FLUCTUATIONS OF THE FRONT IN A ONE DIMENSIONAL MODEL OF $X + Y \to 2X$

FRANCIS COMETS[1,4], JEREMY QUASTEL[2] AND ALEJANDRO F. RAMÍREZ[3,4]

ABSTRACT. We consider a model of the reaction $X + Y \to 2X$ on the integer lattice in which Y particles do not move while X particles move as independent continuous time, simple symmetric random walks. Y particles are transformed instantaneously to X particles upon contact. We start with a fixed number $a \geq 1$ of Y particles at each site to the right of the origin, and define a class of configurations of the X particles to the left of the origin having a finite l^1 norm with a specified exponential weight. Starting from any configuration of X particles to the left of the origin within such a class, we prove a central limit theorem for the position of the rightmost visited site of the X particles.

1. INTRODUCTION

We consider the following microscopic model of a combustive reaction or epidemic on the integer lattice $\mathbb{Z}$: There are two types of particles; X particles, which move as independent, continuous–time, symmetric, nearest neighbor random walks of total jump rate 2; and Y particles which do not move. Initially the Y particles occupy sites $1, 2, \ldots$, with a fixed number $a \geq 1$ of Y particles at each site. Initially there is at least one X particle at 0, and any distribution of X particles at sites $\ldots, -2, -1$ such that $\sum_{x \leq 0} \eta(0, x) e^{\theta x} < \infty$, where $\theta > 0$ is a parameter that will be chosen small and $\eta(0, x)$ is the number of X particles at $x \in \mathbb{Z}$ at time 0. When an X particle jumps to a site where there are Y particles, all a of them immediately become X particles and start moving as rate 2 continuous time symmetric random walks. We are interested in the asymptotic behavior of the rightmost site r_t visited by the X particles up to time t, which we call the *front*.

Let $\eta(t, x)$ denote the number of X particles at $x \in \mathbb{Z}$ at time $t \geq 0$. Since there are always exactly a of the Y particles at each $x > r_t$ we do not have to keep track of them and we can just think of an X particle as branching into $a + 1$ particles when it jumps to $r + 1$, with the result that there are $a + 1$ particles at the new rightmost visited site, $r + 1$. A naive state space of our Markov process is

$$\mathbb{S} = \{(r, \eta) \ : \ r \in \mathbb{Z}, \eta \in \mathbb{N}^{\{\cdots, r-1, r\}}\},$$

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with an infinitesimal generator acting over local functions given by,

$$\mathcal{L}f(r,\eta) = \sum_{x,x+e\le r} \eta(x)(f(r,\eta-\delta_x+\delta_{x+e}) - f(r,\eta))$$

$$+\eta(r)(f(r+1,\eta-\delta_r+(a+1)\delta_{r+1}) - f(r,\eta)).$$

where δ_x denotes the configuration with one particle at x. Nevertheless, to avoid anomalies involving an explosion on the number of particles per site, we will take as the state space of our process,

$$\mathbb{S}'_\theta = \{(r,\eta) \in \mathbb{S} \ : \ \sum_{x\le r} e^{\theta(x-r)}\eta(x) < \infty\}.$$

$\mathbb{S}'_\theta$ with, for example the metric $d((r,\eta),(r',\eta')) = |r-r'| + \sum_{x\le 0} e^{\theta x}|\eta(x+r) - \eta'(x+r')|$, is a Polish space.

We will show (see Section 2 and 6) that if initially $(r,\eta) \in \mathbb{S}'_\theta$, with $r=0$ and $\eta(0,0) \ge 1$, then $(r_t,\eta(t)) \in \mathbb{S}'_\theta$ and furthermore the process is Feller. In [11] it is shown, for certain initial conditions, that there exists v, $0<v<\infty$, such that a.s.,

$$\lim_{t\to\infty} r_t/t = v.$$

We will give an alternate proof in dimension $d=1$ using the regeneration time method (see Section 6) which works for arbitrary initial data in $\mathbb{S}'_\theta$. Note that this could also be proved using the sub-additive ergodic theorem.

Our main results are:

Theorem 1. *(Central limit theorem) For $\theta > 0$ small enough, there exists σ^2 non-random, $0 < \sigma^2 < \infty$, and independent of the the initial conditions $(0,\eta) \in \mathbb{S}'_\theta$, such that*

$$B_t^\epsilon := \epsilon^{1/2}\left(r_{\epsilon^{-1}t} - \epsilon^{-1}vt\right), \qquad t \ge 0, \tag{1}$$

converges in law as $\epsilon \to 0$ to Brownian motion with variance σ^2.

