Ellipticity criteria for ballistic behavior of random walks in random environment

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Abstract

We introduce ellipticity criteria for random walks in i.i.d. random environments under which we can extend the ballisticity conditions of Sznitman's and the polynomial effective criteria of Berger, Drewitz and Ramírez originally defined for uniformly elliptic random walks. We prove under them the equivalence of Sznitman's (T') condition with the polynomial effective criterion $(P)_M$, for M large enough. We furthermore give ellipticity criteria under which a random walk satisfying the polynomial effective criterion, is ballistic, satisfies the annealed central limit theorem or the quenched central limit theorem.

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1 Introduction

We introduce ellipticity criteria for random walks in random environment which enable us to extend to environments which are not necessarily uniformly elliptic the ballisticity conditions for the uniform elliptic case of Sznitman [Sz02] and of Berger, Drewitz and Ramírez [BDR12], their equivalences and some of their consequences [SZ99, Sz01, Sz02, RAS09, BZ08].

For $x \in \mathbb{R}^d$, denote by $|x|_1$ and $|x|_2$ its l_1 and l_2 norm respectively. Call $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\} = \{e_1, \ldots, e_{2d}\}$ the canonical vectors with the convention that $e_{d+i} = -e_i$ for $1 \leq i \leq d$ and let $\mathcal{P} := \{p(e) : p(e) \geq 0, \sum_{e \in U} p(e) = 1\}$. An *environment* is an element ω of the *environment space* $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ so that $\omega := \{\omega(x) : x \in \mathbb{Z}^d\}$, where $\omega(x) \in \mathcal{P}$. We denote the components of $\omega(x)$ by $\omega(x, e)$. The *random walk in the environment ω starting from x* is the Markov chain $\{X_n : n \geq 0\}$ in $\mathbb{Z}^d$ with law $P_{x,\omega}$ defined by the condition $P_{x,\omega}(X_0 = x) = 1$ and the transition probabilities

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$$P_{x,\omega}(X_{n+1} = x + e | X_n = x) = \omega(x, e)$$

for each $x \in \mathbb{Z}^d$ and $e \in U$. Let $\mathbb{P}$ be a probability measure defined on the environment space Ω endowed with its Borel σ-algebra. We will assume that $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. under $\mathbb{P}$. We will call $P_{x,\omega}$ the *quenched law* of the random walk in random environment (RWRE) starting from x, while $P_x := \int P_{x,\omega} d\mathbb{P}$ the *averaged* or *annealed law* of the RWRE starting from x.

We say that the law $\mathbb{P}$ of the RWRE is *elliptic* if for every $x \in \mathbb{Z}^d$ and $e \in U$ one has that $\mathbb{P}(\omega(x, e) > 0) = 1$. We say that $\mathbb{P}$ is *uniformly elliptic* if there exists a constant $\kappa > 0$ such that for every $x \in \mathbb{Z}^d$ and $e \in U$ it is true that $\mathbb{P}(\omega(x, e) \geq \kappa) = 1$. Given $l \in \mathbb{S}^{d-1}$ we say that the RWRE is *transient in direction* l if

$$P_0(A_l) = 1,$$

where

$$A_l := \{\lim_{n\to\infty} X_n \cdot l = \infty\}$$

We say that it is *ballistic in direction* l if P_0-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

The following is conjectured (see for example [Sz04]).

Conjecture 1.1 *Let $l \in \mathbb{S}^{d-1}$. Consider a random walk in a uniformly elliptic i.i.d. environment in dimension $d \geq 2$, which is transient in direction l. Then it is ballistic in direction l.*

Some partial progress towards the resolution of this conjecture has been made in [Sz01, Sz02, DR11, DR12, BDR12]. In 2001 and 2002 Sznitman in [Sz01, Sz02] introduced a class of ballisticity conditions under which he could prove the above statement. For each subset $A \subset \mathbb{Z}^d$ define the first exit time from the set A as

$$T_A := \inf\{n \geq 0 : X_n \notin A\}. \tag{1.1}$$

For $L > 0$ and $l \in \mathbb{S}^{d-1}$ define the slab

$$U_{l,L} := \{x \in \mathbb{Z}^d : -L \leq x \cdot l \leq L\}. \tag{1.2}$$

Given $l \in \mathbb{S}^{d-1}$ and $\gamma \in (0, 1)$, we say that condition $(T)_\gamma$ in direction l (also written as $(T)_\gamma|l$) is satisfied if there exists a neighborhood $V \subset \mathbb{S}^{d-1}$ of l such that for all $l' \in V$

$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log P_0(X_{T_{U_{l',L}}} \cdot l' < 0) < 0.$$

Condition $(T')|l$ is defined as the fulfillment of condition $(T)_\gamma|l$ for all $\gamma \in (0, 1)$. Sznitman [Sz02] proved that if a random walk in an i.i.d. uniformly elliptic environment satisfies $(T')|l$ then it is ballistic in direction l. He also showed that if $\gamma \in (0.5, 1)$, then $(T)_\gamma$ implies (T'). In 2011, Drewitz and Ramírez [DR11] showed that there is a $\gamma_d \in (0.37, 0.39)$ such that if $\gamma \in (\gamma_d, 1)$, then $(T)_\gamma$ implies (T'). In 2012, in [DR12], they were able to show that for dimensions $d \geq 4$, if $\gamma \in (0, 1)$, then $(T)_\gamma$ implies (T'). Recently in [BDR12], Berger, Drewitz and Ramírez introduced a polynomial ballisticity condition, weakening further the conditions $(T)_\gamma$. The condition is effective, in the sense that it can a priori be verified explicitly for a given environment. To define it, for each $L, L', \tilde{L} > 0$ and $l \in \mathbb{S}^{d-1}$ consider the box

$$B_{l,L',L,\tilde{L}} := R\left((-L', L) \times \left(-\tilde{L}, \tilde{L}\right)^{d-1}\right) \cap \mathbb{Z}^d,$$

where R is a rotation of $\mathbb{R}^d$ defined by the condition

$$R(e_1) = l. \tag{1.3}$$

Let also

$$L_0 := \frac{2}{3} 3^{29d}. \tag{1.4}$$

Given $M \geq 1$, we say that condition $(P)_M$ in direction l is satisfied (also written as $(P)_M|l$) if for every $L \geq L_0$, $L' \leq \frac{5}{4}L$ and $\tilde{L} \leq 72L^3$ one has the following upper bound for the probability that the walk does not exit the box $B_{l,L',L,\tilde{L}}$ through its front side

$$P_0(X_{T_{B_{l,L',L,\tilde{L}}}} \cdot l < L) \leq \frac{1}{L^M}.$$

In [BDR12], Berger, Drewitz and Ramírez prove that every random walk in an i.i.d. uniformly elliptic environment which satisfies $(P)_M$ for $M \geq 15d + 5$ is necessarily ballistic.

On the other hand, it is known (see for example Sabot-Tournier [ST11]) that in dimension $d \geq 2$, there exist elliptic random walks which are transient in a given direction but not ballistic in that direction. The purpose of this paper is to investigate to which extent can the assumption of uniform ellipticity be weakened. To do this we introduce several classes of ellipticity conditions on the environment. Define

$$F_e := \sup \left\{ \alpha \geq 0 : \mathbb{E}\left[e^{\alpha \log \frac{1}{\omega(0,e)}} \right] < \infty \right\}.$$

Let $\alpha \geq 0$. We say that the law of the environment satisfies the *ellipticity condition* $(E)_\alpha$ if for every $e \in U$ we have that

$$\min_{e \in U} F_e > \alpha.$$

The first main result of this paper is the following one.

Theorem 1.1 *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d + 5$. Assume that the environment satisfies the ellipticity condition $(E)_0$. Then the polynomial condition $(P)_M|l$ is equivalent to $(T')|l$.*

In this paper we go further from Theorem 1.1, and we obtain assuming (T'), good enough tail estimates for the distribution of the regeneration times of the random walk. Let us recall that there exists an *asymptotic direction* if the limit

$$\hat{v} := \lim_{n \to \infty} \frac{X_n}{|X_n|_2}$$

exists P_0-a.s. The polynomial condition $(P)_M$ implies the existence of an asymptotic direction (see for example Simenhaus [Sim07]). Whenever the asymptotic direction exists, let us define the half space

$$H_{\hat{v}} := \{l \in \mathbb{S}^{d-1} : l \cdot \hat{v} \geq 0\}.$$

Let $\beta > 0$. We say that the law of the environment satisfies the *ellipticity condition* $(E')_\beta$ if there exists an $\bar{\alpha} := \{\alpha(e) : e \in U\} \in (0, \infty)^{2d}$ such that

$$\kappa = \kappa(\bar{\alpha}) := 2 \sum_{e'} \alpha(e') - \sup_{e \in U} (\alpha(e) + \alpha(-e)) > \beta \tag{1.5}$$

and

$$\text{for every } e \in U \text{ we have that} \quad \mathbb{E}\left[e^{\sum_{e' \neq e} \alpha(e') \log \frac{1}{\omega(0,e')}} \right] < \infty \tag{1.6}$$

Note that $(E')_\beta$ implies $(E)_\alpha$ for

$$0 < \alpha < \min_{e \in U} \alpha(e). \tag{1.7}$$

Furthermore, we say that the ellipticity condition $(E')_\beta$ is satisfied *towards the asymptotic direction* if there exists an $\bar{\alpha}$ satisfying (1.5) and (1.6) and such that

$$\text{there exists } \alpha_1 > 0 \text{ such that } \alpha(e) = \alpha_1 \text{ for } e \in H_{\hat{v}}, \text{ while } \alpha(e) \leq \alpha_1 \text{ for } e \notin H_{\hat{v}}. \tag{1.8}$$

The second main result of this paper is the following theorem.

Theorem 1.2 (Law of large numbers) *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$ and the ellipticity condition $(E')_1$ towards the asymptotic direction (cf. (1.5), (1.6) and (1.8)). Then the random walk is ballistic in direction l and there is a $v \in \mathbb{R}^d$, $v \neq 0$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad P_0 - a.s.$$

We have directly the following two corollaries, the second one following from Hölder's inequality.

Corollary 1.1 *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$ and that there is an $\alpha > \frac{1}{4d-2}$ such that*

$$\sup_{e\in U} \mathbb{E}\left[e^{\alpha \sum_{e'\neq e} \log \frac{1}{\omega(0,e')}}\right] < \infty.$$

Then the random walk is ballistic in direction l.

Corollary 1.2 *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$ and the ellipticity condition $(E)_{1/2}$. Then the random walk is ballistic in direction l.*

Theorems 1.1 and Theorem 1.2 give ellipticity criteria for ballistic behavior for general random walks in i.i.d. environments, which as we will see below, should not be far from optimal criteria. Indeed, the value 1 of condition $(E')_1$ in Theorem 1.2 is optimal: within the context of random walks in Dirichlet random environments (RWDRE), it is well known that there are examples of walks which satisfy $(E')_\beta$ for β smaller but arbitrarily close to 1, towards the asymptotic direction (cf. 1.8), which are transient but not ballistic in a given direction (see [Sa11a, Sa11b, ST11]). We can also explicitly construct such examples in analogy to the random conductance model studied by Fribergh in [F11]. In fact, for every $\epsilon > 0$, one can construct an environment such that condition $(E')_{1-\epsilon}$ is satisfied towards the asymptotic direction, but the walk is transient in direction e_1 but not ballistic in direction e_1. Let ϕ be any random variable taking values on the interval $(0, 1/4)$ and such that the expected value of $\phi^{-1/2}$ is infinite, while for every $\epsilon > 0$, the expected value of $\phi^{-(1/2-\epsilon)}$ is finite. Let X be a Bernoulli random variable of parameter 1/2. We now define $\omega_1(0,e_1) = 2\phi$, $\omega_1(0,-e_1) = \phi$, $\omega_1(0,-e_2) = \phi$ and $\omega_1(0,e_2) = 1-4\phi$ and $\omega_2(0,e_1) = 2\phi$, $\omega_2(0,-e_1) = \phi$, $\omega_2(0,e_2) = \phi$ and $\omega_2(0,-e_2) = 1-4\phi$. We then let the environment at site 0 be given by the random variable $\omega(0) := 1_X(1)\omega_1(0) + 1_X(0)\omega_2(0)$. This environment has the property that traps can appear, where the random walk gets caught in an edge, as shown shown in Figure 1 and it does satisfy $(E')_{1-\epsilon}$ towards the asymptotic direction. Furthermore, it is not difficult to check that the random walk in this random environment is transient in direction e_1 but not ballistic. It will be shown in a future work, that this environment satisfies the polynomial condition $(P)_M$ for $M \geq 15d+5$.

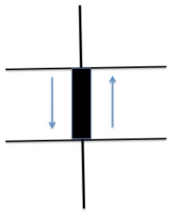

Figure 1: A trap produced by an elliptic environment which does not satisfy $(E')_1$.

As mentioned above, similar examples of random walks in elliptic i.i.d. random environment which are transient in a given direction but not ballistic have been exhibited within the context of the Dirichlet environment. Here, the environment is chosen i.i.d. with a Dirichlet distribution at each site $D(\beta_1,\ldots,\beta_{2d})$ of parameters $\beta_1,\ldots,\beta_{2d}>0$ (see for example [Sa11a, Sa11b, ST11]), the parameter β_i being associated with the direction e_i. For a random walk in Dirichlet random environment, condition $(E')_1$ is equivalent to

$$\lambda := 2\sum_{j=1}^{2d}\beta_j - \sup_{i:1\le i\le 2d}(\beta_i+\beta_{i+d}) > 1. \tag{1.9}$$

This is the characterization of ballisticity given by Sabot in [Sa11b] for random walks in random Dirichlet environments in dimension $d\ge 3$. Tournier in [T11] proved that if $\lambda\le 1$, then the RWDRE is not ballistic in any direction. Sabot in [Sa11b], showed that if $\lambda>1$, and if there is an $i=1,\ldots,d$ such that $\beta_i\ne\beta_{i+d}$, then the random walk is ballistic. It is thus natural to wonder to what general condition corresponds (not restricted to random Dirichlet environments), the characterization of Sabot and Tournier. In section 2, we will see that there are several formulations of the necessary and sufficient condition for ballisticity of Sabot and Tournier for RWDRE (cf. (1.9)), but which are not equivalent for general RWRE. Among these formulations, the following one is the weakest one in general. We say that condition (ES) is satisfied if

$$\max_{i:1\le i\le d}\mathbb{E}\left[\frac{1}{1-\omega(0,e_i)\omega(e_i,-e_i)}\right]<\infty.$$

We have furthermore the following proposition whose proof will be presented in section 2.

Proposition 1.1 *Consider a random walk in a random environment. Assume that condition (ES) is not satisfied. Then the random walk is not ballistic.*

We will see in the proof of Proposition 1.1 how important is the role played by the edges depicted in Figure 1 which play the role of traps.

Another consequence of Theorem 1.1 and the machinery that we develop to estimate the tails of the regeneration times is the following theorem.

Theorem 1.3 *Consider a random walk in an i.i.d. environment in dimensions $d\ge 2$. Let $l\in\mathbb{S}^{d-1}$ and $M\ge 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$.*

a) **(Annealed central limit theorem)** *If $(E')_1$ is satisfied then*

$$\epsilon^{1/2}(X_{[\epsilon^{-1}n]}-[\epsilon^{-1}n]v)$$

converges in law under P_0 as $\epsilon\to 0$ to a Brownian motion with non-degenerate covariance matrix.

b) **(Quenched central limit theorem)** *If $(E')_{88d}$ is satisfied, then $\mathbb{P}$-a.s. we have that*

$$\epsilon^{1/2}(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v)$$

converges in law under $P_{0,\omega}$ as $\epsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

Part (*b*) of the above Theorem is based on a result of Rassoul-Agha and Seppäläinen [RAS09], which gives as a condition so that an elliptic random walk satisfies the quenched central limit theorem that the regeneration times have moments of order higher than $176d$. As they point out in their paper, this particular lower bound on the moment should not have any meaning and it is likely that it could be improved. For example, Berger and Zeitouni in [BZ08], also prove the quenched central limit theorem under lower order moments for the regeneration times but under the assumption of uniform ellipticity. It should be possible to extend their methods to elliptic random walks in order to improve the moment condition of part (*b*) of Theorem 1.3.

It is possible using the methods developed in this paper, to obtain slowdown large deviation estimates for the position of the random walk in the spirit of the estimates obtained by Sznitman in [Sz00] for the case where the environment is plain nestling (see also [Ber12]), under hypothesis of the form

$$\mathbb{E}\left[e^{(\log \omega(0,e))^{\beta}}\right] < \infty,$$

for some $\beta > 1$. Nevertheless, the estimates we would obtain would not be sharp, in the sense that we would obtain an upper bound for the probability that at time n the random walk is slowed down of the form

$$e^{-(\log n)^{\beta'(d)}},$$

where $\beta'(d) > 1$, but $\beta'(d) < d$ (as discussed in [Sz04] and shown in [Ber12], the exponent d is optimal). We have therefore not included them in this article.

The proof of Theorem 1.1 requires extending the methods that have already been developed within the context of random walks in uniformly elliptic random environments. Its proof is presented in section 3. To do this, we first need to show as in [BDR12], that the polynomial condition $(P)_M$ for $M \geq 15d+5$, implies the so called effective criterion, defined by Sznitman in [Sz02] for random walks in uniformly elliptic environments, and extended here for random walks in random environments satisfying condition $(E)_0$. Two renormalization methods are employed here, which need to take into account the fact that the environment is not necessarily uniformly elliptic. These are developed in subsections 3.1 and 3.2. In subsection 3.4 it is shown, following [Sz02], that the effective criterion implies condition (T'). The adaptation of the methods of [BDR12] and [Sz02] from uniformly elliptic environments to environments satisfying some of the ellipticity conditions that have been introduced is far from being straightforward.

The proof of Theorems 1.2 and 1.3, is presented in sections 4 and 5. In section 4, an atypical quenched exit estimate is derived which requires a very careful choice of the renormalization method, and includes the definition of an event which we call the *confinement event*, which ensures that the random walk will be able to find a path to an exit column where it behaves as if the environment was uniformly elliptic. In section 5, we derive the moments estimates of the regeneration time of the random walk using the atypical quenched exit estimate of section 4. Here, condition $(E')_1$ towards the asymptotic direction is required, and appears as the possibility of finding an appropriate path among $4d-2$ possibilities connecting two points in the lattice.

2 Notation and preliminary results

Here we will fix up the notation of the paper and will introduce the main tools that will be used. In subsection 2.2 we will prove Proposition 1.1. Its proof is straightforward, but instructive.

2.1 Setup and background

Throughout the whole paper we will use letters without sub-indexes like c, ρ or κ to denote any generic constant, while we will use the notation $c_{3,1}, c_{3,2}, \ldots, c_{4,1}, c_{4,2}, \ldots$ to denote the specific constants which appear in each section of the paper. Thus, for example $c_{4,2}$ is the second constant of section 4. On the other hand, we will use c_1, c_2, c_3, c_4, c_1' and c_2' for specific constants which will appear several times in several sections. Let $c_1 \geq 1$ be any constant such that for any pair of points $x, y \in \mathbb{Z}^d$, there exists a nearest neighbor path between x and y with less than

$$c_1 \max\{|x-y|_2, 1\} \tag{2.1}$$

sites. Given $U \subset \mathbb{Z}^d$, we will denote its outer boundary by

$$\partial U := \{x \notin U : |x-y|_1 = 1, \text{ for some } y \in U\}.$$

We define $\{\theta_n : n \geq 1\}$ as the canonical time shift on $\mathbb{Z}^{d^{\mathbb{N}}}$. For $l \in \mathbb{S}^{d-1}$ and $u \geq 0$, we define the times

$$T_u^l := \inf\{n \geq 0 : X_n \cdot l \geq u\} \tag{2.2}$$

and

$$\tilde{T}_u^l := \inf\{n \geq 0 : X_n \cdot l \leq u\}.$$

Throughout, we will denote any nearest neighbor path with n steps joining two points $x, y \in \mathbb{Z}^d$ by $(x_1, x_2, \ldots, x_n)$, where $x_1 = x$ and $x_n = y$. Furthermore, we will employ the notation

$$\Delta x_i := x_{i+1} - x_i, \tag{2.3}$$

for $1 \leq i \leq n-1$, to denote the directions of the jumps through this path. Finally, we will call $\{t_x : x \in \mathbb{Z}^d\}$ the canonical shift defined on Ω so that for $\omega = \{\omega(y) : y \in \mathbb{Z}^d\}$,

$$t_x(\omega) = \{\omega(x+y) : y \in \mathbb{Z}^d\}. \tag{2.4}$$

Let us now define the concept of *regeneration times* with respect to direction l. Let

$$a > 2\sqrt{d} \tag{2.5}$$

and

$$D^l := \min\{n \geq 0 : X_n \cdot l < X_0 \cdot l\}.$$

Define $S_0 := 0$, $M_0 := X_0 \cdot l$,

$$S_1 := T^l_{M_0+a}, \quad R_1 := D^l \circ \theta_{S_1},$$

$$M_1 := \sup\{X_n \cdot l : 0 \leq n \leq R_1\},$$

and recursively for $k \geq 1$,

$$S_{k+1} := T^l_{M_k+a}, \quad R_{k+1} := D^l \circ \theta_{S_{k+1}} + S_{k+1},$$

$$M_{k+1} := \sup\{X_n \cdot l : 0 \leq n \leq R_{k+1}\}.$$

Define the *first regeneration time* as

$$\tau_1 := \min\{k \geq 1 : S_k < \infty, R_k = \infty\}.$$

The condition (2.5) on a will be eventually useful to prove the non-degeneracy of the covariance matrix of part (a) of Theorem 1.3. Now define recursively in n the $(n+1)$-st regeneration time τ_{n+1} as $\tau_1(X_\cdot) + \tau_n(X_{\tau_1+\cdot} - X_{\tau_1})$. Throughout the sequel, we will occasionally write $\tau_1^l, \tau_2^l, \ldots$ to emphasize the dependence of the regeneration times with respect to the chosen direction. It is a standard fact (see for example

Sznitman and Zerner [SZ99]) to show that the sequence $((\tau_1, X_{(\tau_1+\cdot)\wedge\tau_2} - X_{\tau_1}), (\tau_2 - \tau_1, X_{(\tau_2+\cdot)\wedge\tau_3} - X_{\tau_2}), \ldots)$ is independent and except for its first term also i.i.d. with the same law as that of τ_1 with respect to the conditional probability measure $P_0(\cdot|D^l = \infty)$. This implies the following theorem (see Zerner [Z02] and Sznitman and Zerner [SZ99] and Sznitman [Sz00]).

Theorem 2.1 (Sznitman and Zerner [SZ99], Zerner [Z02], Sznitman [Sz00]) *Consider a RWRE in an elliptic i.i.d. environment. Let $l \in \mathbb{S}^{d-1}$ and assume that there is a neighborhood V of l such that for every $l' \in V$ the random walk is transient in the direction l'. Then there is a deterministic v such that P_0-a.s. one has that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v.$$

Furthermore, the following are satisfied.

a) If $E_0[\tau_1] < \infty$, the walk is ballistic and $v \neq 0$.

b) If $E_0[\tau_1^2] < \infty$ we have that

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v\right)$$

converges in law under P_0 to a Brownian motion with non-degenerate covariance matrix.

