Quenched invariance principle for random walk in time-dependent balanced random environment

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Abstract

We prove a quenched central limit theorem for balanced random walks in time dependent ergodic random environments. We assume that the environment satisfies appropriate ergodicity and ellipticity conditions. The proof is based on the use of a maximum principle for parabolic difference operators.

1 Introduction

We consider balanced random walks in time-dependent elliptic random environments. Under a mild ergodicity assumption on the law of the environment and a moment condition on the jump probabilities of the random walk, we prove a quenched central limit theorem (QCLT). Our results extend previous results of Lawler [L82] and of Guo and Zeitouni [GZ10] and are based on the use of a new maximum principle for parabolic difference operators. Furthermore, they can be considered as a version of discrete homogenization of stochastic parabolic operators in non-divergence form without uniform ellipticity (for homogenization results in a similar PDE settings, we refer to [AS14, Lin15]).

We state our results in both a continuous and a discrete time setting.

1.1 Discrete time RWRE

Let U be any finite subset of $\mathbb{Z}^d$ which will be called the *jump range*. Let $\kappa \geq 0$ and define $\mathcal{P} = \mathcal{P}(\kappa) := \{v = \{v(e) : e \in U\} : \inf_{e\in U} v(e) > \kappa, \sum_{e\in U} v(e) = 1\}$. Consider a discrete time stochastic process $\omega := \{\omega_n : n \in \mathbb{N}\}$ with state space $\Omega := \mathcal{P}^{\mathbb{Z}^d}$, so that $\omega_n := \{\omega_n(x) : x \in \mathbb{Z}^d\}$ with $\omega_n(x) := \{\omega_n(x, e) : e \in U\} \in \mathcal{P}$. We call $\Omega^{\mathbb{N}}$ the *environmental space* while an element

$$\omega \in \Omega^{\mathbb{N}} \tag{1.1}$$

a *discrete time environment*. Note that throughout this construction the set U is fixed and is not dependent on ω. Let us denote by $\mathbb{P}$ the corresponding law of the process defined on the space $\Omega^{\mathbb{N}}$ and $\mathbb{E}_{\mathbb{P}}$ its expectation. Given $\omega \in \Omega^{\mathbb{N}}$, $x \in \mathbb{Z}^d$ and $n \in \mathbb{Z}$ consider the random walk $\{X_m : m \geq 0\}$ with a law $P_{x,n,\omega}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ defined through $P_{x,n,\omega}(X_n = x) = 1$ and the transition probabilities

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$$P_{x,n,\omega}(X_{n+k+1} = y + e | X_{n+k} = y) = \omega_{n+k}(y, e),$$

for $k \geq 0$, $y \in \mathbb{Z}^d$ and $e \in U$. We call this process a *discrete time random walk in time-dependent random environment* and call $P_{x,n,\omega}$ the *quenched law* of the random walk starting from x at time n. When $\kappa = 0$ we will say that the environment is *elliptic*, while if $\kappa > 0$ that it is *uniformly elliptic*. Whenever the environment ω is time independent so that $\omega_n = \omega_{n+1}$ for $n \geq 0$, we will call the above random walk a *discrete time random walk in static random environment* and its corresponding law starting from ω at site x, just $P_{x,\omega}$.

Given any topological space T, we will denote by $\mathcal{B}(T)$ the corresponding Borel sets. Throughout we will make the following ergodicity assumption. Note that we do not demand the environment to be necessarily ergodic under time shifts. For $x \in \mathbb{Z}^d$ and $n \geq 0$ define the transformation $\theta_{n,x} : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ by $(\theta_{n,x}\omega)_m(y) := \omega_{n+m}(y + x)$ for all $m \geq 0$ and $y \in \mathbb{Z}^d$. Throughout, we will assume that $\mathbb{P}$ is stationary under the action of the $\{\theta_{n,x} : n \geq 0, x \in \mathbb{Z}^d\}$. Let now $Z \subset \{(n,e) : n \geq 1, e \in U\}$. We will say that $\{\theta_{n,e} : (n,e) \in Z\}$ is an ergodic family of transformations acting on the space $(\Omega^{\mathbb{N}}, \mathcal{B}(\Omega^{\mathbb{N}}), \mathbb{P})$ or an *ergodic family of transformations with respect to* $\mathbb{P}$, if whenever $A \in \mathcal{B}(\Omega^{\mathbb{N}})$ is such that $\theta_{n,e}^{-1}A = A$ for every $(n,e) \in Z$, then $\mathbb{P}(A)$ is 0 or 1. Note that $\{\theta_{n,e} : (n,e) \in Z\}$ is an ergodic family of transformations with respect to $\mathbb{P}$ if and only if $\{\theta_{n,e} : (n,e) \in \langle Z \rangle\}$ is an ergodic family of transformations with respect to $\mathbb{P}$, where $\langle Z \rangle$ is the subset of $\mathbb{N} \times \mathbb{Z}^d$ whose elements can be expressed as a finite sum of elements of Z.

In the case of a static random environment, note that for $x \in \mathbb{Z}^d$, we can drop the time subscript from $\theta_{n,x}$, and write simply $\theta_x : \Omega \to \Omega$, which is defined as $(\theta_x\omega)(y) := \omega(x + y)$ for $y \in \mathbb{Z}^d$. In this case we will say that $\{\theta_x : x \in Z\}$ is an ergodic family of transformations acting on the space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ or an *ergodic family of transformations with respect to* $\mathbb{P}$, if whenever $A \in \mathcal{B}(\Omega)$ is such that $\theta_z^{-1}A = A$ for every $z \in \mathbb{Z}^d$, then $\mathbb{P}(A)$ is 0 or 1.

Define the *environment viewed from the random walk* or *environmental process* as the discrete time process

$$\bar{\omega}_n := \theta_{n,X_n}\omega$$

for $n \geq 0$, with state space $\Omega^{\mathbb{N}}$ and initial state $\omega \in \Omega^{\mathbb{N}}$. We call P_ω the corresponding law and

$$E_\omega \tag{1.2}$$

its expectation. In general, if ω is distributed according to some law μ, we define $P_\mu := \int P_\omega d\mu$. We will say that μ is an invariant measure for the environmental process if the law of $\bar{\omega}_n$ under P_μ is independent of n for $n \geq 0$.

Let $D \subset U$. We say that a random walk in random environment with law $\mathbb{P}$ is *balanced in the directions in* D if for every $x \in \mathbb{Z}^d$ and $n \geq 0$

$$\mathbb{P}\left(\sum_{e \in D} e\omega_n(x,e) = 0\right) = 1.$$

If a random walk is balanced in the directions in U, we will just say that it is *balanced*.

Let us now recursively define the range of the random walk after n steps as $U_1 := U$, while for $n \geq 1$

$$U_{n+1} := \left\{y \in \mathbb{Z}^d : y = x + e \text{ for some } x \in U_n \text{ and } e \in U\right\}.$$

Let also for $n \geq 1$ and $x, y \in \mathbb{Z}^d$

$$p_n(x,y) := P_{x,0,\omega}(X_n = y).$$

Now set

$$V_n(x) := \{p_n(x, x+z)z : z \in U_n\} \tag{1.3}$$

Throughout, given a subset $V \subset \mathbb{R}^d$, we will denote by *conv*(V) its convex hull and by $|V|$ its volume. Define for $n \geq 1$

$$\varepsilon_n(x) := \left(p_n(x,x)\,|conv(V_n(x))|\right)^{1/(d+1)}. \tag{1.4}$$

Denote also by $\{e_1, \ldots, e_d\}$ the canonical generators of $\mathbb{Z}^d$.

Theorem 1.1 *Consider a discrete time balanced random walk in a time dependent random environment with law $\mathbb{P}$. Suppose that the family of shifts $\{\theta_{1,e} : e \in U\}$ is ergodic and that*

$$\inf_{n\geq 1} \mathbb{E}_{\mathbb{P}}\left[\varepsilon_n^{-(d+1)}(0)\right] < \infty. \tag{1.5}$$

Then, the following are satisfied.

(i) The environmental process has a unique invariant probability measure ν which is absolutely continuous with respect to $\mathbb{P}$.

(ii) $\mathbb{P}$-a.s., under $P_{0,0,\omega}$, the sequence $X_{[n\cdot]}/\sqrt{n}$ converges in law to a Brownian motion with non-degenerate covariance matrix $A = \{a_{i,j} : 1 \leq i, j \leq d\}$, where

$$a_{i,j} := \sum_{e\in U}(e\cdot e_i)(e\cdot e_j)\int \omega_0(0,e)d\nu.$$

Theorem 1.1 extends the static version of the quenched central limit theorem proved by Lawler [L82] for uniformly elliptic environments and by Guo and Zeitouni [GZ10] for elliptic environments. Other recent related results for random walks in balanced static environments include Berger and Deuschel [BD14] and Baur [Ba14]. On the other hand, it should be pointed out that several results exist proving quenched central limit theorem for random walks in time-dependent environments, but in general under mixing condition which are stronger that our ergodicity assumption (see for example [DKL08]).

Remark 1.1 *For time-independent random environment, a similar criteria as* (1.5) *for $n = 1$ for QCLT is obtained by Guo and Zeitouni [GZ10], where no stay-put is required, i.e, $p_1(x,x) = 0$. However, in the time-dependent case, if there is not enough stay-put, then the random walker may not "feel" certain parts of the environment due to the parity of the space-time, which destroys our QCLT. See Section 7 for a counterexample.*

Remark 1.2 *The non-degeneracy of the matrix A of part* (ii) *of Theorem 1.1 follows from the fact that for any vector $u = (u_1, \ldots, u_d)$ one has that*

$$u\cdot Au = \sum_{e\in U}(e\cdot u)^2\int \omega_0(0,e)d\nu > 0,$$

since the vectors of U generate $\mathbb{R}^d$ and ν is absolutely continuous with respect to $\mathbb{P}$, so that A is a positive definite matrix.

