation of an environment p_0 according to (1.5). Let us write the transition kernel of the environmental process [c.f. (1.3)] as

$$R = R_0 + \epsilon A, \tag{3.1}$$

where R_0 is the transition kernel of the deterministic environment p_ϵ [c.f. (1.6)], so that for $f : \Omega \to \mathbb{R}$ we have

$$R_0 f(\omega) := \sum_{e\in V} p_\epsilon(e) f(\theta_e \omega),$$

and

$$Af(\omega) := \sum_{e\in V} \bar{\xi}(0,e) f(\theta_e \omega).$$

The invariant measure $\mathbb{Q}$ satisfies the equality

$$\int R f d\mathbb{Q} = 0, \tag{3.2}$$

for every continuous function $f : \Omega \to \mathbb{R}$. If we assume that the Radon-Nikodym derivative $h := \frac{d\mathbb{Q}}{d\mathbb{P}}$ exists and that it has an analytic expansion

$$h = \sum_{i=0}^{\infty} \epsilon^i h_i,$$

obviously $h_0 = 1$, and substituting this expansion and R [c.f. (3.1)] into (3.2), and matching powers, we conclude that for each $i \geq 0$ one has that

$$h_{i+1} = (R_0^*)^{-1} A^* h_i, \tag{3.3}$$

where for any linear operator L, L^* denotes its adjoint with respect to the measure $\mathbb{P}$. Now, note that

$$(R_0^*)^{-1} f(\omega) = \sum_{z\in\mathbb{Z}^d} g^{p_\epsilon^*}(0,z) f(\theta_z \omega) = \sum_{z\in\mathbb{Z}^d} g^{p_\epsilon}(z,0) f(\theta_z \omega).$$

where we recall that $p_\epsilon^*(e) := p_\epsilon(-e)$ for $e \in V$ [c.f.(1.8)]. Furthermore,

$$A^* f(\omega) := \sum_{e\in V} \bar{\xi}(-e,e) f(\theta_e \omega).$$

From the recursion (3.3), and the fact that $\sum_{e\in V} \bar{\xi}(z,e) = 0$, it follows that

$$h_1(\omega) = \sum_{z\in\mathbb{Z}^d, e\in V} g^{p_\epsilon^*}(0,z)\bar{\xi}(z-e,e) = \sum_{z\in\mathbb{Z}^d, e\in V} \bar{\xi}(z,e) g^{p_\epsilon^*}(0,z+e)$$
$$= \sum_{z\in\mathbb{Z}^d, e\in V} \bar{\xi}(z,e) J_{p_\epsilon^*}(z+e),$$

which is the factor of the first order term in the expansion (1.10).

4. Invariant measure as a geometric Cesàro limit

In analogy with the fact that limit points of Cesàro averages of the environmental process give rise to invariant measures (see for example [7]), here we will show that when such an average is done according to a geometric stopping time, its limit points are still invariant measures.

For each $\delta \in (0,1)$, let us consider the Green function of the random walk before a stopping time τ_δ with geometric distribution of parameter $1-\delta$, independent of the random walk and of the environment, defined for a given environment $\omega \in \Omega$ and sites $x, y \in \mathbb{Z}^d$ as

$$g_\delta^\omega(x,y) := E'_{x,\omega}\left[\sum_{n=0}^{\tau_\delta - 1} 1_y(X_n)\right],$$

where the expectation $E'_{x,\omega}$ is taken both over the random walk and over the random variable τ_δ. Define now the probability measure μ_δ on Ω as the unique probability measure such that for every continuous function $f : \Omega \to \mathbb{R}$ one has that

$$\int f d\mu_\delta = \frac{\sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[g_\delta^\omega(0,x) f(\theta_x\omega)\right]}{\sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[g_\delta^\omega(0,x)\right]}. \tag{4.1}$$

For the following proposition, we do not require the environment of the random walk to be elliptic, nor any other assumption on the environment.

Proposition 4. *Consider a random walk in random environment. Then, each accumulation point of the set of measures $\{\mu_\delta : \delta > 0\}$ [c.f. (4.1)] as δ goes to 1 is an invariant measure of the environmental process.*

Proof. Note that for each $\delta > 0$, as in the proof of Proposition 2 of Sabot [12], one can prove that for every continuous function $f : \Omega \to \mathbb{R}$ the following identity is satisfied

$$\frac{E'_0\left[\sum_{k=1}^{\tau_\delta - 1} f\left(\theta_{X_k}\omega\right)\right]}{E[\tau_\delta]} = \frac{\sum_{y\in\mathbb{Z}^d} \mathbb{E}\left[g_\delta^\omega(0,y) f\left(\theta_y\omega\right)\right]}{\sum_{y\in\mathbb{Z}^d} \mathbb{E}\left[g_\delta^\omega(0,y)\right]}, \tag{4.2}$$

where the notation E'_0 denotes taking the expectation with respect to the annealed law of the random walk and with respect to τ_δ, while E means the expectation with respect to τ_δ. Let μ be an accumulation point of $\{\mu_\delta : \delta > 0\}$ as $\delta \to 1$. Then, there exists a sequence $\{\delta_k : k \geq 1\}$ such that $\lim_{k\to\infty} \delta_k = 1$ and such that $\lim_{k\to\infty} \mu_k := \lim_{k\to\infty} \mu_{\delta_k} = \mu$ weakly. Using the Markov property of the quenched random walk, one can deduce that for all natural m one has that

$$E_{0,\omega}[Rf(\theta_{X_m}\omega)] = R^{m+1} f(\omega),$$

where R is the transition kernel defined in (1.3). Hence, by (4.2) and the definition (1.3) we see that

