

ALMOST EXPONENTIAL DECAY FOR THE EXIT PROBABILITY FROM SLABS OF BALLISTIC RWRE

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ABSTRACT. It is an open question, whether or not in dimensions $d \geq 2$ any random walk in an i.i.d. uniformly elliptic random environment (RWRE) which is directionally transient is ballistic. The ballisticity conditions for RWRE interpolate between directional transience and ballisticity and have served to quantify the gap which would be needed in order to answer affirmatively this conjecture. Two important ballisticity conditions introduced by Sznitman [Sz01, Sz02] are conditions (T') and (T) , quantifying the decay of the exit probability through the back side of a slab, the first one demanding a stretched exponential decay while the second one an exponential one. It is believed that (T') implies (T) . In this article we show that (T') implies at least an *almost* (in a sense to be made precise) exponential decay.

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1. INTRODUCTION

The relationship between directional transience and ballisticity for random walks in random environment is one of the most challenging open questions within the field of random media. In the case of random walks in an i.i.d. random environment, several ballisticity conditions have been introduced which quantify the exit probability of the random walk through a given side of a slab as its width L grows, with the objective of understanding the above relation. Examples of these ballisticity conditions include Sznitman's (T') and (T) conditions [Sz01, Sz02]: given a slab of width L orthogonal to l , condition (T') in direction l is the requirement that the annealed exit probability of the walk through the side of the slab in the half-space $\{x : x \cdot l < 0\}$, decays faster than

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e^{-CL^γ} for all $\gamma \in (0, 1)$ and some constant $C > 0$, while condition (T) in direction l is the requirement that the decay is exponential e^{-CL} . It is believed that condition (T') , is equivalent to condition (T) . In this article we prove that condition (T') implies an *almost* exponential decay (see Theorem 1.2 for the precise meaning of this statement) of the corresponding exit probabilities. Our proof relies on a recursive renormalization scheme, where the a careful choice of fastly growing scales enables us to obtain the result. We use the equivalence between condition (T') [Sz02] and the $d \geq 2$ dimensional version of Solomon's criterion [So75], known as the effective criterion [Sz02].

Let us introduce the random walk in random environment model. For $x \in \mathbb{Z}^d$ denote its Euclidean norm by $|x|_2$. Let $V := \{e \in \mathbb{Z}^d : |e|_2 = 1\}$ be the set of canonical vectors. Introduce the set \mathcal{P} whose elements are $2d$ -vectors $p(e)_{e \in \mathbb{Z}^d, |e|=1}$ such that

$$p(e) \geq 0, \text{ for all } e \in V, \quad \sum_{e \in \mathbb{Z}^d, |e|=1} p(e) = 1.$$

We define an environment $\omega := \{\omega(x) : x \in \mathbb{Z}^d\}$ as an element of $\Omega := \mathcal{P}^{\mathbb{Z}^d}$, where for each $x \in \mathbb{Z}^d$, $\omega(x) = \{\omega(x, e) : e \in V\} \in \mathcal{P}$. Consider a probability measure \mathbb{P} on Ω endowed with its canonical product σ -algebra, so that an environment is now a random variable such that the coordinates $\omega(x)$ are i.i.d. under \mathbb{P} . The random walk in the random environment ω starting from $x \in \mathbb{Z}^d$ is the canonical Markov Chain $\{X_n : n \geq 0\}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ with *quenched law* $P_{x, \omega}$ starting from x , defined by the transition probabilities for each $e \in \mathbb{Z}^d$ with $|e| = 1$ by

$$P_{x, \omega}(X_{n+1} = X_n + e | X_0, \dots, X_n) = \omega(X_n, e)$$

and

$$P_{x, \omega}(X_0 = x) = 1.$$

The *averaged* or *annealed law*, P_x , is defined as the semi-direct product measure

$$P_x = \mathbb{P} \times P_{x, \omega}$$

on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$. Whenever there is a $\kappa > 0$ such that

$$\inf_{e, x} \omega(x, e) \geq \kappa \quad \mathbb{P} - a.s.$$

we will say that the law \mathbb{P} of the environment is *uniformly elliptic*.

For the statement of the result, we need some further definitions. For each subset $A \subset \mathbb{Z}^d$ we define the first exit time of the random walk from A as

$$T_A := \inf\{n \geq 0 : X_n \notin A\}.$$

Fix a vector $l \in \mathbb{S}^{d-1}$ and $u \in \mathbb{R}$ then define the half-spaces $H_{u,l}^- := \{x \in \mathbb{Z}^d : x \cdot l < u\}$, $H_{u,l}^+ := \{x \in \mathbb{Z}^d : x \cdot l > u\}$,

$$T_u^l := T_{H_{u,l}^-} = \inf\{n \geq 0, X_n \cdot l \geq u\}$$

and

$$\tilde{T}_u^l := T_{H_{u,l}^+} = \inf\{n \geq 0, X_n \cdot l \leq u\}.$$

For $\gamma \in (0, 1]$, we say that condition $(T)_\gamma|l$ holds with respect to direction $l \in \mathbb{S}^{d-1}$, if

$$\limsup_{L \rightarrow \infty} L^{-\gamma} \log P_0(\tilde{T}_{-L}^{l'} < T_L^{l'}) < 0,$$

for all l' in some neighborhood of l . Furthermore, we define $(T')|l$ as the requirement that condition $(T)_\gamma|l$ is satisfied for all $\gamma \in (0, 1)$ and condition $(T)|l$ as the requirement that $(T)_1|l$ is satisfied. In [Sz02], Sznitman proved that when $d \geq 2$ for every $\gamma \in (0.5, 1)$, $(T)_\gamma|l$ is equivalent to $(T')|l$. This equivalence was improved in [DR11] and [DR12] culminating with the work of Berger, Drewitz and Ramírez who in [BDR14] showed that for any $\gamma \in (0, 1)$, condition $(T)_\gamma|l$ implies $(T')|l$. As a matter of fact, in [BDR14], an effective ballisticity condition, which requires polynomial decay was introduced. To define this condition, consider $L, \tilde{L} > 0$ and $l \in \mathbb{S}^{d-1}$ and the box

$$B_{l,L,\tilde{L}} := R \left((-L, L) \times (-\tilde{L}, \tilde{L})^{d-1} \right) \cap \mathbb{Z}^d,$$

where R is a rotation defined by

$$R(e_1) = l. \tag{1}$$

Given $M \geq 1$ and $L \geq 2$, we say that the polynomial condition $(P)_M$ in direction l (also denoted by $(P)_M|l$) is satisfied on a box of size L if there exists and $\tilde{L} \leq 70L^3$ such that

$$P_0 \left(X_{T_{B_{l,L,\tilde{L}}}} \cdot l < L \right) \leq \frac{1}{L^M}.$$

Berger, Drewitz and Ramírez proved in [BDR14] that there exists a constant c_0 such that whenever $M \geq 15d + 5$, the polynomial condition

$(P)_M|l$ on a box of size $L \geq c_0$ is equivalent to condition $(T')|l$ (see also Lemma 3.1 of [CR14]). On the other hand, the following is still open.