Remark 1.3 *Condition (1.5) of Theorem 1.1 is always satisfied if the environment is uniformly elliptic ($\kappa > 0$) and the range U of the walk satisfies $|conv(U)| > 0$. Indeed, since the environment is balanced, we see that $0 \in conv(U)$, so that for some set of non-negative λ_e, $e \in U$, not all equal to 0, we have that $\sum_{e\in U}\lambda_e e = 0$. Moreover, the coefficients λ_e can always be chosen as integer numbers, as all $e \in U$ have integer coordinates. Therefore, the random walk returns to the origin after $N = \sum_{e\in U}\lambda_e$ steps with a probability larger than κ^N, so that $p_N(0) \geq \kappa^N$. On the other hand, the fact that $|conv(U)| > 0$ implies that $|conv(V_N)| > 0$, c.f. (1.3), is also bounded by some positive constant so that ε_N, c.f. (1.4), is bounded from below by some positive constant.*

1.2 Continuous time RWRE

We can also formulate a continuous time version of Theorem 1.1. Recall that U is a finite subset of $\mathbb{Z}^d$. Define

$$\mathcal{Q} := \{v = \{v(e) : e \in U\} : 0 < \inf_{e \in U} v(e)\}.$$

Note that we do not assume any upper bound on the transition rates $v(\cdot) \in \mathcal{Q}$. Consider a continuous time stochastic process

$$\omega := \{\omega_t : t \geq 0\} \in D([0,\infty); \mathfrak{H})$$

with state space $\mathfrak{H} \subset \mathcal{Q}^{\mathbb{Z}^d}$ so that $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ with $\omega_t(x) := \{\omega_t(x,e) : e \in U\} \in \mathcal{Q}$. We call ω a *continuous time environment*. Let us denote by $\mathbb{Q}$ the law of this process defined on $D([0,\infty); \mathfrak{H})$. For a given environment $\omega \in D([0,\infty); \mathfrak{H})$, consider the continuous time random walk defined by the generator

$$L_s f(x) := \sum_{e \in U} \omega_s(x,e)(f(x+e) - f(x)),$$

where $s \geq 0$ and $f : \mathbb{Z}^d \to \mathbb{Z}^d$ is bounded. We call this process a *continuous time random walk in a time-dependent random environment* and denote for each $x \in \mathbb{Z}^d$ by $P^c_{x,t,\omega}$ the law on $D([0,\infty); \mathbb{Z}^d)$ of this random walk starting from x at time t. We call $P^c_{x,t,\omega}$ the *quenched law* of the random walk starting from x.

For the continuous time environment, we will make the following ergodicity assumption. For each $s \geq 0$ and $x \in \mathbb{Z}^d$ define the transformation

$$\theta_{s,x} : D([0,\infty); \mathfrak{H}) \to D([0,\infty); \mathfrak{H})$$

by $(\theta_{s,x}\omega)_t(y) := \omega_{t+s}(x+y)$. We assume that the law of the environment $\mathbb{Q}$ is stationary under the action of the shifts $\{\theta_{s,x} : s \geq 0 : x \in U\}$. We say that the shifts $\{\theta_{s,x} : s > 0, x \in U\}$ form an ergodic family of transformations acting on the space $(D([0,\infty); \mathfrak{H}), \mathcal{B}(D([0,\infty); \mathfrak{H})), \mathbb{Q})$ or an *ergodic family of transformations with respect to* $\mathbb{Q}$ if whenever $A \in \mathcal{B}(D([0,\infty); \mathfrak{H}))$ is such that $\theta^{-1}_{s,x}A = A$ for every $s > 0$ and $x \in U$, then $\mathbb{Q}(A)$ is 0 or 1.

We will say that the continuous time random walk in time-dependent random environment with law $\mathbb{Q}$ is *balanced* if for every $t \geq 0$, $x \in \mathbb{Z}^d$ we have that

$$\sum_{e \in U} e\omega_t(x,e) = 0 \qquad \mathbb{Q} - a.s. \tag{1.6}$$

As in the discrete-time case, we can also define the *environment viewed from the random walk* or the *environmental process* as the continuous time process

$$\bar{\omega}_t := \theta_{t,X_t}\omega$$

for $t \geq 0$. We will assume that it also has state space $\mathfrak{H}$ so that it is a stochastic process defined in $D([0,\infty); \mathfrak{H})$ with starting point $\omega \in D([0,\infty); \mathfrak{H})$. Call P^c_ω the corresponding law. In general, if ω is distributed according to some law μ, we define $P^c_\mu := \int P^c_\omega d\mu$. We will say that μ is an invariant measure for the environmental process if the law of $\bar{\omega}_t$ under P^c_μ is independent of t for $t \geq 0$.

For each $(x,t) \in \mathbb{Z}^d \times [0,\infty)$, let

$$U_{x,t} = \{\omega_t(x,e)e : e \in U\} \tag{1.7}$$

and

$$\varepsilon(x,t) = \varepsilon_\omega(x,t) := (|conv(U_{x,t})|)^{1/(d+1)}, \tag{1.8}$$
$$\upsilon(x,t) = \upsilon_\omega(x,t) := \sum_{e \in U} \omega_t(x,e).$$

We will denote by $\mathbb{E}_{\mathbb{Q}}$ the expectation with respect to the law $\mathbb{Q}$ of the environment and write $\varepsilon = \varepsilon(0,0)$ and $\upsilon = \upsilon(0,0)$.

Theorem 1.2 *Consider a continuous time balanced random walk in time-dependent random environment with law $\mathbb{Q}$. Suppose that the family of shifts $\{\theta_{s,x} : s > 0, x \in U\}$ is ergodic. Assume that*

$$E_{\mathbb{Q}}\left[\frac{\upsilon^{d+1}+1}{\varepsilon^{d+1}}\right] < \infty, \tag{1.9}$$

Then, the following are satisfied.

(i) The environmental process has a unique invariant measure which is absolutely continuous with respect to $\mathbb{Q}$.

(ii) $\mathbb{Q}$-a.s. under $P^c_{0,0,\omega}$ the sequence $X_{t\cdot}/\sqrt{t}$ converges, as $t \to \infty$, in law on the Skorokhod space $D([0,\infty);\mathbb{R}^d)$ to a Brownian motion with non-degenerate diagonal covariance matrix.

1.3 Applications of the QCLTs

Theorem 1.2 can be applied to derive quenched central limit theorems for balanced random walks driven by some interacting particle systems. An example of this situation is a random walk moving among the zero-range process. Given a function $g : \mathbb{N} \to [0,\infty)$ satisfying $g(k) > g(0) = 0$ for all $k > 0$, the zero-range process can be constructed as a Markov process describing the movement of particles on the lattice $\mathbb{Z}^d$, so that if at a site $x \in \mathbb{Z}^d$ and time $t \geq 0$ there are $\eta_t(x)$ particles, a particle jumps uniformly to the nearest neighboring sites of x at a rate $g(\eta_t(x))$. The infinitesimal generator L of this interacting particle system is defined by its action on functions $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ depending on a finite number of coordinates of $\eta = \{\eta(x) : x \in \mathbb{Z}^d\} \in \mathbb{N}^{\mathbb{Z}^d}$ by

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}^d:|x-y|_1=1} g(\eta(x))(f(\eta^{x,y}) - f(\eta)),$$

where

$$\eta^{x,y}(z) := \begin{cases} \eta(x) - 1 & \text{if} \quad \mathrm{z = x} \\ \eta(y) + 1 & \text{if} \quad \mathrm{z = y} \\ \eta(z) & \text{if} \quad \mathrm{z \neq x, y} \end{cases}$$

Under the condition

$$\sup_{k\in\mathbb{N}} |g(k+1) - g(k)| < \infty,$$

this process is well defined whenever the initial condition $\eta \in S$, where

$$S := \{\eta \in \mathbb{N}^{\mathbb{Z}^d} : \sum_{x\in\mathbb{Z}^d} \eta(x)\alpha(x) < \infty\}, \tag{1.10}$$

and

$$\alpha(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} p_n(0,x),$$

with $p_n(0,x)$ the probability that a discrete time simple symmetric random walk starting from 0 is at position x at time n (see [A82] for this construction). The above process is called *zero-range process*, and we will denote by P_η its law on $D([0,\infty);S)$ starting from $\eta \in S$. This process has a family of invariant measures (see also [A82]) defined through the partition function $Z : [0,\infty) \to [0,\infty)$ by

$$Z(\alpha) = \sum_{k\geq 0} \frac{\alpha^k}{g(1)\cdots g(k)}.$$

Define

$$\alpha^* := \sup_{\alpha\in[0,\infty)} \{\alpha : Z(\alpha) < \infty\}.$$

Assume also that

$$\lim_{\alpha \to \alpha^*} Z(\alpha) = \infty.$$

Now, for each $0 \le \alpha < \alpha^*$ define the product probability measure μ_α on $\mathbb{N}^{\mathbb{Z}^d}$ with its Borel σ-algebra, with marginals given by

$$\mu_\alpha(k) = \frac{1}{Z(\alpha)} \frac{\alpha^k}{g(1)\cdots g(k)}.$$

As a matter of fact $\mu_\alpha(S) = 1$, so that we can define the measure

$$P_\alpha := \int P_\eta d\mu_\alpha(\eta).$$

Let us assume that the function g is non-decreasing, so that for each $0 \le \alpha < \alpha^*$, the invariant measure μ_α is also extremal [A82]. Therefore, for $\alpha \in [0, \alpha^*)$, the shifts $\{\theta_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ form an ergodic family of transformations with respect to P_α. For each $e \in \{e_1, \ldots, e_d\}$, fix a finite range function $u(e, \cdot) : S \to (0, \infty)$: in other words, there is an $R > 0$ such that for each $\eta \in S$, $u(e, \eta)$ depends only on $\eta(x)$ for $|x|_1 \le R$. Define for $e \in \{e_1, \ldots, e_d\}$,

$$u(-e, \cdot) := u(e, \cdot)$$

Now, define the stochastic process for $t \ge 0$, $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ where for $x \in \mathbb{Z}^d$, we define

$$\omega_t(x) := \{u(e, \theta_x \eta_t) : e \text{ such that } |\mathrm{e}|_1 = 1\}.$$

Let us call $\omega := \{\omega_t : t \ge 0\}$ a *an environment generated by an attractive zero-range process.* We then have the following immeadiate corollary to Theorem 1.2.