$$\int Rf\, d\mu_k = \frac{E_0'\left[\sum_{m=1}^{\tau_{\delta_k}-1} Rf(\theta_{X_m}\omega)\right]}{E[\tau_{\delta_k}]} = \frac{\mathbb{E}E\left[\sum_{m=1}^{\tau_{\delta_k}-1} R^{m+1} f(\omega)\right]}{E[\tau_{\delta_k}]}$$
$$= \frac{\mathbb{E}E\left[\sum_{m=1}^{\tau_{\delta_k}-1} R^{m} f(\omega)\right]}{E[\tau_{\delta_k}]} + \frac{1}{E[\tau_{\delta_k}]} \cdot \mathbb{E}E\left[R^{\tau_{\delta_k}} f\right] - \frac{1}{E[\tau_{\delta_k}]} \cdot \mathbb{E}[Rf]$$
$$= \int f\, d\mu_k + \frac{1}{E[\tau_{\delta_k}]} \cdot \mathbb{E}E\left[R^{\tau_{\delta_k}} f\right] - \frac{1}{E[\tau_{\delta_k}]} \cdot \mathbb{E}[Rf].$$

Taking the limit when $k \to \infty$ and using the fact that the last two terms tend to zero as $k \to \infty$ by the boundeness of f and the fact that $\lim_{k\to\infty} E\left[\tau_{\delta_k}\right] = \infty$, we conclude that

$$\int Rf d\mu = \int f d\mu.$$

□

To state the next proposition, we recall some of the so called *ballisticity conditions*, which have been important in the study of random walks in random environments with nonvanishing velocity. Given $\gamma \in (0, 1)$ and $l \in \mathbb{S}^d$, we say that condition $(T)_\gamma$ (see [14]) in direction l is satisfied, if there exists a neighborhood V of l in $\mathbb{S}^d$ such that for every $l' \in V$ one has that

$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log P_0 \left(X_{T_{U_{L,l'}}} \cdot l' < 0 \right) < 0,$$

where $U_{L,l}$ is a slab defined by

$$U_{L,l} := \left\{ x \in \mathbb{Z}^d : -L \leq x \cdot l \leq L \right\}.$$

We now say that condition (T') in direction l is satisfied if $(T)_\gamma$ in direction l is satisfied for every $\gamma \in (0, 1)$. On the other hand, we say that the polynomial condition $(P)_M$ in direction l is satisfied (see [3]) if for all $L \geq c_0$, where

$$c_0 = 2^{3(d-1)} \wedge \exp\left\{ 2 \left(\ln 90 + \sum_{j=1}^{\infty} \frac{\ln j}{2^j} \right) \right\},$$

one has that

$$P_0(X_{T_{B_L}} \cdot l < L) \leq \frac{1}{L^M},$$

where

$$B_L := \left\{ x \in \mathbb{Z}^d : -\frac{L}{2} \leq x \cdot l \leq L, |\pi_l x|_\infty \leq 25L^3 \right\},$$

and $\pi_l x$ is the orthogonal projection of x on the subspace perpendicular to l.

Proposition 5. *Consider a random walk in a uniformly elliptic random environment satisfying the polynomial condition $(P)_M$ for $M \geq 15d+5$. Then, $\mu := \lim_{\delta\to 1^-} \mu_\delta$ exists and is an invariant measure for the environmental process. Furthermore, the law of the environmental process at time n, converges in distribution to μ as $n \to \infty$.*

Proof. Let us first note that by Theorem 1 of [3], the polynomial condition $(P)_M$ with $M \geq 15d+5$ implies condition (T') of [14]. On the other hand, Theorem 3.1 of [16], which is formulated under the assumption that Kalikow's condition is satisfied, is still valid if Kalikow's condition is replaced by condition (T'). Therefore, since $\lim_{\delta\to 1} \tau_\delta = \infty$ in probability, we know by Theorem 3.1 of [16] that there exists an invariant measure μ of the environmental process such that in probability

$$\lim_{\delta\to 1} \frac{1}{\tau_\delta} E_0 \left[\sum_{m=1}^{\tau_\delta - 1} f(\theta_{X_k}\omega) \right] = \int f d\mu.$$

Since $\tau_\delta / E[\tau_\delta]$ converges in distribution to an exponential random variable S of parameter 1, it follows that in distribution

$$\lim_{\delta\to 1} \frac{1}{E[\tau_\delta]} E_0 \left[\sum_{m=1}^{\tau_\delta - 1} f(\theta_{X_k}\omega) \right] = S \int f d\mu.$$

Hence,

$$\lim_{\delta\to 1} \frac{1}{E[\tau_\delta]} E_0' \left[\sum_{m=1}^{\tau_\delta - 1} f(\theta_{X_k}\omega) \right] = \int f d\mu,$$

which proves the claim.

□

5. Green function expansion

To prove Theorem 1, we will extend the method presented by Sabot in [12], starting with perturbative estimates for the Green function of the random walk. To do this, we need first the following lemma, which we will use several times. We recall the definition of the ellipticity constant given in (1.2).

Lemma 6. *For each $\delta \in (0,1)$, $e \in V$, $y, z \in \mathbb{Z}^d$, with $y \neq z$ and $\omega \in \Omega$, we have that*

$$g_\delta^\omega(y,z) \geq \delta\kappa g_\delta^\omega(y, z+e). \tag{5.1}$$

Proof. It is enough to note that for all $y, z \in \mathbb{Z}^d$ one has that

$$g_\delta^\omega(y,z) = \delta_{y,z} + \delta \sum_{e\in V} g_\delta^\omega(y, z+e)\,\omega(z+e, e),$$

and then use the fact that the environment is uniformly elliptic with ellipticity constant κ. □

The main result of this section is the following lemma which extends Lemma 1 of [12] for perturbations at one site of the Green function, to perturbation at multiple sites.