Conjecture 1.1. *Consider a random walk in a uniformly elliptic random environment in dimension $d \geq 2$ and $l \in \mathbb{S}^{d-1}$. Then, condition $(T)|l$ is equivalent to $(T')|l$.*

To quantify how far are we presently from proving Conjecture 1.1, we will introduce now a family of intermediate conditions between conditions (T') and (T) . Let $\gamma(L) : [0, \infty) \rightarrow [0, 1]$, with $\lim_{L \rightarrow \infty} \gamma(L) = 1$. Let $l \in \mathbb{S}^d$. We say that condition $(T)_{\gamma(L)}|l$ is satisfied if

$$\limsup_{L \rightarrow \infty} L^{-\gamma(L)} \log P_0(\tilde{T}'_{-L} < T'_L) < 0, \quad (2)$$

for l' in a neighborhood of l . We will call $\gamma(L)$ the *effective parameter* of condition $(T)_{\gamma(L)}$. Note that condition (T) is actually equivalent to $(T)_{\gamma(L)}$ with an effective parameter given by

$$\gamma(L) = 1 - \frac{C}{\log L}, \quad (3)$$

for any constant $C \geq 0$. In 2002 Sznitman [Sz02] was able to prove that (T') implies $(T)_{\gamma(L)}$ with effective parameter

$$\gamma(L) = 1 - \frac{C}{\log L} \sqrt{\log L}, \quad (4)$$

for some constant $C > 0$.

In this paper, we are able to show that condition (T') implies condition $(T)_{\gamma(L)}$ with an effective parameter $\gamma(L)$ which is closer to the effective parameter for condition (T) given by (3). This is the first result since the introduction of condition (T') by Sznitman in 2002, which would give an indication that Conjecture 1.1 is true. To state it, let us introduce some notation. Throughout, for each $n \geq 1$, we will use the standard notation

$$\overbrace{\log \circ \cdots \circ \log}^n x,$$

for the composition of the logarithm function n times with itself, for all x in its domain, where the n superscript means that the composition is performed n times.

Theorem 1.2. *Let $d \geq 2$, $l \in S^{d-1}$ and $M \geq 15d + 5$. Assume that condition $(P)_M|l$ is satisfied on a box of size $L \geq c_0$. Then there exists a constant $C > 0$ and a function $n(L) : [0, \infty) \rightarrow \mathbb{N}$ satisfying $\lim_{L \rightarrow \infty} n(L) = \infty$, such that condition $(T)_{\gamma(L)}|l$, c.f. (2), is satisfied with an effective parameter $\gamma(L)$ given by*

$$\gamma(L) = 1 - \frac{C}{\log L} \overbrace{\log \circ \cdots \circ \log}^{n(L)} L. \quad (5)$$

Remark 1.3. Note that the decay given by the effective parameter (5) of Theorem 1.2 is equivalent to the decay

$$\limsup_{L \rightarrow \infty} \frac{\overbrace{\log \circ \cdots \circ \log}^{n(L)-1} L}{L} \log P_0(\tilde{T}'_{-L} < T'_L) < 0,$$

for l' in a neighborhood of l .

Let us remark that a priori, even if $n(L) \rightarrow \infty$ as $L \rightarrow \infty$, it might happen that the composition of the logarithm $n(L)$ time is bounded. Nevertheless, in the case of Theorem 1.2, it turns out that

$$\lim_{L \rightarrow \infty} \overbrace{\log \circ \cdots \circ \log}^{n(L)} L = \infty.$$

Theorem 1.2 will be proven in the next section, but some remarks are in order. The strategy followed in the proof, roughly speaking, is to improve the iterative procedure used by Sznitman in [Sz02], to prove $(T)_{\gamma(L)}$ with $\gamma(L)$ given by (4), through the so called effective criterion introduced by Sznitman in [Sz02]. The iterative procedure used in [Sz02], in spirit is a renormalization argument, where the idea is to control the exit probability of the walk recursively from an initial scale L_0 to the final size of the slab $L > L_0$ passing through a sequence of intermediate scales $L_0 < L_1 < \dots < L_k = L$. To go from scale L_0 to scale L_1 , a slab of width L_1 is subdivided into overlapping slabs of width L_0 , and the walk is looked at its exit times from successive slabs of width L_0 . Essentially, at these times the walk looks like a one dimensional random walk in random environment, for which one can control its exit probabilities through the expected value of ρ , where ρ is close to the quotient between the probability to exit a slab of width L_0 through its left side and the probability to exit it through its right side. Here, a triggering assumption is needed, which in our case is the effective criterion of Sznitman [Sz02] (the effective criterion is implied by

the polynomial condition introduced by Berger, Drewitz and Ramírez in [BDR14]). This first step is the content of Proposition 2.1. A similar strategy is then used to pass from scale L_k to scale L_{k+1} for $k \geq 1$ (see Lemma 2.2). Nevertheless, reducing the movement of the random walk to a one dimensional walk, has a cost, which is a polynomial factor appearing in the recursion relations, and which somehow is the reason why one cannot go from the initial scale L_0 directly to L in one step. In this paper, we modify Sznitman's argument, choosing a sequence of scales where L_{k+1} is much larger than L_k compared to Sznitman's approach, allowing us to work with a smaller number of intermediate steps in the recursion relation. The use of this new sequence of scales, produces at some points important difficulties in the proof which have to be properly handled.

2. PROOF OF THEOREM 1.2

Throughout the rest of this section, we prove Theorem 1.2. Firstly, in subsection 2.1, we will introduce the basic notation which will be needed to implement the renormalization scheme, and we will recall a basic result of Sznitman which provides a bound for quantities involving the exit probability through the unlikely side of boxes which are inspired in techniques used for one-dimensional random walks in random environment. In the second subsection, we will introduce a growth condition which will limit the maximal way in which the scales on the recursive scheme can grow, while still giving a useful recurrence. In the third subsection we will choose an adequate sequence of scales satisfying the condition of subsection 2.2, and for which one can make computations. Finally, in subsection 2.4, Theorem 1.2 will be proven using the scales constructed in subsection 2.3 through the use of the effective criterion [Sz02].