Theorem 2. *(Ergodic theorem) Consider the process as seen from the front, $\tau_{-r_t}\eta(t)$. For $\theta > 0$ small enough, there exist exactly two invariant measures: One supported on the configuration with no particles, and another, μ_∞. The domain of attraction of the first consists of exactly the configuration with no particles. Any nontrivial configuration in $\mathbb{S}'_\theta$ is in the domain of the second; if we denote by μ_t the distribution of the process $\tau_{-r_t}\eta(t)$, then $\mu_t \to \mu_\infty$ in the sense of weak convergence of probability measures.*

The model we are studying has been considered in the physics literature (see [9] and references therein). Recently there has been a resurgence of interest in such models because, especially in one and two dimensions, strong deviations from mean field behavior were detected experimentally.

Mathematically much less is known. [4] studies a model with at most one particle per site in which particles jump to neighboring sites at rate $\gamma/2$ and create particles at empty neighboring sites at rate $\frac{1}{2}$. Considering initial configurations with a rightmost particle, it is shown that viewed from the rightmost particle, the process has a unique invariant measure. Therefore the position of the rightmost particle grows linearly, in fact with a computable speed.

A discrete time version of our model is known in the probability literature as the "frog model". Shape theorems have been obtained for the model on $\mathbb{Z}^d$ using methods based on the sub-additive ergodic theorem (see [1] and [11] for the continuous time version and [2] where the initial configuration of the Y particles is random). We prove the corresponding result for arbitrary initial conditions in $\mathbb{S}'_\theta$ (see Section 6) which could alternately be obtained with such methods. However, using the method of regeneration times we are able to obtain in addition the central limit theorem for the position of the front and the ergodic theorem for the law of the process as seen from the front. The disadvantage of the method is that it appears at the present time to be restricted to one dimensional systems.

In [7], Kesten and Sidoravicius consider a model in which the Y particles move as well. Let D_X and D_Y denote the jump rates of the two types. If $D_X = D_Y > 0$ they prove a shape theorem in $\mathbb{Z}^d$. When $D_X \neq D_Y$ they can only obtain a linear upper bound. Note that [8] observed experimentally that for one dimensional models of this type with exclusion, the speed does not depend on $D_Y \geq 0$ but only on D_X (as long $D_X > 0$).

One of the aspects which makes these type of problems difficult is that the process as seen from the front does not converge exponentially fast to its equilibrium. For example, starting from one X particle at the origin, the probability that the rightmost occupied site up to time t is still at the origin decays with $\mathcal{O}(t^{-1/2})$. Hence, with such an initial condition, if μ_t is the law of the environment seen from the the front at time t and μ_∞ the (nontrivial) invariant measure of the process seen from the front,

$$||\mu_t - \mu_\infty||_{TV} \geq \mathcal{O}(t^{-1/2}),$$

indicating that we are in the gap-less case. In the physics literature such fronts are called *pulled fronts* [12].

In [5] we considered a preliminary model in which there was a threshold: Any particle which jumps to a site with M particles is immediately killed. That model lacks the sub-additivity of the present model. On the other hand, it is considerably easier in that case to define the renewal structure. The unboundedness in the particle configurations makes it particularly difficult to set up the renewal structure. Essentially one has to show that at the regeneration time, one is not in a bad situation in which there are an unusually large number of particles around. Nevertheless, if uniform estimates on the initial conditions are not obtained, then there is no finiteness of the first and second moments of the corresponding regeneration times. Therefore, to prove Theorem 1, we have defined regeneration times in terms of a modified renewal structure which provides a global control on the number of particles per site far from the front. This difficulty in constructing regeneration times appears to be very common when dealing with dynamic environments (see, for example, [3]).