Lemma 7. *Consider an environment $\omega \in \Omega$. For $B \subset \mathbb{Z}^d$ consider an environment ω^B which is a perturbation of ω in each of the points in B. In particular, we have for each $e \in V$ that*

$$\omega^B(x,e) := \begin{cases} \omega(x,e) & \text{if} \quad x \notin B, \\ \omega(x,e) + \Delta_x\omega(e) & \text{if} \quad x \in B, \end{cases}$$

for some $\{\Delta_x\omega(e) : e \in V\} \in (-1,1)^V$ for each $x \in B$. Let $0 < \delta < 1$. Then, for each $\delta \in (0,1)$ and $y, y' \in \mathbb{Z}^d$, one has that

$$\left| g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \right| \leq c_3\, g_\delta^{\omega^B}(y,y') \tag{5.2}$$

and

$$\begin{aligned} &\Bigg| g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \\ &- \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e)\left[\delta g_\delta^\omega(x+e,y') - g_\delta^\omega(x,x)\right] \Bigg| \\ &\leq \frac{\left(2d \sup_{e\in V, x\in B} |\Delta_x\omega(e)|\right)^2}{\kappa^3} \left[1 + \frac{n-1}{(\delta\,\kappa)^{\rho(B)}}\right](1+c_3)n\ g_\delta^{\omega^B}(y,y'), \end{aligned} \tag{5.3}$$

where

$$c_3 := \frac{2dn \sup_{e\in V, x\in B} |\Delta_x\omega(e)|}{\kappa^2}\left[\frac{2d \sup_{e\in V, x\in B} |\Delta_x\omega(e)|}{\kappa^2} + 1\right]^{n-1}, \tag{5.4}$$

and $\rho(B)$ and n, are the diameter and the cardinality of B, respectively.

Proof. Let us denote by B_k the set $\{x_1, \ldots, x_k\}$, with $B_n = B$ and $B_0 = \emptyset$. We can see, as in the proof of Lemma 1 of Sabot [12], that for each $y, y' \in \mathbb{Z}^d$, for all $k = 0, 1, \ldots, n-1$, the following inequality is satisfied

$$\left| g_\delta^{\omega^{B_{k+1}}}(y,y') - g_\delta^{\omega^{B_k}}(y,y') \right| \leq \frac{2d \sup_{e\in V} |\Delta_{k+1}\omega(e)|}{\kappa^2} g_\delta^{\omega^{B_{k+1}}}(y,y'), \tag{5.5}$$

where $\Delta_m = \Delta_{x_m}$ for each $m = 1, \ldots, n$. From this, it is easy to deduce that

$$\left| g_\delta^{\omega^B}(y, y') - g_\delta^\omega(y, y') \right| \le \frac{2d \sup_{e \in V,\, x \in B} |\Delta_x \omega(e)|}{\kappa^2} \sum_{k=0}^{n-1} g_\delta^{\omega^{B_{k+1}}}(y, y'). \tag{5.6}$$

On the other hand, iterating the upper bound (5.5) a finite number of times, one can deduce that for all $k = 1, \ldots, n$, one has that

$$g_\delta^{\omega^{B_k}}(y, y') \le \left[\frac{2d \sup_{e \in V,\, x \in B} |\Delta_x \omega(e)|}{\kappa^2} + 1 \right]^{n-k} g_\delta^{\omega^B}(y, y').$$

Substituting this last upper bound into (5.6) we get (5.2). Meanwhile, in order to prove (5.3), we will use (5.2) and the two following inequalities, which are valid in any environment ω,

$$|\delta g_\delta^\omega(z + e, z) - g_\delta^\omega(z, z)| \le \frac{1}{\kappa}, \quad \forall z \in \mathbb{Z}^d,\ e \in V, \tag{5.7}$$

and

$$|\delta g_\delta^\omega(z + e, y') - g_\delta^\omega(z, y')| \le \frac{1}{\kappa^2} \frac{g_\delta^\omega(z, y')}{g_\delta^\omega(z, z)}, \quad \forall z, y' \in \mathbb{Z}^d,\ e \in V \tag{5.8}$$

(for more details, see Lemma 1 of [12]). Now, through standard Green operator expansions, we can obtain the following second order expansion of $g_\delta^{\omega^B}$,

$$\begin{aligned} g_\delta^{\omega^B}(y, y') - g_\delta^\omega(y, y') - \sum_{x \in B} g_\delta^\omega(y, x) \sum_{e \in V} \Delta_x \omega(e) \left[\delta g_\delta^\omega(x + e, y') - g_\delta^\omega(x, x) \right] \\ = \textstyle\sum_{x \in B} \sum_{e \in V} g_\delta^\omega(y, x) \Delta_x \omega(e) \left(\delta g_\delta^\omega(x + e, x) - g_\delta^\omega(x, x) \right) \\ \times \textstyle\sum_{z \in B} \sum_{e' \in V} \Delta_z \omega(e') \left(\delta g_\delta^{\omega^B}(z + e', y') - g_\delta^{\omega^B}(z, y') \right). \end{aligned} \tag{5.9}$$