In 2009, both Rassoul-Agha and Seppäläinen in [RAS09] and Berger and Zeitouni in [BZ08] were able to prove a quenched central limit theorem under good enough moment conditions on the regeneration times. The result of Rassoul-Agha and Seppäläinen which does not require a uniform ellipticity assumption is the following one.

Theorem 2.2 (Rassoul-Agha and Seppäläinen [RAS09]) *Consider a RWRE in an elliptic i.i.d. environment. Let $l \in \mathbb{S}^{d-1}$ and let τ_1 be the corresponding regeneration time. Assume that*

$$E_0[\tau_1^p] < \infty,$$

for some $p > 176d$. Then $\mathbb{P}$-a.s. we have that

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v\right)$$

converges in law under $P_{0,\omega}$ to a Brownian motion with non-degenerate covariance matrix.

We now define the *n-th regeneration radius* as

$$X^{*(n)} := \max_{\tau_{n-1}\leq k\leq\tau_n} |X_k - X_{\tau_{n-1}}|_1.$$

The following theorem was stated and proved without using uniform ellipticity by Sznitman as Theorem A.2 of [Sz02], and provides a control on the lateral displacement of the random walk with respect to the asymptotic direction. We need to define for $z \in \mathbb{R}^d$

$$\pi(z) := z - (z \cdot \hat{v})\hat{v}.$$

Theorem 2.3 (Sznitman [Sz02]) *Consider a RWRE in an elliptic i.i.d. environment satisfying condition $(T)_\gamma|l$. Let $l \in \mathbb{S}^{d-1}$ and $\gamma \in (0,1)$. Then, for any $c > 0$ and $\rho \in (0.5, 1)$,*

$$\limsup_{u\to\infty} u^{-(2\rho-1)\wedge(\gamma\rho)} \log P_0\left(\sup_{0\leq n\leq T_u^l} |\pi(X_n)| \geq cu^\rho\right) < 0,$$

where T_u^l is defined in (2.2).

Define the function $\gamma_L : [2,\infty) \to \mathbb{R}$ as

$$\gamma_L := \frac{\log 2}{\log\log L}. \tag{2.6}$$

Given $l \in \mathbb{S}^{d-1}$, we say that condition $(T)_0$ in direction l (also written as $(T)_0|l)$ is satisfied if there exists a neighborhood $V \subset \mathbb{S}^{d-1}$ of l such that for all $l' \in V$

$$\limsup_{L\to\infty} \frac{1}{L^{\gamma_L}} \log P_0(X_{T_{U_{l',L}}} \cdot l' < 0) < 0,$$

where the slabs $U_{l',L}$ are defined in (1.2).

Throughout this paper we will also need the following generalization of an equivalence proved by Sznitman [Sz02], for the case $\gamma \in (0,1)$ and which does not require uniform ellipticity. It is easy to extend Sznitman's proof to include the case $\gamma = 0$.

Theorem 2.4 (Sznitman [Sz02]) *Consider a RWRE in an elliptic i.i.d. environment. Let $\gamma \in [0,1)$ and $l \in S^{d-1}$. Then the following are equivalent.*

(i) Condition $(T)_\gamma|l$ is satisfied.

(ii) $P_0(A_l) = 1$ and if $\gamma > 0$ we have that $E_0[\exp\{c(X^{(1)})^\gamma\}] < \infty$ for some $c > 0$, while if $\gamma = 0$ we have that $E_0[\exp\{c(X^{*(1)})^{\gamma_L}\}] < \infty$ for some $c > 0$.

(iii) There is an asymptotic direction $\hat{v}$ such that $l \cdot \hat{v} > 0$ and for every l' such that $l' \cdot \hat{v} > 0$ one has that $(T)_\gamma|l'$ is satisfied.

The following corollary of Theorem 2.4 will be important.

Corollary 2.1 (Sznitman [Sz02]) *Consider a RWRE in an elliptic i.i.d. environment. Let $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$. Assume that $(T)_\gamma|_l$ holds. Then there exists a constant c such that for every L and $n \geq 1$ one has that*

$$P_0\left(X^{*(n)} > L\right) \leq \frac{1}{c} e^{-cL^\gamma}. \tag{2.7}$$

2.2 Comments and proof of Proposition 1.1

Here we will show that $(E')_1$ implies (ES). We will do this passing through another ellipticity condition. We say that condition (ES') is satisfied if there exist non-negative real numbers $\alpha_1, \ldots, \alpha_d$ and $\alpha'_1, \ldots, \alpha'_d$ such that

$$\min_{1\leq i\leq d} (\alpha_i + \alpha'_i) > 1$$

and

$$\max_{1\leq i\leq d} \mathbb{E}\left[\left(\frac{1}{1-\omega(0,e_i)}\right)^{\alpha_i}\right] < \infty \quad \text{and} \quad \max_{1\leq i\leq d} \mathbb{E}\left[\left(\frac{1}{1-\omega(0,e_{i+d})}\right)^{\alpha'_i}\right] < \infty. \tag{2.8}$$

We have the following lemma.

Lemma 2.1 *Consider a random walk in an i.i.d. random environment. Then condition $(E')_1$ implies (ES') which in turn implies (ES). Furthermore, for a random walk in a random Dirichlet environment, (ES) and (ES') are equivalent to $\lambda > 1$ (cf. (1.9)).*

Proof. It is easy to check that that for random Dirichlet environments (ES') and (ES) are equivalent to $\lambda > 1$. We therefore first prove that (ES') implies (ES). Note first that by the independence between $\omega(0, e_i)$ and $\omega(e_i, -e_i)$, (2.8) is equivalent to

$$\max_{1\leq i\leq d} \mathbb{E}\left[\left(\frac{1}{1-\omega(0,e_i)}\right)^{\alpha_i}\left(\frac{1}{1-\omega(e_i,-e_i)}\right)^{\alpha_i'}\right] < \infty.$$

Then it is enough to prove that for each pair of real numbers u_1, u_2 in $(0, 1)$ one has that

$$\frac{1}{1-u_1u_2} \leq \frac{1}{(1-u_1)^{\alpha}(1-u_2)^{\alpha'}} \tag{2.9}$$

for any α, $\alpha' \geq 0$ such that $\alpha + \alpha' > 1$. Now if we denote by $v_1 = 1 - u_1$ and $v_2 = 1 - u_2$ then (2.9) is equivalent to

$$v_1v_2 + v_1^{\alpha}v_2^{\alpha'} \leq v_1 + v_2. \tag{2.10}$$

But (2.10) follows easily by our conditions on v_1, v_2, α and α'. To prove that $(E')_1$ implies (ES'), we choose for each $1 \leq i \leq d$

$$\alpha_i = \sum_{\alpha(e)\neq\alpha(e_i)} \alpha(e), \quad \alpha_i' = \sum_{\alpha(e)\neq\alpha(e_{i+d})} \alpha(e).$$

Note in particular that

$$\alpha_i + \alpha_i' > 1, \quad \forall i \in \{1, \dots, d\}. \tag{2.11}$$

Then, $(E')_1$ and the monotonicity of the function $\log x$ imply that

$$\mathbb{E}\left(e^{-\sum_{e\neq e_i}\alpha(e)\log\omega(0,e)}\right) < \infty, \quad \alpha_i \log \sum_{e\neq e_i} \omega(0,e) \geq \sum_{e\neq e_i} \alpha(e)\log\omega(0,e). \tag{2.12}$$

for each $1 \leq i \leq d$. Then (ES') follows by (2.11) and (2.12).

∎

Let us now prove Proposition 1.1. If the random walk is not transient in any direction, there is nothing to prove. So assume that the random walk is transient in a direction l and hence the corresponding regeneration times are well defined. Essentially, we will exhibit a trap as the one depicted in Figure 1, in the edge $(0, e_i)$. Define the first exit time of the random walk form the edge $(0, e_i)$, so that

$$F := \min\{n \geq 0 : X_n \notin (0, e_i)\}.$$

We then have for every $k \geq 0$ that

$$P_{x,\omega}(F = 2k + 2) = \omega_1^{k+1}\omega_2^k(1 - \omega_2),$$

and

$$P_{x,\omega}(F = 2k + 1) = \omega_1^k\omega_2^k(1 - \omega_1).$$

Hence,

$$P_{x,\omega}(F > 2k) = (\omega_1\omega_2)^k$$

and

$$\sum_{k=0}^{\infty} P_{x,\omega}(F_x > 2k) = \frac{1}{1-\omega_1\omega_2}. \tag{2.13}$$

This proves that under the annealed law,

$$E_0(F) = \infty.$$

We can now show using the strong Markov property under the quenched measure and the i.i.d. nature of the environment, that for each natural $m > 0$, the time $T_m := \min\{n \geq 0 : X_n \cdot l > m\}$ can be bounded from below by a sequence $F_1, \ldots, F_m$ of random variables which under the annealed measure are i.i.d. and distributed as F. This proves that P_0-a.s. $T_m/m \to \infty$ which implies that the random walk is not ballistic in direction l.

3 Equivalence between the polynomial ballisticity condition and (T')

Here we will prove Theorem 1.1, establishing the equivalence between the polynomial condition $(P)_M$ and condition (T'). To do this, we will pass through both the effective criterion and an version of condition $(T)_\gamma$ which corresponds to the choice of $\gamma = \gamma_L$ according to (2.6) (see [BDR12]). Now, to prove Theorem 1.1, we will first show in subsection 3.1 that $(P)_M$ implies $(T)_0$ for $M \geq 15d + 5$. In subsection 3.2, we will prove that $(T)_0$ implies a weak kind of an atypical quenched exit estimate. In these first two steps, we will generalize the methods presented in [BDR12] for random walks satisfying condition $(E)_0$. In subsection 3.3, we will see that this estimate implies the effective criterion. Finally, in subsection 3.4, we will show that the effective criterion implies (T'), generalizing the method presented by Sznitman [Sz02], to random walks satisfying $(E)_0$.

Before we continue, we will need some additional notation. Let $l \in \mathbb{S}^{d-1}$. Let $L, L' > 0$, $\tilde{L} > 0$,

$$B(R, L, L', \tilde{L}) := R\left((-L, L') \times (-\tilde{L}, \tilde{L})^{d-1}\right) \cap \mathbb{Z}^d \tag{3.1}$$

and

$$\partial_+ B(R, L, L', \tilde{L}) := \partial B \cap \left\{x \in \mathbb{Z}^d : x \cdot l \geq L', |R(e_j) \cdot x| < \tilde{L}, \text{ for each } 2 \leq j \leq d\right\}.$$

Here R is the rotation defined by (1.3). When there is no risk of confusion, we will drop the dependence of $B(R, L, L', \tilde{L})$ and $\partial_+ B(R, L, L', \tilde{L})$ with respect to R, L, L' and $\tilde{L}$ and write B and $\partial_+ B$ respectively. Let also,

$$\rho_B := \frac{P_{0,\omega}(X_{T_B} \notin \partial_+ B)}{P_{0,\omega}(X_{T_B} \in \partial_+ B)} = \frac{q_B}{p_B},$$

where $q_B := P_{0,\omega}(X_{T_B} \notin \partial_+ B)$ and $p_B := P_{0,\omega}(X_{T_B} \in \partial_+ B)$ and for $0 < \alpha < \min_{e \in U} F_e$,

$$\eta_\alpha := \sup_{e \in U} \mathbb{E}\left[\left(\frac{1}{\omega(0, e)}\right)^\alpha\right]. \tag{3.2}$$

3.1 Polynomial ballisticity implies $(T)_0$

Here we will prove that the Polynomial ballisticity condition implies $(T)_0$. To do this, we will use a multi-scale renormalization scheme as presented in Section 3 of [BDR12]. Let us note that [BDR12] assumes that the walk is uniformly elliptic.

Proposition 3.1 *Let $M > 15d + 5$ and $l \in \mathbb{S}^{d-1}$. Assume that conditions $(P)_M|l$ and $(E)_0$ are satisfied. Then $(T)_0|l$ holds.*

Let us now to prove Proposition 3.1. Let $N_0 \geq \frac{3}{2} L_0$, where L_0 is defined in (1.4). For $k \geq 0$, define recursively the scales

$$N_{k+1} := 3(N_0 + k)^2 N_k.$$

Define also for $k \geq 0$ and $x \in \mathbb{R}^d$ the boxes

$$B(x, k) := \left\{y \in \mathbb{Z}^d : -\frac{N_k}{2} < (y - x) \cdot l < N_k, \ |(y - x) \cdot R(e_i)| < 25 N_k^3 \text{ for } 2 \leq i \leq d\right\}$$

and their *middle frontal part*

$$\tilde{B}(x,k) := \left\{y \in \mathbb{Z}^d : N_k - N_{k-1} \le (y-x)\cdot l < N_k,\ |(y-x)\cdot R(e_i)| < N_k^3 \text{ for } 2 \le i \le d\right\}$$

with the convention that $N_{-1} := 2N_0/3$. We also define the *the front side*

$$\partial_+ B(x,k) := \{y \in \partial B(x,k) : (y-x)\cdot l \ge N_k\},$$

the *back side*

$$\partial_- B(x,k) := \{y \in \partial B(x,k) : (y-x)\cdot l \le -\frac{N_k}{2}\},$$

and the *lateral sides*

$$\partial_l B(x,k) := \{y \in \partial B(x,k) : |(y-x)\cdot R(e_i)| \ge 25N_k^3 \text{ for } 2 \le i \le d\}.$$

We need to define for each $n, m \in \mathbb{N}$ the sub-lattices

$$\mathcal{L}_{n,m} := \{x \in \mathbb{Z}^d : [x\cdot l] \in n\mathbb{Z},\ [x\cdot R(e_j)] \in m\mathbb{Z},\ \text{for } 2 \le j \le d\}$$

and refer to the elements of

$$\mathcal{B}_k := \left\{B(x,k) : x \in \mathcal{L}_{N_{k-1}-1, N_k^3-1}\right\}$$

as *boxes of scale* k. When there is no risk of confusion, we will denote a typical element of this set by B_k or simply B and its middle part as $\tilde{B}_k$ or $\tilde{B}$. Furthermore, we have

$$\cup_{B\in\mathcal{B}_k} \tilde{B} = \mathbb{Z}^d,$$

which will be an important property that we will be useful. In this subsection, it is enough to assume a weaker condition than $(P)_M|l$. The following lemma is straightforward, so its proof will be omitted.

Lemma 3.1 *Let $M > 0$ and $l \in \mathbb{S}^{d-1}$. Assume that condition $(P)_M|l$ is satisfied. Then, whenever $N_0 \ge \frac{2}{3}L_0$ one has that*

$$\sup_{x\in\tilde{B}_0} P_x\left(X_{T_{B_0}} \notin \partial_+ B_0\right) < N_0^{-M}.$$

We now say that box $B \in \mathcal{B}_0$ is *good* if

$$\sup_{x\in\tilde{B}_0} P_{x,\omega}\left(X_{T_{B_0}} \notin \partial_+ B_0\right) < N_0^{-5}.$$

Otherwise, we say that the box $B \in \mathcal{B}_0$ is *bad*. The following lemma appears in [BDR12] as Lemma 3.3.

Lemma 3.2 *Let $M > 0$ and $l \in \mathbb{S}^{d-1}$. Assume that $(P)_M|_l$ holds. Then for all $B_0 \in \mathcal{B}_0$ and $N_0 \ge \frac{2}{3}L_0$,*

$$\mathbb{P}(B_0 \text{ is good}) \ge 1 - 2^{d-1}N_0^{3d+3-M}.$$

Proof. Note that

$$\mathbb{P}(B_0 \text{ is bad}) \le \sum_{x\in\tilde{B}_0} \mathbb{P}\left(P_{x,\omega}\left(X_{T_{B_0}} \notin \partial_+ B_0\right) \ge N_0^{-5}\right). \tag{3.3}$$

Now by Markov's inequality we have for $x \in \tilde{B}_0$ that

$$\mathbb{P}\left(P_{x,\omega}\left(X_{T_{B_0}} \notin \partial_+ B_0\right) \ge N_0^{-5}\right) \le N_0^5 \sup_{x\in\tilde{B}_0} P_x\left(X_{T_{B_0}} \notin \partial_+ B_0\right). \tag{3.4}$$

Now, with the help of Lemma 3.1, (3.3), (3.4) and from a routine counting argument we obtain

$$\mathbb{P}(B_0 \text{ is bad}) \le 2^{d-1} N_0^{3d+3-M}.$$

∎

Now, we want to extend the concept of good and bad boxes of scale 0 to boxes of any scale $k \ge 1$. To do this, due to the lack of uniform ellipticity, we need to modify the notion of good and bad boxes for scales $k \ge 1$ presented in Berger, Drewitz and Ramírez [BDR12]. Consider a box Q_{k-1} of scale $k-1 \ge 1$. For each $x \in \tilde{Q}_{k-1}$ we associate a natural number n_x and a self-avoiding path $\pi^{(x)} := (\pi_1^{(x)}, \dots, \pi_{n_x}^{(x)})$ starting from x so that $\pi_1^{(x)} = x$, such that $(\pi_{n_x}^{(x)} - x) \cdot l \ge N_{k-2}$ and so that

$$c_{3,1} N_{k-2} \le n_x \le c_{3,2} N_{k-2},$$

for some pair of constants $c_{3,1}$ and $c_{3,2}$. Now, let

$$\xi := \frac{1}{2} e^{-\frac{c_{3,2} \log \eta_\alpha + 9d}{c_{3,1}}}. \tag{3.5}$$

We say that the box $Q_{k-1} \in \mathcal{B}_{k-1}$ is *elliptically good* if for each $x \in \tilde{Q}_{k-1}$ one has that

$$\sum_{i=1}^{n_x} \log \frac{1}{\omega(\pi_i^{(x)}, \Delta\pi_i^{(x)})} \le n_x \log\left(\frac{1}{\xi}\right).$$

Otherwise the box is called *elliptically bad*. We can now recursively define the concept of good and bad boxes. For $k \ge 1$ we say that a box $B_k \in \mathcal{B}_k$ is *good*, if the following are satisfied:

(a) There is a box $Q_{k-1} \in \mathcal{B}_{k-1}$ which is elliptically good.

(b) Each box $C_{k-1} \in \mathcal{B}_{k-1}$ of scale $k-1$ satisfying $C_{k-1} \cap Q_{k-1} \ne \emptyset$ and $C_{k-1} \cap B_k \ne \emptyset$ is elliptically good.

(c) Each box $B_{k-1} \in \mathcal{B}_{k-1}$ of scale $k-1$ satisfying $B_{k-1} \cap Q_{k-1} = \emptyset$ and $B_{k-1} \cap B_k \ne \emptyset$, is good.

Otherwise, we say that the box B_k is *bad*. Now we will obtain an important estimate on the probability that a box of scale $k \ge 1$ is good, corresponding to Lemma 3.4 of [BDR12]. Nevertheless, note that here we have to deal with our different definition of good and bad boxes due to the lack of uniform ellipticity. Let

$$c_{3,3} := c_{3,1} \log \frac{1}{\xi} - c_{3,2} \log \eta_\alpha - 9d = c_{3,1} \log 2 > 0.$$

We first need the following estimate.

Lemma 3.3 *For each $k \ge 1$ we have that*

$$\mathbb{P}(B_k \text{ is not elliptically good}) \le e^{-c_{3,3} N_{k-1}}. \tag{3.6}$$

Proof. By translation invariance and using Chebychev's inequality as well as independence, we have that for any $\alpha > 0$

$$\begin{aligned} \mathbb{P}(B_k \text{ is not elliptically good}) &\le \textstyle\sum_{x \in \tilde{B}_k} \mathbb{P}\left(\sum_{i=1}^{n_x} \log \frac{1}{\omega(\pi_i^{(x)}, \Delta\pi_i^{(x)})} > n_x \log\left(\frac{1}{\xi}\right)\right) \\ &\le N_{k-1} N_k^{3(d-1)} e^{-N_{k-1}\left(c_{3,1}\alpha \log\left(\frac{1}{\xi}\right) - c_{3,2} \log \eta_\alpha\right)} \\ &\le e^{-N_{k-1}\left(c_{3,1} \log\left(\frac{1}{\xi}\right) - c_{3,2} \log \eta_\alpha - 9d\right)} \end{aligned}$$

where $N_{k-1} N_k^{3(d-1)}$ is an upper bound for $|\tilde{B}_k|$ and we have used the inequality $N_k \le 12 N_{k-1}^3$. But this expression can be bounded by e^{9dN_k} due to our choice of N_0. Then for any $\alpha > 0$, using the definition of ξ in (3.5), we have that

$$\mathbb{P}(B_k \text{ is not elliptically good}) \le e^{-c_{3,3}N_{k-1}}.$$

∎

We can now state the following lemma giving an estimate for the probability that a box of scale $k \ge 0$ is bad. We will use Lemma 3.1.

Lemma 3.4 *Let $l \in \mathbb{S}^{d-1}$, $M \ge 15d + 5$, and assume that $(P)_M|l$ is satisfied. Then for $N_0 \ge \frac{3}{2}L_0$ one has that for all $k \ge 0$ and all $B_k \in \mathcal{B}_k$,*

$$\mathbb{P}(B_k \text{ is good}) \ge 1 - e^{-2^k}.$$

Proof. By Lemma 3.2 we see that

$$\mathbb{P}(B_0 \text{ is bad}) \le e^{-c_{3,0}},$$

where

$$c'_{3,0} := \log \frac{N_0^{M-3d-3}}{2^{d-1}}.$$

We will show that this implies for all $k \ge 1$ that

$$P_0(B_k \text{ is bad}) \le e^{-c'_{3,k}2^k}, \tag{3.7}$$

for a sequence of constants $\{c'_{3,k} : k \ge 0\}$ defined recursively by

$$c'_{3,k+1} := c'_{3,k} - \frac{\log\left(3^{16d}(N_0+k)^{12d}\right)}{2^{k+1}}. \tag{3.8}$$

We will now prove (3.7) using induction on k. To simplify notation, we will denote by q_k for $k \ge 0$, the probability that the box B_k is bad. Assume that (3.7) is true for some k, $k \ge 0$. Let A be the event that all boxes of scale k that intersect B_{k+1} are elliptically good, and B the event that each pair of bad boxes of scale k have a non-empty intersection. Note that the event $A \cap B$ implies that the box B_{k+1} is good. Therefore, the probability q_{k+1} that the box B_{k+1} is bad is bounded by the probability that there are at least two bad boxes B_k which intersect B_{k+1} plus the probability that there is at least one elliptically bad box of scale k, so that by Lemma 3.3, for each $k \ge 0$ one has that

$$q_{k+1} \le m_k^2 q_k^2 + m_k e^{-c_{3,3}N_k}, \tag{3.9}$$

where m_k is the total number of bad boxes of scale k that intersect B_{k+1}. Now note that

$$\sqrt{2}m_k \le 3^{8d}(N_0+k)^{6d}. \tag{3.10}$$

But by the the fact that $c_{3,3}N_k \ge c'_{3,k}2^{k+1}$ for $k \ge 0$ we have that

$$e^{-c_{3,3}N_k} \le e^{-c'_{3,k}2^{k+1}}.$$

Hence, substituting this estimate and estimate (3.10) back into (3.9) and using the induction hypothesis, we conclude that

$$q_{k+1} \le 3^{16d}(N_0+k)^{12d}e^{-c'_{3,k}2^{k+1}} = e^{-c'_{3,k+1}2^{k+1}}.$$

Now note that the recursive definition (3.8) implies that

$$c'_{3,k} \ge \log \frac{N_0^{M-3d-3}}{2^{d-1}} - \sum_{k=0}^{\infty} \frac{\log\left(3^{16d}(N_0+k)^{12d}\right)}{2^{k+1}}.$$

Using the inequality $\log(a+b) \le \log a + \log b$ valid for $a, b \ge 1$, we see that

$$\sum_{k=0}^{\infty} \frac{\log\left(3^{16d}(N_0+k)^{12d}\right)}{2^{k+1}} \le 16d\log 3 + 12d\log N_0 + 12d.$$

From these estimates we see that whenever $M \ge 15d+5$ and

$$\log N_0 - \log 2^{d-1}3^{16d}e^{12d+1} \ge 0, \tag{3.11}$$

then for every $k \ge 0$, one has that $c'_{3,k} \ge 1$. But (3.11) is clearly satisfied for $N_0 \ge 3^{29d}$.