Regeneration time methods were already used by Kesten in [6] to study the invariant measure of an i.i.d. environment as seen from a one dimensional random walk on that environment (RWRE). Our approach to define the regeneration times in terms a sequence of stopping times is inspired in the methods presented in [14] for multidimensional RWRE. At a heuristic level, regeneration occurs each time the front moves forward and the particles behind it never catch it up later on. After such a time, the behaviour of the front depends only on the a newly created particles sitting at the front at that time, but not on those behind the front at that time. The idea is to find an increasing sequence $\{\kappa_n : n \geq 1\}$ of regeneration times, having independent increments and such that the probability of the event $\{\kappa_n > t\}$ decreases fast enough as $t \to \infty$ providing good enough integrability conditions. As in [5], in order to estimate the tails of the regeneration times, it is useful to decouple particles initially on the front from those behind it. Nevertheless, a crucial difficulty and difference in the construction of the sequence of stopping times with respect to [5], is that in this model the number of X particles per site is not bounded. This requires a control in terms of some norm of the size of the cloud of particles behind the front. To do so, we introduce at each time $t \geq 0$, an *exponential norm* depending on the parameter θ and on an integer z, which is given by $\sum_{x \leq r_t} e^{\theta(x-r_t)} \eta_z(t,x)$. Here, $\eta_z(t,x)$ is the number of X particles at site x and at time t which originated from some branching (of an X particle) at some site $y \leq z$. This is a measure of the magnitude of the density of particles from r_t to $-\infty$, which originated from some site $y \leq z$. We then define a stopping time S depending on an integer length L, as the first hitting time to a site of the form $r_0 + jL$, $j \geq 1$, such that the exponential norm of the particles originating to the left of $r_0 + (j-1)L$ is small enough. In [5], the corresponding stopping time was defined simply as the first time the front advances L steps to the right. One of the main difficulties of our proof, is to show that the tails of the law of S provide good enough integrability conditions for the corresponding regeneration times and the associated position of the front. We are able to do this only for small values of θ and large values of L: we obtain polynomially decaying tails of a degree which increases linearly with L for the regeneration times $\{\kappa_n : n \geq 2\}$. It is conceivable that for a fixed value of L, the optimal bound for the corresponding regeneration times is indeed of power law type (see [13] for a discussion of this problem within the context of transient multidimensional RWRE).

In the next section, we will define the notion of exponential norm, and the labeled and auxiliary processes, which will be needed subsequently to define the renewal structure. In Section 3, the renewal structure is defined, following the algorithmic approach of [14]. Here it is proved that the regeneration times, define sequences with independent increments, and except for the first term, are identically distributed. This is used in Section 4, to prove the law of large numbers, the central limit theorem in Theorem 1, and Theorem 2. In Section 5, the crucial estimates which ensure the finiteness of the second moments of the i.i.d. sequences defined through the regeneration times are derived. Of particular importance is Lemma 20, which shows that the tails of the stopping time S are small enough. Finally, in Section 6, it is proved that the process is Feller on $\mathbb{S}'_\theta$. Note that in related models (see [7]) it is not known whether the Feller property holds. Throughout the paper a generic constant will be denoted by C.

2. Setup and preliminary definitions

The process will be constructed out of a large collection of independent, continuous time, symmetric, simple random walks, each with jump rate 2. For each site $x \leq r$, we have a countable collection of these: $\{Y_{x,1}, Y_{x,2}, \ldots\}$. For each site $x > r$, we need only a of them: $\{Y_{x,1}, \ldots, Y_{x,a}\}$. Assume that $Y_{x,i}(0) = x$.

First we construct the process for finite initial conditions (r, η) i.e., those in which η has only a finite number of particles.

For each $x \leq r$, and $i \leq \eta(x)$, let $Z_{x,i}(t) = Y_{x,i}(t)$. Let τ_1 be the first time that one of the random walks $Z_{x,i}(t)$, $x \leq r$, hits $r+1$. For $0 \leq t < \tau_1$, let $r_t = r$ and $\eta(z,t) = \sum_{x \leq r,\ i} 1(Z_{x,i}(t) = z)$.

At time τ_1 we add a particles, $\{Z_{r+1,1}, \ldots, Z_{r+1,a}\}$, where $Z_{r+1,i}(t) = Y_{r+1,i}(t-\tau_1)$. Let τ_2 be the first time that one of the random walks $Z_{x,i}(t)$, $x \leq r+1$, hits $r+2$. For $\tau_1 \leq t < \tau_2$, let $r_t = r+1$ and $\eta(z,t) = \sum_{x \leq r+1,\ i} 1(Z_{x,i}(t) = z)$.

Continuing in this way, we define the process $\{(r_t, \eta(t) : t \geq 0\}$ for finite initial conditions and the sequence of stopping times $\{\tau_n : n \geq 1\}$. In Section 6 we will show that the definition makes sense. In particular, one has to show that $\lim_{n\to\infty} \tau_n = \infty$ with probability one.