Hence, with the help of (5.7),(5.8) and (5.9), we can see that

$$\begin{aligned} &\left| g_\delta^{\omega^B}(y, y') - g_\delta^\omega(y, y') - \sum_{x \in B} g_\delta^\omega(y, x) \sum_{e \in V} \Delta_x \omega(e) \left(\delta g_\delta^\omega(x + e, y') - g_\delta^\omega(x, x) \right) \right| \\ &\le A \textstyle\sum_{x, z \in B} \frac{g_\delta^\omega(y, x) g_\delta^{\omega^B}(z, y')}{g_\delta^{\omega^B}(z, z)} \\ &= A \left(\textstyle\sum_{z \in B} \frac{g_\delta^\omega(y, z) g_\delta^{\omega^B}(z, y')}{g_\delta^{\omega^B}(z, z)} + \sum_{\substack{x, z \in B \\ x \neq z}} \frac{g_\delta^\omega(y, x) g_\delta^{\omega^B}(z, y')}{g_\delta^{\omega^B}(z, z)} \right) \\ &\le A \left(1 + \tfrac{n-1}{(\delta\kappa)^{\rho(B)}} \right) \left(\textstyle\sum_{z \in B} \frac{g_\delta^\omega(y, z) g_\delta^{\omega^B}(z, y')}{g_\delta^{\omega^B}(z, z)} \right), \end{aligned} \tag{5.10}$$

where we define

$$A := \frac{1}{\kappa^3} \left(2d \sup_{e \in V,\, z \in B} |\Delta_z \omega(e)| \right)^2,$$

and where in the last step, for each z we selected a non-random nearest neighbor self-avoiding path from z to x and we used the inequality (5.1) of Lemma 6. Now, with the help of (5.2), for each $z \in B$ one can deduce that

$$\frac{g_\delta^\omega(y,z)}{g_\delta^{\omega^B}(y,z)} \le 1 + c_3, \tag{5.11}$$

where c_3 is defined in (5.4). Thus, we can substitute (5.11) into (5.10) to conclude that

$$\begin{aligned}&\left| g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') - \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e)\left[\delta g_\delta^\omega(x+e,y') - g_\delta^\omega(x,x)\right]\right| \\ &\quad \le \tfrac{1}{\kappa^3}\left(2d\sup_{e\in V,\, z\in B}|\Delta_z\omega(e)|\right)^2 \times \left(1+\tfrac{n-1}{(\delta\kappa)^{\rho(B)}}\right)(1+c_3)n\, g_\delta^{\omega^B}(y,y').\end{aligned}$$

□

6. Local function expansions

In this section we will derive in Proposition 8 which follows, an asymptotic expansion in the perturbation parameter ϵ for certain expectations of a given local function f, involving the Green function of the random walk. In fact, these expectations are with respect to the so called Kalikow environment [9]. Throughout, we fix a transition kernel $p_0 \in \mathcal{P}_0$.

Proposition 8. *Let A be a finite fixed subset of $\mathbb{Z}^d$. Consider a continuous function f defined on $\Omega_{p_0,\epsilon}$, which depends only on sites located at A. Let $\eta > 0$. Then, there exists an $\epsilon_0 > 0$, and a constant $c_4 = c_4(\eta)$, such that for all $0 < \epsilon \le \epsilon_0$, whenever δ is close enough to 1, there is a function h_δ such that for each $y \in \mathbb{Z}^d$ the following identity is satisfied.*

$$\begin{aligned}&\frac{\mathbb{E}\left[g_\delta^\omega(0,y)f(\theta_y\omega)\right]}{\mathbb{E}\left[g_\delta^\omega(0,y)\right]} \\ &= \mathbb{E}[f] + \epsilon\frac{1}{\mathbb{E}\left[g_\delta^\omega(0,y)\right]}\sum_{z\in A}\sum_{e\in V}\mathrm{Cov}\left[\xi(z,e),f\right]J_{p_\epsilon^*}(z+e)\mathbb{E}\left[g_\delta^\omega(0,z+y)\right] \\ &\qquad + \mathbb{E}[h_\delta f]\times O(\epsilon^{2-\eta}),\end{aligned} \tag{6.1}$$

where h_δ satisfies

$$\Big|\mathbb{E}[|h_\delta|f]\Big| \le \mathbb{E}\Big[|f|\Big], \tag{6.2}$$

$|O(\epsilon^{2-\eta})| \le c_4\epsilon^{2-\eta}$ and $J_{p_\epsilon^*}(x)$ is defined in (1.7).

Let us now prove Proposition 8. For each subset $B \subset \mathbb{Z}^d$, we will define the following perturbation of a given environment $\omega \in \Omega$,

$$\omega^B(x,e) := \begin{cases} \omega(x,e) & \text{if} \quad x \notin B \\ p_\epsilon(e) & \text{if} \quad x \in B. \end{cases}$$

Note that trivially we can get

$$\frac{\mathbb{E}\Big[g_\delta^\omega(0,y) f\left(\theta_y \omega\right)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} = \frac{\mathbb{E}\Big[g_\delta^\omega(0,y) \bar{f}\left(\theta_y \omega\right)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} + \mathbb{E}[f], \tag{6.3}$$

where $\bar{f} = f - \mathbb{E}[f]$. Next, using the independence between $g_\delta^{\omega^{A+y}}$ and $f \circ \theta_y$ and the fact that $\bar{f}$ is a centered random variable, we can see that

$$\frac{\mathbb{E}\Big[g_\delta^\omega(0,y) \bar{f}\left(\theta_y \omega\right)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} = \frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y)\Big) \bar{f}\left(\theta_y \omega\right)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}.$$

Thus, using inequality (5.3) of Lemma 7, we can deduce that

$$\begin{aligned} &\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y)\Big) \bar{f}(\theta_y \omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} \\ &= \frac{\epsilon\, \mathbb{E}\Big[\sum_{z \in A+y} \sum_{e \in V} g_\delta^{\omega^{A+y}}(0,z) \bar{\xi}(z,e) \Big(\delta g_\delta^{\omega^{A+y}}(z+e,y) - g_\delta^{\omega^{A+y}}(z,z)\Big) \bar{f}(\theta_y \omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} \\ &\qquad + \frac{\mathbb{E}[g_\delta^\omega(0,y) \bar{f}(\theta_y \omega) \times O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}. \end{aligned} \tag{6.4}$$

where $O_1(\epsilon)$ satisfies the inequality

$$|O_1(\epsilon^2)| \le \frac{8d^2}{\kappa^3} \left[1 + \frac{n-1}{(\delta\, \kappa)^{\rho(A)}}\right] (1 + c_5) n \epsilon^2, \tag{6.5}$$

n is the cardinality of A and here c_5 is defined by (see (1.1), (1.4) and (5.4))

$$c_5 = c_5(d, \kappa, A) := \frac{2dn\epsilon}{\kappa^2} \left[\frac{2d\epsilon}{\kappa^2} + 1\right]^{n-1}.$$