2.1. Preliminaries and notation. The proof of Theorem 1.2 will follow the renormalization method used by Sznitman to prove Proposition 2.3 of [Sz02]. The idea is to use a renormalization procedure which somehow mimics a computation for a one-dimensional random walk in random environment, where one goes from one scale to the next (larger) one through formulas where the exit probabilities of the random walk through slabs at the smaller scales are involved.

Following Sznitman we introduce boxes transversal to direction l , which are specified in terms of $\mathcal{B} = (R, L, L', \tilde{L})$, where L, L', \tilde{L} are positive numbers and R is the rotation defined in (1). The box attached to \mathcal{B} , is

$$B := R((-L, L') \times (-\tilde{L}, \tilde{L})^{d-1}) \cap \mathbb{Z}^d$$

and the positive part of its boundary is defined as

$$\partial_+ B := \partial B \cap \{x \in \mathbb{Z}^d, x \cdot l \geq L', |R(e_i) \cdot x| < \tilde{L}, i \geq 2\}.$$

We can now define the following random variable depending on a given specification \mathcal{B} , analogous to the quotient in dimension $d = 1$ between the probability to jump to the left and the probability to jump to the right [SW69, So75], for $\omega \in \Omega$ as

$$\rho_{\mathcal{B}}(\omega) := \frac{q_{\mathcal{B}}(\omega)}{p_{\mathcal{B}}(\omega)},$$

where

$$q_{\mathcal{B}}(\omega) := P_{0,\omega}(X_{T_{\mathcal{B}}} \notin \partial_+ B) =: 1 - p_{\mathcal{B}}(\omega).$$

The first step in the renormalization procedure will be to control the moments of $\rho_{\mathcal{B}}$ at the two first scales. To this end, consider positive numbers

$$3\sqrt{d} < L_0 < L_1, \quad 3\sqrt{d} < \tilde{L}_0 < \tilde{L}_1$$

along with the box-specifications

$$\mathcal{B}_0 := (R, L_0 - 1, L_0 + 1, \tilde{L}_0)$$

and

$$\mathcal{B}_1 := (R, L_1 - 1, L_1 + 1, \tilde{L}_1).$$

It is convenient to introduce now the notation

$$q_0 := q_{\mathcal{B}_0}, p_0 := p_{\mathcal{B}_0}, \quad q_1 := q_{\mathcal{B}_1}, p_1 := p_{\mathcal{B}_1},$$

and

$$\rho_0 := \rho_{\mathcal{B}_0}, \rho_1 := \rho_{\mathcal{B}_1}. \tag{6}$$

Let also

$$N_0 := \frac{L_1}{L_0} \quad \text{and} \quad \tilde{N}_0 := \frac{\tilde{L}_1}{\tilde{L}_0}.$$

We will also need to introduce the constant

$$c_1(d) = c_1 := \sqrt{d}.$$

Note that for each pair of points $x, y \in \mathbb{Z}^d$, there exists a nearest neighbor path joining them which has less than $c_1|x - y|_2$ steps.

Let us now recall the following Proposition of Sznitman [Sz02].

Proposition 2.1. *There exist $c_2(d) > 3\sqrt{d}$, $c_3(d), c_4(d) > 1$, such that when $N_0 \geq 3, L_0 \geq c_2, \tilde{L}_1 \geq 48N_0\tilde{L}_0$, for each $a \in (0, 1]$ one has that*

$$\mathbb{E} \left[\rho_1^{\frac{a}{2}} \right] \leq c_3 \left\{ \kappa^{-10c_1L_1} \left(c_4 \tilde{L}_1^{d-2} \frac{L_1^3}{\tilde{L}_0^2} \tilde{L}_0 \mathbb{E}[q_0] \right)^{\frac{\tilde{L}_1}{12N_0\tilde{L}_0}} + \sum_{0 \leq m \leq N_0+1} \left(c_4 \tilde{L}_1^{d-1} \mathbb{E}[\rho_0^a] \right)^{\frac{[N_0]+m-1}{2}} \right\}. \quad (7)$$

2.2. The maximal growth condition on scales. We next recursively iterate inequality (7) at different scales which will increase as fast as possible, in the sense that a certain induction condition should enable us to push forward the recursion.

We next recursively iterate inequality (7) at different scales which will increase as fast as possible, in the sense that a certain induction hypothesis should enable us to push forward the recursion. Let

$$v := 8, \quad \alpha := 240$$

and introduce two sequences of scales L_k, \tilde{L}_k $k \geq 0$, such that

$$L_0 \geq c_2, \quad 3\sqrt{d} < \tilde{L}_0 \leq L_0^3 \quad (8)$$

and for $k \geq 0$

$$N_k \geq 7, \quad L_{k+1} = N_k L_k, \quad \tilde{L}_{k+1} = N_k^3 \tilde{L}_k, \quad (9)$$

as well as box-specifications

$$\mathcal{B}_k := (R, L_k - 1, L_k + 1, \tilde{L}_k).$$

Note that

$$\tilde{L}_{k+1} = \left(\frac{L_k}{L_0} \right)^3 \tilde{L}_0. \quad (10)$$

Introduce also the notation for the respective attached random variables

$$\rho_k := \rho_{\mathcal{B}_k}.$$

Throughout, we will adopt the notation

$$u_0 := \frac{3(d-1)}{L_0 \log \frac{1}{\kappa}}, \quad (11)$$

and for $k \geq 1$,

$$u_k := \frac{u_0}{v^k}. \quad (12)$$

We also let

$$c_5 := 2c_3c_4.$$

Condition (G). We say that the scales $L_k, N_k, k \geq 0$ satisfy condition (G) if

$$u_k N_k \geq \alpha c_1 \text{ for } k \geq 0, \quad (13)$$

and if

$$c_5 N_{k+1}^{3(d-1)} L_{k+1}^{3d-1} \kappa^{u_{k+1} L_{k+1}} \leq 1 \text{ for } k \geq 0. \quad (14)$$

Let us now state the following lemma which generalizes Lemma 2.2 of Sznitman ([Sz02]), for scales satisfying condition (G). For completeness we include its proof.