Corollary 1.1 *Consider a continuous time random walk $\{X_t : t \ge 0\}$ in an environment ω generated by an attractive zero-range process, with law P_α for some $\alpha \in [0, \alpha^*)$. Assume that*

$$\int \frac{\left(\sum_{e \in U} u(e, \eta)\right)^{d+1} + 1}{\prod_{i=1}^d u(e_i, \eta)} d\mu_\alpha < \infty. \tag{1.11}$$

Then P_α-a.s. the random walk $\{X_t : t \ge 0\}$ satisfies a quenched central limit theorem with nondegenerate covariance matrix.

The condition (1.11) is satisfied in the case in which the function u is bounded from above and below. Furthemore, Corollary 1.1 includes the case of a second class particle in the zero-range process, solved by Saada in [Sa90], where nevertheless the main problem we face here, which is the construction of the invariant measure, is not present.

Theorem 1.1 can be applied to derive quenched central limit theorems for a certain class of random walks in static random environment. In order to give a simple example, we will consider random walks in $\mathbb{Z}^2$. Given $x \in \mathbb{Z}^2$, we will write $x = (y, z)$, so that y and z are the coordinates of x. Consider then a two-dimensional random walk $\{X_n = (Y_n, Z_n) : n \ge 0\}$ in static random environment with a law $\mathbb{P}$ and jump range $U = \{e \in \mathbb{Z}^d : |e|_1 = 1\} = \{e_1, e_2, -e_1, -e_2\}$, where for $x \in \mathbb{Z}^2$, $|x|_1 := |y| + |z|$ denotes its l_1-norm. Let us note that given $\omega \in \Omega$ and $X_n = x = (y, z)$, the process $\{Y_n : n \ge 0\}$ will jump from $Y_n = y$ to $Y_n = y + e$ with $e = e_1, -e_1$ or 0, with the following probabilities,

$$\omega_Y(x, e) := \begin{cases} \omega(x, e_1) & \text{if} \quad e = e_1 \\ \omega(x, -e_1) & \text{if} \quad e = -e_1 \\ \omega(x, e_2) + \omega(x, -e_2) & \text{if} \quad e = 0. \end{cases} \tag{1.12}$$

Similarly, given $\omega \in \Omega$ and $X_n = x = (y, z)$, the process $\{Z_n : n \ge 0\}$ will jump from $Z_n = z$ to $Z_n = y + e$ with $e = e_2, -e_2$ or 0, with the following probabilities,

$$\omega_Z(x,e) := \begin{cases} \omega(x,e_2) & \text{if} \quad e = e_2 \\ \omega(x,-e_2) & \text{if} \quad e = -e_2 \\ \omega(x,e_1) + \omega(x,-e_1) & \text{if} \quad e = 0. \end{cases} \tag{1.13}$$

Given an environment ω, we will say that $\{Y_n : n \geq 0\}$ is a *autonomous random walk in a static random environment* if for every $x_1 = (y_1, z_1)$ and $x_2 = (y_2, z_2) \in \mathbb{Z}^d$ with $y_1 = y_2$ and $e \in \{e_1, -e_1, 0\}$ one has that

$$\bar{\omega}_Y(y_1, e) := \omega_Y(x_1, e) = \omega_Y(x_2, e). \tag{1.14}$$

In other words, the jump probabilities (1.12) do not depend on the z coordinate of the site $x = (y, z)$. Note that $\bar{\omega}_Y := \{\bar{\omega}_Y(y) : y \in \mathbb{Z}\}$ with $\bar{\omega}_Y(y) := \{\bar{\omega}_Y(y,e) : e \in \{e_1, -e_1, 0\}\}$ defines an environment in $\mathbb{Z}$.

For $x = (y, z) \in \mathbb{Z}^2$ define the projections $\pi_1, \pi_2 : \mathbb{Z}^2 \to \mathbb{Z}$ by $\pi_1 x = y$ and $\pi_2 x = z$. We say that an environment $\omega \in \Omega$ defines a *random walk with coordinates with independent jumps* if for every $x \in \mathbb{Z}^d$ and $e \in U$ one has that

$$\omega(x, e) = \omega_Y(x, \pi_1 e)\omega_Z(x, \pi_2 e). \tag{1.15}$$

Now, given a law $\mathbb{P}$ defined on the environmental space Ω, we will say that $\mathbb{P}$ defines a *autonomous random walk in a static random environment* for the first coordinate if for $\mathbb{P}$-a.a. $\omega \in \Omega$ this is satisfied. Similarly, we will say that $\mathbb{P}$ defines a *random walk with coordinates with independent jumps* if for $\mathbb{P}$-a.a. $\omega \in \Omega$ this happens. Now, suppose that under $\mathbb{P}$, $\{Y_n : n \geq 0\}$ is a autonomous random walk in a static environment. For each $y \in \mathbb{Z}$, we define $\hat{y} \in \mathbb{Z}^2$ by the relations

$$\pi_1 \hat{y} = y \tag{1.16}$$

$$\pi_2 \hat{y} = 0. \tag{1.17}$$

We will say that *there is an absolutely continuous invariant measure for the random walk* $\{Y_n : n \geq 0\}$ if there exists a probability measure $\mathbb{Q}$ which is absolutely continuous with respect to $\mathbb{P}$ and such that for every bounded and continuous $g : \Omega \to \mathbb{R}$ one has that

$$\int g(\omega) d\mathbb{Q} = \sum_{e \in \{e_1, -e_1, 0\}} \int \omega_Y(0, e) g(\theta_{\hat{e}}\omega) d\mathbb{Q}.$$

Corollary 1.2 *Consider a random walk $\{X_n = (Y_n, Z_n) : n \geq 0\}$ in a static uniformly elliptic random environment with law $\mathbb{P}$. Assume that $\mathbb{P}$ is translation invariant and ergodic under the action of the shifts $\theta_{(e_1,0)}$ and $\theta_{(-e_1,0)}$. Suppose that the following are satisfied.*

- *(a)* **(autonomous first coordinate)** *Under $\mathbb{P}$, $\{Y_n : n \geq 0\}$ is a autonomous random walk.*
- *(b)* **(coordinates with independent jumps)** *The random walk $\{X_n : n \geq 0\}$ has coordinates with independent jumps under $\mathbb{P}$.*
- *(c)* **(absolutely continuous invariant measure for the first coordinate of the random walk)** *There is an absolutely continuous invariant measure with respect to $\mathbb{P}$ for the random walk $\{Y_n : n \geq 0\}$.*
- *(d)* **(quenched CLT for the first coordinate of the random walk)** *$\mathbb{P}$-a.s., the random walk $\{Y_n : n \geq 0\}$ satisfies a quenched central limit theorem, so that there is a a deterministic v_1 such that $\mathbb{P}$-a.s. under $P_{0,\omega}$, the sequence $(Y_{[n\cdot]} - nv_1)/\sqrt{n}$ converges in law to a Brownian motion with non-degenerate covariance matrix.*
- *(e)* **(balancedness of the second coordinate of the random walk)** *The random walk $\{Z_n : n \geq 0\}$ is balanced.*

Then the following are satisfied.

(i) The environment seen from the random walk $\{X_n\}$ has an invariant measure which is absolutely continuous with respect to $\mathbb{P}$, and hence satisfies a law of large numbers with velocity v.

(ii) $\mathbb{P}$-a.s. under $P_{0,\omega}$, the sequence $(X_{[n\cdot]} - nv)/\sqrt{n}$ converges in law to a Brownian motion with non-degenerate covariance matrix.

1.4 Organization of the article

Since the proofs of Theorem 1.1 and 1.2 are similar, most of the subsequent sections of this paper will give the details of the proof of the discrete time Theorem 1.1, while an outline of the proof of the continuous time Theorem 1.2 is given in section 5. In section 2.1, we state the version of Kozlov's theorem that will be used to construct the absolutely continuous invariant measure for the discrete time random walk. In subsection 2.2, the parabolic maximum principle for general meshes is stated while its proof is deferred to subsection 4. Both Kozlov theorem and the parabolic maxima inequality are subsequently used in section 3 to prove Theorem 1.1. Corollary 1.2 is proved in section 6. In section 7, we give an example that the ergodicity hypothesis of Theorem 1.1 cannot be changed to a weaker kind of total ergodicity.