∎

The next lemma establishes that the probability that a random walk exits a box B_k through its lateral or back side is small if this box is good.

Lemma 3.5 *There is a constant $c_{3,4} > 0$ such that for each $k \ge 0$ and $B_k \in \mathcal{B}_k$ which is good one has*

$$\sup_{x \in \tilde{B}_k} P_{x,\omega}\left(X_{T_{B_k}} \notin \partial_+ B_k\right) \le e^{-c_{3,4}N_k}.$$

Proof. Let us first note that for each $k \ge 0$,

$$P_{x,\omega}\left(X_{T_{B_k}} \notin \partial_+ B_k\right) \le P_{x,\omega}\left(X_{T_{B_k}} \in \partial_- B_k\right) + P_{x,\omega}\left(X_{T_{B_k}} \in \partial_l B_k\right).$$

We denote by $p_k := \sup_{x\in\tilde{B}_k} P_{x,\omega}\left(X_{T_{B_k}} \in \partial_l B_k\right)$ and $r_k := \sup_{x\in\tilde{B}_k} P_{x,\omega}\left(X_{T_{B_k}} \in \partial_- B_k\right)$. We will first show by induction on k that

$$p_k \le e^{-c''_{3,k}N_k} \quad \text{and} \tag{3.12}$$

$$r_k \le e^{-c''_{3,k}N_k}, \tag{3.13}$$

where

$$c''_{3,k} := \frac{5\log N_0}{N_0} - \sum_{j=1}^{k} \frac{\log 27(N_0+j)^4}{N_{j-1}} - \sum_{j=1}^{k} \frac{5N_{j-1} + \log 24 + 6d(\log\xi)^2 N_{j-1}}{N_j},$$

and ξ is defined in (3.5). The case $k=0$ follows easily by the definition of good box at scale 0 with

$$c''_{3,0} := \frac{5\log N_0}{N_0}.$$

Now, we assume that (3.12) and (3.13) hold for some $k \ge 0$ and will show that this implies that (3.12) is satisfied for $k+1$. Let κ_1 be the first time that the random walk exits some fixed box of scale k whose middle part frontal part contains the point x. Define recursively for every $n \ge 1$, κ_{n+1} as the first time after time κ_n such that the random walk exits some fixed box of scale k whose middle frontal part contains the point X_{κ_n}. We choose these fixed boxes arbitrarily. We now define the *rescaled random walk* $\{Y_n : n \ge 0\}$ as

$$Y_0 := x \quad \text{and} \quad Y_n := X_{\kappa_n},$$

for $n \ge 1$. Since the box B_{k+1} is good, we know that there exists a box $Q_k \in \mathcal{B}_k$ such that every box of scale k, intersecting B_{k+1} but not Q_k, is good. Let us now define for each $k \ge 1$ the collection of sets

$$\mathcal{S}_k := \begin{array}{l}\{B_k \in \mathcal{B}_k : B_k \cap B_{k+1} \neq \emptyset, \text{ and } \forall\, i \in \{2,\dots,d\}, x\cdot R(e_i) = y\cdot R(e_i), \\ \quad \text{for some } x \in B_k,\, y \in Q_k\}.\end{array}$$

In words, this is the collection of boxes of scale k which have at least one point whose component orthogonal to l coincides with the component orthogonal to l of some point in Q_k. Now, define the strip

$$S_k := \bigcup_{B_k \in \mathcal{S}_k} B_k.$$

(See Figure 2)

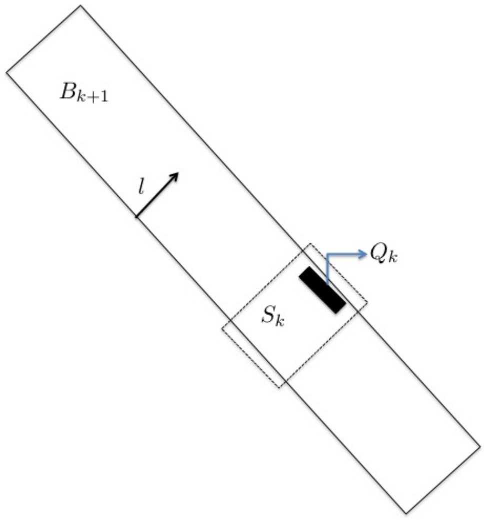

Figure 2: The bad box Q_k with its strip S_k.

Let m_1 be the first time that the random walk $\{Y_n\}$ is at a distance larger than $7N_{k+1}^3$ from the strip S_k and from the sides $\partial_l B_{k+1}$ of the box B_{k+1},

$$m_1 := \inf\left\{n \geq 0 : \mathrm{dist}(Y_n, S_k) \geq 7N_{k+1}^3 \quad \text{and} \quad \mathrm{dist}(Y_n, \partial_l B_{k+1}) \geq 7N_{k+1}^3\right\}.$$

Let m_2 be the first time that the random walk $\{Y_n\}$ exits the box B_{k+1} so that

$$m_2 := \inf\{n \geq 0 : Y_n \notin B_{k+1}\}$$

and note that on the event $\left\{X_{T_{B_{k+1}}} \in \partial_l B_{k+1}\right\}$ one has that

$$m_1 < m_2 < \infty.$$

Also, define

$$m_3 := \{n > m_1 : Y_n \in S_k\}.$$

Define

$$J_k := \frac{3N_{k+1}/2}{N_{k-1}} + 1.$$

This is the minimal number of steps needed by the random walk $\{Y_n\}$ to exit the box B_{k+1} through its front side. Then, we have that a.s. on the event $\left\{X_{T_{B_{k+1}}} \in \partial_l B_{k+1}\right\}$ the following inequality is satisfied

$$m_2 \wedge m_3 - m_1 \geq \frac{7N_{k+1}^3}{25N_k^3} \geq \frac{4}{25} J_k \frac{N_{k+1}}{N_k} + 1.$$

Now note that starting from Y_{m_1} if the random walk $\{Y_n\}$ consecutively exits J_k boxes of scale k through their front side, it would leave the box B_{k+1} through $\partial_+ B_{k+1}$. Therefore, by the induction hypothesis we have that

$$P_{Y_{m_1},\omega}(Y_j \in B_k, \text{ for all } 1 \le j \le J_k) \le J_k e^{-c''_{3,k} N_k}.$$

Thus, by the Markov property we get that

$$P_{x,\omega}(X_{T_{B_{k+1}}} \in \partial_l B_{k+1}) \le \left(e^{-c''_{3,k} N_k + \log J_k}\right)^{N_{k+1}/N_k} \le e^{-c''_{3,k+1} N_{k+1}}.$$

This completes the proof of (3.12) for k.

Recall the definition of r_k. We will now assume that (3.12) and (3.13) hold for some $k \ge 0$ and will show that (3.13) is satisfied for $k+1$. Define

$$L_{Q_k} := \inf\{l \cdot z : z \in Q_k\} - N_{k-1} \qquad R_{Q_k} := \sup\{l \cdot z : z \in Q_k\} + \frac{3}{2} N_k,$$

where Q_k is a box of scale k in B_{k+1} with the property that any other box of scale k which does not intersect it but which intersect B_{k+1} is good, while any other box of scale k which does intersect it but which intersects B_{k+1} is elliptically good. We will define a one dimensional random walk which at most sites has a very strong drift to the right (towards the front side of the box) whenever it is at any site $x \in \mathbb{Z} \backslash ([L_{Q_k}, R_{Q_k}] \cap \mathbb{Z})$: we define $\{Z_n : n \ge 0\}$ as a random walk which at each unit time, if it is at site $x \in \mathbb{Z} \backslash ([L_{Q_k}, R_{Q_k}] \cap \mathbb{Z})$, it jumps N_{k-1} steps to the right with probability $1 - e^{-c''_{3,k} N_k}$ and $\frac{3}{2} N_k$ steps to the left with probability $e^{-c''_{3,k} N_k}$, while if it as a site $x \in \mathbb{Z} \cap [L_{Q_k}, R_{Q_k}]$ it jumps N_{k-1} steps to the right with probability $\xi^{N_{k-1}}$ and $\frac{3}{2} N_k$ steps to the left with probability $1 - \xi^{N_{k-1}}$. We will call P_z the law of this random walk starting from $z \in \mathbb{Z}$. Let us call H_k the first hitting time of the random walk to the strip defined by L_{Q_k} and R_{Q_k} so that

$$H_k := \inf\{n \ge 0 : X_n \cdot l \in [L_{Q_k}, R_{Q_k}]\}.$$

Coupling in the natural way the random walk $\{X_n\}$ with the random walk $\{Z_n\}$, now note that

$$\begin{aligned} &\sup\nolimits_{x \in \tilde{B}_{k+1}} P_{x,\omega}(X_{T_{B_{k+1}}} \in \partial_- B_{k+1}) \\ &\le \sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(H_k \le T_{\partial_l B_{k+1}} \wedge T_{\partial_+ B_{k+1}}) \times \sup_{z \in [L_{Q_k}, R_{Q_k}] \cap \mathbb{Z}} P_z\left(T_{-\left(R_{Q_k} + \frac{N_{k+1}}{2}\right)} < T_{N_{k+1} - R_{Q_k}}\right). \end{aligned} \tag{3.14}$$

But,

$$\begin{aligned} &\sup\nolimits_{x \in \tilde{B}_{k+1}} P_{x,\omega}(H_k \le T_{\partial_l B_{k+1}} \wedge T_{\partial_+ B_{k+1}}) \\ &\le \sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(X_{T_{B_{k+1}}} \in \partial_l B_{k+1}) + \sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(H_k \le T_{\partial_l B_{k+1}} \wedge T_{\partial_+ B_{k+1}}, T_{\partial B_{k+1}} \ne T_{\partial_l B_{k+1}}). \end{aligned}$$

Now, by the estimate already done concerning the probability to exit the box B_{k+1} through the sides, we know that the first term is bounded from above by $e^{-c''_{3,k+1} N_{k+1}}$. For the second term, we couple the random walk to the random walk $\{Z_n\}$ previously defined. It is easy to see that $\{Z_n\}$ can be coupled to a random walk $\{Z'_n\}$ which jumps $\frac{3}{2} N_k$ steps to the right with probability $(1 - e^{-c''_{3,k} N_k})^{3N_k/(2N_{k-1})}$ and $\frac{3}{2} N_k$ steps to the left with probability $1 - (1 - e^{-c''_{3,k} N_k})^{3N_k/(2N_{k-1})}$. Now, the probability that a random walk which jumps one step to the right with probability p and one to the left with probability q to exit the interval $[-a, b] \cap \mathbb{Z}$ through $-a$, where $a, b \in \mathbb{Z}$ is given by

$$q^a \frac{p^b - q^b}{p^{b+a} - q^{b+a}}.$$

Applying the above formula with $a = (N_{k+1} - N_k - R_{Q_k})/(3N_k/2)$, $b = 2$, $p = (1 - e^{-c''_{3,k} N_k})^{3N_k/(2N_{k-1})}$ and $q = 1 - p$ we get that for $N_0 \ge \log \frac{1}{\xi}$,

$$\sup_{x \in \tilde{B}_k} P_{x,\omega}(H_k \le T_{\partial_l B_k} \wedge T_{\partial_+ B_k}, T_{\partial B_k} \ne T_{\partial_l B_k}) \le e^{-\frac{c''_{3,k}}{4}(N_{k+1} - N_k - R_{Q_k})}.$$

We will now find an upper bound for the second factor of (3.14). Let $z \in [L_{Q_k}, R_{Q_k}] \cap \mathbb{Z}$ and define the events

$$D^+ := \left\{ T_{N_{k+1}-R_{Q_k}} < T_z \circ \theta_1(Z) \right\} \quad \text{and} \quad D^- := \left\{ T_{-\left(R_{Q_k}+\frac{N_{k+1}}{2}\right)} < T_z \circ \theta_1(Z) \right\}.$$

It is straightforward to see that

$$\sup_{z \in [L_{Q_k}, R_{Q_k}] \cap \mathbb{Z}} P_z \left(T_{-\left(R_{Q_k}+\frac{N_{k+1}}{2}\right)} < T_{N_{k+1}-R_{Q_k}} \right) \leq \sup_{z \in \left[L_{Q_k}, R_{Q_k}\right] \cap \mathbb{Z}} \frac{P_z(D^-)}{P_z(D^+)}.$$

Now, by the fact that the box Q_k and those which intersect it are elliptically good, we conclude as in [BDR12] that for N_0 large enough,

$$P_z(D^+) \geq \frac{1}{2} \xi^{c_{3,2} 4 N_k},$$

where ξ is defined in (3.5). On the other hand, by the strong Markov property we conclude that

$$P_z(D^-) \leq 3 \left(e^{-c''_{3,k} N_k} \right)^{\frac{L_{Q_k}+N_{k+1}/2}{N_k}}.$$

From here we see that

$$\sup_{x \in \tilde{B}_k} P_{x,\omega} \left(X_{T_{B_k}} \in \partial_- B_k \right) \leq e^{-c''_{3,k} N_k}.$$

It is easy to check that

$$c_{3,4} := \inf_k c''_{3,k} > 0.$$

∎

We can now repeat the last argument of Proposition 2.1 of [BDR12], which does not require uniform ellipticity, to finish the proof of Proposition 3.1.

3.2 Condition $(T)_0$ implies a weak atypical quenched exit estimate

In this subsection we will prove that the condition $(T)_0$ implies a weak atypical quenched exit estimate. Throughout, we will denote by B the box

$$B := B(R, L, L, L), \tag{3.15}$$

as defined in (3.1), with R the rotation which maps e_1 to l. Let

$$\epsilon_L := \frac{1}{(\log \log L)^2}.$$

Proposition 3.2 *Let $l \in \mathbb{S}^{d-1}$. Assume that the ellipticity condition $(E)_0$ and that $(T)_0|l$ are fulfilled. Then, for each function $\beta_L : (0, \infty) \to (0, \infty)$ and each $c > 0$ there exists $c_{3,11} > 0$ such that*

$$\mathbb{P} \left(P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-cL^{\beta_L + \epsilon_L}} \right) \leq \frac{1}{c_{3,11}} e^{-c_{3,11} L^{\beta_L}} \tag{3.16}$$

where B is the box defined in (3.15).

Let us now prove Proposition 3.2. Let $\rho > 0$. We will perform a one scale renormalization analysis involving boxes of side $\rho L^{\frac{\epsilon_L}{d+1}}$ which intersect the box B. Without loss of generality, we assume that e_1 belongs to the intersection of the half-spaces so that

$$e_1 \in \{x \in \mathbb{Z}^d : x \cdot l \geq 0\} \tag{3.17}$$

and

$$e_1 \in \{x \in \mathbb{Z}^d : x \cdot \hat{v} \geq 0\}. \tag{3.18}$$

Define the hyperplane perpendicular to direction e_1 as

$$H := \{x \in \mathbb{R}^d : x \cdot e_1 = 0\}. \tag{3.19}$$

We will need to work with the projection on the direction l along the hyperplane H defined for $z \in \mathbb{Z}^d$ as

$$P_l z := \left(\frac{z \cdot e_1}{l \cdot e_1} \right) l, \tag{3.20}$$

and the projection of z on H along l defined by

$$Q_l z := z - P_l z. \tag{3.21}$$

Let $r > 0$ be a fixed number which will eventually be chosen large enough. For each $x \in \mathbb{Z}^d$ and n define the *mesoscopic box*

$$D_n(x) := \{y \in \mathbb{Z}^d : -n < (y - x) \cdot e_1 < n, \, -rn \leq |Q_l(y - x)|_\infty \leq rn\},$$

and their front boundary

$$\partial^+ D_n(x) := \{y \in \partial D_n(x) : (y - x) \cdot e_1 \geq n\}.$$

Define the set of mesoscopic boxes intersecting B as

$$\mathcal{D} := \{D_n(x) \text{ with } x \in \mathbb{Z}^d : D_n(x) \cap B \neq \emptyset\}.$$

From now on, when there is no risk of confusion, we will write D instead of D_n for a typical box in $\mathcal{D}$. Also, let us set $n := \rho L^{\frac{\epsilon_L}{d+1}}$. We now say that a box $D(x) \in \mathcal{D}$ is *good* if

$$P_{x,\omega}(X_{T_{D(x)}} \in \partial_+ D(x)) \geq 1 - \frac{1}{L}. \tag{3.22}$$

Otherwise we will say that $D(x)$ is *bad.*

Lemma 3.6 *Let $l \in \mathbb{S}^{d-1}$ and $M > 15d + 5$. Consider a RWRE satisfying condition $(P)_M|l$ and the ellipticity condition $(E)_0$. Then, there is a $c_{3,5}$ such that for $r \geq c_{3,5}$ one has that*

$$\limsup_{L \to \infty} L^{-\frac{\epsilon_L \gamma_L}{d+1}} \log \mathbb{P}(D(0) \text{ is bad}) < 0. \tag{3.23}$$

Proof. By (3.22) and Markov inequality we have that

$$\mathbb{P}\left(D(0) \text{ is bad}\right) \leq \mathbb{P}\left(P_{0,\omega}(X_{T_{D(0)}} \notin \partial_+ D(0)) > \frac{1}{L} \right) \leq L P_0\left(X_{T_{D(0)}} \notin \partial_+ D(0)\right). \tag{3.24}$$

Now, by Proposition 3.1 of Section 3.1, we know that the polynomial condition $(P)_M|l$ and the ellipticity condition $(E)_\alpha$ imply $(T)_0|l$. But by Theorem 2.4, and the fact that e_1 is in the half spaces determined by l and $\hat{v}$ (see (3.17) and (3.18)), we can conclude that $(T)_0|l$ implies $(T)_0|_{e_1}$. On the other hand, it is straightforward to check that there are constants $c_{3,5}$, $c_{3,6} > 0$ such that for $r \geq c_{3,5}$, $(T)_0|_{e_1}$ implies that

$$P_0\left(X_{T_{D(0)}} \notin \partial_+ D(0)\right) \leq \frac{1}{c_{3,6}} e^{-c_{3,6} L^{\frac{\epsilon_L \gamma_L}{d+1}}}.$$

Substituting this back into inequality (3.24) we see that (3.23) follows.

∎

For each m such that $0 \le m \le \left\lceil \frac{2L(l\cdot e_1)}{n} \right\rceil$ define the *block* R_m as the collection of mesoscopic boxes (see Figure 3)

$$R_m := \{D(x) \in \mathcal{D} : \text{ for some } x \text{ such that } x \cdot e_1 = nm\}. \tag{3.25}$$

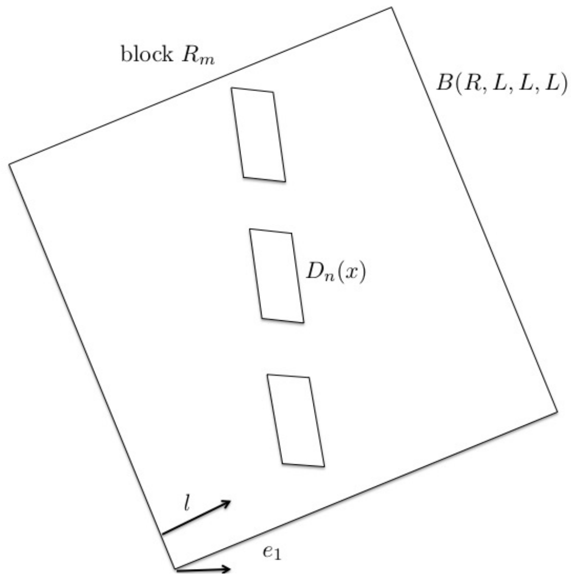

Figure 3: A box B with a set of inner boxes $D_n(x)$, which belong to a block R_m.

The collection of these blocks is denoted by $\mathcal{R}$. We will say that a block R_m is *good* if every box $D \in R_m$ is good. Otherwise, we will say that the block R_m is *bad*. Now, for each $x \in R_m$ we associate a self-avoiding path $\pi^{(x)}$ such that

(a) The path $\pi^{(x)} = (\pi_1^{(x)}, \dots, \pi_{2n+1}^{(x)})$ has $2n$ steps.

(b) $\pi_1^{(x)} = x$ and the end-point $\pi_{2n+1}^{(x)} \in R_{m+1}$.

(c) Whenever $D(x)$ does not intersect $\partial_+ B$, the path $\pi^{(x)}$ is contained in B. Otherwise, the end-point $\pi_{2n+1}^{(x)} \in \partial_+ B$.