Lemma 2.2. Consider scales $L_k, N_k, k \geq 0$, such that condition (G) is satisfied. Then, whenever $L_0 \geq c_2$, $3\sqrt{d} \leq \tilde{L}_0 \leq L_0^3$, and $a_0 \in (0, 1]$, we have that

$$\varphi_0 := c_4 \tilde{L}_1^{d-1} L_0 \mathbb{E}[\rho_0^{a_0}] \leq \kappa^{u_0 L_0}. \quad (15)$$

then for all $k \geq 0$,

$$\varphi_k := c_4 \tilde{L}_{k+1}^{d-1} L_k \mathbb{E}[\rho_k^{a_k}] \leq \kappa^{u_k L_k}. \quad (16)$$

with

$$a_k = a_0 2^{-k}, \quad u_k = u_0 v^{-k}.$$

Proof. As in the proof of Lemma 2.2 of [Sz02], we can conclude by Proposition 2.1 that if $L_0 \geq c_2$ (note that by the choice of N_k in (9), the other conditions of Proposition 2.1 are satisfied) we have that for $k \geq 0$,

$$\varphi_{k+1} \leq c_3 c_4 \tilde{L}_{k+2}^{d-1} L_{k+1} \left\{ \kappa^{-10c_1 L_{k+1}} \varphi_k^{\frac{N_k^2}{12}} + \sum_{0 \leq m \leq N_{k+1}} \varphi_k^{\frac{[N_k]+m-1}{2}} \right\}. \quad (17)$$

We will now prove inequality (16) by induction on k using inequality (17). Since inequality (15) is identical to inequality (16) with $k = 0$, the induction hypothesis is satisfied for $k = 0$. We assume now that it is true for $k > 0$, along with inequality (13) of assumption (G) and conclude that

$$\kappa^{-10c_1 L_{k+1}} \varphi_k^{\frac{N_k^2}{24}} \leq \kappa^{-10c_1 L_{k+1}} \kappa^{N_k^2 \frac{L_k u_k}{24}} \leq 1. \quad (18)$$

Therefore, using (18) and the fact that $[N_k] - 1 \geq \frac{N_k}{2}$ because $N_k \geq 7$ we see that

$$\begin{aligned} \varphi_{k+1} &\leq c_3 c_4 \tilde{L}_{k+2}^{d-1} L_{k+1} \left\{ \varphi_k^{\frac{N_k^2}{24}} + L_{k+1} \varphi_k^{\frac{N_k}{4}} \right\} \\ &\leq c_5 \tilde{L}_{k+2}^{d-1} L_{k+1}^2 \varphi_k^{\frac{N_k}{8}} \varphi_k^{\frac{N_k}{8}}, \end{aligned} \quad (19)$$

where we recall that $c_5 = 2c_3 c_4$. Now, by the induction hypothesis (16) we see that

$$\varphi_k^{\frac{N_k}{8}} \leq \kappa^{u_{k+1} L_{k+1}}.$$

Substituting this into (19), we see that it is enough now to show that

$$c_5 \tilde{L}_{k+2}^{d-1} L_{k+1}^2 \varphi_k^{\frac{N_k}{8}} \leq 1.$$

But this is true, by (14) of condition (G), the induction hypothesis and the inequality $\tilde{L}_{k+1} \leq L_{k+1}^3$ for $k \geq 0$ which follows by induction starting from (8). Indeed, using these facts,

$$c_5 \tilde{L}_{k+2}^{d-1} L_{k+1}^2 \varphi_k^{\frac{N_k}{8}} \leq c_5 N_{k+1}^{3(d-1)} L_{k+1}^{3d-1} \kappa^{u_{k+1} L_{k+1}} \leq 1,$$

which ends the proof. \square

2.3. An adequate choice of fast-growing scales. We will now construct a sequence of scales $\{L_k : k \geq 0\}$ which satisfy condition (G), and for which Lemma 2.2 will eventually imply Theorem 1.2. This is not the fastest possible growing sequence of scales, but somehow it captures the best possible choice of $\gamma(L)$.

Let $\{f_k : k \geq 1\}$ be a sequence of functions from $[0, \infty)$ to $[0, \infty)$ defined recursively as

$$f_0(x) := 1,$$

$$f_1(x) := v^x$$

and for $k \geq 1$,

$$f_{k+1}(x) := f_k \circ f_1(x).$$

Let now, for $k \geq 0$,

$$N_k := \frac{\alpha c_1 f_{\lfloor \frac{k+2}{2} \rfloor} \left(\left\lceil \frac{k+1}{2} \right\rceil \right)}{u_0 f_{\lfloor \frac{k+1}{2} \rfloor} \left(\left\lceil \frac{k}{2} \right\rceil \right)}. \quad (20)$$

According to display (9), we have the following formula valid for $k \geq 0$,

$$L_{k+1} = f_{\lfloor \frac{k+2}{2} \rfloor} \left(\left\lceil \frac{k+1}{2} \right\rceil \right) \left(\frac{\alpha c_1}{u_0} \right)^{k+1} L_0. \quad (21)$$

Lemma 2.3. *The condition*

$$u_k N_k \geq \alpha c_1 \quad \text{for } k \geq 0$$

(c.f. (13) of condition (G)) is equivalent to

$$\frac{f_{\lfloor \frac{k+2}{2} \rfloor} \left(\left\lceil \frac{k+1}{2} \right\rceil \right)}{f_{\lfloor \frac{k+1}{2} \rfloor} \left(\left\lceil \frac{k}{2} \right\rceil \right) v^k} \geq 1 \quad \text{for } k \geq 0, \quad (22)$$

Proof. Note that (22) can be easily verified for $k = 0, 1$ and 2 . Therefore it is enough to prove inequality (22) for $k \geq 3$. For this purpose, we will first show that for all positive integers n , and $a, b \in [1, \infty)$, we have that

$$f_n(a+b) \geq f_n(a)f_n(b). \quad (23)$$

To prove (23), suppose that

$$A := \{n \in \mathbb{N} : f_n(a+b) < f_n(a)f_n(b) \text{ for some } a, b \geq 1\} \neq \emptyset.$$

Let m be the smallest element of A and remark that m is greater than 1. Also, note that

$$f_m(a+b) < f_m(a)f_m(b)$$

for some $a, b \geq 1$. However, note that for $a, b \geq 1$ one has that

$$v^{a+b} \geq v^a + v^b.$$

Furthermore, for each $k \geq 0$, the function $f_k(\cdot)$ is increasing. Therefore,

$$\begin{aligned} f_{m-1}(v^a)f_{m-1}(v^b) &= f_m(a)f_m(b) \\ &> f_m(a+b) = f_{m-1}(v^{a+b}) \geq f_{m-1}(v^a + v^b). \end{aligned}$$