2 Two preliminary tools

Here we state two theorems that will be used to prove Theorem 1.1. The first is a version of a well known theorem of Kozlov for time dependent random walks, while the second is the parabolic maximum principle for general meshes, whose proof is given in Section 4. The parabolic maximum principle is a crucial tool to construct the absolutely continuous invariant measure of part (i) of Theorem 1.1, while Kozlov's theorem is required to derive part (ii) of the same theorem.

2.1 Kozlov's theorem

The proof of Theorem 1.1 will require the following version of Kozlov's theorem [Ko85] for time dependent random walks.

Theorem 2.1 *Consider a random walk in a time-dependent elliptic random environment which has a law $\mathbb{P}$. Assume that $\{\theta_{1,z} : z \in U\}$ is an ergodic family of transformations with respect to $\mathbb{P}$. Assume that there exists an invariant measure ν for the environmental process, which is absolutely continuous with respect to $\mathbb{P}$. Then, the following are satisfied:*

(i) ν is equivalent to $\mathbb{P}$.

(ii) The environment as seen from the random walk with initial law ν is ergodic.

(iii) ν is the unique probability measure for the environmental process which is absolutely continuous with respect to $\mathbb{P}$.

Proof. Since the proof is similar to the case of random walks in static random environments, we will stress the steps which are different (see [BS02] for example for a proof of the theorem for static random walk in random environment).

To prove part (i), let f be the Radon-Nikodym derivative of ν with respect to $\mathbb{P}$ and define $E := \{f = 0\}$. We will prove that $\mathbb{P}(E) = 0$. Using the fact that ν is invariant, we can conclude that $\mathbb{P}$-a.s. for every $z \in U$,

$$1_E(\omega) \geq \sum_{z' \in U} \omega_0(0, z') 1_E(\theta_{1,z'}\omega) \geq \omega_0(0, z) 1_E(\theta_{1,z}\omega).$$

From the ellipticity assumption and the fact that $1_E(\omega)$ and $1_E(\theta_{1,z}\omega)$ only take the values 0 and 1 we see that for each $z \in U$, $\mathbb{P}$-a.s.

$$1_E(\omega) \geq 1_E(\theta_{1,z}\omega).$$

Now since $\mathbb{P}(E) = \mathbb{P}(\theta_{1,z}^{-1}E)$, we conclude that for each $z \in U$, $\mathbb{P}$-a.s.

$$1_E = 1_{\theta_{1,z}^{-1}E}. \tag{2.18}$$

By a similar reasoning we conclude that for every $n \geq 0$ and $z \in U_n$,

$$1_E = 1_{\theta_{n,z}^{-1}E}.$$

Now define

$$\tilde{E} := \cap_{m=0}^{\infty} \cup_{k=m}^{\infty} \cup_{y \in U_k} \theta_{k,y}^{-1} E.$$

Note that for every $z \in U$, we have that

$$\tilde{E} = \theta_{1,z}^{-1} \tilde{E}.$$

Since $\mathbb{P}$ is ergodic with respect to the family of space-time shifts $\{\theta_{1,z} : z \in U\}$, and $\tilde{E}$ differs from E on a set of measure 0, we conclude that

$$\mathbb{P}(E) = \mathbb{P}(\tilde{E}) \in \{0, 1\}.$$

But $\int_{E^c} f d\mathbb{P} = \int f d\mathbb{P} = 1$ implies that $\mathbb{P}(E^c) > 0$, so that necessarily $\mathbb{P}(E) = 0$.

To prove part (ii) as in the static case (see [BS02]) it is possible to prove that for every $A \in \mathcal{B}(\Gamma^{\mathbb{N}})$ such that $T^{-1}A = A$, where $T : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the one step time shift on the space of environments, the process

$$\phi(\bar{\omega}_n) := P_{\bar{\omega}_n}(A),$$

is a martingale and also there is a set $B \in \mathcal{B}(\Gamma)$ such that

$$\phi = 1_B.$$

We then show that $\mathbb{P}$-a.s. for each $z \in U$, the inequality

$$1_B(\omega) \geq \sum_{z' \in U} \omega_0(0, z') 1_B(\theta_{1,z'}\omega) \geq \omega_0(0, z) 1_B(\theta_{1,z}\omega)$$

is satisfied. Using an argument similar to the one employed in part (i) we now see that $\nu(B) \in \{0, 1\}$, which proves that $P_\nu(A) = \nu(B) \in \{0, 1\}$.

The uniqueness of ν stated in part (iii) can be obtained following exactly the same argument as in the static case [BS02].

∎

2.2 A new maximum principle for parabolic difference operators on general meshes

The quenched central limit theorem for balanced random walks in static random environments [L82] can be proved using lattice versions of the maximum principles for elliptic operators of Aleksandrov-Bakel'man [A63, B61] (see also [P66]) for elliptic partial differential equations (see Papanicolaou and Varadhan [PV82] for an application to prove a QCLT for diffusions with random coefficients .). The maximum principle for elliptic difference operators were proved by Kuo and Trudinger in a series of papers (see for example [KT90]).

Nevertheless, to prove Theorem 1.1, we will need a parabolic maximum principle. Within the context of diffusions, this was first established by Krylov [Kr76], and subsequently a discrete version for general meshes proved by Kuo and Trudinger in [KT93, KT95, KT98]. Here we state a new parabolic maximum principle, Theorem 2.2, for difference operators and prove it in section 4 using a geometric approach.

We firstly introduce some notation. Given $x \in \mathbb{Z}^d$, we denote by $|x|_2$ its l_2 norm. For $x_0 \in \mathbb{Z}^d$, $R > 0$, let

$$B_R(x_0) := \{x \in \mathbb{Z}^d : |x - x_0|_2 \leq R\},$$

Consider a balanced time dependent environment $a = \{a_n : n \geq 0\} \in \Omega^{\mathbb{N}}$ (c.f. (1.1)). For any finite subset $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{Z}$, define its *parabolic boundary* by

$$\mathcal{D}^p := \{(x,n) \notin \mathcal{D} : \quad a_{n+1}(y, x-y) > 0 \text{ for some } (y, n+1) \in \mathcal{D}\}.$$

Define the parabolic operator

$$\mathcal{L}_a u(x,n) := \sum_{z \in U} a_n(x,z)(u(x+z, n-1) - u(x, n-1)) - (u(x,n) - u(x,n-1)).$$

For a real function g defined on $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{Z}$ and $p > 0$ define

$$||g||_{\mathcal{D},p} := \left(\sum_{(x,n) \in \mathcal{D}} |g(x,n)|^p \right)^{1/p}. \tag{2.19}$$

Set,

$$U_{x,n} := \{a_n(x,z)z : z \in U\}$$

$$v(x,n) := |conv(U_{x,n})|$$

and define

$$\varepsilon_a(x,n) := \left(a_n(x,0) \frac{v(x,n)}{|U|} \right)^{1/(d+1)}.$$

Theorem 2.2 *Assume that $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{Z}$ is a finite set and $\mathcal{D} \cup \mathcal{D}^p \subset B_R \times [0,T]$ for some $R, T > 0$. Let u be a function on $\mathcal{D} \cup \mathcal{D}^p$. If u satisfies*

$$\mathcal{L}_a u \geq -f \qquad \text{in} \quad \mathcal{D} \tag{2.20}$$

for some function f on $\mathcal{D}$, then

$$\max_{\mathcal{D}} u \leq \max_{\mathcal{D}^p} u + C R^{d/(d+1)} \|f/\varepsilon_a\|_{\mathcal{D},d+1},$$

where $C = C(U,d)$ is a constant.

Remark 2.1 *The elliptic version of Theorem 2.2 was implicitly obtained in [KT00, (44)]. However, there is a minor gap in its proof. That is, [KT00, Lemma 3] is not true for general non-symmetric convex bodies. This can be fixed by symmetrization (and using the balanced assumption), see* (4.32).

3 Proof of theorem 1.1

It is easy to check, as in the case of random walks in static balanced random environments, that part (i) of Theorem 1.1 implies, through Theorem 2.1, part (ii) (see [L82]). We therefore will concentrate on the proof of part (i).

Throughout this section, we fix a balanced environment $\omega \in \Omega$, so that for all $x \in \mathbb{Z}^d$ and $n \geq 0$,

$$\sum_{e \in U} e\omega_n(x,e) = 0$$

Let N be an even natural number. By (1.5), we know that there is a k such that the random walk returns to its starting point after k steps and such that

$$\mathbb{E}_{\mathbb{P}} \left[\varepsilon_k^{-(d+1)} \right] < \infty. \tag{3.21}$$

Let us first assume that $k = 1$. We introduce for $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ the equivalence classes

$$\overline{(x,n)} := (x,n) + (2N+1)\mathbb{Z}^d \times (N^2+1)\mathbb{Z}. \tag{3.22}$$

In addition we define the periodized version $\omega^{(N)}$ of ω so that $\omega_m^{(N)}(y) = \omega_n(x)$ for $(y,m) \in \mathbb{Z}^d \times \mathbb{Z}$ and

$$(x,n) \in B_N := \{z \in \mathbb{Z}^d : |z|_\infty \le N\} \times \{n' : 1 \le n' \le N^2 + 1\}$$

such that $\overline{(y,m)} = \overline{(x,n)}$. Set

$$\Omega_{N,\omega} = \Omega_N := \{\theta_{n,x}\omega^{(N)} : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}.$$

It is straightforward to see that the process $(\theta_{n,X_n}\omega^{(N)})_{n\ge 0}$ is a Markov chain with a finite state space Ω_N and has an invariant measure $\nu_N \ll \mathbb{P}_N$, where

$$\mathbb{P}_N := \frac{1}{(N^2+1)} \frac{1}{(2N+1)^d} \sum_{(x,n)\in B_N} \delta_{\theta_{n,x}\omega^{(N)}}.$$

Although it will not be used in this proof, note that the invariant measure ν_N is of the form

$$\nu_N = \sum_{(x,n)\in B_N} \phi_N(x,n)\delta_{\theta_{n,x}\omega^{(N)}},$$

where ϕ_N is also the density of the invariant measure of the random walk $\{\overline{(X_n,n)} : n \ge 0\}$, c.f. (3.22), on B_N with periodic boundary conditions, induced by $\{X_n : n \ge 0\}$. Using this fact, we can write explicitly the equation solved by ϕ_N, from where it is possible to use analytic tools to derive properties of ν_N (see [E00]).