Define next J as the total number of bad boxes of the collection $\mathcal{D}$ and define

$$G_1 := \{\omega \in \Omega : J \le L^{\beta_L + \frac{d}{d+1}\epsilon_L}\}.$$

We will now denote by $\{m_1, \dots m_N\}$ a generic subset of $\{0, \dots, |\mathcal{R}|-1\}$ having N elements. Let $\xi \in (0,1)$. Define

$$G_2 := \left\{ \omega \in \Omega : \sup_{N,\{m_1,\dots,m_N\}} \sum_{j=1}^{N} \sup_{x_j \in R_{m_j}} \sum_{i=1}^{2n} \log \frac{1}{\omega(\pi_i^{(x_j)}, \Delta\pi_i^{(x_j)})} \le 2n \log\left(\frac{1}{\xi}\right) L^{\beta_L + \frac{d}{d+1}\epsilon_L} \right\},$$

where the first supremum runs over $N \le L^{\beta_L + \frac{d}{d+1}\epsilon_L}$ and all subsets $\{m_1, \dots, m_N\}$ of the set of blocks. Now, we can say that

$$\mathbb{P}\left(p_B \le e^{-cL^{\beta_L+\epsilon_L}}\right) \le \mathbb{P}\left(p_B \le e^{-cL^{\beta_L+\epsilon_L}}, G_1 \cap G_2\right) + \mathbb{P}(G_1^c) + \mathbb{P}(G_2^c). \tag{3.26}$$

Let us now show that the first term on the right-hand side of (3.26) vanishes. Indeed, on the event $G_1 \cap G_2$, the probability p_B is bounded from below by the probability that the random walk exits every mesoscopic box from its front side. Since $\omega \in G_1$, the random walk will have to do this for at most $L^{\beta_L + \frac{d}{d+1}\epsilon_L}$ bad boxes. On each bad box $D(x)$ it will follow the path $\pi^{(x)}$ defined above. But then on the event G_2, we have a control on the product of the probability of traversing all these paths through the bad boxes. Hence, applying the strong Markov property and using the definition of good box, we conclude that for fixed ξ there is a $c_{3,7} > 0$ such that for $0 < \rho \leq c_{3,7}$ and on the event $G_1 \cap G_2$,

$$p_B \geq e^{-2L^{\beta_L+\epsilon_L}\rho \log\left(\frac{1}{\xi}\right)} \left(1 - \frac{1}{L}\right)^L > e^{-cL^{\beta_L+\epsilon_L}}.$$

Let us now estimate the term $\mathbb{P}(G_1^c)$ of (3.26). Note first that the set $\mathcal{D}$ of mesoscopic boxes can be divided into less that $2^d r^{d-1}\rho^d L^{\frac{d\epsilon_L}{d+1}}$ collections of boxes, whose union is $\mathcal{D}$ and each collection has only disjoint boxes. Let us call M the number of such collections. We also denote by $\mathcal{D}_i$ and J_i, where $1 \leq i \leq M$, the i-th collection and the number of bad boxes in such a collection respectively. We then have that

$$\mathbb{P}(G_1^c) \leq \sum_{i=1}^{M} \mathbb{P}\left(J_i \geq \frac{1}{M} L^{\beta_L + \frac{d}{d+1}\epsilon_L}\right). \tag{3.27}$$

Now, by Chebychev inequality

$$\begin{aligned} \mathbb{P}\left(J_i \geq \tfrac{1}{M} L^{\beta_L + \frac{d}{d+1}\epsilon_L}\right) &\leq e^{-\frac{L^{\beta_L + \frac{d}{d+1}\epsilon_L}}{M}} \mathbb{E}[e^{J_i}] \\ = e^{-\frac{L^{\beta_L + \frac{d}{d+1}\epsilon_L}}{M}} &\textstyle\sum_{n=0}^{|\mathcal{D}_i|} \binom{|\mathcal{D}_i|}{n} (ep_L)^n (1-ep_L)^{|\mathcal{D}_i|-n} \left(\frac{1-p_L}{1-ep_L}\right)^{|\mathcal{D}_i|-n}, \end{aligned} \tag{3.28}$$

where p_L is the probability that a box is bad. Now the last factor of each term after the summation of the right-hand side of (3.28) is bounded by

$$\left(\frac{1-p_L}{1-ep_L}\right)^{|\mathcal{D}_i|},$$

which clearly tends to 1 as $L \to \infty$ by the fact that $|\mathcal{D}_i| \leq c_{3,8} L^d$, the definition of ϵ_L and by Lemma 3.6 for some $c_{3,8} > 0$. Thus, there is a constant $c_{3,9} > 0$ such that

$$\mathbb{P}\left(J_i \geq \frac{1}{M} L^{\beta_L + \frac{d}{d+1}\epsilon_L}\right) \leq c_{3,9} e^{-\frac{L^{\beta_L + \frac{d}{d+1}\epsilon_L}}{M}}.$$

Substituting this back into (3.27) we hence see that

$$\mathbb{P}(G_1^c) \leq c_{3,9}(2\rho)^d r^{d-1} L^{\frac{d}{d+1}\epsilon_L} e^{-\frac{L^{\beta_L}}{(2\rho)^d r^{d-1}}}. \tag{3.29}$$

Let us now bound the term $\mathbb{P}(G_2^c)$ of (3.26). Define $\beta_L' := \beta_L + \frac{d}{d+1}\epsilon_L$. Note that for each $0 < \alpha < \min_e F_e$ one has that

$$\begin{aligned} \mathbb{P}(G_2^c) \leq \textstyle\sum_{N=1}^{L^{\beta_L'}} \mathbb{P}\Big(\exists \{m_1, \ldots, m_N\} &\text{ and } x_j \in R_{m_j} \text{ such that} \\ \textstyle\sum_{j=1}^{N} \sum_{i=1}^{2n} \log \frac{1}{\omega\left(\pi_i^{(x_j)}, \Delta\pi_i^{(x_j)}\right)} &> 2n \log\left(\tfrac{1}{\xi}\right) L^{\beta_L}\Big) \\ \leq \textstyle\sum_{N=1}^{L^{\beta_L'}} \binom{|\mathcal{R}|}{N} r^{d-1} (2\rho)^{L^{\beta_L'}} &e^{(\log L)\frac{\epsilon_L}{d+1} L^{\beta_L'}} e^{(\log \eta_\alpha) 2nL^{\beta_L'} - 2\alpha n \log\left(\frac{1}{\xi}\right) L^{\beta_L'}} \\ \leq L^{\beta_L'} \lceil \tfrac{2L(l\cdot e_1)}{n} \rceil^{L^{\beta_L'}} r^{d-1} (2\rho)^{L^{\beta_L'}} &e^{(\log L)\frac{\epsilon_L}{d+1} L^{\beta_L'}} e^{(\log \eta_\alpha) 2nL^{\beta_L'} - 2\alpha n \log\left(\frac{1}{\xi}\right) L^{\beta_L'}}. \end{aligned}$$

It now follows that for ξ such that $\log\left(\frac{1}{\xi^{2\alpha}\eta_\alpha^3}\right) > 0$ one can find a constant $c_{3,10}$ such that

$$\mathbb{P}(G_2^c) \leq \frac{1}{c_{3,10}} e^{-c_{3,10} L^{\beta_L + \epsilon_L}}. \tag{3.30}$$

Substituting back (3.29) and (3.30) into (3.26) we end up the proof of Proposition 3.2.

3.3 Condition $(T)_0$ implies the effective criterion

Here we will introduce a generalization of the effective criterion introduced by Sznitman in [Sz02] for RWRE, dropping the assumption of uniformly ellipticity and replacing it by the ellipticity condition $(E)_0$. Let $l \in \mathbb{S}^{d-1}$ and $d \geq 2$. We will say that the *effective criterion in direction* l holds if

$$c_2(d) \inf_{L \geq c_3, 3\sqrt{d} \leq \tilde{L} < L^3} \inf_{\alpha > 0} \inf_{0 < a \leq \alpha} \left\{ \Upsilon^{3(d-1)} \tilde{L}^{d-1} L^{3(d-1)+1} \mathbb{E}[\rho_B^a] \right\} < 1, \tag{3.31}$$

where

$$B = B(R, L-2, L+2, \tilde{L}) \text{ and } \Upsilon := \max\left\{ \frac{\alpha}{24}, \left(\frac{2c_1}{c_1 - 1} \right) \log \eta_\alpha^2 \right\}, \tag{3.32}$$

while $c_2(d)$ and $c_3(d)$ are dimension dependent constants that will be introduced in subsection 3.4. Note that in particular, the effective criterion in direction l implies that condition $(E)_0$ is satisfied. Here we will prove the following proposition.

Proposition 3.3 *Let $l \in \mathbb{S}^{d-1}$. Assume that the ellipticity condition $(E)_0$ and that $(T)_0|l$ are fulfilled. Then, the effective criterion in direction l is satisfied.*

To prove Proposition 3.3, we begin defining the following quantities

$$\beta_1(L) := \frac{\gamma_L}{2} = \frac{\log 2}{2 \log\log L}$$

$$\sigma(L) := \frac{\gamma_L}{3} = \frac{\log 2}{3 \log\log L}$$

$$a := L^{-\sigma(L)}.$$

We will write ρ instead of ρ_B, where B is the box defined in (3.32) (see 3.1) with $\tilde{L} = L^2$. Following [BDR12], it is convenient to split $\mathbb{E}\rho^a$ according to

$$\mathbb{E}\rho^a = \mathcal{E}_0 + \sum_{j=1}^{n-1} \mathcal{E}_j + \mathcal{E}_n \tag{3.33}$$

where

$$n := n(L) := \left\lceil \frac{4(1 - \gamma_L/2)}{\gamma_L} \right\rceil + 1,$$

$$\mathcal{E}_0 := \mathbb{E}\left(\rho^a, p_B > e^{-cL^{\beta_1}} \right),$$

$$\mathcal{E}_j := \mathbb{E}\left(\rho^a, e^{-cL^{\beta_{j+1}}} < p_B \leq e^{-cL^{\beta_j}} \right)$$

for $j \in \{1, \ldots, n-1\}$, and

$$\mathcal{E}_n := \mathbb{E}\left(\rho^a, p_B \leq e^{-cL^{\beta_n}} \right)$$

with parameters

$$\beta_j(L) := \beta_1(L) + (j-1)\frac{\gamma_L}{4},$$

for $2 \leq j \leq n(L)$. We will now estimate each of the n terms appearing in (3.33). For the first $n-1$ terms, we now state two lemmas proved by Berger, Drewitz and Ramírez in [BDR12], whose proofs we omit. The following lemma is a consequence of Jensen's inequality.

Lemma 3.7 *Assume that $(T)_0$ is satisfied. Then*

$$\mathcal{E}_0 \le e^{cL^{\frac{\gamma_L}{6}} - L^{\frac{2}{3}\gamma_L(1+o(1))}}$$

as $L \to \infty$.

The second lemma follows from Proposition 3.2.

Lemma 3.8 *Assume that the weak atypical quenched exit estimate (3.16) is satisfied. Then there exists a constant $c_{3,12} > 0$ such that for all L large enough and all $j \in \{1, \ldots, n-1\}$ one has that*

$$\mathcal{E}_j \le \frac{1}{c_{3,12}} e^{cL^{\left(\frac{1}{6}+\frac{j}{4}\right)\gamma_L} - c_{3,12}L^{\left(\frac{1}{4}+\frac{j}{4}\right)\gamma_L - \epsilon(L)}}.$$

In [BDR12], where it is assumed that the environment is uniform elliptic, one has that $\mathcal{E}_n = 0$. Nevertheless, since here we are not assuming uniform ellipticity this is not the case.

Lemma 3.9 *Assume that $(E)_0$ and $(T)_0$ are satisfied. Then there exists a constant $c_{3,16} > 0$ such that for all L large enough we have*

$$\mathcal{E}_n \le \frac{1}{c_{3,16}} e^{-c_{3,16}L^{1-\epsilon(L)}}.$$

Proof. Choose $0 < \alpha < \min_e F_e$. Consider a nearest neighbor self-avoiding path $(x_1, \ldots, x_m)$ from 0 to $\partial_+ B$, so that $x_1 = 0$ and $x_m \in \partial_+ B$, $x_1, \ldots, x_{m-1} \in B$ and which has the minimal number of steps m. Then,

$$\begin{aligned}\mathbb{E}\left[\rho^a, p_B \le e^{-cL^{\beta_n}}\right] &\le \mathbb{E}\left[e^{\frac{\alpha}{2}\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)}}, \textstyle\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)} > \frac{3m}{\alpha}\log\eta_\alpha\right] \\ &+\mathbb{E}\left[e^{a\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)}}, \textstyle\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)} \le \frac{3m}{\alpha}\log\eta_\alpha, p_B \le e^{-cL^{\beta_n}}\right], \end{aligned} \tag{3.34}$$

where in the first line, we have used that for any $\alpha > 0$, $a \le \frac{\alpha}{2}$ for L large. Now, using Cauchy-Schwartz inequality, Chebyshev inequality and (3.16), we can see that the right-hand side of (3.34) is smaller than

$$\begin{aligned}\mathbb{E}\left[e^{\alpha\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)}}\right]^{1/2} \mathbb{P}\left(\textstyle\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)} > \frac{3m}{\alpha}\log\eta_\alpha\right)^{1/2} + e^{\frac{3am}{\alpha}\log\eta_\alpha}\mathbb{P}\left(p_B \le e^{-cL^{\beta_n}}\right) \\ \le e^{-m\log\eta_\alpha} + \frac{1}{c_{3,13}} e^{\frac{3am}{\alpha}\log\eta_\alpha - c_{3,13}L^{\beta_n(L)-\epsilon(L)}},\end{aligned} \tag{3.35}$$

for some constant $c_{3,13} > 0$. Now, using the fact that there are constants $c_{3,14}$ and $c_{3,15}$ such that

$$c_{3,14}L \le m \le c_{3,15}L,$$

we can substitute (3.35) into (3.34) to conclude that there is a constant $c_{3,16}$ such that

$$\mathbb{E}\left[\rho^a, p_B \le e^{-cL^{\beta_n}}\right] \le \frac{1}{c_{3,16}} e^{-c_{3,16}L^{1-\epsilon(L)}}.$$

∎

It is now straightforward to conclude the proof of Proposition 3.3 using the estimates of Lemmas 3.7, 3.8 and 3.9.

3.4 The effective criterion implies (T')

We will prove that the generalized effective criterion and the ellipticity condition $(E)_0$ imply (T'). To do this, it is enough to prove the following.

Proposition 3.4 *Throughout choose $0 < \alpha < \min_e F_e$. Let $l \in \mathbb{S}^{d-1}$ and $d \geq 2$. If the effective criterion in direction l holds then there exists a constant $c_{3,28} > 0$ and a neighborhood V_l of direction l such that for all $l' \in V_l$ one has that*

$$\overline{\lim_{L\to\infty}} L^{-1} e^{c_{3,28}(\log L)^{1/2}} \log P_0 \left[\tilde{T}^{l'}_{-\tilde{b}L} < T^{l'}_{bL}\right] < 0, \;\; \text{for all } b, \tilde{b} > 0.$$

In particular, if (3.31) is satisfied, condition $(T')|l$ is satisfied.

To prove this proposition, we will follow the same strategy used by Sznitman in [Sz02] to prove Proposition 2.3 of that paper under the assumption of uniform ellipticity. Firstly we need to define some constants. Let

$$c'_1(d,\alpha) := 13 + \frac{24d}{\alpha} + \frac{24d + 12\log\eta_\alpha}{2\log\eta_\alpha},$$

$$c'_2(d,\alpha) := c_1 c'_1,$$

and

$$c_4(d,\alpha) := \frac{48c'_2}{\alpha},$$

where c_1 is defined in (2.1). Define for $k \geq 0$ the sequence $\{N_k : k \geq 0\}$ by

$$N_k := \frac{c_4}{u_0} 8^k, \tag{3.36}$$

where $u_0 \in (0,1)$. Let $L_0, \tilde{L}_0, L_1$ and $\tilde{L}_1$ be constants such that

$$3\sqrt{d} \leq \tilde{L}_0 \leq L_0^3, \quad L_1 = N_0 L_0 \quad \text{and } \tilde{L}_1 = N_0^3 \tilde{L}_0. \tag{3.37}$$

Now, for $k \geq 0$ define recursively the sequences $\{L_k : k \geq 0\}$ and $\{\tilde{L}_k : k \geq 0\}$ by

$$L_{k+1} := N_k L_k, \quad \text{and} \quad \tilde{L}_{k+1} := N_k^3 \tilde{L}_k. \tag{3.38}$$

It is straightforward to see that for each $k \geq 1$

$$L_k = \left(\frac{c_4}{u_0}\right)^k 8^{\frac{k(k-1)}{2}} L_0, \quad \tilde{L}_k = \left(\frac{L_k}{L_0}\right)^3 \tilde{L}_0. \tag{3.39}$$

Furthermore, we also consider for $k \geq 0$ the box

$$B_k := B(R, L_k - 1, L_k + 1, \tilde{L}_k),$$

and the positive part of its boundary $\partial_+ B_k$, and will use the notations $\rho_k = \rho_{B_k}, p_k = p_{B_k}, q_k = q_{B_k}$ and $n_k = [N_k]$. Following Sznitman [Sz02], we introduce for each $i \in \mathbb{Z}$

$$\mathcal{H}_i := \{x \in \mathbb{Z}^d, \exists\, x' \in \mathbb{Z}^d, |x - x'| = 1, (x \cdot l - iL_0)(x' \cdot l - iL_0) \leq 0\}. \tag{3.40}$$

We also define the function $I : \mathbb{Z}^d \to \mathbb{Z}$ by

$$I(x) := i, \text{ for } x \text{ such that } x \cdot l \in \left[iL_0 - \frac{L_0}{2}, iL_0 + \frac{L_0}{2}\right).$$

Consider now the successive times of visits of the random walk to the sets $\{\mathcal{H}_i : i \in \mathbb{Z}\}$, defined recursively as

$$V_0 := 0, \quad V_1 := \inf\{n \geq 0 : \, X_n \in \mathcal{H}_{I(X_0)+1} \cup \mathcal{H}_{I(X_0)-1}\}$$

and

$$V_{k+1} := V_k + V_1 \circ \theta_{V_k}, \; k \geq 0.$$

For $\omega \in \Omega$, $x \in \mathbb{Z}^d$, $i \in \mathbb{Z}$, let

$$\widehat{q}(x,\omega) := P_{x,\omega}[X_{V_1} \in \mathcal{H}_{I(x)-1}] \tag{3.41}$$

while $\widehat{p}(x,\omega) := 1 - \widehat{q}(x,\omega)$, and

$$\widehat{\rho}(i,\omega) := \sup\left\{ \frac{\widehat{q}(x,\omega)}{\widehat{p}(x,\omega)} : x \in \mathcal{H}_i, \; \sup_{2 \leq j \leq d} |R(e_j) \cdot x| < \tilde{L}_1 \right\}. \tag{3.42}$$

We consider also the stopping time

$$\tilde{T} := \inf\left\{ n \geq 0 : \sup_{2 \leq j \leq d} |X_n \cdot R(e_j)| \geq \tilde{L}_1 \right\},$$

and the function $f : \{n_0+2, n_0+1, \ldots\} \times \Omega \to \mathbb{R}$ defined by

$$f(n_0+2,\omega) := 0, \quad f(i,\omega) := \sum_{m=i}^{n_0+1} \prod_{j=m+1}^{n_0+1} \widehat{\rho}(j,\omega)^{-1}, \quad \text{for} \quad i \leq n_0+1.$$

We will frequently write $f(n)$ instead $f(n,\omega)$. Let us now proceed to prove Proposition 3.4. The following proposition corresponds to the first step in an induction argument which will be used to prove Proposition 3.4.

Proposition 3.5 *Let $\alpha > 0$. Let $L_0, L_1, \tilde{L}_0$ and $\tilde{L}_1$ be constants satisfying (3.37), with $N_0 \geq 7$. Then, there exist $c_{3,17}, c_{3,18}(d), c_{3,19}(d) > 0$ such that for $L_0 \geq c_{3,17}$, $a \in (0,\alpha]$, $u_0 \in [\xi^{L_0/d}, 1]$, $0 < \xi < \frac{1}{\eta_\alpha^{2/\alpha}}$ and*

$$N_0 \leq \frac{1}{L_0}\left(\frac{e}{\xi}\right)^{L_0}, \tag{3.43}$$

the following is satisfied

$$\mathbb{E}[\rho_1^{a/2}] \leq c_{3,18}\left\{ \xi^{-c_2' L_1}\left(c_{3,19}\tilde{L}_1^{(d-2)}\frac{L_1^3}{L_0^2}\tilde{L}_0\mathbb{E}[q_0]\right)^{\frac{\tilde{L}_1}{12N_0\tilde{L}_0}} \right.$$
$$\left. + \sum_{m=0}^{N_0+1}\left(c_{3,19}\tilde{L}_1^{(d-1)}\mathbb{E}[\rho_0^a]\right)^{\frac{N_0+m-1}{2}} + e^{-c_1 L_1 \log\frac{1}{\xi^\alpha \eta_\alpha^2}} \right\}. \tag{3.44}$$

Proof. The following inequality is stated and proved in [Sz02] by Sznitman without using any kind of uniform ellipticity assumption (inequality (2.18) in [Sz02]). For every $\omega \in \Omega$

$$P_{0,\omega}\left(\tilde{T}^l_{1-L_1} < \tilde{T} \wedge T^l_{L_1+1}\right) \leq \frac{f(0)}{f(1-n_0)}. \tag{3.45}$$

Consider now the event

$$G := \{\omega : P_{0,\omega}\left(\tilde{T} \leq \tilde{T}^l_{1-L_1} \wedge T^l_{L_1+1}\right) \leq \xi^{(c_1'-1)c_1 L_1}\},$$

and write

$$\mathbb{E}[\rho_1^{a/2}] = \mathbb{E}[\rho_1^{a/2}, G] + \mathbb{E}[\rho_1^{a/2}, G^c]. \tag{3.46}$$

The first term $\mathbb{E}[\rho_1^{a/2}, G]$ of (3.46), can in turn be decomposed as

$$\mathbb{E}[\rho_1^{a/2}, G] = \mathbb{E}[\rho_1^{a/2}, G, A_1] + \mathbb{E}[\rho_1^{a/2}, G, A_1^c], \tag{3.47}$$

where we have defined

$$A_1 := \{\omega \in \Omega : f(2-n_0) - f(0) \geq f(1-n_0)\xi^{(c_1'-1)c_1 L_1}, f(0) \geq f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\}.$$

Furthermore, note that

$$A_1^c \subset A_2 \cup A_3,$$

where

$$A_2 := \{\omega \in \Omega : f(2-n_0) - f(0) < f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\}, \text{ while}$$
$$A_3 := \{\omega \in \Omega : f(0) < f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\}.$$

Therefore,

$$\mathbb{E}[\rho_1^{a/2}] \leq \mathbb{E}[\rho_1^{a/2}, G, A_1] + \mathbb{E}[\rho_1^{a/2}, A_2] + \mathbb{E}[\rho_1^{a/2}, G, A_3] + \mathbb{E}[\rho_1^{a/2}, G^c]. \tag{3.48}$$

We now subdivide the rest of the proof in several steps corresponding to an estimation for each one of the terms in inequality (3.48).