This contradicts the minimality of m and hence $A = \emptyset$ which proves (23). Back to (22), note that

$$\frac{f_{\lfloor \frac{k+2}{2} \rfloor}(\lfloor \frac{k+1}{2} \rfloor)}{f_{\lfloor \frac{k+1}{2} \rfloor}(\lfloor \frac{k}{2} \rfloor)v^k} \geq \frac{f_{\lfloor \frac{k+2}{2} \rfloor}(\lfloor \frac{k+1}{2} \rfloor - 1) f_{\lfloor \frac{k+2}{2} \rfloor}(1)}{f_{\lfloor \frac{k+1}{2} \rfloor}(\lfloor \frac{k}{2} \rfloor) v^k} \geq \frac{f_{\lfloor \frac{k+2}{2} \rfloor}(1)}{v^k} \geq 1,$$

where the first inequality was gotten using (23), the second one is a consequence of the inequality

$$\frac{f_{\lfloor \frac{k+2}{2} \rfloor}(\lfloor \frac{k+1}{2} \rfloor - 1)}{f_{\lfloor \frac{k+1}{2} \rfloor}(\lfloor \frac{k}{2} \rfloor)} \geq 1,$$

valid for $k \geq 3$, and which can be proved in a straightforward fashion if we divide the argument according to whether k is even or odd, and the last inequality comes from the fact that

$$f_{\lfloor \frac{k+2}{2} \rfloor - 1}(1) - k \geq 0 \quad \text{for } k \geq 3. \quad (24)$$

Now, it is easy to verify inequality (24) when $k = 3$ and $k = 4$. Furthermore, the left hand of (24) is increasing as a function of $k \geq 2$ for k odd. Similarly, it is increasing for $k \geq 2$ for k even. We can therefore conclude, using induction that (24) is satisfied. This completes the proof of (22). \square

Using Lemma 2.3 we can now obtain the following important lemma which gives conditions on the growth of a sequence of scale which ensure that (G) is satisfied.

Lemma 2.4. *There exists a constant $c_6(d)$ such that when $L_0 \geq c_6$, the scales $\{L_k : k \geq 0\}$ and $\{N_k : k \geq 0\}$ defined by (21) and (20) satisfy condition (G).*

Proof. By Lemma 2.3 we know that (13) of condition (G) is satisfied. We therefore just prove inequality (14) of condition (G). We need to show that there exists a constant $c(d, \kappa)$, such that whenever $L_0 \geq c(d, \kappa)$, for all $k \geq 0$ one has that

$$c_5 N_{k+1}^{3(d-1)} L_{k+1}^{3d-1} \kappa^{u_{k+1} L_{k+1}} \leq 1. \quad (25)$$

We will first show that there exists $c_7(d, \kappa) = c_7(d) > 0$, such that whenever $L_0 \geq c_7$, one has that for $k \geq 0$,

$$N_{k+1}^{3(d-1)} \kappa^{\frac{u_{k+1} L_{k+1}}{3}} \leq 1. \quad (26)$$

Now (26) is equivalent to

$$3(d-1) \log_v \left(\frac{\alpha c_1}{u_0} \frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right) - \frac{L_0 u_0 f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v u_0} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)}{3} \leq 0.$$

Therefore, (26) is equivalent to the bound for $k \geq 0$,

$$L_0 \geq \frac{\frac{9(d-1)}{u_0} \log_v \left(\frac{\alpha c_1}{u_0} \frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v u_0} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)}. \quad (27)$$

Let us focus on right-hand side of inequality (27). Note that it can be split as

$$\frac{\frac{9(d-1)}{u_0} \log_v \left(\frac{\alpha c_1}{u_0} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v u_0} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)} + \frac{\frac{9(d-1)}{u_0} \log_v \left(\frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v u_0} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)}. \quad (28)$$

Let us now try to find an upper bound for this expression independent on u_0 (or equivalently, on L_0). By the definition of u_0 (c.f. (11)) note that for $k \geq 0$ and $L_0 \geq \frac{3(d-1)}{\log \frac{1}{\kappa}}$ one has that,

$$\frac{1}{u_0} \frac{1}{\left(\frac{\alpha c_1}{v u_0} \right)^{k+1}} = \frac{1}{\left(\frac{\alpha c_1}{v u_0} \right)^k} \frac{1}{\left(\frac{\alpha c_1}{v} \right)} \leq \frac{1}{\left(\frac{\alpha c_1}{v} \right)^{k+1}}.$$

Substituting this into (28) we see that it is bounded from above by

$$\frac{9(d-1) \log_v \left(\frac{\alpha c_1}{u_0} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)} + \frac{9(d-1) \log_v \left(\frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)}. \quad (29)$$

Note that only the left-most term of (29) depends on L_0 . Choose a constant $c_8(d, \kappa) = c_8(d) > 1$, such that if $L_0 \geq c_8$

$$\log_v \left(\frac{\alpha c_1}{u_0} \right) \leq L_0 \frac{\log_v \left(\frac{1}{\kappa} \right)}{d-1}. \quad (30)$$

Then, when $L_0 \geq c_8$, we see using the fact that the left-most term of (29) is a decreasing function of $k \geq 0$ and from inequality (30), that it can be bounded from above by

$$L_0 \frac{9v}{\alpha c_1} \leq L_0 \frac{72}{240} \leq \frac{L_0}{3}. \quad (31)$$

Thus, whenever $L_0 \geq c_8$, from (28), (29) and (31), we see that (27) is satisfied if

$$L_0 \geq \frac{3}{2} \frac{9(d-1) \log_v \left(\frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \left(\frac{\alpha c_1}{v} \right)^{k+1} \log_v \left(\frac{1}{\kappa} \right)}. \quad (32)$$

Therefore, in order to prove (26) it is enough to show that the right hand side of inequality (32) is bounded. To do this, it is enough to prove that the expression

$$\frac{\log_v \left(\frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)}$$

is bounded. Now,

$$\frac{\log_v \left(\frac{f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} \leq \frac{\log_v \left(f_{\lceil \frac{k+3}{2} \rceil}(\lceil \frac{k+2}{2} \rceil) \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)}. \quad (33)$$

Let us now remark that if k is even, then $\lceil \frac{k+3}{2} \rceil = \lceil \frac{k+2}{2} \rceil$ and $\lceil \frac{k+1}{2} \rceil = \lceil \frac{k+2}{2} \rceil - 1$. Therefore, in this case, the right-hand side of inequality (33) is smaller than

$$\frac{f_{\lceil \frac{k+2}{2} \rceil - 1}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+2}{2} \rceil - 1)} = \frac{f_{\lceil \frac{k+2}{2} \rceil - 1}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil - 1}(v^{\lceil \frac{k+2}{2} \rceil - 1})}.$$

But, since for k fixed, the function $f_k(\cdot)$ is increasing, and since for $k \geq 0$ we have that

$$v^{\lceil \frac{k+2}{2} \rceil - 1} \geq \left\lfloor \frac{k+2}{2} \right\rfloor,$$

we see that the right-hand side of inequality (33) is bounded. Hence, for k even the right-most term of (33) is bounded by a constant $c_9(d, \kappa) = c_9(d) > 0$.