Using now the independence between $\bar{\xi}(z,e)$ for $z \in A+y$ and the Green function $g_\delta^{\omega^{A+y}}$, we can see by (6.4) that

$$\begin{aligned} &\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y)\Big) \bar{f}(\theta_y \omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} \\ &= \epsilon \sum_{\substack{z \in A \\ e \in V}} \operatorname{Cov}\Big[\xi(z,e), f(\omega)\Big] \frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y) \Big(\delta g_\delta^{\omega^{A+y}}(z+y+e,y) - g_\delta^{\omega^{A+y}}(z+y,z+y)\Big)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} \\ &\qquad + \frac{\mathbb{E}[g_\delta^\omega(0,y) \bar{f}(\theta_y \omega) \times O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}. \end{aligned} \tag{6.6}$$

In addition, with the help of (5.2) of Lemma 7, we can thanks to the development of (6.6) conclude that

$$\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y)-g_\delta^{\omega^{A+y}}(0,y)\Big)\bar f(\theta_y\omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}$$

$$= \epsilon\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]\frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big(\delta g_\delta^{\omega^{A+y}}(z+y+e,y)-g_\delta^{\omega^{A+y}}(z+y,z+y)\Big)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,y)\Big]}$$

$$+\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]\frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big(\delta g_\delta^{\omega^{A+y}}(z+y+e,y)-g_\delta^{\omega^{A+y}}(z+y,z+y)\Big)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,y)\Big]}$$

$$\times\frac{\mathbb{E}[g_\delta^\omega(0,y)O_2(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}+\frac{\mathbb{E}[g_\delta^\omega(0,y)\bar f(\theta_y\omega)O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}, \tag{6.7}$$

where

$$|O_2(\epsilon^2)|\le\left(\frac{4dn}{\kappa^2}\left[\frac{2d\epsilon}{\kappa^2}+1\right]^{n-1}\right)\epsilon^2. \tag{6.8}$$

Now, to express the second term of (6.7) in terms of $J_{p_\epsilon^*}$ [c.f. (1.7)], we will require a lemma which is a variation of Lemma 3 of [12]. For $v\in\mathbb{Z}^d$, define

$$\phi^\epsilon(v):=\prod_{i=1}^d\left(\sqrt{\frac{p_\epsilon(-e_i)}{p_\epsilon(e_i)}}\right)^{v_i}, \tag{6.9}$$

where v_i are the coordinates of v. Also, for each $z\in A$, $e\in V$ and $y\in\mathbb{Z}^d$, define

$$J_e^\delta(y,z):=\frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big(\delta g_\delta^{\omega^{A+y}}(z+y+e,y)-g_\delta^{\omega^{A+y}}(z+y,z+y)\Big)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big]}. \tag{6.10}$$

Lemma 9. *Assume that the measure $\mathbb{P}$ satisfies the local drift condition* **(LD)** *[c.f. (1.9)]. Let $\eta>0$. Then there exists a constant $c_6=c_6(\eta)>0$ and $\epsilon_0>0$ such that for each $\epsilon\le\epsilon_0$ we have that for all $z\in A$, $e\in V$ and $y\in\mathbb{Z}^d$ one has that*

$$\overline{\lim_{\delta\to1}}\,|J_e^\delta(y,z)-J_{p_\epsilon^*}(z+e)|\le c_6\phi^\epsilon(z+e)\epsilon^{1-\eta}. \tag{6.11}$$

Proof. We will just give an outline of the proof, stressing the steps where modifications have to be made with respect to the proof of Lemma 3 of [12]. For each $z\in A$, $y\in\mathbb{Z}^d$ we define

$$\tilde{\mathbb{P}}:=\frac{g_\delta^{\omega^A}(0,y+z)}{\mathbb{E}\Big[g_\delta^{\omega^A}(0,y+z)\Big]}\mathbb{P}.$$

Now, using a generalized version of a result of Kalikow [9], stated in [12], we can see that

$$J_e^\delta(y,z)=\delta g_\delta^{\tilde\omega}(z+y+e,y)-g_\delta^{\tilde\omega}(z+y,z+y),$$

where $g_\delta^{\tilde\omega}$ denotes the Green function of Kalikow random walk, defined by its transition probabilities $\tilde\omega(x,e)$ given by

$$\tilde\omega(x,e):=\frac{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^A}(y+z,x)\omega^A(x,e)\Big]}{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^A}(y+z,x)\Big]},$$

for each $x\in\mathbb{Z}^d$ and $e\in V$. Here $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. It is easy to verify that

$$\tilde\omega(x,e)=\begin{cases}\mathbb{E}[\omega(x,e)]+\epsilon\frac{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^A}(y+z,x)\bar\xi(x,e)\Big]}{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^A}(y+z,x)\Big]} & \text{if}\quad x\notin A,\\ \mathbb{E}[\omega(x,e)] & \text{if}\quad x\in A.\end{cases}\tag{6.12}$$