Step 1: estimate of $\mathbb{E}[\rho_1^{a/2}, G, A_1]$. Here we estimate the first term of display (3.47). To do this, we can follow the argument presented by Sznitman in Section 2 of [Sz02], to prove that inequality (3.45) implies that there exist constant $c_{3,20}(d)$ such that

$$\mathbb{E}\left[\rho_1^{a/2}, G, A_1\right] \leq 2\sum_{m=0}^{n_0+1}\left(c_{3,20}(d)\tilde{L}_1^{(d-1)}\mathbb{E}[\rho_0^a]\right)^{\frac{n_0+m-1}{2}}. \tag{3.49}$$

Indeed on $G \cap A_1$ and with the help of (3.45) one gets that

$$\begin{aligned}\rho_1 &= \frac{P_{0,\omega}[\tilde{T}^l_{-L_1+1} < \tilde{T} \wedge T^l_{L_1+1}] + P_{0,\omega}[\tilde{T} \leq \tilde{T}^l_{-L_1+1} \wedge T^l_{L_1+1}]}{1 - P_{0,\omega}[\tilde{T}^l_{-L_1+1} < \tilde{T} \wedge T^l_{L_1+1}] - P_{0,\omega}[\tilde{T} \leq \tilde{T}^l_{-L_1+1} \wedge T^l_{L_1+1}]} \\ &\leq \frac{f(0) + f(1-n_0)\xi^{(c_1'-1)c_1L_1}}{\left(f(1-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1L_1}\right)_+} \leq \frac{2f(0)}{\left(f(1-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1L_1}\right)_+},\end{aligned} \tag{3.50}$$

where in the first inequality we have used the fact that $\omega \in G$, while in the second that $\omega \in A_1$. Regarding the term in the denominator in the last expression, we can use the definition of the function f and obtain

$$\begin{aligned} &f(1-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1L_1} \\ &= \textstyle\prod_{j=2-n_0}^{n_0+1} \widehat{\rho}(j,\omega)^{-1} + f(2-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1L_1} \\ &\geq \textstyle\prod_{j=2-n_0}^{n_0+1} \widehat{\rho}(j,\omega)^{-1},\end{aligned}$$

where we have used that $\omega \in A_1$ in the last inequality. Substituting this estimate in (3.50), we conclude that for $\omega \in G \cap A_1$ one has that

$$\rho_1 \leq 2\prod_{j=2-n_0}^{n_0+1}\widehat{\rho}(j,\omega)f(0) = 2\sum_{m=0}^{n_0+1}\prod_{j=2-n_0}^{m}\widehat{\rho}(j,\omega). \tag{3.51}$$

At this point, using (3.51), the fact that $(u+v)^{a/2} \leq u^{a/2} + v^{a/2}$ for $u, v \geq 0$, the fact that $\{\widehat{\rho}(j,\omega), j \text{ even}\}$ and $\{\widehat{\rho}(j,\omega), j \text{ odd}\}$ are two collections of independent random variables and the Cauchy-Schwartz's inequality, we can assert that

$$\begin{aligned}
&\mathbb{E}[\rho_1(\omega)^{a/2}, G, A_1] \\
\leq\ & 2 \sum_{0\le m\le n_0+1} \mathbb{E}\left[\prod_{1-n_0<j\le m} \widehat{\rho}(j,\omega)^{a/2}\right] \\
\leq\ & 2 \sum_{0\le m\le n_0+1} \mathbb{E}\left[\prod_{\substack{1-n_0<j\le m\\ j\, is\, even}} \widehat{\rho}(j,\omega)^{a}\right]^{1/2} \mathbb{E}\left[\prod_{\substack{1-n_0<j\le m\\ j\, is\, odd}} \widehat{\rho}(j,\omega)^{a}\right]^{1/2} \\
=\ & 2 \sum_{0\le m\le n_0+1} \prod_{1-n_0<j\le m} \mathbb{E}\left[\widehat{\rho}(j,\omega)^{a}\right]^{1/2}.
\end{aligned}$$

In view of (3.41) one gets easily that for $i \in \mathbb{Z}$ and $x \in \mathcal{H}_i$,

$$\widehat{p}(x,\omega) \geq p_0 \circ t_x(\omega),$$

where the canonical shift $\{t_x : x \in \mathbb{Z}^d\}$ has been defined in (2.4). Hence, for $i \in \mathbb{Z}$ and $x \in \mathcal{H}_i$,

$$\frac{\widehat{q}(x)}{\widehat{p}(x)} \leq \rho_0 \circ t_x.$$

Following Sznitman [Sz02] with the help of (3.40) the estimate (3.49) follows.

Step 2: estimate of $\mathbb{E}[\rho_1^{a/2}, A_2]$. Here we will prove the following estimate for the second term of inequality (3.48),

$$\mathbb{E}[\rho^{a/2}, A_2] \leq 4e^{-c_1 L_1 \log \frac{1}{\xi^\alpha \eta_\alpha^2}}.$$

By the definition of c_1 (see (2.1)), we know that necessarily there exists a path with less than $c_1(L_1 + 1 + \sqrt{d})$ steps between the origin and $\partial_+ B_1$. Therefore, for $L_0 \geq 1 + \sqrt{d}$, there is a nearest neighbor self-avoiding path $(x_1, \ldots, x_n)$ with n steps from the origin to $\partial_+ B_1$, such that $2c_1 L_1 \leq n \leq 2c_1 L_1 + 1$, $x_1, \ldots, x_n \in B_1$ and $x_n \cdot l \geq L_1 + 1$. Thus, for every $r \geq 0$ we have that

$$\rho_1^r \leq \frac{1}{p_1^r} \leq e^{r \sum_{i=1}^n \log \frac{1}{\omega(x_i, \Delta x_i)}}, \tag{3.52}$$

where $\Delta x_i := x_{i+1} - x_i$ for $1 \leq i \leq n-1$ as defined in (2.3). We then have applying inequality (3.52) with $r = a/2$ that

$$\begin{aligned}
\mathbb{E}\left[\rho_1^{a/2}, A_2\right] \leq\ & \mathbb{E}\left[e^{\alpha/2 \sum_1^n \log \frac{1}{\omega(x_i,\Delta x_i)}}, A_2, \textstyle\sum_1^n \log \frac{1}{\omega(x_i,\Delta x_i)} \leq n \log\left(\frac{1}{\xi}\right)\right] \\
&+ \mathbb{E}\left[e^{\alpha/2 \sum_1^n \log \frac{1}{\omega(x_i,\Delta x_i)}}, \textstyle\sum_1^n \log \frac{1}{\omega(x_i,\Delta x_i)} > n \log\left(\frac{1}{\xi}\right)\right].
\end{aligned} \tag{3.53}$$

Regarding the second term of the right side of (3.53), we can apply the Cauchy-Schwarz inequality, the exponential Chebychev inequality and conclude that and use the fact that the jump probabilities $\{\omega(x_i, \Delta_i) : 1 \leq i \leq n-1\}$ are independent to conclude that

$$\begin{aligned}
&\mathbb{E}\left[e^{\alpha \sum_1^n \log \frac{1}{\omega(x_i,\Delta x_i)}}\right]^{1/2} \mathbb{P}\left(\textstyle\sum_1^n \log \frac{1}{\omega(x_i,\Delta x_i)} > n \log\left(\frac{1}{\xi}\right)\right)^{1/2} \\
&\qquad \leq e^{\left(2 \log \eta_\alpha - \alpha \log\left(\frac{1}{\xi}\right)\right) n/2}.
\end{aligned} \tag{3.54}$$

Meanwhile, note that the first term on the right side of (3.53) can be bounded by

$$e^{\frac{\alpha \log\left(\frac{1}{\xi}\right) n}{2}} \mathbb{P}(A_2). \tag{3.55}$$

Hence, we need an adequate estimate for $\mathbb{P}(A_2)$. Now,

$$\begin{aligned}\mathbb{P}(A_2) &= \mathbb{P}\left(f(2-n_0)-f(0)<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}, A_2\right)\\ &\quad+\mathbb{P}\left(f(2-n_0)-f(0)\geq e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}, A_2\right)\\ &\leq \mathbb{P}\left(f(2-n_0)-f(0)<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right)+\mathbb{P}\left(f(1-n_0)>e^{\frac{(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right). \end{aligned} \tag{3.56}$$

The two terms in the rightmost side of display (3.56) will be estimated by similar methods: in both cases, we will use the fact that $\{\widehat{\rho}(j,\omega), j\,\text{even}\}$ and $\{\widehat{\rho}(j,\omega), j\,\text{odd}\}$ are two collections of independent random variables, the Cauchy-Schwartz's inequality and the Chebyshev inequality. Specifically for the first term of the rightmost side of (3.56) we have that

$$\begin{aligned}\mathbb{P}\left(f(2-n_0)-f(0)<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) &\leq \mathbb{P}\left(\prod_{j=0}^{n_0+1}\widehat{\rho}(j,\omega)^{-1}<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right)\\ &= \mathbb{P}\left(\prod_{j=0}^{n_0+1}\widehat{\rho}(j,\omega)^{\alpha/2}>e^{\frac{(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}\right)\\ &\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}\mathbb{E}\left[\prod_{\substack{j=1,\\ j\,odd}}^{n_0+1}\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2}\mathbb{E}\left[\prod_{\substack{j=0,\\ j\,even}}^{n_0+1}\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2}\\ &= e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}\prod_{j=0}^{n_0+1}\mathbb{E}\left[\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2}. \end{aligned} \tag{3.57}$$

By an estimate analogous to (3.52), we know that for $L_0\geq 1+\sqrt{d}$, for each $j\in\{0,\dots,n_0+1\}$ and each $x\in\mathcal{H}_j$, there exists a nearest neighbor self-avoiding path $(y_1,\dots,y_m)$ with m steps, such that $2c_1L_0\leq m\leq 2c_1L_0+1$, between x and $\mathcal{H}_{j+1}$. Also, $y_1\cdot l,\dots,y_{m-1}\cdot l\in(1-L_0,L_0+1)$ and $y_m\cdot l\geq L_0+1$. Then, in view of (3.38), (3.39), (3.41) and (3.42), we have that for each $j\in\{0,\dots,n_0+1\}$

$$\begin{aligned}\mathbb{E}\left[\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2} &\leq \sum\mathbb{E}\left[\widehat{p}(x,\omega)^{-\alpha}\right]^{1/2}\\ &\leq 2L_1^{3(d-1)}\mathbb{E}\left[e^{\alpha\sum_1^m\log\frac{1}{\omega(y_i,\Delta y_i)}}\right]^{1/2}\leq 2L_1^{3(d-1)}e^{\frac{m\log\eta_\alpha}{2}}, \end{aligned} \tag{3.58}$$

where the summation goes over all $x\in\mathcal{H}_j$ such that $\sup_{2\leq i\leq d}|R(e_i)\cdot x|<\widetilde{L}_1$. Substituting the estimate (3.58) back into (3.57) we see that

$$\begin{aligned}\mathbb{P}\left(f(2-n_0)-f(0)<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) &\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}2^{(n_0+2)}L_1^{3(d-1)(n_0+2)}e^{\frac{(\log\eta_\alpha)m(n_0+2)}{2}}\\ &\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}+\log 2(n_0+2)+3(d-1)\frac{\log L_0N_0}{L_0}L_0(n_0+2)+(\log\eta_\alpha)\frac{(2c_1L_0+1)(n_0+2)}{2}}\\ &\leq e^{-L_1\left(\frac{(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1}{4}-1-6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right)-\log\eta_\alpha(3c_1+1)\right)}, \end{aligned} \tag{3.59}$$

where we have used the fact that for $L_0\geq 2\log c_4$ it is true that $\frac{\log N_0L_0}{L_0}\leq 1+\log\left(\frac{1}{\xi}\right)$ for all $u_0\in[\xi^{L_0/d},1]$. Meanwhile, for the second term of the rightmost side of (3.56), we have that

$$\begin{aligned}\mathbb{P}\left(f(1-n_0)>e^{\frac{(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) &= \mathbb{P}\left(\sum_{k=1-n_0}^{1+n_0}\prod_{j=k+1}^{n_0+1}\widehat{\rho}(j,\omega)^{-1}>e^{\frac{(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right)\\ &\leq \sum_{k=1-n_0}^{1+n_0}\mathbb{P}\left(\prod_{j=k+1}^{n_0+1}\widehat{\rho}(j,\omega)^{-\alpha/2}>\frac{e^{(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)/4}}{(2n_0+1)^{\alpha/2}}\right)\\ &= e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}(2n_0+1)^{\alpha/2}\sum_{k=1-n_0}^{1+n_0}\prod_{j=k+1}^{n_0+1}\mathbb{E}\left[\widehat{\rho}(j,\omega)^{-\alpha}\right]^{1/2} \end{aligned}$$

In analogy to (3.58), we can conclude that $\mathbb{E}\left[\widehat{\rho}(j,\omega)^{-\alpha}\right]^{1/2} \leq 2L_1^{3(d-1)}e^{\frac{(\log\eta_\alpha)m}{2}}$. Therefore, for $L_0 \geq 2\log c_4$ we see that

$$\mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right)$$

$$\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}(2n_0+1)^{\alpha/2}\sum_{k=1-n_0}^{1+n_0} 2^{n_0+1-k}L_1^{3(d-1)(n_0+1-k)}e^{(\log\eta_\alpha)(2c_1L_0+1)(n_0+1-k)}$$

$$\leq e^{\frac{-(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1L_1}{4}+\left(\frac{\alpha+2}{2}\right)\log(2n_0+1)+2n_0\log 2+6(d-1)n_0\left(\frac{\log L_0N_0}{L_0}\right)L_0+(4c_1L_0n_0+2n_0)\log\eta_\alpha}$$

$$\leq e^{-L_1\left(\frac{(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1}{4}-1-6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right)-(4c_1+1)(\log\eta_\alpha)\right)}. \tag{3.60}$$

Now, in view of (3.55), (3.56), (3.59) and (3.60) the first term on the right side of (3.53) is bounded by

$$2e^{-L_1\left(\frac{(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1}{4}-2\alpha c_1\log\left(\frac{1}{\xi}\right)-1-6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right)-(4c_1+1)(\log\eta_\alpha)\right)}. \tag{3.61}$$

Now, since $c_1' \geq 13+\frac{24d}{\alpha}+\frac{24d+12\log\eta_\alpha}{\alpha\log\frac{1}{\xi}}$, we conclude that

$$\frac{(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1}{4} - 2\alpha c_1\log\left(\frac{1}{\xi}\right) - 1 - 6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right) - (4c_1+1)(\log\eta_\alpha)$$
$$\geq \alpha c_1\log\left(\frac{1}{\xi}\right) - 2c_1\log\eta_\alpha,$$

and therefore, by (3.54) and (3.61) we have that

$$E\left[\rho_1^{a/2}, A_2\right] \leq 4e^{-c_1L_1\log\frac{1}{\xi^\alpha\eta_\alpha^2}}. \tag{3.62}$$

Step 3: estimate of $\mathbb{E}[\rho_1^{a/2}, G, A_3]$. Here we will estimate the third term of the inequality (3.48). Specifically we will show that

$$\mathbb{E}[\rho_1^{a/2}, G, A_3] \leq 2e^{-c_1L_1\log\frac{1}{\xi^\alpha\eta_\alpha^2}}. \tag{3.63}$$

This upper bound will be almost obtained as the previous case, where we achieved (3.62). Indeed, in analogy to the development of (3.50) in Step 3, one has that for $\omega \in G$,

$$\rho_1 \leq \frac{f(0)+f(1-n_0)\xi^{(c_1'-1)c_1L_1}}{(f(1-n_0)-f(0)-\xi^{(c_1'-1)c_1L_1}f(1-n_0))_+}.$$

But, if $\omega \in A_3$ also, one easily gets that $0<\rho_1\leq 1$ if $L_0 \geq \frac{\alpha\log 4}{2\log\eta_\alpha}$. Thus,

$$\mathbb{E}\left[\rho_1^{a/2}, G, A_3\right] \leq \mathbb{P}(A_3).$$

Therefore, since $c_1' \geq 13+\frac{24d}{\alpha}+\frac{24d+12\log\eta_\alpha}{\alpha\log\frac{1}{\xi}}$, it is enough to prove that

$$\mathbb{P}(A_3) \leq 2e^{-L_1\left(\frac{(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1}{4}-1-6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right)-(4c_1+1)(\log\eta_\alpha)\right)}. \tag{3.64}$$

To justify this inequality, note that

$$\mathbb{P}(A_3) \leq \mathbb{P}\left(f(0) < e^{\frac{-(c_1'-1)c_1L_1\log\frac{1}{\xi}}{2}}\right) + \mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1L_1\log\frac{1}{\xi}}{2}}\right)$$
$$\leq \mathbb{P}\left(\prod_{j=1}^{n_0+1}\widehat{\rho}(j,\omega)^{-1} < e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) + \mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1L_1\log\frac{1}{\xi}}{2}}\right),$$

and hence we are in a very similar situation as in (3.56) and development in (3.57) and (3.60), from where we derive (3.64).

Step 4: estimate of $\mathbb{E}[\rho_1^{a/2}, G^c]$. Here we will prove that there exist constants $c_{3,21}(d)$ and $c_{3,22}(d)$ such that

$$\mathbb{E}[\rho_1^{a/2}, G^c] \leq c_{3,21}\xi^{-c_1' c_1 L_1}\left(c_{3,22}\widetilde{L}_1^{(d-2)}\frac{L_1^3}{L_0^2}\widetilde{L}_0\mathbb{E}[q(0)]\right)^{\frac{\widetilde{L}_1}{12N_0\widetilde{L}_0}} + e^{-c_1 L_1 \log\frac{1}{\xi^\alpha\eta_\alpha^2}}. \tag{3.65}$$

Firstly, we need to consider the event

$$A_4 := \left\{\omega \in \Omega : P_{0,\omega}\left(T^l_{L_1+1} \leq \widetilde{T} \wedge \widetilde{T}^l_{1-L_1}\right) \geq \xi^{2c_1 L_1}\right\}.$$

In the case that $\omega \in G^c \cap A_4$, the walk behaves as if effectively it satisfies a uniformly ellipticity condition with constant $\kappa = \xi$, so that we can follow exactly the same reasoning presented by Sznitman in [Sz02] leading to inequality (2.32) of that paper, showing that there exist constants $c_{3,21}(d)$, $c_{3,22}(d)$ such that whenever $\widetilde{L}_1 \geq 48N_0\widetilde{L}_0$ one has that

$$\mathbb{E}\left[\rho_1^{a/2}, G^c, A_4\right] \leq \xi^{-c_1 L_1}\mathbb{P}(G^c) \leq c_{3,21}\xi^{-c_1' c_1 L_1}\left(c_{3,22}\widetilde{L}_1^{(d-2)}\frac{L_1^3}{L_0^2}\widetilde{L}_0\mathbb{E}[q(0)]\right)^{\frac{\widetilde{L}_1}{12N_0\widetilde{L}_0}}. \tag{3.66}$$

The second inequality of (3.66) does not use any uniformly ellipticity assumption. It would be enough now to prove that

$$\mathbb{E}(\rho_1^{a/2}, A_4^c) \leq e^{-c_1 L_1 \log\frac{1}{\xi^\alpha\eta_\alpha^2}}. \tag{3.67}$$

To do this we will follow the reasoning presented in Step 2. Namely, for $L_0 \geq 1 + \sqrt{d}$, there is a nearest neighbor self-avoiding path $(x_1, \ldots, x_n)$ with n steps from 0 to $\partial_+ B_1$ such that $2c_1 L_1 \leq n \leq 2c_1 L_1 + 1$, $x_1, \ldots, x_n \in B_1$ and $x_n \cdot l \geq L_1 + 1$. Therefore

$$A_4^c \subset \left\{\omega \in \Omega : \prod_1^n \omega(x_i, \Delta x_i) < \xi^n\right\} = \left\{\omega \in \Omega : \sum_1^n \log\frac{1}{\omega(x_i, \Delta x_i)} > n\log\frac{1}{\xi}\right\},$$

so that

$$\begin{aligned}\mathbb{E}\left[\rho_1^{a/2}, A_4^c\right] &\leq \mathbb{E}\left[e^{\alpha/2\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)}}, \textstyle\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\frac{1}{\xi}\right] \\ &\leq \mathbb{E}\left[e^{\alpha\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)}}\right]^{1/2}\mathbb{P}\left(\textstyle\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\frac{1}{\xi}\right)^{1/2} \\ &\leq e^{(2\log\eta_\alpha - \alpha\log\frac{1}{\xi})c_1 L_1},\end{aligned}$$

which proves (3.67) and finishes Step 4.

Step 5: conclusion. Combining the estimates (3.49) of step 1, (3.62) of step 2, (3.63) of step 3 and (3.65) of step 4, we have (3.44). ∎

We will now prove a corollary of Proposition 3.5, which will imply Proposition 3.4. For this, it will be important to note that the statement of Proposition 3.5 is still valid if given $k \geq 1$ we change L_0 by L_k, L_1 by L_{k+1}, $\widetilde{L}_0$ by $\widetilde{L}_k$ and $\widetilde{L}_1$ by $\widetilde{L}_{k+1}$. In effect, to see this, it is enough to note that inequality (3.43) is satisfied with these replacements. Define

$$c_{3,23} := e^{-\frac{4c_1\log\eta_\alpha}{(c_1-1)\alpha}}.$$

Corollary 3.1 *Let* $0 < \xi < \min\{c_{3,23}, e^{-1/24}\}$ *and* $\alpha > 0$. *Let* $\{L_k : k \geq 0\}$ *and* $\{\widetilde{L}_k : k \geq 0\}$ *be sequences satisfying (3.36), (3.37) and (3.38). Then there exists* $c_{3,25}(d,\alpha) > 0$*, such that when for some* $L_0 \geq c_{3,25}$*,* $a_0 \in (0,\alpha]$*,* $u_0 \in [\xi^{L_0/d}, 1]$*, it is true that*

$$\phi_0 := c_{3,19}\widetilde{L}_1^{d-1}L_0\mathbb{E}[\rho_0^{a_0}] \leq \xi^{\alpha u_0 L_0}, \tag{3.68}$$

then for all $k \geq 0$*,*

$$\phi_k := c_{3,19}\widetilde{L}_{k+1}^{d-1}L_k\mathbb{E}[\rho_k^{a_k}] \leq (k+1)\xi^{\alpha u_k L_k}, \tag{3.69}$$

with $a_k := a_0 2^{-k}$*,* $u_k := u_0 8^{-k}$*.*

Proof. We will use induction in k to prove (3.69). By hypothesis we only need to show (3.69) for $n = k+1$ assuming that (3.69) holds for $n = k$. To do this, with the help of Proposition 3.5 we have that for any $k \geq 0$

$$\mathbb{E}[\rho_{k+1}^{a_{k+1}}] \leq c_{3,18} \Bigg\{ \xi^{-c_2' L_{k+1}} \left(c_{3,19} \widetilde{L}_{k+1}^{(d-2)} \frac{L_{k+1}^3}{L_k^2} \widetilde{L}_k \mathbb{E}[q_k] \right)^{\frac{\widetilde{L}_{k+1}}{12 N_k \widetilde{L}_k}}$$
$$+ \sum_{0 \leq m \leq N_k+1} \left(c_{3,19} \widetilde{L}_{k+1}^{(d-1)} \mathbb{E}[\rho_k^{a_k}] \right)^{\frac{[N_k]+m-1}{2}} + e^{-c_1 L_{k+1} \log \frac{1}{\xi^\alpha \eta_\alpha^2}} \Bigg\},$$

so that, for $k \geq 0$ and with the help of (3.38)

$$\phi_{k+1} \leq c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} \left\{ \xi^{-c_2' L_{k+1}} \phi_k^{N_k^2/12} + \sum_{0 \leq m \leq N_k+1} \phi_k^{\frac{N_k+m-1}{2}} \right\}$$
$$+ c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} e^{-c_1 L_{k+1} \log \frac{1}{\xi^\alpha \eta_\alpha^2}}.$$