Note that since $\mathbb{P}$ is ergodic under the action of $\{\theta_{1,x} : x \in U\}$ which is a subset of the transformations generated by $\theta_{1,0}$ and $\theta_{0,x}, x \in U$, by the multidimensional ergodic theorem (see [DF88, Theorem VIII.6.9]),

$$\lim_{N\to\infty} \mathbb{P}_N = \mathbb{P} \qquad \mathbb{P} - a.s.$$

Define the stopping times $\tau_0 = 0$,

$$\tau_{j+1} = \inf\{i > \tau_j : |X_i - X_{\tau_j}|_\infty > N \text{ or } i - \tau_j > N^2\}, \quad j \ge 0, \tag{3.23}$$

Lemma 3.1 *There exists a constant $c_2 > 0$ such that for all $c \ge c_2$, there is an N_0 such that for $N \ge N_0$ we have that*

$$\sup_{x\in\mathbb{Z}^d, n\ge 0, \xi\in\Omega_N} E_{x,n,\xi}\left[\left(1 - \frac{c}{N^2}\right)^{\tau_1}\right] \le \frac{1}{2}. \tag{3.24}$$

Proof: The proof follows the lines of [GZ10, Lemma 4]. Since $\{X_n : n \ge 0\}$ is a martingale, by Doob's martingale inequality, for any $1 \le K < N^2$,

$$\begin{aligned} P_{x,n,\xi}[\tau_1 \le K] &\le 2\sum_{i=1}^d P_{x,n,\xi}(\max_{0<n\le K} X_n(i)^+ > N) \\ &\le \frac{2}{N}\sum_{i=1}^d E_{x,n,\xi}[X_K^+(i)] \le \frac{2dC_U\sqrt{K}}{N}, \end{aligned}$$

where $X_n(i)$ denotes the i-th coordinate of X_n and $C_U = \max\{|e| : e \in U\}$. Hence for every $c > 0$ we have that

$$E_\xi\left[\left(1 - \frac{c}{N^2}\right)^{\tau_1}\right] \le \left(1 - \frac{c}{N^2}\right)^K + \frac{2dC_U\sqrt{K}}{N}.$$

Taking $K = (\frac{N}{8dC_U})^2$ it follows that for c large enough whenever N is large enough then inequality (3.24) is satisfied. ∎

In accordance with the notation of (1.2), we denote by E_ξ the expectation associated to the law P_ξ of the environmental process starting from ξ, c.f. (1.2). Denote by $S = S_{\omega^{(N)}}$ the transition semigroup of the environment Markov chain $\{\xi_n : n \geq 0\}$ defined for each $\xi \in \Omega_N$ as $\xi_n := \theta_{n,X_n}\xi$ for $n \geq 0$. That is, for every function g on Ω_N and $\xi \in \Omega_N$, for $k \geq 0$ define

$$S^k g(\xi) := E_\xi \left[g\left(\xi_k\right)\right], \quad \forall k \in \mathbb{N}.$$

Since ν_N is the invariant measure for the environment Markov chain, we have

$$\int g d\nu_N = \int S^k g d\nu_N, \quad \forall k \in \mathbb{N}.$$

Let c_2 be the constant as in Lemma 3.1. Putting $\rho = \rho(\omega, N) := 1 - \frac{c_2}{N^2} \in (0,1)$, we see that

$$\begin{aligned} (1-\rho)^{-1} \textstyle\int g d\nu_N &= \textstyle\sum_{k=0}^{\infty} \rho^k \int S^k g d\nu_N \\ \leq \max_{\xi \in \Omega_N} \textstyle\sum_{k=0}^{\infty} \rho^k S^k g(\xi) &= \max_{\xi\in\Omega_N} E_\xi \left[\textstyle\sum_{k=0}^{\infty} \rho^k g(\xi_k)\right]. \end{aligned} \tag{3.25}$$

On the other hand, for c large enough we have that

$$\begin{aligned} \max_{\xi\in\Omega_N} E_\xi \left[\textstyle\sum_{k=0}^{\infty} \rho^k g(\xi_k)\right] &\leq \textstyle\sum_{m=0}^{\infty} \max_{\xi\in\Omega_N} E_\xi \left[\rho^{\tau_m} \sum_{k\in[\tau_m,\tau_{m+1})} g(\xi_k)\right] \\ &\leq \textstyle\sum_{m=0}^{\infty} \left(\max_{\xi\in\Omega_N} E_\xi[\rho^{\tau_1}]\right)^m \max_{\xi\in\Omega_N} E_\xi \left[\sum_{k=0}^{\tau_1 - 1} g(\xi_k)\right] \\ \leq \tfrac{1}{1-\max_{\xi\in\Omega_N} E_\xi[\rho^{\tau_1}]} \max_{\xi\in\Omega_N} E_\xi \left[\textstyle\sum_{k=0}^{\tau_1-1} g(\xi_k)\right] &\leq 2 \max_{\xi\in\Omega_N} E_\xi \left[\textstyle\sum_{k=0}^{\tau_1-1} g(\xi_k)\right] \end{aligned} \tag{3.26}$$

where in the last and second to last inequality we have used the fact that for c large enough, we can apply inequality (3.24) of Lemma 3.1. Recall that B_N^p denotes the parabolic boundary of B_N. Now, for any $(x,n) \in B_N \cup B_N^p$ and $\xi \in \Omega_N$, define $E_\xi^{(x,n)} := E_{\theta_{x,n}\xi}$ and

$$f_\xi(x,n) := E_{(x,N^2+1-n,\xi)} \left[\sum_{k=0}^{\tau-1} g(\xi_k)\right]$$

where $\tau = \inf\{i \geq 0 : |X_i| > N \text{ or } i > N^2\}$. Then f_ξ satisfies

$$\begin{cases} \mathcal{L}_a f_\xi(x,n) = -G_\xi(x,n), & \text{if } (x,n) \in B_N \\ f_\xi(x) = 0, & \text{if } (x,n) \in B_N^p, \end{cases}$$

where $G_\xi(x,n) := g(\theta_{x,N^2+1-n}\xi)$. We can now apply Theorem 2.2 to conclude that

$$\begin{aligned} \max_{\xi\in\Omega_N} f_\xi(x,n) = \max_{\xi\in\Omega_N} E_\xi \left[\sum_{0\leq k\leq \tau-1} g(\xi_k)\right] &= \max_{\xi\in\Omega_N} f_\xi(0, N^2+1) \\ &\leq \max_{\xi\in\Omega_N} C N^{d/(d+1)} \|G_\xi/\varepsilon_1\|_{B_N,d+1} \\ &= C N^2 \|g/\varepsilon_1\|_{L^{d+1}(\mathbb{P}_N)}, \end{aligned} \tag{3.27}$$

where the norm $||\cdot||_{B_N,d+1}$ is defined in (2.19). Therefore, combining (3.25), (3.26) and (3.27), we conclude that for some constant $C > 0$,

$$\int g d\nu_N \leq C \|g/\varepsilon_1\|_{L^{d+1}(\mathbb{P}_N)}.$$

Using the compactness of Ω and Prohorov's theorem, we can extract a subsequence ν_{N_k} of ν_N which converges weakly to some limit ν as $k \to \infty$. Then, by the ergodic theorem and the assumption $E_{\mathbb{P}}[1/\varepsilon_1^{d+1}] < \infty$, c.f. (3.21), we would conclude that

$$\int g d\nu \leq C \|g/\varepsilon_1\|_{L^{d+1}(\mathbb{P})} \quad \text{for any bounded measurable function } g \text{ on } \Omega.$$

Note that the above inequality implies that ν is absolutely continuous with respect to the probability measure μ defined by

$$d\mu := \frac{1}{\mathbb{E}_{\mathbb{P}}\left[\varepsilon_1^{-(d+1)}(0,0)\right]}\frac{1}{\varepsilon_1^{d+1}(0,0)}d\mathbb{P}.$$

Since μ is by definition absolutely continuous with respect to $\mathbb{P}$, we conclude that $\nu \ll \mathbb{P}$. Now, note also that Theorem 2.1 ensures that ν is unique.