Using twice (5.11), we can deduce that

$$\begin{aligned}\frac{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^A}(y+z,x)\bar\xi(x,e)\Big]}{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^A}(y+z,x)\Big]}&=\frac{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)\bar\xi(x,e)\Big]}{\tilde{\mathbb{E}}\Big[g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)\Big]}+O(\epsilon)\\ &=\frac{\mathbb{E}\Big[g_\delta^{\omega^A}(0,y+z)\,g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)\,\bar\xi(x,e)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^A}(0,y+z)\,g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)\Big]}+O(\epsilon)\\ &=O(\epsilon),\end{aligned}\tag{6.13}$$

where in the last step, we used the independence between $g_\delta^{A\cup\{x\}}$ and $\bar\xi(x,e)$, and the fact that $|O(\epsilon)|\le c_7\epsilon$, where c_7 is a constant, which depends on κ,d and cardinality of A (see (5.4)). Now, from (6.12) and (6.13), we can deduce that for each $x\in\mathbb{Z}^d$ and $e\in V$ the following identity is satisfied

$$\tilde\omega(x,e)=\mathbb{E}[\omega(x,e)]+\epsilon^2\Delta\omega(x,e),$$

where $\Delta\omega(x,e)$ is uniformly bounded in x,e,y,z,δ,ϵ. The following steps of the proof are then identical to steps 2 and 3 of Lemma 3 of [12]. □

We can now continue with the proof of Proposition 8. Note that using the definition (6.10), we can rewrite (6.7) as

$$\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y)-g_\delta^{\omega^{A+y}}(0,y)\Big)\bar f(\theta_y\omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}$$

$$= \epsilon\sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_e^\delta(y,z)\times\frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,y)\Big]}$$

$$+\sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_e^\delta(y,z)\times\frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}\times\frac{\mathbb{E}[g_\delta^\omega(0,y)O_2(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$

$$+\frac{\mathbb{E}[g_\delta^\omega(0,y)\bar f(\theta_y\omega)O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}. \tag{6.14}$$

On the other hand, with the help of (6.11) of Lemma 9, it follows that for each $\eta>0$ we can choose δ_0 such that for $\delta\in(\delta_0,1)$ one has that

$$|J_e^\delta(y,z)-J_{p_\epsilon^*}(z+e)|\le 2c_6\phi^\epsilon(z+e)\epsilon^{1-\eta}. \tag{6.15}$$

Hence, for $\delta\in(\delta_0,1)$ we conclude from (6.14) that

$$\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y)-g_\delta^{\omega^{A+y}}(0,y)\Big)\bar f(\theta_y\omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}$$

$$= \epsilon\sum_{z\in A}\sum_{e\in V}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\times\frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,y)\Big]}$$

$$+O_3(\epsilon^{2-\eta})\sum_{z\in A}\sum_{e\in V}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]$$

$$+\sum_{z\in A}\sum_{e\in V}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\ \frac{\mathbb{E}[g_\delta^\omega(0,y)O_4(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$

$$+\sum_{z\in A}\sum_{e\in V}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]\ \frac{\mathbb{E}[g_\delta^\omega(0,y)O_5(\epsilon^{3-\eta})]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$

$$+\frac{\mathbb{E}[g_\delta^\omega(0,y)\bar f(\theta_y\omega)O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}. \tag{6.16}$$

where we have used the fact that the expression $\frac{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,z+y)\Big]}{\mathbb{E}\Big[g_\delta^{\omega^{A+y}}(0,y)\Big]}$ can be bounded by $\frac{1}{(\delta\,\kappa)^{\rho(A)}}$ choosing a non-random nearest neighbor self-avoiding path from y to $z+y$ and using (5.1) of Lemma 6 at most $\rho(A)$ times, and where

$$|O_3(\epsilon^{2-\eta})|\le\frac{2c_6c_8\epsilon^{2-\eta}}{(\delta\kappa)^{\rho(A)}},\quad |O_4(\epsilon^2)|\le\frac{\left(\frac{4dn}{\kappa^2}\left[\frac{2d\epsilon}{\kappa^2}+1\right]^{n-1}\right)}{(\delta\kappa)^{\rho(A)}}\epsilon^2, \tag{6.17}$$

$$|O_5(\epsilon^{3-\eta})| \leq 2\frac{\left(\frac{4dn}{\kappa^2}\left[\frac{2d\epsilon}{\kappa^2}+1\right]^{n-1}\right)}{(\delta\kappa)^{\rho(A)}}c_6c_8\epsilon^{3-\eta}. \tag{6.18}$$

and

$$c_8 := \sup_{z\in A,\, e\in V} |\phi^\epsilon(z+e)|.$$

In addition, if we use again (5.2) of Lemma 7, we can say that

$$\frac{\mathbb{E}\Big[{g_\delta^\omega}^{A+y}(0,z+y)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,z+y)\Big]} = 1 + \frac{\mathbb{E}\Big[g_\delta^\omega(0,z+y)O_6(\epsilon)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,z+y)\Big]}, \tag{6.19}$$

and

$$\frac{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}{\mathbb{E}\Big[{g_\delta^\omega}^{A+y}(0,y)\Big]} = \frac{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big(1+O_7(\epsilon)\Big)\Big]}, \tag{6.20}$$

provided that for each $\tau > 0$ one has that

$$|O_i(\tau)| \leq c_3\tau, \quad \forall i = 6,7. \tag{6.21}$$

Using (6.19) and (6.20) for the first term of the right-hand side of (6.16) and using once more (6.15), we see that for $\delta \geq \delta_0$ one has that

$$\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y)-{g_\delta^\omega}^{A+y}(0,y)\Big)\bar{f}(\theta_y\omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}$$

$$= \epsilon\frac{1}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]}\sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}\,\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\mathbb{E}\Big[g_\delta^\omega(0,z+y)\Big]$$