Since $\xi < c_{3,23}$, we can assert that $c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} e^{-c_1 L_{k+1} \log \frac{1}{\xi^\alpha \eta_\alpha^2}} \leq \xi^{\alpha u_{k+1} L_{k+1}}$. Hence, we only need to prove that

$$c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} \left\{ \xi^{-c_2' L_{k+1}} \phi_k^{N_k^2/12} + \sum_{0 \leq m \leq N_k+1} \phi_k^{\frac{N_k+m-1}{2}} \right\} \leq (k+1) \xi^{\alpha u_{k+1} L_{k+1}} \tag{3.70}$$

Firstly, note that for L_0 large enough by the induction hypothesis, (3.38) and the fact that $\xi < e^{-\frac{1}{24}}$

$$\xi^{-c_2' L_{k+1}} \phi_k^{N_k^2/24} \leq \xi^{-c_2' L_{k+1}} (k+1)^{\frac{N_k^2}{24}} \xi^{\frac{\alpha u_k N_k^2 L_k}{24}}$$
$$\leq e^{c_2' \left(\log \frac{1}{\xi} + \frac{1}{24} - \frac{c_4 \log \frac{1}{\xi}}{24} \right) L_{k+1}} \leq 1.$$

Substituting this estimate back into (3.70) and using the hypothesis induction again, we obtain that

$$c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} \left\{ \xi^{-c_2' L_{k+1}} \phi_k^{N_k^2/12} + \sum_{0 \leq m \leq N_k+1} \phi_k^{\frac{N_k+m-1}{2}} \right\}$$
$$\leq c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} \left\{ \phi_k^{N_k^2/24} + (N_k+2) \phi_k^{N_k/4} \right\}$$
$$\leq c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} N_{k+1} \phi_k^{N_k/4}$$
$$\leq c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} \phi_k^{N_k/8} (k+1)^{N_k/8} \xi^{\alpha u_k L_k N_k/8}$$
$$= c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} \phi_k^{N_k/8} (k+1)^{N_k/8-1} (k+1) \xi^{\alpha u_{k+1} L_{k+1}},$$

where $c_{3,24} := 2 c_{3,18} c_{3,19}$. Thus, in order to show that $\phi_{k+1} \leq (k+2) \xi^{\alpha u_{k+1} L_{k+1}}$ it is enough to prove that

$$c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/8-1} \phi_k^{N_k/8} \leq 1. \tag{3.71}$$

First, note that by the induction hypothesis,

$$c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/8-1} \phi_k^{N_k/8} \leq c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/4-1} \xi^{6 c_2' L_k}. \tag{3.72}$$

From (3.37), (3.38) and (3.39), we can say that

$$
\begin{aligned}
& c_{3,24}\widetilde{L}_{k+2}^{(d-1)}L_{k+2}(k+1)^{N_k/4-1}\xi^{6c_2' L_k} \\
&= c_{3,24}\left(\tfrac{L_{k+2}}{L_0}\right)^{3(d-1)}\widetilde{L}_0^{(d-1)}L_{k+2}(k+1)^{N_k/4-1}\xi^{6c_2' L_k} \\
&\le c_{3,24}L_{k+2}^{3(d-1)+1}(k+1)^{N_k/4-1}\xi^{6c_2' L_k} \\
&= c_{3,24}(N_{k+1}N_k)^{3d-2}L_k^{3d-2}(k+1)^{N_k/4-1}\xi^{6c_2' L_k} \\
&\le c_{3,24}8^{3d-2}L_k^{3d-2}(k+1)^{N_k/4-1}\xi^{c_1 L_k}N_k^{6d}\xi^{(6c_1'-1)c_1 L_k}. \qquad (3.73)
\end{aligned}
$$

But, note that

$$
c_{3,24}8^{3d-2}L_k^{3d-2}(k+1)^{N_k/4-1}\xi^{c_1 L_k} \le 1.
$$

for L_0 large enough. Hence, substituting this estimate back into (3.73) and (3.72) we deduce that

$$
c_{3,24}\widetilde{L}_{k+2}^{(d-1)}L_{k+2}(k+1)^{N_k/8-1}\phi_k^{N_k/8} \le N_k^{6d}\xi^{(6c_1'-1)c_1 L_k} \le N_k^{6d}\xi^{77c_1 L_k},
$$

by our choice of c_1'. Finally, choosing L_0 large enough, the expression $N_k^{6d}\xi^{77c_1L_k} \le 1$ for all $k \ge 1$. In the case of $k = 0$, we have that

$$
\left(\frac{c_4}{u_0}\right)^{6d}\xi^{77c_1L_0} \le u_0^{-6d}\xi^{6L_0} \le 1
$$

by our assumption on u_0. Then (3.71) follows and thus we get (3.69) by induction and choosing $L_0 \ge c_{3,25}$ for some constant $c_{3,25} > 0$.

∎

The following corollary implies Proposition 3.4. Since such a derivation follows exactly the argument presented by Sznitman in [Sz02], we omit it.

Corollary 3.2 *Let $l \in \mathbb{S}^{d-1}$, $d \ge 2$ and $\Upsilon = \max\left\{\frac{\alpha}{24}, \left(\frac{2c_1}{c_1-1}\right)\log\eta_\alpha^2\right\}$. Then, there exist constants $c_{3,26} = c_{3,26}(d) > 0$ and $c_{3,27} = c_{3,27}(d) > 0$ such that if the following inequality is satisfied*

$$
c_{3,26}(d)\inf_{L_0\ge c_{3,27}, 3\sqrt{d}\le\tilde{L}_0<L_0^3}\ \inf_{0<a\le\alpha}\left\{\Upsilon^{3(d-1)}\tilde{L}_0^{d-1}L_0^{3(d-1)+1}\mathbb{E}[\rho_B^a]\right\} < 1, \qquad (3.74)
$$

where $B = B(R, L_0-1, L_0+1, \tilde{L}_0)$, then there exists a constant $c_{3,28} > 0$ such that

$$
\overline{\lim_{L\to\infty}}\, L^{-1}e^{c_{28}(\log L)^{1/2}}\log P_0\left[\tilde{T}^l_{-\tilde{b}L} < T^l_{bL}\right] < 0, \quad \textit{for all } b, \tilde{b} > 0.
$$

Proof. If (3.74) holds then there is a $\xi > 0$ such that

$$
c_{3,26}(d)\inf_{L_0\ge c_{3,27}, 3\sqrt{d}\le\tilde{L}_0<L_0^3}\ \inf_{0<a\le\alpha}\left\{\left(\alpha\log\frac{1}{\xi}\right)^{3(d-1)}\tilde{L}_0^{d-1}L_0^{3(d-1)+1}\mathbb{E}[\rho_B^a]\right\} < 1, \qquad (3.75)
$$

with $\xi < \{c_{3,23}, e^{-1/24}\}$. Then, by (3.36) and (3.37),

$$
\widetilde{L}_1^{d-1}L_0 = \left(\frac{c_4}{u_0}\right)^{3(d-1)}\widetilde{L}_0^{d-1}L_0.
$$

Now, the maximum of $u_0^{3(d-1)}\xi^{\alpha u_0 L_0}$, as a function of u_0 for $u_0 \in [\xi^{\frac{L_0}{d}}, 1]$, is given by $c_{3,29}(d)\left(\alpha L_0 \log\frac{1}{\xi}\right)^{-3(d-1)}$ for $u_0 = \frac{3(d-1)}{\alpha L_0\log\frac{1}{\xi}}$, when L_0 is large enough, where $c_{3,29}(d) := \left(\frac{3(d-1)}{e}\right)^{3(d-1)}$. Thus if (3.75) holds, (3.68) holds as well. Hence, applying Corollary 3.1 we can say that (3.69) is true for all $k \ge 0$. The same reasoning used by Sznitman in [Sz02] to derive Proposition 2.3 of that paper gives the estimate

$$P_0\left(\tilde{T}^l_{-\tilde{b}L} < T^l_{bL}\right) \le \left(|C| + \frac{bL}{L_k} + 1\right)\xi^{\frac{\alpha}{2}u_k L_k}$$
$$\le e^{-\widetilde{b}Le^{-c_{3,28}(\log\,\widetilde{b}L)^{\frac{1}{2}}}},$$

for some constant $c_{3,28} > 0$ and L large enough, and where we have chosen $u_0 = \frac{3(d-1)}{\alpha L_0 \log\frac{1}{\xi}}$,

∎

4 An atypical quenched exit estimate

Here we will prove a crucial atypical quenched exit estimate for tilted boxes, which will subsequently enable us in section 5 to show that the regeneration times of the random walk are integrable. Let us first introduce some basic notation.

Without loss of generality, we will assume that e_1 is contained in the open half-space defined by the asymptotic direction so that

$$\hat{v}\cdot e_1 > 0.$$

Recall the definition of the hyperplane perpendicular to direction e_1 in (3.19) so that

$$H := \{x \in \mathbb{R}^d : x \cdot e_1 = 0\}.$$

Let $P := P_{\hat{v}}$ (see (3.20)) be the projection on the asymptotic direction along the hyperplane H defined for $z \in \mathbb{Z}^d$

$$Pz := \left(\frac{z\cdot e_1}{\hat{v}\cdot e_1}\right)\hat{v},$$

and $Q := Q_l$ (see (3.21)) be the projection of z on H along $\hat{v}$ so that

$$Qz := z - Pz.$$

Now, for $x \in \mathbb{Z}^d$, $\beta > 0$, $\varrho > 0$ and $L > 0$, define the *tilted boxes with respect to the asymptotic direction* $\hat{v}$ as

$$B_{\beta,L}(x) := \left\{y \in \mathbb{Z}^d : -L^\beta < (y-x)\cdot e_1 < L;\ \|Q(y-x)\|_\infty < \varrho L^\beta\right\}. \tag{4.1}$$

and their *front boundary* by

$$\partial^+ B_{\beta,L}(x) := \{y \in \partial B_{\beta,L}(x) : y\cdot e_1 - x\cdot e_1 = L\}.$$

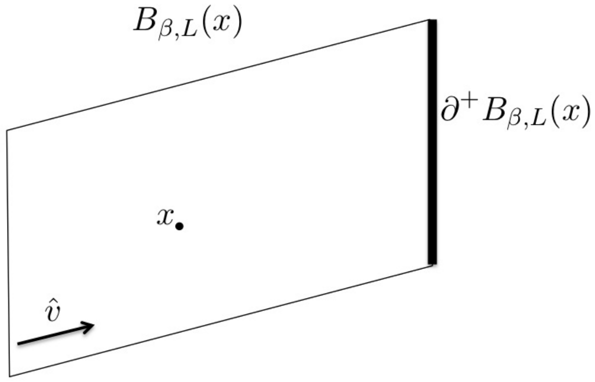

Figure 4: The box $B_{\beta,L}(x)$.

See Figure 4 for a picture of the box $B_{\beta,L}$ and its front boundary.

Proposition 4.1 *Let $\alpha > 0$ and assume that $\eta_\alpha < \infty$ as defined in (3.2). Let $M \geq 15d + 5$ and assume that $(P)_M|l$ is satisfied. Let $\beta_0 \in (1/2, 1)$, $\beta \in \left(\frac{\beta_0+1}{2}, 1\right)$ and $\zeta \in (0, \beta_0)$. Then, for each $\kappa > 0$ we have that*

$$\limsup_{L\to\infty} L^{-g(\beta_0,\beta,\zeta)} \log \mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \leq e^{-\kappa L^\beta}\right) < 0,$$

where

$$g(\beta_0, \beta, \zeta) := \min\{\beta + \zeta, 3\beta - 2 + (d-1)(\beta - \beta_0)\}. \tag{4.2}$$

We will now prove Proposition 4.1 following ideas similar in spirit from those presented by Sznitman in [Sz02].

4.1 Preliminaries

Firstly we need to define an appropriate mesoscopic scale to perform a renormalization analysis. Let $\beta_0 \in (0.5, 1)$, $\beta \in (\beta_0, 1)$ and $\chi := \beta_0 + 1 - \beta \in (\beta_0, 1]$. Define

$$L_0 := \frac{L - \varrho L^{\beta_0}}{[L^{1-\chi}]}.$$

Now, for each $x \in \mathbb{R}^d$ we consider the *mesoscopic box*

$$\tilde{B}(x) := \left\{y \in \mathbb{Z}^d : -L^{\beta_0} < (y-x)\cdot e_1 < \varrho L_0;\ \|y - x - P(y-x)\|_\infty < (1+\varrho)L^{\beta_0}\right\},$$

and its *central part*

$$\tilde{C}(x) := \left\{y \in \mathbb{Z}^d : 0 \leq (y-x)\cdot e_1 < \varrho L_0;\ \|y - x - P(y-x)\|_\infty < L^{\beta_0}\right\}.$$

Define also

$$\partial^+ \tilde{B}(x) := \{y \in \partial \tilde{B}(x) : y \cdot e_1 - x \cdot e_1 = \varrho L_0\}$$

and

$$\partial^+\tilde{C}(x) := \{y \in \partial\tilde{C}(x) : y \cdot e_1 - x \cdot e_1 = \varrho L_0\}.$$

We now say that a box $\tilde{B}(x)$ is *good* if

$$\sup_{x \in \tilde{C}(x)} P_{x,\omega}\left(X_{T_{\tilde{B}(x)}} \notin \partial_+\tilde{B}(x)\right) < \frac{1}{2},$$

Otherwise the box is called *bad*. At this point, by Theorem 1.1 proved in section 3, we have the following version of Theorem 2.3 (Theorem A.2 of Sznitman [Sz02]).

Theorem 4.1 *Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d + 5$. Consider an elliptic RWRE satisfying condition $(P)_M|l$. Then, for any $c > 0$ and $\rho \in (0.5, 1)$,*

$$\limsup_{u \to \infty} u^{-(2\rho-1)} \log P_0\left(\sup_{0 \leq n \leq T_u^{e_1}} |X_n - P(X_n)| \geq cu^\rho\right) < 0,$$

where $T_u^{e_1}$ is defined in (2.2).

The following lemma is an important corollary of Theorem 4.1.

Lemma 4.1 *Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d + 5$. Consider an elliptic RWRE satisfying condition $(P)_M|l$. Then*

$$\limsup_{L \to \infty} L^{-(\beta+\beta_0-1)} \log \mathbb{P}(\tilde{B}(0) \text{ is bad}) < 0. \tag{4.3}$$

Proof. By Chebyschev's inequality we have that

$$\begin{aligned}&\mathbb{P}(\tilde{B}(0) \text{ is bad})\\ &\leq 2^{d-1}L_0L^{\beta_0(d-1)}\left(P_0\left(\sup\nolimits_{0 \leq n \leq T^{\hat{v}}_{\varrho L_0}} |X_n - PX_n| \geq (1+\varrho)L^{\beta_0}\right) + P_0\left(\tilde{T}^{\hat{v}}_{-L^{\beta_0}} < \infty\right)\right).\end{aligned}$$

By Theorem 4.1, the first summand can be estimated as

$$\limsup_{L \to \infty} L^{-(\beta+\beta_0-1)} \log P_0\left(\sup_{0 \leq n \leq T^{\hat{v}}_{\varrho L_0}} |X_n - PX_n| \geq (1+\varrho)L^{\beta_0}\right) < 0.$$

To estimate the second summand, since $(P)_M|l$ is satisfied, by Theorem 1.1 and the equivalence given by Theorem 2.4, we can chose γ close enough to 1 so that $\gamma\beta_0 \geq \beta_0 + \beta - 1$ and such that

$$\limsup_{L \to \infty} L^{-\gamma\beta_0} \log P_0\left(\tilde{T}^{\hat{v}}_{-(1+\varrho)L^{\beta_0}} < \infty\right) < 0.$$

∎

Let $k_1, \ldots, k_d \in \mathbb{Z}$. From now on, we will use the notation $x = (k_1, \ldots, k_d) \in \mathbb{R}^d$ to denote the point

$$x = k_1 \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v} + \sum_{j=2}^d 2k_j(1+\varrho)L^{\beta_0}e_j.$$

Define the following set of points which will correspond to the centers of mesoscopic boxes.

$$\mathcal{L} := \left\{x \in \mathbb{R}^d : x = (k_1, \ldots, k_d) \text{ for some } k_1, \ldots, k_d \in \mathbb{Z}\right\}.$$

We will use subsequently the following property of the lattice $\mathcal{L}$: there exist 2^d disjoint sub-lattices $\mathcal{L}_1, \dots, \mathcal{L}_{2^d}$ such that $\mathcal{L} = \cup_{i=1}^{2^d} \mathcal{L}_i$ and for each $1 \le i \le 2^d$, the sub-lattice $\mathcal{L}_i$ corresponds to the centers of mesoscopic boxes which are pairwise disjoint. Let $\mathcal{L}_0$ be the set defined by

$$\mathcal{L}_0 := \{x = (k_1, \dots, k_d) \in \mathcal{L} : k_1 = 0\}.$$

For each $x \in \mathcal{L}_0$ we define the *column* of mesoscopic boxes as

$$C_x := \bigcup_{k_1=-1}^{[L^{1-\chi}]} \tilde{B}\left(x + k_1 \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v}\right)$$

See Figure 5 for a picture of the column C_x, for some $x \in \mathcal{L}_0$.

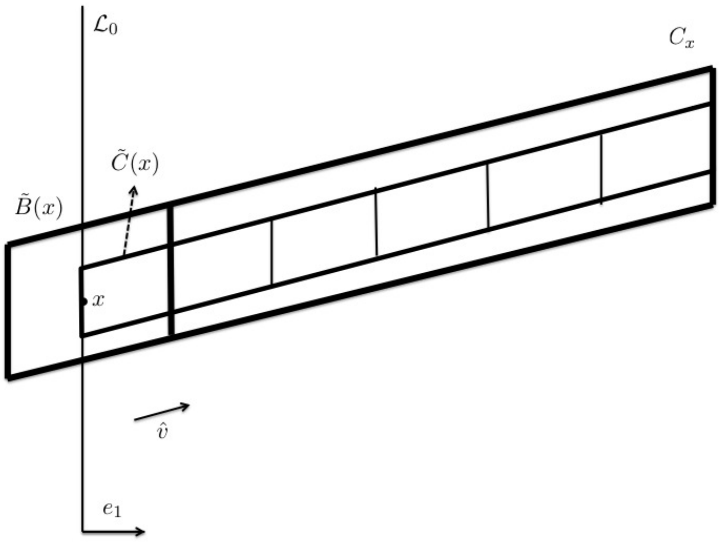

Figure 5: A box $\tilde{B}$ with its corresponding middle part $\tilde{C}$, which belongs to the column C_x.

The collection of these columns will be denoted by $\mathcal{C}$. Define now for each $C_x \in \mathcal{C}$ and $-1 \le k \le [L^{1-\chi}]$ define

$$\partial_{k,1} C_x := \partial_+ \tilde{C}\left(x + k \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v}\right) \quad \text{and} \quad \partial_{k,2} C_x := \partial_+ \tilde{B}\left(x + k \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v}\right) \setminus \partial_{k,1} C_x.$$

For each point $y \in \partial_{k,1} C_x$ we assign a path $\pi^{(k)} = \{\pi_1^{(k)}, \dots, \pi_{n_1}^{(k)}\}$ with $n_1 := \left[2c_1 \frac{\varrho}{\hat{v} \cdot e_1} L_0\right]$ steps from y to $\partial_{k+1,1} C_x$, so that $\pi_1^{(k)} = y$ and $\pi_{n_1}^{(k)} \in \partial_{k+1,1} C_x$. For each point $z \in \partial_{k,2} C_x$ we assign a path $\bar{\pi}^{(k)} = \{\bar{\pi}_1^{(k)}, \dots, \bar{\pi}_{n_2}^{(k)}\}$ with $n_2 := \left[2c_1 \varrho L^{\beta_0}\right]$ steps from z to $\partial_{k,1} C_x$, so that $\bar{\pi}_1^{(k)} = z$ and $\bar{\pi}_{n_2}^{(k)} \in \partial_{k,1} C_x$. We will also use the notation $\{m_1, \dots, m_N\}$ to denote some subset of $\{-1, \dots, [L^{1-\chi}]\}$ with N elements.

Let $x \in \mathcal{L}_0$ and $\xi > 0$. A column of boxes $C_x \in \mathcal{C}$ will be called *elliptically good* if it satisfies the following two conditions

$$\sup_{N \le \left[\frac{L^\beta}{L_0}\right]} \sup_{\{m_1, \dots, m_N\}} \sum_{j=1}^{N} \sup_{y_{m_j} \in \partial_{m_j,1} C_x} \sum_{i=1}^{n_1} \log \frac{1}{\omega(\pi_i^{(m_j)}, \Delta \pi_i^{(m_j)})} \le 2c_1 \frac{\varrho}{\hat{v} \cdot e_1} \log\left(\frac{1}{\xi}\right) L^\beta \tag{4.4}$$

and

$$\sum_{k=-1}^{[L^{1-\chi}]} \sup_{z_k \in \partial_{k,2} C_x} \sum_{i=1}^{n_2} \log \frac{1}{\omega(\bar{\pi}_i^{(k)} \Delta \bar{\pi}_i^{(k)})} \le 2c_1 \varrho \log\left(\frac{1}{\xi}\right) L^\beta. \tag{4.5}$$

If neither (4.4) nor (4.5) is satisfied, we will say that the column C_x is *elliptically bad.*

Lemma 4.2 *For any $x \in \mathcal{L}_0$, $\beta \geq \frac{\beta_0+1}{2}$ and $\xi > 0$ such that $\log \frac{1}{\xi^{2\alpha}\eta_\alpha^3} > 0$ we have that*

$$\limsup_{L\to\infty} L^{-\beta} \log \mathbb{P}\left(C_x \textit{ is elliptically bad}\right) < 0 \tag{4.6}$$

Proof. Let us first note that $\frac{L^\beta}{L_0} \geq 1$ by our condition on β. Now, it is clear that

$$\mathbb{P}\left(C_x \text{ is elliptically bad}\right) \leq \mathbb{P}\left((4.4)\,\text{is not satisfied}\right) + \mathbb{P}\left((4.5)\,\text{is not satisfied}\right) \tag{4.7}$$

Regarding the first term on the right of (4.7) and since $2\beta - \beta_0 - 1 < \beta - \beta_0 < \beta$ we have that

$$\begin{aligned}\mathbb{P}\left((4.4)\,\text{is not satisfied}\right) &\leq \textstyle\sum_{N=1}^{[\frac{L^\beta}{L_0}]} \mathbb{P}\Big(\exists\{m_1,\ldots,m_N\} \text{ and } y_{m_j} \in \partial_{m_j,1}C_x \,\text{such that}\\ &\qquad \textstyle\sum_{j=1}^{N}\sum_{i=1}^{n_1} \log \frac{1}{\omega\left(\pi_i^{(m_j)},\Delta\pi_i^{(m_j)}\right)} > 2c_1 \frac{\varrho}{\hat{v}\cdot e_1} \log\left(\frac{1}{\xi}\right) L^\beta\Big)\\ &\leq \frac{L^\beta}{L_0} L^{(\beta-\beta_0)\frac{L^\beta}{L_0}} e^{(\log L)\beta_0(d-1)L^{\beta-\beta_0}} e^{2(\log\eta_\alpha)c_1\frac{\varrho}{\hat{v}\cdot e_1}L^\beta} e^{-2c_1\frac{\varrho}{\hat{v}\cdot e_1}\left(\alpha\log\frac{1}{\xi}\right)L^\beta} \leq e^{-c_{4,1}L^\beta}\end{aligned} \tag{4.8}$$

for some constant $c_{4,1} > 0$ if L is large enough and $\log \frac{1}{\xi^{2\alpha}\eta_\alpha^3} > 0$.