Suppose now that k is odd. Then $\lceil \frac{k+3}{2} \rceil = \lceil \frac{k+2}{2} \rceil + 1$ and $\lceil \frac{k+1}{2} \rceil = \lceil \frac{k+2}{2} \rceil$. Therefore, in this case, the right-hand side of inequality (33) is equal to

$$\frac{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+2}{2} \rceil)} = 1,$$

so that we can conclude that the right-hand side of inequality (33) is bounded, and hence that there is constant $c_{10}(d, \kappa) = c_{10}(d) > 0$ which is an upper bound for the right-hand side of inequality (27). We can hence conclude, taking $c_7(d) = \max\{c_9(d), c_{10}(d)\}$, that when $L_0 \geq c_7(d)$, then (26) holds.

As a second step to prove (25), we will show that it is possible to find a positive constant $c_{11}(d, \kappa) = c_{11}(d)$ such that when $L_0 \geq c_{11}$ one has that for all $k \geq 0$,

$$L_{k+1}^{3d-1} \kappa^{\frac{u_{k+1} L_{k+1}}{3}} \leq 1. \quad (34)$$

Inserting the definition (21) that defines L_k into this inequality, we see that it is enough to prove that

$$(3d-1) \log_v(L_{k+1}) - \frac{\log_v\left(\frac{1}{\kappa}\right) u_0 \left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) L_0}{3} \leq 0. \quad (35)$$

If we show that for all $k \geq 0$, $L_0 \geq \frac{\log_v(L_{k+1}) 3(3d-1)}{\log_v\left(\frac{1}{\kappa}\right) u_0 \left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)}$, we

have a proof of (35). But the right-hand side of this inequality can be written as

$$\frac{3(3d-1) \log_v \left[L_0 \left(\frac{\alpha c_1}{u_0}\right)^{k+1} \right]}{\log_v\left(\frac{1}{\kappa}\right) u_0 \left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)} + \frac{3(3d-1) \log_v \left(f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil) \right)}{f_{\lceil \frac{k+2}{2} \rceil}(\lceil \frac{k+1}{2} \rceil)}.$$

We need to establish a control with respect to L_0 in this expression. Only the first term depends on L_0 so we concentrate on the first term. Now, this term is decreasing with k . Therefore, it is smaller than

$$\frac{3(3d-1) \log_v \left[L_0 \left(\frac{\alpha c_1}{u_0}\right) \right]}{\log_v\left(\frac{1}{\kappa}\right) \left(\frac{\alpha c_1}{v}\right)} = \frac{3(3d-1) \log_v \left(\frac{L_0^2 \alpha c_1 \log\left(\frac{1}{\kappa}\right)}{3(d-1)} \right)}{\log_v\left(\frac{1}{\kappa}\right) \left(\frac{\alpha c_1}{v}\right)}$$

From this last expression, it is clear that we can choose a constant $c_{12}(d, \kappa) = c_{12}(d) > 0$ such that whenever $L_0 \geq c_{12}(d)$ one has that

$$\frac{3(3d-1) \log_v \left[L_0 \left(\frac{\alpha c_1}{u_0} \right)^{k+1} \right]}{\log_v \left(\frac{1}{\kappa} \right) u_0 \left(\frac{\alpha c_1}{u_0 v} \right)^{k+1} f_{\lceil \frac{k+2}{2} \rceil} \left(\lceil \frac{k+1}{2} \rceil \right)} \leq \frac{L_0}{3}. \quad (36)$$

Therefore, if $L_0 \geq c_{12}(d)$ and if

$$L_0 \geq \frac{3}{2} \frac{3(3d-1) \log_v \left(f_{\lceil \frac{k+2}{2} \rceil} \left(\lceil \frac{k+1}{2} \rceil \right) \right)}{f_{\lceil \frac{k+2}{2} \rceil} \left(\lceil \frac{k+1}{2} \rceil \right)}, \quad (37)$$

we would have (34), whenever we could prove that the right hand side of (37) is bounded independently of $k \geq 0$. This can be proven in analogy to the previous computations made to show that the right-hand side of (32) is bounded. We have thus established the existence of a constant $c_{11}(d)$ such that (34) is satisfied whenever $L_0 \geq c_{11}(d)$.

On the other hand it is obvious that there is a constant $c_{13}(d)$, such that when $L_0 \geq c_{13}(d)$, for $k \geq 0$,

$$c_5 \kappa^{\frac{u_{k+1} L_{k+1}}{3}} \leq 1.$$

Finally, in order for inequality (14) of condition (G) to be fulfilled, it is enough to take $c_6(d) := \max\{c_7(d), c_{11}(d), c_{13}(d)\}$. \square

2.4. The effective criterion implies Theorem 1.2. We continue now showing how Lemma 2.2 with the appropriate choice of scales, enables us to use the effective criterion (see Theorem 2.4 of [Sz02] where it was introduced) to prove the decay of Theorem 1.2. Let us define for $x \in \mathbb{Z}^d$,

$$|x|_{\perp} := \max\{|x \cdot R(e_i)| : 2 \leq i \leq d\}.$$

Also, define for each $x \in \mathbb{Z}^d$, the canonical translation on the environments $t_x : \Omega \rightarrow \Omega$ as

$$t_x(\omega)(y) := \omega(x+y) \quad \text{for } y \in \mathbb{Z}^d.$$

For the statement of the following proposition and its proof, we will use the shorthand notation for each n ,

$$\log_8^{(n)}(L) := \overbrace{\log_8 \circ \cdots \circ \log_8}^n(L).$$

Proposition 2.5. *There exist $c_{15}(d) > 1$, $c_{14}(d) \geq 3\sqrt{d}$ such that whenever $L_0 \geq c_{14}$, $3\sqrt{d} \leq \tilde{L}_0 \leq L_0^3$, and for the box specification $\mathcal{B}_0 = (R, L_0 - 1, L_0 + 1, \tilde{L}_0)$, the condition*

$$c_{15} \left(\log \left(\frac{1}{\kappa} \right) \right)^{3(d-1)} \tilde{L}_0^{d-1} L_0^{3d-2} \inf_{a \in (0,1]} \mathbb{E}[\rho_0^a] < 1, \quad (38)$$

is satisfied (recall the definition of ρ_0 in (6)), then there exist a constant $c > 0$ and a function $n(L) : [0, \infty) \rightarrow \mathbb{N}$, with $n(L) \rightarrow \infty$ as $L \rightarrow \infty$, such that

$$\limsup_{L \rightarrow \infty} L^{-1} \exp\{c \log_8^{n(L)} L\} \log P_0(T_L^l \leq \tilde{T}_{-L}^l) < 0. \quad (39)$$

Remark 2.6. The assumption (38) of Proposition 2.5, is called the *effective criterion*, and was introduced by Sznitman in [Sz02].