$$+\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\frac{\mathbb{E}\Big[O_8(\epsilon^2)g_\delta^\omega(0,z+y)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,z+y)\Big]}$$
$$-\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\frac{\mathbb{E}\Big[O_9(\epsilon^2)g_\delta^\omega(0,y)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big(1+O_7(\epsilon)\Big)\Big]}$$
$$-\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\frac{\mathbb{E}\Big[O_8(\epsilon^2)g_\delta^\omega(0,z+y)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,z+y)\Big]}\frac{\mathbb{E}\Big[O_9(\epsilon)g_\delta^\omega(0,y)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big(1+O_7(\epsilon)\Big)\Big]}$$
$$+O_3(\epsilon^{2-\eta})\textstyle\sum_{z\in A}\sum_{e\in V}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]$$
$$+\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]J_{p_\epsilon^*}(z+e)\ \frac{\mathbb{E}[g_\delta^\omega(0,y)O_4(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$
$$+\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}\Big[\xi(z,e),f(\omega)\Big]\ \frac{\mathbb{E}[g_\delta^\omega(0,y)O_5(\epsilon^{3-\eta})]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$
$$+\frac{\mathbb{E}[g_\delta^\omega(0,y)\bar f(\theta_y\omega)O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]},\tag{6.22}$$

where, by (6.21), we know that for each $\tau>0$

$$|O_i(\tau)|\le\frac{c_3}{(\delta\kappa)^{\rho(A)}}\tau,\quad\forall i=8,9.\tag{6.23}$$

Defining

$$h_1:=\sum_{\substack{z\in A\\ e\in V}}\bar\xi(z,e)J_{p_\epsilon^*}(z+e)\times\frac{\mathbb{E}\Big[O_8(\epsilon^2)g_\delta^\omega(0,z+y)\Big]}{E\Big[g_\delta^\omega(0,z+y)\Big]}.$$
$$h_2:=-\sum_{\substack{z\in A\\ e\in V}}\bar\xi(z,e)J_{p_\epsilon^*}(z+e)\times\frac{\mathbb{E}\Big[O_9(\epsilon^2)g_\delta^\omega(0,y)\Big]}{E\Big[g_\delta^\omega(0,y)(1+O_7(\epsilon))\Big]}.$$

$$h_3:=-\sum_{\substack{z\in A\\ e\in V}}\bar\xi(z,e)J_{p_\epsilon^*}(z+e)\times\frac{\mathbb{E}\Big[O_8(\epsilon^2)g_\delta^\omega(0,z+y)\Big]}{E\Big[g_\delta^\omega(0,z+y)\Big]}\times\frac{\mathbb{E}\Big[O_9(\epsilon)g_\delta^\omega(0,y)\Big]}{E\Big[g_\delta^\omega(0,y)(1+O_7(\epsilon))\Big]}.$$
$$h_4:=\sum_{\substack{z\in A\\ e\in V}}\bar\xi(z,e)O_3(\epsilon^{2-\eta}).$$

$$h_5 := \sum_{\substack{z\in A\\ e\in V}} \bar\xi(z,e) J_{p_\epsilon^*}(z+e) \times \frac{\mathbb{E}\Big[O_4(\epsilon^2) g_\delta^\omega(0,y)\Big]}{E\Big[g_\delta^\omega(0,y)\Big]}.$$

$$h_6 := \sum_{\substack{z\in A\\ e\in V}} \bar\xi(z,e) \times \frac{\mathbb{E}\Big[O_5(\epsilon^{3-\eta}) g_\delta^\omega(0,y)\Big]}{E\Big[g_\delta^\omega(0,y)\Big]}$$

$$h_7 := \frac{g_\delta^\omega(-y,0)\tilde O_1(\epsilon^2)}{\mathbb{E}[g_\delta^\omega(0,y)]}, \quad \text{with} \quad \tilde O_1(\epsilon^2)(\omega) := O_1(\epsilon^2)\Big(\theta_{(-y)}\omega\Big).$$

$$\text{and} \quad h_8 := -\frac{\mathbb{E}[g_\delta^\omega(0,y) O_1(\epsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]},$$

we can rewrite (6.22) in the following way

$$\begin{aligned} &\frac{\mathbb{E}\Big[\Big(g_\delta^\omega(0,y) - {g_\delta^\omega}^{A+y}(0,y)\Big)\bar f(\theta_y\omega)\Big]}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} \\ &= \epsilon \frac{1}{\mathbb{E}\Big[g_\delta^\omega(0,y)\Big]} \sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}\Big[\xi(z,e), f(\omega)\Big] J_{p_\epsilon^*}(z+e)\mathbb{E}\Big[g_\delta^\omega(0,z+y)\Big] \\ &\qquad + E[h_\delta f], \end{aligned} \tag{6.24}$$

where $h_\delta := \sum_{i=1}^8 h_i$. Now, in order to show that (6.2) is satisfied, it is necessary to justify that for any $z \in A$ and $e \in V$, $J_{p_\epsilon^*}(z+e)$ is bounded. If we use (6.5), (6.8), (6.17), (6.18), (6.21), (6.23) and the fact that for each $e \in V$ and $z \in A$, $\bar\xi(z,e)$ is bounded by 2, it is easy to deduce that there exists $\epsilon_0 > 0$ and an constant $c_9 = c_9(\eta)$ such that for all $0 < \epsilon \le \epsilon_0$, whenever δ is close enough to 1, the following inequality is satisfied for all $1 \le i \le 8$,

$$\Big|\mathbb{E}[|h_i|f]\Big| \le \Big|O(\epsilon^{2-\eta})\Big|\mathbb{E}[|f|], \quad \text{with} \ \Big|O(\epsilon^{2-\eta})\Big| \le c_9\epsilon^{2-\eta}. \tag{6.25}$$

In the case of h_7, if we apply independence and use (5.4), (5.11) and (6.5) , one can deduce that for $0 < \epsilon \le \epsilon_0$ exists a constant $c_{10} > 0$ such that

$$\begin{aligned} \Big|\mathbb{E}[|h_7|f]\Big| &\le \frac{1}{\mathbb{E}[g_\delta^\omega(0,y)]} \times \mathbb{E}\left[{g_\delta^\omega}^{A+y}(0,y)\cdot|f(\theta_y\omega)|\cdot\frac{g_\delta^\omega(0,y)}{{g_\delta^\omega}^{A+y}(0,y)}|O_1(\epsilon^2)|\right] \\ &\le \Big|O(\epsilon^2)\Big|\mathbb{E}[|f|], \end{aligned} \tag{6.26}$$

where $|O(\epsilon^2)| \le c_{10}\epsilon^2$. Finally, with the help of (6.25) and (6.26), Proposition 8 is easily proven.