Similarly for the rightmost term of (4.7) we have that,

$$\begin{aligned}&\mathbb{P}\left((4.5)\,\text{is not satisfied}\right)\\ &\leq \mathbb{P}\left(\exists\, z_k \in \partial_{k,2}C_x \,\text{such that} \textstyle\sum_{k=-1}^{[L^{1-\chi}]}\sum_{i=1}^{n_2}\log\frac{1}{\omega(\bar{\pi}_i^{(k)}\Delta\bar{\pi}_i^{(k)})} > 2c_1\varrho\log\left(\frac{1}{\xi}\right)L^\beta\right)\\ &\qquad \leq e^{\log L(\varrho\beta_0(d-1)L^{1-\chi})} e^{2(\log\eta_\alpha)c_1\varrho L^\beta} e^{-2c_1\varrho\left(\alpha\log\frac{1}{\xi}\right)L^\beta} \leq e^{-c_{4,2}L^\beta}\end{aligned} \tag{4.9}$$

for some constant $c_{4,2} > 0$ if L is large enough and $\log \frac{1}{\xi^{2\alpha}\eta_\alpha^3} > 0$. Substituting (4.8) and (4.9) back into (4.7), (4.6) follows. ∎

The proof Proposition 4.1 will reduced to the control of the probability of the three events: the first one, corresponding to subsection 4.2, gives a control on the number of bad boxes; the second one, corresponding to subsection 4.3, gives a control on the number of elliptically good columns; the third one, corresponding to subsection 4.4, gives a control on the probability that the random walk can find an appropriate path which leads to an elliptically good column.

4.2 Control on the number of bad boxes

We will need to consider only the mesoscopic boxes which intersect the box $B_{\beta,L}(0)$ and whose k_1 index is larger than or equal to -1. We hence define the collection of mesoscopic boxes

$$\mathcal{B} := \left\{\tilde{B}(x) : \tilde{B}(x) \cap B_{\beta,L}(0) \neq \emptyset, x = (k_1,\ldots,k_d),\ k_1,\ldots,k_d \in \mathbb{Z},\ k_1 \geq -1\right\}$$

In addition, we call the number of bad mesoscopic boxes in $\mathcal{B}$,

$$N(L) := \left|\left\{\tilde{B} \in \mathcal{B} : \tilde{B} \text{ is bad}\right\}\right|,$$

and for each $1 \leq i \leq 2^d$, call the number of bad mesoscopic boxes in $\mathcal{B}$ with centers in the sub-lattice $\mathcal{L}_i$ as

$$N_i(L) := \left|\left\{\tilde{B}(x) \in \mathcal{B} : \tilde{B}(x) \text{ is bad and } x \in \mathcal{L}_i\right\}\right|,$$

Define

$$G_1 := \left\{ \omega \in \Omega : N(L) \le \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)} L^{\beta}}{2(1+\varrho)^{d-1} L_0} \right\}.$$

Lemma 4.3 *Assume that* $\beta > \frac{\beta_0+1}{2}$. *Then, there is a constant* $c_{4,3} > 0$ *such that for every* $L > 1$ *we have that*

$$\mathbb{P}(G_1^c) \le e^{-c_{4,3} L^{3\beta-2+(d-1)(\beta-\beta_0)}}.$$

Proof. Note that the number of columns intersecting the box $B_{\beta,L}(0)$ is equal to

$$\left\lceil \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{(1+\varrho)^{d-1}} \right\rceil.$$

Hence, whenever $\omega \in G_1$, necessarily there exist at least $\left\lceil \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{2(1+\varrho)^{d-1}} \right\rceil$ columns each one with at most $\left\lceil \frac{L^\beta}{L_0} \right\rceil$ bad boxes. Let us take $m_1 := \left\lceil \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)} L^\beta}{2(1+\varrho)^{d-1} L_0} \right\rceil$ and $m_2 := |\mathcal{B}| = \left\lceil \frac{\varrho^{d-2} L^{d(\beta-\beta_0)}}{(1+\varrho)^{d-1}} + \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{(1+\varrho)^{d-1}} \right\rceil$. Now, using the fact that the mesoscopic boxes in each sub-lattice $\mathcal{L}_i$, $1 \le i \le 2^d$, are disjoint, and the estimate (4.3) of Lemma 4.1, we have by independence that there exists a constant $c_{4,3} > 0$ such that for every $L \ge 1$,

$$\begin{aligned} \mathbb{P}(G_1^c) &= \mathbb{P}\left(N(L) \ge m_1\right) \le \sum_{i=1}^{2^d} \mathbb{P}\left(N_i(L) \ge \tfrac{m_1}{2^d}\right) \\ &\le \sum_{i=1}^{2^d} \sum_{n=m_1/2^d}^{m_2} \binom{m_2}{n} \mathbb{P}\left(\tilde{B}(0) \text{ is bad}\right)^n \le 2^d \sum_{n=m_1/2^d}^{m_2} m_2^n e^{-nL^{\beta+\beta_0-1}} \\ &\le e^{-c_{4,3} L^{\beta+\beta_0-1+(d-1)(\beta-\beta_0)+2\beta-\beta_0-1}} \le e^{-c_{4,3} L^{3\beta-2+(d-1)(\beta-\beta_0)}}. \end{aligned}$$

Note that in the second to last inequality we have used the fact that $2\beta + \beta_0 - 2 > 0$ which is equivalent to the condition $\beta > \frac{2-\beta_0}{2}$. Now, this last condition is implied by the requirement $\beta > \frac{\beta_0+1}{2}$.

∎

4.3 Control on the number of elliptically bad columns

Let $m_3 := \left\lceil \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{2(1+\varrho)^{d-1}} \right\rceil$ and define the event that any sub-collection of the set of columns of cardinality less than or equal to m_3 has at least one elliptically good column

$$G_2 := \{\omega \in \Omega : \forall \mathcal{D} \subset \mathcal{C},\, |\mathcal{D}| \ge m_3,\, \exists\, C_x \in \mathcal{D} \text{ such that } C_x \text{ is elliptically good}\}. \quad (4.10)$$

Here we will prove the following lemma.

Lemma 4.4 *There is a constant* $c_{4,4} > 0$ *such that for every* $L \ge 1$,

$$\mathbb{P}(G_2^c) \le e^{-c_{4,4} L^{\beta+(d-1)(\beta-\beta_0)}}.$$

Proof. Note that the total number of columns intersecting the box $B_{\beta,L}$ is equal to

$$m_4 := \left\lceil \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{(1+\varrho)^{d-1}} \right\rceil.$$

Using the fact that the events $\{C_x \text{ is elliptically bad}\}$, $\{C_y \text{ is elliptically bad}\}$ are independent if $x \ne y$, since these columns are disjoint, we conclude that there is a constant $c_{4,4} > 0$ such that for all $L \ge 1$,

$$\begin{aligned} \mathbb{P}(G_2^c) &= \mathbb{P}\left(\exists\, \mathcal{D} \subset \mathcal{C},\, |\mathcal{D}| \ge m_3 \text{ such that } \forall C_x \in \mathcal{D},\ C_x \text{ is elliptically bad}\right) \\ &\le \sum_{n=m_3}^{m_4} m_4^n \mathbb{P}\left(C_x \text{ is elliptically bad}\right)^n \le e^{-c_{4,4} L^{\beta+(d-1)(\beta-\beta_0)}}, \end{aligned}$$

where in the last inequality we have used the estimate (4.6) of Lemma 4.2 which provides a bound for the probability of a column to be elliptically bad.

∎

4.4 The confinement event

Here we will obtain an adequate estimate for the probability that the random walk hits an elliptically good column. We will need to introduce some notation, corresponding to the the box where the random walk will move before hitting the elliptically good column and a certain class of hyperplanes of this region. Let first $\zeta \in (0, \beta_0)$, a parameter which gives the order of width of the box $\bar{B}_{\zeta,\beta,L}$ where the random walk will be able to find a reasonable path to the elliptically good column, so that

$$\bar{B}_{\zeta,\beta,L} := \{x \in \mathbb{Z}^d : -L^\zeta \le x \cdot e_1 \le L^\zeta, \|x - Px\|_\infty < L^\beta\}.$$

Note that this box is contained in $B_{\beta,L}(0)$ and that it also contains the starting point 0 of the random walk. Define now for each $0 \le z \le L^\zeta$, the hyperplane

$$H_z := \left\{x \in \bar{B}_{\zeta,\beta,L} : x \cdot e_1 = z\right\},$$

and consider the two collection of hyperplanes defined as

$$\mathcal{H}^+ = \left\{H_z : z \in \mathbb{Z}, 0 \le z \le L^\zeta\right\} \quad \text{and} \quad \mathcal{H}^- = \left\{H_z : z \in \mathbb{Z}, -L^\zeta \le z < 0\right\}.$$

Whenever there is no risk of confusion, we will drop the subscript from H_z writing H instead. Let $r := [2\varrho L^\beta]$. Now, for each $H \in \mathcal{H}^+ \cup \mathcal{H}^-$ and each j such that $e_j \ne \pm e_1$, we will consider the set of paths Π_j with r steps defined by $\pi = \{\pi_1, \ldots, \pi_r\} \in \Pi_j$ if and only if

$$\pi \subset H \quad \text{and} \quad \pi_{i+1} - \pi_i = e_j.$$

In other words, π is contained in the hyperplane H and it has steps which move only in the direction e_j. We now say that an hyperplane $H \in \mathcal{H}^+ \cap \mathcal{H}^-$ is *elliptically good* if for all paths $\pi \in \cup_{j \ne 1, d+1} \Pi_j$ one has that

$$\sum_{i=1}^{r} \log \frac{1}{\omega(\pi_i, \Delta\pi_i)} \le 2\varrho \log\left(\frac{1}{\xi}\right) L^\beta. \tag{4.11}$$

Otherwise H will be called *elliptically bad* (See Figure 6).

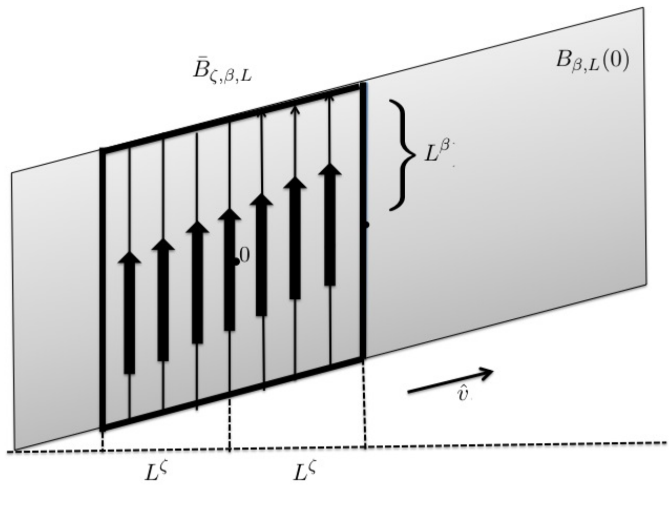

Figure 6: The box $\bar{B}_{\zeta,\beta,L}$. The arrows indicate the uniform ellipticity condition given by (4.11), which implies that each hyperplane is elliptically good.

From a routine counting argument and applying Chebyshev inequality, note that for each $H \in \mathcal{H}^+ \cup \mathcal{H}^-$ and $\xi > 0$ such that $\log \frac{1}{\xi^\alpha \eta_\alpha^2} > 0$ there is a constant $c_{4,5} > 0$ such that

$$\mathbb{P}\left(H \text{ is elliptically bad}\right) \le e^{-c_{4,5} L^\beta}. \tag{4.12}$$

Now choose a rotation $\hat{R}$ such that $\hat{R}(e_1) = \hat{v}$. Let $\hat{v}_j := \hat{R}(e_j)$ for $j \ge 2$. We now want to make a construction analogous to the one which led to the concept of elliptically good hyperplane. But now, we would need to define hyperplanes perpendicular to the directions $\{\hat{v}_j\}$ which are not necessarily equal to a canonical vector. Therefore, we will work here with strips, instead of hyperplanes. For each $z \in \mathbb{Z}$ even and $k \in \{2, \ldots, d\}$ consider the strip $I_{k,z} := \{x \in \bar{B}_{\zeta,\beta,L}(0) : z - 1 < x \cdot \hat{v}_j < z + 1\}$. Consider also the two sets of strips, $\mathcal{I}_k^+$ and $\mathcal{I}_k^-$ defined by

$$\mathcal{I}_k^+ := \left\{ I_{k,z} : z \text{ even}, 0 \le z \le \varrho L^\beta \right\} \quad \text{and} \quad \mathcal{I}_k^- := \left\{ I_{k,z} : z \text{ even}, -\varrho L^\beta \le z < 0 \right\}.$$

Whenever there is no risk of confusion, we will drop the subscripts from a strip $I_{k,z}$ writing I instead. We will need to work with the set of canonical directions which are contained in the closed positive half-space defined by the asymptotic direction, so that

$$U^+ := \{e \in U : e \cdot \hat{v} \ge 0\}.$$

Let $s := \left[2c_1 \frac{L^\varsigma}{\hat{v} \cdot e_1}\right]$. For each $I \in \mathcal{I}_k^+ \cup \mathcal{I}_k^-$ and each $y \in I$ we associate a path $\hat{\pi} = \{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$, with $s \le n \le s + 1$, which satisfies

$$\hat{\pi} \subset I_{j,z}$$

and

$$\hat{\pi}_{i+1} - \hat{\pi}_i \in U^+ \quad \text{for } 1 \le i \le n-1, \quad \hat{\pi}_n \in H_{[L^\varsigma]}.$$

Note that by the fact that the strip I has a Euclidean width 1, it is indeed possible to find a path satisfying these conditions and also that such a path is not necessarily unique. We will call $\hat{\Pi}_k$ such a set of paths associated to all the points of the strip I. Now, a strip $I \in \mathcal{I}_k^+ \cup \mathcal{I}_k^-$ will be called *elliptically good* if for all paths $\hat{\pi} \in \hat{\Pi}_k$ one has that

$$\sum_{i=1}^{n} \log \frac{1}{\omega\left(\hat{\pi}_i, \Delta\hat{\pi}_i\right)} \le \log\left(\frac{1}{\xi}\right) n \tag{4.13}$$

Otherwise I will be called *elliptically bad* (See Figure 7).

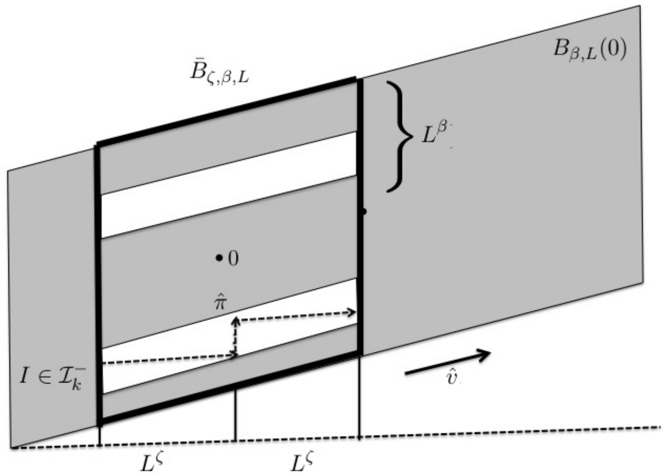

Figure 7: In each strip $\mathcal{I}$, every path π chosen previously satisfies the uniform ellipticity condition given by (4.13). Then $\mathcal{I}$ is elliptically good.

As before, from a routine counting argument and by Chebyshev inequality, note that for each $k \in \{2, \ldots, d\}$, $I \in \mathcal{I}_k^+ \cup \mathcal{I}_k^-$ and $\xi > 0$ which satisfies $\log \frac{1}{\xi^\alpha \eta_\alpha^2} > 0$, there exists a constant $c_{4,6} > 0$ such that

$$\mathbb{P}\left(I \text{ is elliptically bad}\right) \leq e^{-c_{4,6} L^{\zeta}}. \tag{4.14}$$

We now define the *confinement event* as

$$G_3 := \{\omega \in \Omega : \exists\, H_+ \in \mathcal{H}_+, H_- \in \mathcal{H}_-, I_{+,2} \in \mathcal{I}_2^+, \ldots, I_{+,d} \in \mathcal{I}_d^+, I_{-,2} \in \mathcal{I}_2^-, \ldots, I_{-,d} \in \mathcal{I}_d^- \\ \text{such that } H_+, H_-, I_{+,2}, \ldots, I_{+,d}, I_{-,2}, \ldots, I_{-,d} \text{ are elliptically good}\}.$$

We can now state the following lemma which will eventually give a control on the probability that the random walk hits an elliptically good column.

Lemma 4.5 *There is a constant $c_{4,7} > 0$ such that for every $L \geq 1$,*

$$\mathbb{P}\left(G_3^c\right) \leq e^{-c_{4,7} L^{\beta+\zeta}}. \tag{4.15}$$

Proof. Note that

$$\mathbb{P}\left(G_3^c\right) \leq \mathbb{P}\left(\bigcap_{H \in \mathcal{H}^+} \{H \text{ is elliptically bad}\}\right) + \mathbb{P}\left(\bigcap_{H \in \mathcal{H}^-} \{H \text{ is elliptically bad}\}\right) \\ + \sum_{k=2}^{d} \mathbb{P}\left(\bigcap_{I \in \mathcal{I}_k^+} \{I \text{ is elliptically bad}\}\right) + \sum_{k=2}^{d} \mathbb{P}\left(\bigcap_{I \in \mathcal{I}_k^-} \{I \text{ is elliptically bad}\}\right),$$

Now, inequality (4.15) follows using the estimate (4.12) for the probability that a hyperplane is elliptically bad, the estimate (4.14) for the probability that a strip is elliptically bad, applying independence and translation invariance.

∎

4.5 Proof of Proposition 4.1

Firstly, note that for any $\kappa > 0$,

$$\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \le e^{-\kappa L^\beta}\right) \le \mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) + \mathbb{P}(G_3^c)$$
$$+\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \le e^{-\kappa L^\beta}, G_1, G_2, G_3\right). \tag{4.16}$$

Let us begin bounding the first three terms of the right-hand side of (4.16). Let $\zeta \in (0, \beta_0)$ and $\beta > \frac{\beta_0+1}{2}$. By Lemma 4.3 of subsection 4.2, Lemma 4.4 of subsection 4.3 and Lemma 4.5 of subsection 4.4 we have that there is a constant $c_{4,8} > 0$ such that

$$\mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) + \mathbb{P}(G_3^c) \le \frac{1}{c_{4,8}} e^{-c_{4,8} L^{3\beta-2+(d-1)(\beta-\beta_0)}} + \frac{1}{c_{4,8}} e^{-c_{4,8} L^{\beta+(d-1)(\beta-\beta_0)}} + \frac{1}{c_{4,8}} e^{-c_{4,8} L^{\beta+\zeta}}. \tag{4.17}$$

Since $\beta < 1$ is equivalent to $\beta + (d-1)(\beta-\beta_0) > 3\beta - 2 + (d-1)(\beta-\beta_0)$, the sum in (4.17) can be bounded as

$$\mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) + \mathbb{P}(G_3^c) \le \frac{1}{c_{4,9}} e^{-c_{4,9} L^{g(\beta,\beta_0,\zeta)}}, \tag{4.18}$$

for some constant $c_{4,9} > 0$ and where $g(\beta, \beta_0, \zeta) := \min\{\beta + \zeta, 3\beta - 2 + (d-1)(\beta - \beta_0)\}$.

We will now prove that the fourth term of the right-hand side of inequality (4.16) satisfies for L large enough

$$\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \le e^{-\kappa L^\beta}, G_1, G_2, G_3\right) = 0. \tag{4.19}$$

In fact, we will show that for L large enough on the event $G_1 \cap G_2 \cap G_3$ one has that

$$P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) > e^{-\kappa L^\beta}. \tag{4.20}$$

We will prove (4.20) showing that the walk can exit $B_{\beta,L}(0)$ through $\partial^+ B_{\beta,L}(0)$ choosing a strategy which corresponds to paths which go through an elliptically good column. This implies, in particular, that the walk exit successively of boxes $\tilde{B}(x)$ through $\partial^+ \tilde{B}(x)$. The event G_1 implies that there exist at least $m_3 = \left[\frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{2(1+\varrho)^{d-1}}\right]$ columns each one with at most $\left[\frac{L^\beta}{L_0}\right]$ of bad boxes. Meanwhile, the event G_2 asserts that in any collection of columns with cardinality m_3 or more, there is at least one elliptically good column. Therefore, on the event $G_1 \cap G_2$ there exists at least one elliptically good column D with at most L^β/L_0 bad boxes. Thus, on $G_1 \cap G_2$ we have that for any point $y \in D$ and $\xi > 0$,

$$P_{y,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \ge \left(\frac{1}{2}\right)^{L^{\beta-\beta_0}+1} \xi^{2c_1 \frac{\varrho}{\hat{v}\cdot e_1} L^\beta} \xi^{2c_1 \varrho L^\beta}, \tag{4.21}$$

where the first factor is a bound for the probability that the random walk exits all the good boxes of the column through their front side, while the second factor is a bound for the probability that the walk traverses each bad box (whose number is at most L^β/L_0) exiting through its front side and following a path with at most $\frac{2c_1\rho L_0}{\hat{v}\cdot e_1}$ steps and is given by the condition (4.4) for elliptically good columns, while the third factor is a bound for the probability that once the walk exits a box (whose number is at most $L^{\beta-\beta_0}+1$) it moves through its front boundary to the central point of this front boundary following a path with at most $[2c_1\rho L_0^\beta]$ steps and is given by the condition (4.5) for elliptically good columns.

Now, the confinement event G_3 ensures that with a high enough probability the random walk will reach the elliptically good column D which has at most L^β/L_0 bad boxes. More precisely, a.s. on G_3, the random walk reaches either an elliptically good hyperplane $H \in \mathcal{H}_+ \cup \mathcal{H}_-$, an elliptically good strip $I \in \mathcal{I}_2^+ \cup \cdots \cup \mathcal{I}_d^+$ or an elliptically good strip $I \in \mathcal{I}_2^- \cup \cdots \cup \mathcal{I}_d^-$ (recall the definitions of elliptically good hyperplanes and strips given in (4.11) and (4.13) of subsection 4.3). Now, once the walk reaches either an elliptically good hyperplane or strip, we know by (4.11) or (4.13), choosing an appropriate path that the probability that it hits the column D is at least $\xi^{c_{4,10}\varrho L^\beta}$ for some constant $c_{4,10}>0$. Thus, we know that there is a constant $c_{4,10}>0$ such that

$$P_{0,\omega}\left(\text{the walk reaches } D\cap \bar{B}_{\zeta,\beta,L}(0)\right)\geq \xi^{c_{4,10}\varrho L^\beta}. \tag{4.22}$$

Therefore, combining (4.21) and (4.22), we conclude that there is a constant $c_{4,11}>0$ such that for all $\varrho\in(0,1)$ on the event $G_1\cap G_2\cap G_3$ the following estimate is satisfied,

$$P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}}\in\partial^+B_{\beta,L}(0)\right)>e^{-c_{4,11}\rho L^\beta}.$$

Hence, choosing ϱ sufficiently small, we have that on $G_1\cap G_2\cap G_3$,

$$P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}}\in\partial^+B_{\beta,L}(0)\right)>e^{-\kappa L^\beta} \tag{4.23}$$

for L larger than a deterministic constant depending only on ϱ. This proves (4.19).