In the case in which (3.21) is satisfied for $k > 1$, by the same argment as in the case in which $k = 1$, we can construct an invariant measure ν_k which is absolutely continuous with respect to $\mathbb{P}$, for the environmental process looked at times which are multiples of k, defined for $n \geq 0$ by

$$\bar{\omega}_n^{(k)} := \theta_{nk,X_{nk}}\omega.$$

We will now show how to construct from ν_k an invariant measure ν which is absolutely continuous with respect to $\mathbb{P}$, for the environmental process $\{\bar{\omega}_n : n \geq 0\}$. Define for every bounded and continuous function g, the measure ν by

$$\int g d\nu := \sum_{i=0}^{k-1}\int R^i g d\nu_k,$$

where $R : \Omega \to \Omega$ is defined for $\omega \in \Omega$ by

$$Rg(\omega) := \sum_{e\in U}\omega_0(0,e)\theta_{1,X_1}\omega$$

Then note that

$$\int Rg d\nu = \sum_{i=1}^{k}\int R^i g d\nu_k = \int g d\nu + \int g d\nu - \int R^k g d\nu = \int g d\nu,$$

where the last equality is a consequence of the invariance of ν_k. This proves that ν is an invariant measure for the environmental process. To see that ν is absolutely continuous with respect to $\mathbb{P}$ note that for each measurable A in Ω, with $\mathbb{P}(A) = 0$, we have that

$$\int R^i 1_A d\nu_k \leq \sum_{z\in U_k}\nu_k(\theta_{k,z}^{-1}A) = 0,$$

since the stationarity of $\mathbb{P}$ implies that $\mathbb{P}(\theta_{k,z}^{-1}A) = 0$ which in turn implies by the fact that $\nu_k \ll P$ that $\nu_k(\theta_{k,z}^{-1}A) = 0$. Therefore we conclude that $\nu(A) = 0$ and hence that ν is absolutely continuous with respect to $\mathbb{P}$.

4 Proof of the maximum principle

Here we will prove the maximum principle in Theorem 2.2. Define

$$M := \max_{\mathcal{D}} u.$$

Without loss of generality assume that $M > 0$, $\max_{\mathcal{D}^p} u \leq 0$ and that $f \geq 0$. For each $(x,n) \in \mathcal{D}$ define

$$I_u(x,n) := \left\{p \in \mathbb{R}^d : u(x,n) - u(y,m) \geq p\cdot(x-y) \text{ for all } (y,m) \in \mathcal{D}\cup\mathcal{D}^p \text{ with } m < n\right\}.$$

Let also

$$\Gamma = \Gamma(u,\mathcal{D}) := \{(x,n) \in \mathcal{D} : I_u(x,n) \neq \emptyset\},$$

$$\Gamma^+ = \Gamma^+(u,\mathcal{D}) := \{(x,n) \in \Gamma : R|p|_2 < u(x,n) - p\cdot x \text{ for some } p \in I_u(x,n)\}$$

and

$$\Lambda := \left\{(\xi, h) \in \mathbb{R}^d \times \mathbb{R} : R|\xi|_2 < h < \frac{M}{2}\right\} \subset \mathbb{R}^{d+1}.$$

For $(x, n) \in \mathcal{D}$ define the set

$$\chi(x, n) := \{(p, q - x \cdot p) : p \in I_u(x, n), q \in [u(x, n-1), u(x, n)]\} \subset \mathbb{R}^{d+1}.$$

Step 1. We will first show that

$$\Lambda \subset \chi(\Gamma^+) = \bigcup_{(x,n)\in\Gamma^+} \chi(x, n). \tag{4.28}$$

Indeed, let $(\xi, h) \in \Lambda$, and define for $(x, n) \in \mathcal{D}$,

$$\phi(x, n) := u(x, n) - \xi \cdot x - h.$$

Let $(x_0, n_0) \in \mathcal{D}$ be such that $u(x_0, n_0) = M$. Then, by the definition of Λ, we see that $\phi(x_0, n_0) > 0$ and

$$\phi(x, n) < 0,$$

for $(x, n) \in \mathcal{D}^p$. We now claim that there exists $(x_1, n_1) \in \Gamma^+$ with $n_1 \le n_0$ such that $\phi(x_1, n_1) \ge 0$ and $(\xi, h) \in \chi(x_1, n_1)$. Indeed, for $x \in B_R$, let

$$N_x := \min\{n : (n, x) \in \mathcal{D} \text{ and } \phi(x, n) \ge 0\}$$

and

$$n_1 := \min_{x\in B_R} N_x \le n_0 \le T,$$

with the convention $\min \emptyset = \infty$. Let $x_1 \in B_R$ be such that $n_1 = N_{x_1}$. Thus, for all $(x, n) \in \mathcal{D} \cup \mathcal{D}^p$ with $n \le n_1$

$$u(x, n) - \xi \cdot x < h \le u(x_1, n_1) - \xi \cdot x_1.$$

Hence $\xi \in I_u(x_1, n_1) \ne \emptyset$ and $h + \xi \cdot x_1 \in (u(x_1, n_1 - 1), u(x_1, n_1)]$, which proves the claim and the statement of display (4.28).

Step 2. We will now show that for each $(x, n) \in \Gamma^+$

$$|I_u(x, n)| \le |U|^d \frac{(L_a^* u(x, n))^d}{|conv(U_{x,n})|}, \tag{4.29}$$

where for every $(x, n) \in \Gamma^+$ and function $h(x, n) : \Gamma^+ \to \mathbb{R}$ we define

$$L_a^* h(x, n) := \sum_{z\ne 0} a(n, x, z) \left(h(x, n) - h(x + z, n - 1)\right)$$

and

$$U_{x,n} := \{a(n, x, z)z : z \in U\}.$$

Fix $p \in I_u(x, n)$ and set

$$w(y, m) := u(y, m) - p \cdot y.$$

Then we have that $I_w(x, n) = I_u(x, n)+p$. Furthermore, by definition, if $q \in I_w(x, n)$ and $(y, n-1) \in \mathcal{D} \cup \mathcal{D}^p$ we have that

$$w(x, n) - w(y, n - 1) \ge (x - y) \cdot q.$$

Hence, for each $q \in I_w(x, n)$ and $z \in U_{x,n}$ we have that

$$L_a^* u(x,n) = L_a^* w(x,n) = \sum_{z\neq 0} a(n,x,z)\,(w(x,n) - w(x+z,n-1)) \geq z\cdot q. \tag{4.30}$$

Let now $V_{x,n} := conv(U_{x,n})$ and consider the polar body of $V_{x,n}$, given by $V^o_{x,n} := \{z \in \mathbb{R}^d : z\cdot y \leq 1 \text{ for all } y \in V_{x,n}\}$. Display (4.30) implies that

$$I_w(x,n) \subset L_a^* u(x,n) V^o_{x,n}. \tag{4.31}$$

Using the fact that $\sum_{l\in U_{x,n}} l = 0$, note that if $z \in V^o_{x,n}$, then for each $y \in U_{x,n}$

$$z\cdot(-y) = z\cdot \sum_{l\in U_{x,n}\setminus\{y\}} l \leq |U|.$$

Hence, setting $\tilde{U}_{x,n} := \{\pm y : y \in U_{x,n}\}$ and $\tilde{V}_{x,n} := conv(\tilde{U}_{x,n})$ we see that

$$V^o_{x,n} \subset \left\{ z : z\cdot y \leq |U| \text{ for all } y \in \tilde{U}_{x,n} \right\} = |U| \tilde{V}^o_{x,n}. \tag{4.32}$$

Combining (4.31) with (4.32) we conclude that

$$I_w(x,n) \subset |U| L_a^* u(x,n) \tilde{V}^o_{x,n}.$$

Now, since $\tilde{V}^o_{x,n}$ is a symmetric convex body, by Mahler's inequality [M39], we see that

$$|\tilde{V}^o_{x,n}| \leq \frac{4^d}{|\tilde{V}_{x,n}|},$$

which finishes the proof of (4.29).

Step 3. Here we derive the maximum inequality from steps 1 and 2. Set,

$$\chi(\Gamma^+, x) := \bigcup_{m:(x,m)\in\Gamma^+} \chi(x,m).$$

For each $x \in \mathcal{D}$, define $\rho_x : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\rho_x(y,m) = (y, m + y\cdot x).$$

Let

$$\tilde{\chi}(x,n) := \rho_x \circ \chi(x,n) = I_u(x,n) \times [u(x,n-1), u(x,n)] \subset \mathbb{R}^{d+1}.$$

Then, using the inequality $a^{\frac{1}{d+1}} b^{\frac{d}{d+1}} \leq \frac{a+db}{d+1}$, valid for $a \geq 0$, $b \geq 0$, and the notation $\sum'$ for the sum running from $n=1$ to $n=T$ with $u(x,n) - u(x,n-1)$ and $L_a^* u(x,n)$ positive we see that

$$\begin{aligned}
\left|\chi(\Gamma^+, x)\right| &= \left|\tilde{\chi}(\Gamma^+, x)\right| \\
&\leq \sum{}' (u(x,n) - u(x,n-1))\, |I_u(x,n)|\, 1_{(x,n)\in\Gamma^+} \\
&\leq |U|^d \sum{}' (u(x,n) - u(x,n-1)) \frac{\left(L_a^* u(x,n)\right)^d}{v(x,n)} 1_{(x,n)\in\Gamma^+} \\
&\leq d \sum{}' \left(\frac{a(n,x,x)(u(x,n) - u(x,n-1)) + L_a^* u(x,n)}{(d+1)\varepsilon(x,n)} \right)^{d+1} 1_{(x,n)\in\Gamma^+} \\
&= \sum{}' \left(\frac{-\mathcal{L}_a u(x,n)}{\varepsilon(x,n)} \right)^{d+1} 1_{(x,n)\in\Gamma^+}.
\end{aligned} \tag{4.33}$$

Now, note that

$$|\Lambda| = C \frac{M^{d+1}}{R^d}.$$

Combining this with inequality (4.33) and with (4.28) of Step 1, and using the hypothesis (2.20), we see that

$$C\frac{M^{d+1}}{R^d} \le \sum_{(x,n)\in\mathcal{D}} \frac{1}{\varepsilon^{d+1}}|f|^{d+1}1_{(x,n)\in\Gamma^+}.$$

Therefore,

$$\max_{(x,n)\in\mathcal{D}} u(x,n) \le CR^{\frac{d}{d+1}}\left\|\frac{f}{\varepsilon}\right\|_{\mathcal{D},d}.$$

5 Proof of the continuous time quenched central limit theorem

The proof of Theorem 1.2 follows a strategy similar to that of Theorem 1.1. In other words, since the continuous time random walk is also $\mathbb{Q}$-a.s. a martingale, it suffices to construct an invariant measure for the environmental process which is absolutely continuous with respect to the initial law $\mathbb{Q}$ of the environment. However, unlike the discrete time case, the continuous time process is allowed to jump at unbounded rates. To obtain a QCLT, we need not only to deal with the degeneracy of the ellipticity, but also to control the jump rates. This is achieved by first performing a time change to "slow-down" the original RWRE, and then applying a maximum principle for (continuous-time) parabolic difference operators to construct the invariant measure.