Proof. Let us choose a sequence of scales $\{L_k : k \geq 0\}$ and $\{\tilde{L}_k : k \geq 0\}$ according to displays (21) and (10). With this choice of scales, as in the proof of Proposition 2.3 of Sznitman [Sz02], one can see that there are constants $c_{15}(d)$ and $c_{14} \geq \max\{c_6, c_2\}$ such that if $L_0 \geq c_{14}$ then condition (38) implies condition (15) of Lemma 2.2 with u_0 chosen according to (11). By Lemma (2.4), the chosen scales $\{L_k : k \geq 0\}$ and $\{\tilde{L}_k : k \geq 0\}$ satisfy condition (G). Therefore, since (15) of Lemma (2.2) is satisfied, we know that for all $k \geq 0$, inequality (16) is satisfied. The strategy to prove (39) will be similar to that employed in [Sz02] to prove Proposition 2.3: we will first choose an appropriate k so that L_k approximates a fixed scale L tending to ∞ . Nevertheless, since here we are working with scales which are much larger than those used in [Sz02], we will have to be much more careful with this argument.

Let $L \geq L_0$. Then, there exists a unique integer $k = k(L)$ such that

$$L_k \leq L < L_{k+1}.$$

Note that to prove (39) it is enough to show that there exists a positive constant c_{16} such that for all $L \geq L_0$ one has that

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{16}} \exp \left\{ -c_{16} L \exp \left\{ -\frac{1}{c_{16}} \log_8^{\left(\lceil \frac{k+1}{2} \rceil\right)}(L) \right\} \right\}. \quad (40)$$

In effect, since clearly $k \rightarrow \infty$ as $L \rightarrow \infty$, choosing $n(L) = \lceil \frac{k+1}{2} \rceil$ we have (39).

We will divide the proof of (40) into two cases.

Case 1. Assume that

$$L \leq \frac{2\alpha c_1}{u_0} v^k L_k. \quad (41)$$

Let

$$B := \left\{ x \in \mathbb{Z}^d : |x|_\perp \leq \left\lfloor \frac{L}{L_k} \right\rfloor \tilde{L}_k, x \cdot l \in (-L, L) \right\}.$$

From the inequality $\mathbb{E}[q_k] \leq \mathbb{E}[\rho_k^{a_k}]$, Lemma 2.2 and Chebyshev inequality, we see that if

$$\mathcal{H} := \{ \omega \in \Omega : \exists x \in B \text{ such that } q_k \circ t_x(\omega) \geq \kappa^{\frac{1}{2}u_k L_k} \},$$

then

$$\mathbb{P}(\mathcal{H}) \leq \kappa^{\frac{1}{2}u_k L_k} \frac{|B|}{\tilde{L}_{k+1}^{d-1} L_k}.$$

Note that on \mathcal{H}^c , by the strong Markov property one has that

$$P_{0,\omega}(T_L^l \leq \tilde{T}_{-L}^l) \geq (1 - \kappa^{\frac{1}{2}u_k L_k})^{\left\lfloor \frac{L}{L_k} \right\rfloor + 1}.$$

Therefore, since for $x \in [0, 1]$ and n natural one has that $(1-x)^n \leq n(1-x)$, for L large enough

$$\begin{aligned} P_0(\tilde{T}_{-L}^l < T_L^l) &\leq \left(\frac{|B|}{\tilde{L}_{k+1}^{d-1} L_k} + \frac{L}{L_k} + 1 \right) \kappa^{\frac{1}{2}u_k L_k} \\ &\leq 3 \times 2^d \left(\frac{L}{L_k} \right)^d \kappa^{\frac{1}{2}u_k L_k} \\ &\leq 3 \times 2^d \left(\frac{2\alpha c_1 v^k}{u_0} \right)^d \kappa^{\frac{1}{4}u_k L_k} \leq 1, \end{aligned} \quad (42)$$

where in the third inequality we have used our assumption on L (41). Hence, we can check that there is a constant c_{17} , such that for $k \geq 0$,

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{17}} \exp \left\{ -c_{17} \frac{L_k}{v^k} \right\}. \quad (43)$$

Now, again by our assumption (41), observe that there is a constant c_{18} such that

$$\frac{L_k}{v^k} > c_{18} \frac{L}{v^{2k}}. \quad (44)$$

On the other hand, note that whenever L_0 is chosen so that $L_0 \geq \sqrt{\frac{3(d-1)}{\alpha c_1 \log \frac{1}{\kappa}}}$, we have by the choice of scales given in (21), that for $k \geq 1$

$$f_{\lfloor \frac{k+1}{2} \rfloor} \left(\left\lfloor \frac{k}{2} \right\rfloor \right) \leq L_k \leq L. \quad (45)$$

Repeatedly taking logarithms in (45), we conclude that for $k \geq 1$

$$\frac{k}{4} \leq \left\lfloor \frac{k}{2} \right\rfloor \leq \log_8^{\left(\lfloor \frac{k+1}{2} \rfloor\right)}(L). \quad (46)$$

Then, substituting the inequalities (44) and (46) into (43), we see that there exists a positive constants c_{16} such that for $L \geq L_0$

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{16}} \exp \left\{ -c_{16} L \exp \left\{ -\frac{1}{c_{16}} \log_8^{\left(\lfloor \frac{k+1}{2} \rfloor\right)}(L) \right\} \right\}.$$

Now, (39) follows taking $n(L) = \lfloor \frac{k+1}{2} \rfloor$.