Finally, with the help of (4.16), (4.18) and (4.23) the Proposition 4.1 is proved.

5 Moments of the regeneration time

Here we will prove Theorems 1.2 and 1.3. Our method is inspired on some ideas used by Sznitman to prove Proposition 3.1 of [Sz01], which give tail estimates on the distribution of the regeneration times. Theorem 1.2 will follow from part (a) of Theorem 2.1, while part (a) of Theorem 1.3 from part (b) of Theorem 2.1. Part (b) of Theorem 1.3 will follow from Theorem 2.2. The non-degeneracy of the covariance matrix in the annealed part (a) of Theorem 1.3 can be proven exactly as in section IV of [Sz00] using (2.5).

Proposition 5.1 *Let $l\in\mathbb{S}^{d-1}$, $\beta>0$ and $M\geq 15d+5$. Assume that $(P)_M|l$ is satisfied and that $(E')_\beta$ holds towards the asymptotic direction (cf. (1.5), (1.6), (1.8)). Then*

$$\limsup_{u\to\infty}(\log u)^{-1}\log P_0[\tau_1^{\hat{v}}>u]\leq-\beta.$$

The proof of the above proposition is based on the atypical quenched exit estimate corresponding to Proposition 4.1 of section 4. Some slight modifications in the proof of Proposition 4.1, would lead to a version of it, which could be used to show that Proposition 5.1 remains valid if the regeneration time $\tau_1^{\hat{v}}$ is replaced by τ_1^l for any direction l such that $l\cdot\hat{v}>0$. Note also that Proposition 5.1 implies that whenever $(E')_1$ is satisfied towards the asymptotic direction, then the first regeneration time is integrable. Through part (a) of Theorem 2.1, this implies Theorem 1.2. Similarly we can conclude part (a) of Theorem 1.3. Part (b) of Theorem 1.3 can be derived analogously through Theorem 2.2.

Let us now proceed with the proof of Proposition 5.1. Let us take a rotation $\hat{R}$ in R^d such that $\hat{R}(e_1)=\hat{v}$ and fix $\beta\in\left(\frac{5}{6},1\right)$ and $M>0$. For each $u>0$ define the scale

$$L=L(u):=\left(\frac{1}{4M\sqrt{d}}\right)^{\frac{1}{\beta}}(\log u)^{\frac{1}{\beta}},$$

and the box

$$C_L:=\left\{x\in\mathbb{Z}^d:\frac{-L}{2(\hat{v}\cdot e_1)}\leq x\cdot\hat{R}(e_i)\leq\frac{L}{2(\hat{v}\cdot e_1)},\text{for } 0\leq i\leq 2d\right\}.$$

Throughout the rest of this proof we will continue writing τ_1 instead of $\tau_1^{\hat{v}}$. Now note that

$$P_0(\tau_1>u)\leq P_0\left(\tau_1>u,T_{C_{L(u)}}\leq\tau_1\right)+P_0\left(T_{C_{L(u)}}>u\right), \tag{5.1}$$

where $T_{C_{L(u)}}$ is the first exit time from the set $C_{L(u)}$ defined in (1.1). For the second term of the right-hand side of inequality (5.1), we can use Corollary 2.1, to conclude that for every $\gamma \in (\beta, 1)$ there exists a constant $c_{5,1}$ such that

$$P_0\left(\tau_1 > u, T_{C_{L(u)}} \le \tau_1\right) \le \frac{1}{c_{5,1}} e^{-c_{5,1} L^\gamma(u)}. \tag{5.2}$$

For the first term of the right-hand side of inequality (5.1), following Sznitman [Sz01] we introduce the event

$$F_1 := \left\{ \omega \in \Omega : t_\omega\left(C_{L(u)}\right) > \frac{u}{(\log u)^{\frac{1}{\beta}}} \right\},$$

where for each $A \subset \mathbb{Z}^d$ we define

$$t_\omega(A) := \inf\left\{ n \ge 0 : \sup_x P_{x,\omega}\left(T_A > n\right) \le \frac{1}{2} \right\}.$$

Trivially,

$$P_0\left(T_{C_{L(u)}} > u\right) \le \mathbb{E}\left[F_1^c, P_{0,\omega}\left(T_{C_{L(u)}} > u\right)\right] + \mathbb{P}(F_1). \tag{5.3}$$

To bound the first term of the right-hand side of (5.3), on the event F_1^c we apply the strong Markov property $[(\log u)^{\frac{1}{\beta}}]$ times to conclude that

$$\mathbb{E}\left[F_1^c, P_{0,\omega}\left(T_{C_{L(u)}} > u\right)\right] \le \left(\frac{1}{2}\right)^{[(\log u)^{\frac{1}{\beta}}]}. \tag{5.4}$$

To bound the second term of the right-hand side of (5.3), we will use the fact that for each $\omega \in \Omega$ there exists $x_0 \in C_{L(u)}$ such that

$$P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \le \frac{2|C_{L(u)}|}{t_\omega(C_{L(u)})} \tag{5.5}$$

where for $y \in \mathbb{Z}^d$,

$$\tilde{H}_y = \inf\{n \ge 1 : X_n = y\}.$$

(5.5) can be derived using the fact that for every subset $A \subset \mathbb{Z}^d$ and $x \in A$,

$$E_{x,\omega}(T_A) = \sum_{y \in A} \frac{P_{x,\omega}(H_y < T_A)}{P_{y,\omega}(\tilde{H}_y > T_A)}$$

(see for example Lemma 1.3 of Sznitman [Sz01]). Now note that (5.5) implies

$$\mathbb{P}(F_1) \le \mathbb{P}\left(\omega \in \Omega : \exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \le \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}| \right). \tag{5.6}$$

Choose for each $x \in C_{L(u)}$ a point y_x as any point in $\mathbb{Z}^d$ which is closest to the point $x + \left(\frac{\log u}{2M\sqrt{d}(\hat{v}\cdot e_1)}\right) \hat{v} = x + 2\left(\frac{L^\beta}{\hat{v}\cdot e_1}\right) \hat{v}$. It is straightforward to see that

$$N - 1 \le |y_x - x|_1 \le N + 1,$$

where

$$N := \frac{|\hat{v}|_1 \log u}{2M\sqrt{d}(\hat{v} \cdot e_1)}.$$

Let us call

$$V(y_x) := \{y \in \mathbb{Z}^d : |y_x - y|_1 \leq 4d\},$$

the closed l_1 ball centered at y_x. We furthermore define

$$V'(y_x) := \begin{cases} V(y_x) \cap C_{L(u)} & \text{if } y_x \in C_{L(u)} \\ V(y_x) \cap (C_{L(u)})^c & \text{if } y_x \notin C_{L(u)}. \end{cases} \tag{5.7}$$

Now, there are constants K_1 and K_2 such that for each $1 \leq i \leq 2d$, we can find $4d-2$ different paths $\{\pi^{(i,j)} : 1 \leq j \leq 4d-2\}$, each one with n_j steps, with $\pi^{(i,j)} := \{\pi_1^{(i,j)}, \ldots, \pi_{n_j}^{(i,j)}\}$ for each $1 \leq j \leq 4d-2$ such that the following conditions are satisfied:

(a) **(the paths connect x or $x + e_i$ to $V'(y_x)$)** For each $1 \leq j \leq 2d-1$, the path $\pi^{(i,j)}$ goes from x to $V'(y_x)$, so that $\pi_1^{(i,j)} = x$ and $\pi_{n_i}^{(i,j)} \in V'(y_x)$. For each $2d \leq j \leq 4d-2$, the path $\pi^{(i,j)}$ goes from $x + e_i$ to $V'(y_x)$, so that $\pi_1^{(i,j)} = x + e_i$ and $\pi_{n_j}^{(i,j)} \in V'(y_x)$.

(b) **(the paths start using $4d-2$ directions as shown in Figure 8)** For each $1 \leq j \leq 2d-1$, $\pi_2^{(i,j)} - \pi_1^{(i,j)} = f_j$ where $f_j \in U - \{e_i\}$ and $f_j \neq f_{j'}$ for $1 \leq j \neq j' \leq 2d-1$. For each $2d \leq j \leq 4d-2$, $\pi_2^{(i,j)} - \pi_1^{(i,j)} = g_j$ where $g_j \in U - \{-e_i\}$ and $g_j \neq g_{j'}$ for $2d \leq j \neq j' \leq 4d-2$.

(c) **(the paths intersect at most K_1 times)** For each $1 \leq j \leq 4d-2$, the paths $\pi^{(i,j)}$ and $\pi^{(i,j')}$ intersect at at most K_1 vertexes. Furthermore, after each point of intersection, both paths perform jumps in different directions in $H_{\hat{v}}$.

(d) **(the number of steps of all paths is close to N)** The number of steps n_j of each path satisfies $N - 1 \leq n_j \leq N + K_2$.

(e) **(increments in $H_{\hat{v}}$)** For each $1 \leq j \leq 2d-1$, we have that $\Delta\pi_k^{(i,j)} \in H_{\hat{v}} \cup \{f_j\}$ for $2 \leq k \leq n_j - 1$, while for each $2d \leq j \leq 4d-2$, we have that $\Delta\pi_k^{(i,j)} \in H_{\hat{v}} \cup \{g_j\}$ for $2 \leq k \leq n_j - 1$.

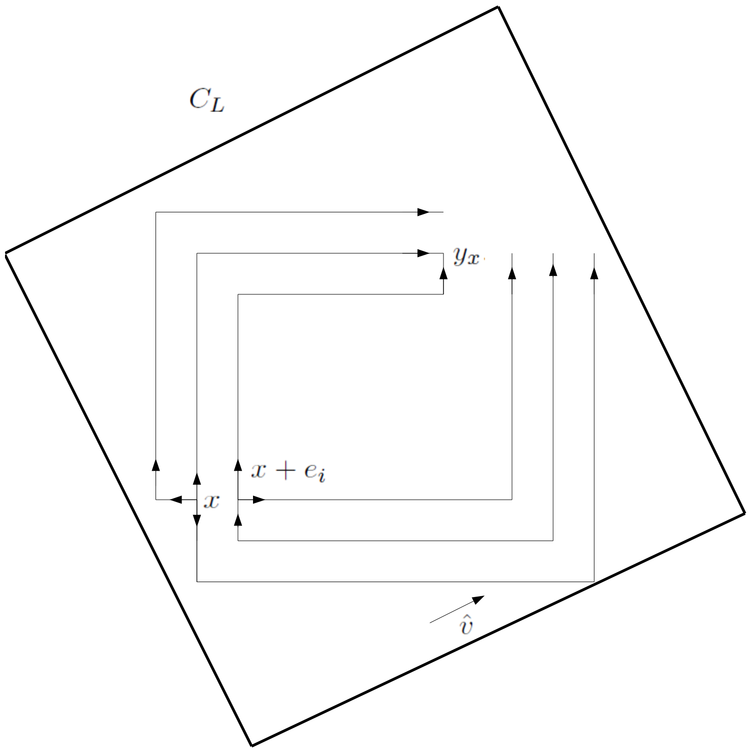

Figure 8: The $4 \times 2 - 2 = 6$ paths from x to $V'(y_x)$ and from $x + e_i$ to $V'(y_x)$ are represented by the arrowed lines. The set $V'(y_x)$ is given by the 5 endpoints of the paths.

In Figure 8 it is seen how can one construct such a set of paths for dimensions $d = 2$ (a similar construction works for dimensions $d \geq 3$). From Figure 8, note that the maximal number of steps of each path is given by $|y_x - x|_1 + 7$, where the 7 corresponds to the extra steps which have to be performed when a path exits the point x (or $x + e_i$) or enters some point in $V'(y_x)$. Let us now introduce for each $1 \leq i \leq 2d$ the event

$$F_{2,i} := \Big\{\omega \in \Omega : \text{for each } x \in C_{L(u)},\ \exists\, j \in \{1, \ldots, 2d\} \text{ such that} \\ \textstyle\sum_{k=1}^{n_j} \log \frac{1}{\omega(\pi_k^{(i,j)}, \Delta\pi_k^{(i,j)})} \leq \frac{2(M-1)(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1} n_j\Big\}$$

and also

$$F_2 := \cap_{i=1}^{2d} F_{2,i}.$$

Then, with the help of (5.6) we have that

$$\mathbb{P}(F_1) \leq \mathbb{P}\left(\exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|, F_2\right) + \mathbb{P}(F_2^c). \tag{5.8}$$

Let us define

$$F_3 := \left\{\omega \in \Omega : \exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|, F_2\right\}.$$

Note that on the event F_3, which appears in the probability of the right-hand side of (5.8), we can use the definition of the event F_2 to join for each $1 \leq i \leq 2d$, x_0 with $V'(y_{x_0})$ using one of the paths $\pi^{(i,j)}$ to conclude that

$$\omega_i e^{-\frac{8(M-1)(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}} u^{-\left(1-\frac{1}{M}\right)} \inf_{z \in V'(y_x)} P_{z,\omega}\left(T_{C_{L(u)}} < H_{x_0}\right) \leq P_{x_0,\omega}\left(T_{C_{L(u)}} < \tilde{H}_{x_0}\right) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|, \tag{5.9}$$

where

$$\omega_i := \omega(x, e_i).$$

The factor ω_i above corresponds to the probability of jumping from x to $x + e_i$ (in the case that the path $\pi^{(i,j)}$ starts from $x + e_i$). Summing up over i in (5.9) and using the equality

$$\sum_{i=1}^{2d} \omega_i = 1, \tag{5.10}$$

we conclude that on the event F_3 one has that

$$e^{-\frac{8(M-1)(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}} u^{-\left(1-\frac{1}{M}\right)} \inf_{z \in V'(y_x)} P_{z,\omega}\left(T_{C_{L(u)}} < H_x\right) \leq 4d \frac{(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|.$$

In particular, on F_3 we can see that for u large enough $V'(y_x) \subset C_{L(u)}$ (see (5.7)). As a result, on F_3 we have that for u large enough

$$\inf_{z \in V'(y_x)} P_{z,\omega}\left(X_{T_{z+U_{\beta,L}}} \cdot e_1 > z \cdot e_1\right) \leq \inf_{z \in V'(y_x)} P_{z,\omega}\left(T_{C_{L(u)}} < H_x\right) \leq \frac{1}{u^{\frac{1}{2M}}} = e^{-2\sqrt{d}L(u)^{\beta}},$$

where

$$U_{\beta,L} := \{x \in \mathbb{Z}^d : -L^{\beta} < x \cdot e_1 < L\}.$$

From this and using the translation invariance of the measure $\mathbb{P}$, we conclude that

$$\mathbb{P}\left(\exists\ x_0\in C_{L(u)} \text{ such that } P_{x_0,\omega}[\tilde{H}_{x_0}>T_{C_{L(u)}}]\le 4d\frac{(\log u)^{\frac{1}{\beta}}}{u}|C_{L(u)}|,F_2\right)$$
$$\le \mathbb{P}\left(\exists\ x_0\in C_{L(u)} \text{ such that } \inf_{z\in V'(y_x)} P_{z,\omega}(X_{T_{z+U_{\beta,L(u)}}}\cdot e_1>z\cdot e_1)\le e^{-2\sqrt{d}L(u)^\beta}\right)$$
$$\le |V'(y_x)||C_{L(u)}|\mathbb{P}\left(P_{0,\omega}\left(X_{T_{U_{\beta,L(u)}}}\cdot e_1>0\right)\le e^{-2\sqrt{d}L(u)^\beta}\right)$$
$$\le |V'(y_x)||C_{L(u)}|\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L(u)}}}\cdot e_1>0\right)\le e^{-2\sqrt{d}L(u)^\beta}\right),$$

where the titled box $B_{\beta,L}$ was defined in (4.1) of section 4. Therefore, we can estimate the first term of the right-hand side of (5.8) using Proposition 4.1 to conclude that there is a constant $c_{5,2}>0$ such that for each $\beta_0\in\left(\frac{1}{2},1\right)$ one has that

$$\mathbb{P}\left(\exists\ x_0\in C_{L(u)} \text{ such that } P_{x_0,\omega}[\tilde{H}_{x_0}>T_U]\le\frac{2(\log u)^{\frac{1}{\beta}}}{u}|C_{L(u)}|,F_2\right)\le\frac{1}{c_{5,2}}e^{-c_{5,2}L(u)^{g(\beta_0,\beta,\zeta)}}, \qquad (5.11)$$

where $g(\beta_0,\beta,\zeta)$ is defined in (4.2) of Proposition 4.1. We next have to control the probability $\mathbb{P}(F_2^c)$. To simplify the notation in the calculations that follow, we drop the super-index i in the paths writing $\pi^{(j)}:=\pi^{(i,j)}$. Furthermore we will define

$$M_1:=2(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}.$$

Then, by the independence property (e) of the paths $\{\pi^{(i,j)}\}$, we can say that

$$\mathbb{P}(F_2^c)\le$$
$$\sum_{i=1}^{2d}\mathbb{P}\left(\exists x\in C_{L(u)} \text{ such that } \forall\ j,\ \log\frac{1}{\omega(\pi_1^{(j)},\Delta\pi_1^{(j)})}+\sum_{k=2}^{n_j}\log\frac{1}{\omega(\pi_k^{(j)},\Delta\pi_k^{(j)})}>M_1n_j\right)$$
$$\le\sum_{i=1}^{2d}\sum_{x\in C_{L(u)}}\mathbb{P}\left(\forall\ 1\le j\le 2d-1,\ \log\frac{1}{\omega(\pi_1^{(j)},\Delta\pi_1^{(j)})}+\sum_{k=2}^{n_j}\log\frac{1}{\omega(\pi_k^{(j)},\Delta\pi_k^{(j)})}>M_1n_j\right)$$
$$\times\mathbb{P}\left(\forall\ 2d\le j\le 4d-2,\ \log\frac{1}{\omega(\pi_1^{(j)},\Delta\pi_1^{(j)})}+\sum_{k=2}^{n_j}\log\frac{1}{\omega(\pi_k^{(j)},\Delta\pi_k^{(j)})}>M_1n_j\right). \qquad (5.12)$$

Now, using Chebychev's inequality, the first step property (b) of the paths, the intersection property (c), the bound on the number of steps (d), $N-1\le n_j$, and the property of the increments (e), we can bound the rightmost-hand side of (5.12) by the following expression, where for $1\le j\le 2d-1$, $\alpha_j:=\alpha_1$ for $f_j\in H_{\hat{v}}$ while $\alpha_j:=\alpha(f_j)\le\alpha_1$ for $f_j\notin H_{\hat{v}}$ (cf. (1.8), and similarly for $2d\le j\le 4d-2$.

$$|C_{L(u)}|\sum_{i=1}^{2d}\mathbb{E}\left[e^{\sum_{j=1}^{2d-1}\alpha_j\log\frac{1}{\omega(0,f_j)}}\right]^{2K_1}e^{-M_1\sum_{j=1}^{2d-1}\alpha_jn_j}\prod_{j=1}^{2d-1}\mathbb{E}\left[e^{\alpha_j\sum_{k=2}^{n_j}\log\frac{1}{\omega\left(\pi_k^{(j)},\Delta\pi_k^{(j)}\right)}}\right]$$
$$\times\mathbb{E}\left[e^{\sum_{j=2d}^{4d-2}\alpha_j\log\frac{1}{\omega(0,g_j)}}\right]^{2K_1}e^{-M_1\sum_{j=2d}^{4d-2}\alpha_jn_j}\prod_{j=2d}^{4d-2}\mathbb{E}\left[e^{\alpha_j\sum_{k=2}^{n_j}\log\frac{1}{\omega\left(\pi_k^{(j)},\Delta\pi_k^{(j)}\right)}}\right]$$
$$\le|C_{L(u)}|\sum_{i=1}^{2d}(\eta_i^+\eta_i^-)^{2K_1}e^{(\log\eta_\alpha)\sum_{j=1}^{4d-2}n_j+}e^{-\kappa\left(1-\frac{1}{M}\right)\log u},$$

where κ is defined in (1.5), η_α is defined in (3.2) for $\alpha<\min_{e\in U}\alpha(e)$ (cf. (1.7)) and where

$$\eta_i^+:=\mathbb{E}\left[e^{\sum_{j=2d}^{4d-2}\alpha_j\log\frac{1}{\omega(0,g_j)}}\right]$$

and

$$\eta_i^- := \mathbb{E}\left[e^{\sum_{j=1}^{2d-1}\alpha_j \log \frac{1}{\omega(0,g_j)}}\right].$$

Then, using the inequality $n_j \le N + K_2$ (property (g) of the paths), we have that

$$\mathbb{P}\left(F_2^c\right) \le |C_{L(u)}| \sum_{i,i'=1}^{2d} (\eta_i^+ \eta_i^-)^{2K_1} e^{(4d-2)(N+K_2)\log \eta_\alpha} e^{-\kappa\left(1-\frac{1}{M}\right)\log u}.$$

Using the definition of N, and of condition $(E')_\beta$ (cf. (1.5) and (1.6)) we see from here that if we choose M large enough, one has that for u large enough

$$\mathbb{P}(F_2^c) \le c_{5,3} u^{-\beta}. \tag{5.13}$$

for some constant $c_{5,3} > 0$. Now note that for each $\beta \in \left(\frac{5}{6}, 1\right)$ there exists a $\beta_0 \in \left(\frac{1}{2}, \beta\right)$ such that for every $\zeta \in \left(0, \frac{1}{2}\right)$ one has that

$$g(\beta, \beta_0, \zeta) > \beta. \tag{5.14}$$

Therefore, substituting (5.11) and (5.13) back into (5.8) and using (5.14) we can see that there is a constant $c_{5,4} > 0$ such that for u large enough

$$\mathbb{P}(F_1) \le c_{5,4} u^{-\beta}. \tag{5.15}$$

Now with the help of (5.3), (5.4) and (5.15) there exists a constant $c_{5,5} > 0$ such that for u large

$$P_0\left(T_{C_{L(u)}} > u\right) \le c_{5,5} u^{-\beta}.$$

Finally, since $\gamma \in (\beta, 1)$ in (5.2), using (5.1) we conclude the proof, since we see that for u large enough

$$P_0(\tau_1 > u) \le c_{5,6} u^{-\beta},$$

for a certain constant $c_{5,6} > 0$. This proves part Proposition 5.1.

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