7. Proof of Theorem 1

In this section we will prove Theorem 1. Note that, with the help of (4.1) and Proposition 8, there exists an $\epsilon_0 > 0$ and a constant c_4 such that for δ close enough to 1, for all $y \in \mathbb{Z}^d$ and $0 < \epsilon \le \epsilon_0$ we have that

$$\int f d\mu_\delta$$

$$= \frac{\sum_{y\in\mathbb{Z}^d}\left(\mathbb{E}[g_\delta^\omega(0,y)]\mathbb{E}[f]+\epsilon\sum_{\substack{z\in A\\e\in V}}\mathrm{Cov}\left[\xi(z,e),f(\omega)\right]J_{p_\epsilon^*}(z+e)\mathbb{E}\left[g_\delta^\omega(0,z+y)\right]+\mathbb{E}[g_\delta^\omega(0,y)]\mathbb{E}[h_\delta f]\right)}{\sum_{y\in\mathbb{Z}^d}\mathbb{E}[g_\delta^\omega(0,y)]}$$

$$= \mathbb{E}[f] + \epsilon\sum_{\substack{z\in A\\e\in V}}\mathrm{Cov}\left[\xi(z,e),f(\omega)\right]J_{p_\epsilon^*}(z+e) + \mathbb{E}[h_\delta f]\times O\left(\epsilon^{2-\eta}\right),$$

where

$$|\mathbb{E}[|h_\delta|f]| \le \mathbb{E}[|f|], \tag{7.1}$$

and $|O(\epsilon^{2-\eta})| \le c_4\epsilon^{2-\eta}$. Taking now the limit when $\delta\to 1^-$, by Proposition 5 we conclude that

$$\int f d\mathbb{Q} = \mathbb{E}[f] + \epsilon\sum_{z\in A,e\in V}\mathrm{Cov}\left[\xi(z,e),f(\omega)\right]J_{p_\epsilon^*}(z+e) + \int f d\mathbb{V},$$

where by (7.1), $\mathbb{V}$ is a signed measure satisfying

$$\left|\int f d|\mathbb{V}|\right| \le \left|O(\epsilon^{2-\eta})\right|\mathbb{E}[|f|],$$

where $|\mathbb{V}|$ is the variation of $\mathbb{V}$. Hence, the restriction of $\mathbb{Q}$ to A is absolutely continuous with respect to $\mathbb{P}_A$, from where we can conclude that there is a function h such that $\mathbb{E}[|h|]<\infty$, and such that for every continuous function f one has that

$$|\mathbb{E}[f|h|]| \le \mathbb{E}[|f|] \tag{7.2}$$

and

$$\int f d\mathbb{Q} = \mathbb{E}[f] + \epsilon\sum_{z\in A,e\in V}\mathrm{Cov}\left[\xi(z,e),f(\omega)\right]J_{p_\epsilon^*}(z+e) + \mathbb{E}[fh]\times O\left(\epsilon^{2-\eta}\right)$$

Now, noting that any function f in L_1 can be approximates by continuous functions, it is easy to check that in fact (7.2) is satisfied for every $f\in L_1$. Therefore, h is bounded, from where we conclude the proof of Theorem 1.

8. Proof of Corollary 2

Here we prove Corollary 2. It is enough to show that $J_{p_\epsilon^*}$ [c.f. (1.7)] is well approximated by $J_{p_0^*}$. By standard Fourier inversion formulas (see for example Spitzer [13]) we can conclude that for each $z\in\mathbb{Z}^d$ and $e\in V$ one has that

$$
\begin{aligned}
&J_{p^*_\epsilon}(z+e)\\
&= \tfrac{1}{(2\pi)^d}\left(\prod_{j=1}^d \left(\tfrac{p^\epsilon(-e_j)}{p^\epsilon(e_j)}\right)^{\frac{z_j+e_j}{2}} - 1\right) \int_{[0,2\pi]^d} \frac{\cos\left(\sum_{j=1}^d (z_j+e_j)x_j\right)}{1-2\sum_{j=1}^d \sqrt{p^\epsilon(e_j)p^\epsilon(-e_j)}\,\cos(x_j)} \prod dx_j\\
&\quad + \tfrac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{\cos\left(\sum_{j=1}^d (z_j+e_j)x_j\right)-1}{1-2\sum_{j=1}^d \sqrt{p^\epsilon(e_j)p^\epsilon(-e_j)}\,\cos(x_j)} \prod dx_j, \qquad (8.1)
\end{aligned}
$$

where for each $1 \le j \le d$, z_j is the j-th coordinates of z. When $p_0 = 0$, we can conclude from (8.1) that

$$
J_{p^*_\epsilon}(z+e) = \begin{cases} J_{p^*_0}(z+e) + O(\epsilon\log\epsilon) & \text{if } \quad d=2\\ J_{p^*_0}(z+e) + O(\epsilon) & \text{if } \quad d\ge 3, \end{cases}
$$

For the case $p_0 \neq 0$, a simpler estimation gives us that for any dimension $d \ge 2$ one has that

$$
J_{p^*_\epsilon}(z+e) = J_{p^*_0}(z+e) + O(\epsilon).
$$

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