Case 2. Let us now assume that

$$L > \frac{2\alpha c_1}{u_0} v^k L_k.$$

Let m_k be the unique integer such that

$$m_k L_k \leq L < (m_k + 1)L_k.$$

By the definition of m_k we have the inequality

$$m_k \geq \frac{\alpha c_1}{u_0} v^k. \quad (47)$$

We will now follow an approach similar to the one employed for *Case 1*, but using a sequence of scales which approximate L with a higher precision than the $\{L_k\}$ sequence. Let us define

$$\begin{aligned} S_1^k &:= m_k L_k, \\ \tilde{S}_1^k &:= m_k^3 \tilde{L}_k, \\ S_2^k &:= m_k^2 L_k, \\ \tilde{S}_2^k &:= m_k^6 \tilde{L}_k, \end{aligned} \quad (48)$$

along with the box-specification $\hat{\mathcal{B}} := (R, S_1^k - 1, S_1^k + 1, \tilde{S}_1^k)$ and the random variable $\hat{\rho}_k$ attached to this box-specification. In analogy with the proof of Lemma 2.2, we will prove that

$$(\tilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\hat{\rho}_k^{a_{k+1}}] \leq \kappa^{u_{k+1} S_1^k}. \quad (49)$$

For the time being, assume that this inequality is true. Let

$$\hat{B} = \left\{ x \in \mathbb{Z}^d : |x|_{\perp} \leq \left\lfloor \frac{L}{S_1^k} \right\rfloor \tilde{S}_1^k, x \cdot l \in (-L, L) \right\}.$$

In analogy with the development of *Case 1*, using (49) we can arrive to the following inequality analogous to (42)

$$P_0[\tilde{T}_{-L}^l < T_L^l] \leq \left(\frac{|\widehat{B}|}{(\tilde{S}_2^k)^{d-1} S_1^k} + \frac{L}{S_1^k} + 1 \right) \kappa^{\frac{1}{2} u_{k+1} S_1^k}.$$

From here we conclude that there is a constant c_{19} such that for $k \geq 0$

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{19}} \exp \left\{ -\frac{c_{19} S_1^k}{v^k} \right\} \quad (50)$$

Now, the computation $S_1^k = m_k L_k = (m_k + 1) L_k - L_k \geq L - \frac{u_0}{2\alpha c_1} v^{-k} L$, replaced at (50), gives us

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{19}} \exp \left\{ -\frac{c_{19} L \left(1 - \frac{u_0}{2\alpha c_1} v^{-k} \right)}{v^k} \right\}$$

So that, there exists c_{20} such that

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{20}} \exp \left\{ -c_{20} \frac{L}{v^k} \right\}$$

Using now (46) we conclude that there is a constant c_{16} such that for $L \geq L_0$ one has that

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leq \frac{1}{c_{16}} \exp \left\{ -c_{16} L \exp \left\{ -\frac{1}{c_{16}} \log_8 \left(\left[\frac{k+1}{2} \right] \right) (L) \right\} \right\}.$$

Choosing $n(L) = \left[\frac{k+1}{2} \right]$ we conclude the proof.

Now, we need to prove (49). Using Proposition 2.1, with $\widehat{\mathcal{B}}$ and \mathcal{B}_k instead of \mathcal{B}_1 and \mathcal{B}_0 , we have:

$$\mathbb{E}[\widehat{\rho}_k^{a_{k+1}}] \leq c_3 \left\{ \kappa^{-10c_1 S_1^k} \varphi_k^{\frac{m_k^2}{12}} + \sum_{0 \leq j \leq m_k+1} \varphi_k^{\frac{m_k+j-1}{2}} \right\}$$

So that

$$(\tilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_{k+1}}] \leq c_3 (S_2^k)^{d-1} S_1^k \left\{ \kappa^{-10c_1 S_1^k} \varphi_k^{\frac{m_k^2}{12}} + \sum_{0 \leq j \leq m_k+1} \varphi_k^{\frac{m_k+j-1}{2}} \right\}$$

Now,

$$\kappa^{-10c_1 S_1^k} \varphi_k^{\frac{m_k^2}{24}} \leq \kappa^{-10c_1 S_1^k} \kappa^{\frac{m_k S_1^k u_k}{24}} \leq 1. \quad (51)$$

where the first inequality follows from inequality (47), the definition (48) of S_1^k and (12) of u_k , and from Lemma 2.4, which enables us to apply inequality (16) of Lemma 2.2, while the second inequality of (51) follows from the fact that $m_k u_k \geq 240c_1$ for $k \geq 0$.

Then, inequality (51) and the fact that $m_k - 1 \geq \frac{m_k}{2}$, imply that

$$(\tilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_{k+1}}] \leq c_3 (\tilde{S}_2^k)^{d-1} S_1^k \left\{ \varphi_k^{\frac{m_k^2}{24}} + S_1^k \varphi_k^{\frac{m_k}{4}} \right\}.$$

So that

$$(\tilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_{k+1}}] \leq 2c_3 (\tilde{S}_2^k)^{d-1} (S_1^k)^2 \varphi_k^{\frac{m_k}{8}} \kappa^{u_{k+1}} S_1^k.$$

Where, it was used the result of Lemma 2.2. Finally, note that to finish the proof we have to show that for $k \geq 0$,

$$2c_3 (\tilde{S}_2^k)^{d-1} (S_1^k)^2 \varphi_k^{\frac{m_k}{8}} \leq 1. \quad (52)$$

By our definitions in (48),

$$(\tilde{S}_2^k)^{d-1} (S_1^k)^2 = m_k^{6d-4} \tilde{L}_k^{d-1} L_k^2.$$

Now, by Lemma 2.4 and its consequence Lemma 2.2, we have that $\varphi_k^{\frac{m_k}{8}} \leq (\kappa^{u_k L_k})^{\frac{m_k}{8}} = \kappa^{u_{k+1} m_k L_k}$. Therefore, the left hand side of inequality (52) is smaller than

$$2c_3 m_k^{6d-4} \tilde{L}_k^{d-1} L_k^2 \kappa^{u_{k+1} m_k L_k}.$$

However, as d is fixed, and k is large, it is clear that

$$\tilde{L}_k^{d-1} L_k^2 \kappa^{\frac{u_{k+1} m_k L_k}{2}} \leq 1$$

and

$$2c_3 m_k^{6d-4} \kappa^{\frac{u_{k+1} m_k L_k}{2}} \leq 1.$$

This completes the proof. \square

It is now easy to check that Proposition 2.5 implies Theorem 1.2 with the function $\log x$ replaced by $\log_8 x$. Indeed, note that (38) is equivalent to the effective criterion. On the other hand, using the fact that for every $x > 0$, $\log x \geq \log_8 x$, we can then obtain Theorem 1.2.

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