Let us state the version of the parabolic maximum principle that we use. Consider a balanced continuous time-dependent environment $\{a_t : t \ge 0\}$, c.f. (1.6), with $a_t := \{a_t(x) : x \in \mathbb{Z}^d\}$ and $a_t(x) := \{a_t(x,e) : e \in U\} \in \mathcal{Q}$. Given any finite set $D \subset \mathbb{Z}^d$ and $T > 0$, we define

$$\mathcal{D} := D \times (0,T].$$

Define the *parabolic boundary* of $\mathcal{D}$ by

$$\mathcal{D}^p := \mathcal{D}^\ell \cup \mathcal{D}^T,$$

where $\mathcal{D}^T = D \times \{0\}$ denotes its *time boundary* and

$$\mathcal{D}^\ell := \{(x,t) \notin \mathcal{D} : a_t(y, x-y) > 0 \text{ for some } (y,t) \in \mathcal{D}\}$$

is the *lateral boundary* of $\mathcal{D}$.

Now, for $u : \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ bounded and differentiable in time for each $x \in \mathbb{Z}^d$, we define the parabolic difference operator

$$\mathcal{L}_a u(x,t) := \sum_{e\in U} a_t(x,e)[u(x+e,t) - u(x,t)] - \partial_t u(x,t).$$

For $p > 0$ and any real-valued function g that is summable on $\mathcal{D} = D \times [0,T]$, define

$$\|g\|_{\mathcal{D},p} := \left(\int_0^T \sum_{x\in D} |g(x,t)|^p \mathrm{d}t\right)^{1/p}.$$

We can now state the maximum principle.

Theorem 5.1 *Assume that a is a balanced environment. Let u be a function on $\mathcal{D} \cup \mathcal{D}^p$ which is differentiable with respect to t in $(0,T]$. Let f be an integrable function in $\mathcal{D}$. Assume that u satisfies*

$$\mathcal{L}_a u \ge f \quad \text{in} \quad \mathcal{D}.$$

Then, there is a constant $C = C(U,d) > 0$ such that

$$\sup_{\mathcal{D}} u \le \sup_{\mathcal{D}^p} u + CR^{d/(d+1)}||f/\varepsilon||_{\mathcal{D},d+1},$$

where $R := diam(D)$ and ε is as defined in (1.4).

Note that the space-time process $(X_t, t)_{t\geq 0}$ is a Markov process on $\mathbb{Z}^d \times \mathbb{R}$ with generator

$$\mathcal{L}^*_\omega u(x,t) := \sum_{e\in U} \omega_t(x,e)(u(x+e,t) - u(x,t)) + \partial_t u(x,t).$$

To show that $(X_t)_{t\geq 0}$ does not explode, i.e, there are only finitely many jumps within finite time, we will first consider a slowed-down process. Recall the definition of υ_ω in (1.8). Let

$$(Y_t, T_t)_{t\geq 0}$$

be the Markov process on $\mathbb{Z}^d \times \mathbb{R}$ with generator $(\upsilon_\omega + 1)^{-1}\mathcal{L}^*_\omega$. That is, at space-time (x,t), the slowed-down process $Y_\cdot$ will jump to $x + e \in x + U$ with rate

$$\omega_t(x,e)/(\upsilon_\omega(x,t) + 1),$$

and the time process $T_\cdot$ will increase at rate $(\upsilon_\omega(x,t)+1)^{-1}$. Note that

$$T_t = \int_0^t \frac{1}{\upsilon_\omega(Y_s, T_s) + 1} \mathrm{d}s \tag{5.34}$$

and

$$X_{T_t} = Y_t. \tag{5.35}$$

Define the stopping times $\tau_0 = \tau_0(Y_\cdot, T_\cdot) = 0$, and

$$\tau_{j+1} = \tau_{j+1}(Y_\cdot, T_\cdot) = \inf\{t > \tau_j : |Y_t - Y_{\tau_j}|_\infty > N \text{ or } T_t - \tau_j > N^2\}.$$

We have the following analogue of Lemma 3.1.

Lemma 5.1 *There exists a constant $c > 0$ such that for all N large and any $\omega \in \Omega$,*

$$E\Big[\big(1 - \frac{c}{N^2}\big)^{\tau_1(Y_\cdot, T_\cdot)}\Big] \leq \frac{1}{2}.$$

Proof: The proof follows similar argument as in Lemma 3.1. Recall that $C_U = \max\{|e| : e \in U\}$. Note that $(Y_t)_{t\geq 0}$ is a martingale and $(|Y_t|^2 - C_U t)_{t\geq 0}$ is a super-martingale. Let $K = \frac{N^2}{2C_U}$. Then, by Doob's L^2-martingale inequality,

$$\begin{aligned} P(\tau_1 \leq K) &= P(\max_{0<t\leq K} |Y_t| > N) \\ &\leq \frac{1}{N^2} E[|Y_K|^2] \leq \frac{C_U K}{N^2} = \frac{1}{2}, \end{aligned}$$

where in the first equality we used the fact that $T_K \leq K < N^2$. ∎

Theorem 5.2 *Assume the same conditions as in Theorem 1.2. Then the environmental process $(\theta_{T_t, Y_t}\omega)_{t\geq 0}$ has a unique invariant probability measure $\bar{\nu}$ which is equivalent to $\mathbb{Q}$.*

Proof: Let $Q_N = \{z \in \mathbb{Z}^d : |z|_\infty \leq N\} \times (0, N^2]$. We introduce on $\mathbb{Z}^d$ the equivalent classes

$$\overline{(x,t)} := (x,t) + (2N+1)\mathbb{Z}^d \times N^2\mathbb{Z}.$$

Fix a balanced environment $\omega \in \mathcal{Q}$, and define its periodized environment $\omega^{(N)}$ so that for any $(x,t) \in Q_N$,

$$\omega^{(N)}_s(y) = \omega_t(x)$$

whenever $\overline{(y,s)} = \overline{(x,t)}$.

Set

$$\Omega_{N,\omega} = \Omega_N := \{\theta_{t,x}\omega^{(N)} : (x,t) \in Q_N\}$$

and let $\mathbb{P}_N = \mathbb{P}_{N,\omega}$ denote the probability measure

$$\mathbb{P}_N(\mathrm{d}\xi) = \sum_{x:(x,t)\in Q_N} 1_{\theta_{t,x}\omega=\xi}\mathrm{d}t.$$

Under the environment $\omega^{(N)}$, recall that Y_t is the slowed-down process. By compactness, there is an invariant probability measure $\nu_N \ll \mathbb{P}_N$ for the Markov chain $(\theta_{T_t,Y_t}\omega^{(N)})_{t\ge 0}$, and it has the form

$$\nu_N(\mathrm{d}\xi) = \sum_{x:(x,t)\in Q_N} \phi_N(x,t)1_{\theta_{t,x}\omega=\xi}\mathrm{d}t,$$

where ϕ_N is the density of the invariant measure of the random walk $(\overline{Y_t, T_t})_{t\ge 0}$ on Q_N.

For $\xi \in \Omega_N$, let $\xi_t := \theta_{T_t,Y_t}\xi$ denote the environment process. By similar arguments as in Section 3, Lemma 5.1 implies

$$\nu_N g \le CN^{-2} \max_{\xi\in\Omega_N} E_\xi \left[\int_0^{\tau_1} g(\xi_t)\mathrm{d}t \middle| Y_0 = 0, T_0 = 0\right].$$

Letting

$$u(x,t) = E_\xi \left[\int_0^{\tau_1} g(\xi_s)\mathrm{d}s \middle| Y_0 = x, T_0 = N^2 - t\right],$$

we have

$$\begin{cases} (\upsilon_\omega + 1)^{-1}\mathcal{L}_\xi u = G_\xi(x,t) & \text{in } Q_N \\ u = 0 & \text{in } Q_N^p, \end{cases}$$

where $G_\xi(x,t) := g(\theta_{x,N^2-t}\xi)$. Then, applying Theorem 5.1 to the operator $\mathcal{L}_\xi$, we get

$$\max_{Q_N} u \le CN^2 \|(\upsilon+1)g/\varepsilon\|_{L^{d+1}(\mathbb{P}_N)}$$

and so

$$\nu_N g \le \|(\upsilon+1)g/\varepsilon\|_{L^{d+1}(\mathbb{P}_N)}.$$

Since, $\lim_{N\to\infty} \mathbb{P}_{N,\omega} = \mathbb{Q}$, $\mathbb{Q}$-a.s. and

$$E_\mathbb{Q}[(\upsilon+1)^{d+1}/\varepsilon_\omega^{d+1}] \le 2^d E_\mathbb{Q}[(\upsilon^{d+1}+1)/\varepsilon_\omega^{d+1}] < \infty,$$

using the ergodic theorem and Kozlov's argument, the conclusion follows. ∎

Corollary 5.1 *Assume the same conditions as in Theorem 1.2. For $\mathbb{Q}$-almost all ω, P_ω-almost surely the process $(X_t)_{t\ge 0}$ does not explode. Moreover, the environmental process $(\theta_{t,X_t}\omega)_{t\ge 0}$ has a unique invariant probability measure ν which is equivalent to $\mathbb{Q}$.*

Proof: Set $\bar\omega_t := \theta_{t,X_t}\omega$. By (5.34), Theorem 5.2 and the ergodic theorem,

$$\lim_{t\to\infty} \frac{T_t}{t} = E_{\bar\nu}[\frac{1}{\upsilon+1}] \in (0,1) \qquad \mathbb{Q}\otimes P_\omega\text{-a.s.}$$

Hence, by (5.35), $(X_t)_{t\ge 0}$ is not explosive. Furthermore, let

$$\mathrm{d}\nu := \frac{N}{\upsilon+1}\mathrm{d}\bar\nu,$$

where $N = (E_{\bar\nu}[\frac{1}{\upsilon+1}])^{-1}$ is a normalization constant. Then ν is the invariant measure of $(\bar\omega_t)_{t\ge 0}$ and it is equivalent to $\mathbb{Q}$. ∎

Theorem 1.2 (i) is proved in the above corollary. As in Theorem 1.1, this implies the invariance principle Theorem 1.2 (ii).

6 Proof of Corollary 1.2

For each $n \geq 0$, denote by $\mathcal{F}_n^Y$ the σ-algebra generated by $\{Y_0, Y_1, \ldots, Y_n\}$, while $\mathcal{F}^Y$ the σ-algebra generated by $\{Y_0, Y_1, \ldots\}$. Now, given $\omega \in \Omega$, define the time dependent environment $\omega^Y := \{\omega_n^Y : n \geq 0\}$ on $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ by

$$\omega_n^Y := \{\omega_n^Y(x) : x \in \mathbb{Z}^d\}$$

for $n \geq 0$ and $\omega_n^Y(x) := \{\omega_n^Y(x, e) : e \in \{e_2, -e_2, 0\}\}$ with

$$\omega_n^Y((y, z), e) := \omega_{D_2}((y + Y_n, z), e)$$

for $y, z \in \mathbb{Z}$ and $f \in \{e_2, -e_2, 0\}$. Furthermore, we define the time dependent environment $\tilde{\omega}^Y := \{\tilde{\omega}_n^Y : n \geq 0\}$ on $\Omega_1 := \mathcal{P}_2^{\mathbb{Z}}$ with $\mathcal{P}_2 := \{v(e), e \in \{e_2, -e_2, 0\} : v(e) \geq 0, \sum_{e \in \{e_2, -e_2, 0\}} v(e) = 1\}$, by

$$\tilde{\omega}_n^Y := \{\tilde{\omega}_n^Y(y) : y \in \mathbb{Z}\}$$

for $n \geq 0$ and $\tilde{\omega}_n^Y(y) := \{\tilde{\omega}_n^Y(y, e) : e \in \{e_2, -e_2, 0\}\}$ with

$$\tilde{\omega}_n^Y(y, e) := \omega_n^Y((0, y), e).$$

Lemma 6.1 *$\mathbb{P}$-a.s. we have that $P_{0,\omega}$-a.s. under the law $P_{0,\omega}(\cdot|\mathcal{F}^Y)$, $\{Z_n : n \geq 0\}$ is a random walk on the lattice $\mathbb{Z}$ in the time dependent environment $\tilde{\omega}^Y$.*

Proof. Fix $0 \leq m \leq n$ and fix two sequences $z_1, \ldots, z_{m+1} \in \mathbb{Z}^{D_2}$ and $y_1, \ldots, y_n \in \mathbb{Z}^{D_1}$ and define $x_i = (y_i, z_i)$ for $1 \leq i \leq m + 1$. Using property (b) of Theorem 1.2 (independence property of the jumps of the coordinates, c.f. (1.15)) and property (a) of Theorem 1.2 (the fact that the first coordinate of the random walk is autonomous, c.f. (1.14)), note that

$$\begin{aligned}
&P_{0,\omega}(Z_{m+1} = z_{m+1}|Z_1 = z_1, \ldots, Z_m = z_m, Y_1 = y_1, \ldots, Y_n = y_n) \\
&= \frac{P_{0,\omega}(Z_1 = z_1, \ldots, Z_m = z_m, Z_{m+1} = z_{m+1}, Y_1 = y_1, \ldots, Y_n = y_n)}{P_{0,\omega}(Z_1 = z_1, \ldots, Z_m = z_m, Y_1 = y_1, \ldots, Y_n = y_n)} \\
&= \frac{\omega_Z(x_m, z_{m+1} - z_m)\omega_Y(x_m, y_{m+1} - y_m)\bar{\omega}_Y(y_{m+1}, y_{m+2} - y_{m+1}) \cdots \bar{\omega}_Y(y_{n-1}, y_n - y_{n-1})}{\omega_Y(x_m, y_{m+1} - y_m)\bar{\omega}_Y(y_{m+1}, y_{m+2} - y_{m+1}) \cdots \bar{\omega}_Y(y_{n-1}, y_n - y_{n-1})} \\
&= \omega_Z(x_m, z_{m-1} - z_m) \\
&= \omega_Z((y_m, z_m), z_{m-1} - z_m),
\end{aligned}$$

which finishes the proof. ∎

Now, by property (c) of Theorem 1.2, we see that there is a measure ν defined on Ω which is invariant for the environmental process $\{\omega_n^Y : n \geq 0\}$ and absolutely continuous with respect to $\mathbb{P}$. Let us call Q_ν the law of this process starting from ν.

By Theorem 2.1 of Kozlov, Q_ν is ergodic under the action of the time shifts $\{\theta_{n,0} : n \geq 0\}$. Now, the random walk $\{Z_n : n \geq 0\}$ has at each time a positive probability of not jumping. Therefore, Q_ν is an invariant measure for the random walk $\{Z_n : n \geq 0\}$ on the time-dependent environment $\tilde{\omega}^Y$, which is ergodic under the action of the shifts $\{\theta_{1,e} : e \in \{e_2, -e_2, 0\}\}$. Note that the environmental process associated to $\{\tilde{\omega}_n^Y : n \geq 0\}$ is defined for $n \geq 0$ as $\theta_{n,Z_n}\tilde{\omega}^Y$. We can then apply part (i) of Theorem 1.1 to conclude that there exists an invariant measure μ for the environmental process associated to $\{\tilde{\omega}_n^Y : n \geq 0\}$, which is absolutely continuous with respect to Q_ν. Note that this environmental process is equal at time n to $\omega((Z_n, Y_n))$. This proves part (i) of Theorem 1.2.

Furthermore, by part (ii) of Theorem 1.1, we conclude that $\mathbb{P}$-a.s, for a.a. trajectories $\{Y_0, Y_1, \ldots\}$, and for all open $B \in C([0, \infty); \mathbb{Z})$,

$$\lim_{n \to \infty} P_{0,\omega}\left(\frac{Z_{[n\cdot]}}{\sqrt{n}} \in B \middle| \mathcal{F}^Y \right) \geq Q(B), \tag{6.36}$$

where Q is the probability that a Brownian motion with a specified covariance matrix belongs to B. We can now conclude that for any pair of open sets $A \in C([0,\infty);\mathbb{Z})$ and $B \in C([0,\infty);\mathbb{Z})$ one has that

$$P_{0,\omega}\left(\frac{Y_{[n\cdot]}}{\sqrt{n}} \in A, \frac{Z_{[n\cdot]}}{\sqrt{n}} \in B\right) = E_{0,\omega}\left(1_{\frac{Y_{[n\cdot]}}{\sqrt{n}} \in A} P_{0,\omega}\left(\frac{Z_{[n\cdot]}}{\sqrt{n}} \in B \middle| \mathcal{F}^Y\right)\right),$$

which by (6.36) has a lim inf which $\mathbb{P}$-a.s. is bounded from below by the probability that a Brownian motion with a specified non-degenerate covariance matrix in dimension d belongs to $A \times B$. Using the fact that any open set in $C([0,\infty) : \mathbb{Z}^d)$ can be expressed as a countable union of sets of the form $A \times B$ we conclude the proof.

7 Counterexample of a random walk in an environment which is not ergodic enough

Here we show that the ergodicity hypothesis of Theorem 1.1 cannot be weakened. Consider a balanced environment in $\mathbb{Z}^2$. Let $A = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$, $B = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}\}$. For any $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$, set

$$\xi_{x,n} = \begin{cases} A & \text{if } |x|_1 + n \text{ is even} \\ B & \text{if } |x|_1 + n \text{ is odd} \end{cases}$$

$$\xi'_{x,n} = \begin{cases} A & \text{if } |x|_1 + n \text{ is odd} \\ B & \text{if } |x|_1 + n \text{ is even.} \end{cases}$$

Define the environment measure $\mathbb{P}$ to be

$$\mathbb{P}(\omega = \xi) = \mathbb{P}(\omega = \xi') = \frac{1}{2}.$$

Noting that $\theta_{x,0}\xi = \xi'$, we see that the measure $\mathbb{P}$ is ergodic under the shifts $\{\theta_{1,x} : x \in \mathbb{Z}^d\}$. However, the QCLT does not hold, since $X_{n\cdot}/\sqrt{n}$ can converge to Brownian motions with different covariance matrices.

Remark 7.1 *The reason why Theorem 1.1 cannot be applied and the quenched central limit theorem fails is that $\mathbb{P}$ is not ergodic under the group of shifts $\{\theta_{1,x} : x \in U\}$.*

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