For general $(r, \eta) \in \mathbb{S}'_\theta$, with arbitrary θ, we construct the process by taking limits of approximations with finite initial conditions. For each $\ell = 1, 2, \ldots$, let $\eta^\ell(x) = 0$ if $x \leq r - \ell$, and $\eta^\ell(x) = \eta(x)$ if $r - \ell < x \leq r$. Consider the process $\{(r^\ell_t, \eta^\ell(t)) : t \geq 0\}$ starting from this finite initial condition. In Section 6 we will prove

Proposition 1. *For every $(r, \eta) \in \mathbb{S}'_\theta$ and $t \geq 0$, $r_t = \lim_{\ell\to\infty} r^\ell_t$ and $\eta(t,x) = \lim_{\ell\to\infty} \eta^\ell(t,x)$ exist, are finite a.s. and $(r_t, \eta(t)) \in \mathbb{S}'_\theta$. The limit is a Markov process with Feller semi-group $P_t f(r,\eta) = E_{r,\eta}[f(r_t, \eta(t))]$ on $C(\mathbb{S}'_\theta)$, where $E_{r,\eta}$ is the expectation associated to the joint law $P_{r,\eta}$ of $\{(r_t, \eta(t)) : t \geq 0\}$.*

2.1. Auxiliary process.

Let
$$M = 4(a+5). \tag{2}$$
Let now $r \in \mathbb{Z}$, define $\nu_0 := 0$ and ν_1 as the first time one of the random walks $\{Y_{r,i} : 1 \leq i \leq a\}$, hits the site $r+1$. Next, define ν_2 as the first time one of the random walks $\{Y_{z,i} : r \leq z \leq r+1, 1 \leq i \leq a\}$ hits the site $r+2$. In general, for $k \geq 2$, we define ν_k as the first time one of the random walks $\{Y_{z,i} : r \vee (r+k-M) \leq z \leq r+k-1, 1 \leq i \leq a\}$, hits the site $r+k$. For $n \in \mathbb{N}$, let
$$\tilde{r}^r_t := r + n, \qquad \text{if} \qquad \sum_{k=0}^{n} \nu_k \leq t < \sum_{k=0}^{n+1} \nu_k.$$
Now, observing that for each $1 \leq j \leq M-1$, the random variables $\{\nu_{Mk+j} : k \geq 1\}$ are independent and have finite moments since $M \geq 3$, we see that a.s. (see also [5]),
$$\lim_{t\to\infty} \tilde{r}^r_t / t =: \alpha > 0. \tag{3}$$

2.2. Labeled process.

We enlarge the state space of the stochastic combustion process so that particles carry labels indicating at which site they originated. Each particle will have a starting position $z \in \mathbb{Z}$ and label $(x, i), x \in \mathbb{Z}, i \in \{1, \ldots, a\}$

describing its birthplace, allowing the possibility that $z \neq x$. Throughout the sequel, we will adopt the convention of calling x the site where the particle *originated*, whereas z the site where the particle was initially.

We fix at time 0, an $r \in \mathbb{Z}$ representing the rightmost visited site, and a subset $\mathcal{I}(0)$ of the labels (x,i) with $x \leq r$, representing the set of labels of particles at time 0. To each one of these labels we assign a position $z = Z_{x,i}(0) \leq r$ which is the position at time $t = 0$ of that particle. The position at time t is $Z_{x,i}(t) = Y_{x,i}(t) + z - x$. Now, the first time a particle jumps to site $r+1$, the labels $\{(r+1,1),\ldots,(r+1,a)\}$ are added to the set of labels of particles. Let us call ρ_1 the time this happens. The trajectories $Z_{r+1,i}(t)$ of these new particles are then equal to $Y_{r+1,i}(t-\rho_1)$ for $t \geq \rho_1$. Similarly, for $k \geq 2$, $\rho_1 + \cdots + \rho_k$ will be the first time a particle jumps to $r+k$ adding at that time the labels $\{(r+k,1),\ldots,(r+k,a)\}$ to $\mathcal{I}$, with trajectories $Z_{r+k,i}(t) = Y_{r+k,i}(t-\rho_1-\cdots\rho_k)$ for $t \geq \rho_1+\cdots\rho_k$.

Now denote by $\mathcal{I}(t)$ the set of labels of particles at time t and by $\mathcal{Z}(t) := \{Z_{x,i}(t) : (x,i) \in \mathcal{I}(t)\}$ their corresponding positions. We assume that initially the set of labels of particles includes at least one with $x = r$. Then, the rightmost visited site is defined as $r_t = \sup\{x : (x,i) \in \mathcal{I}(t)\}$. Call $\mathbb{L}$ the triples $(r,\mathcal{I},\mathcal{Z})$ of integers r, labels $\mathcal{I} \subset \{(x,i) : x \leq r, 1 \leq i \leq a\}$ and position function $\mathcal{Z} : \mathcal{I} \to \{\ldots, r-2, r-1, r\}$. The unlabeled process defined in the previous section is just the particle count

$$\eta(y,t) = \sum_{(x,i)\in\mathcal{I}(t)} 1(Z_{x,i}(t) = y).$$

For $\theta > 0$, let us now denote by $\mathbb{L}_\theta$ the set of triples $(r,\mathcal{I},\mathcal{Z}) \in \mathbb{L}$ such that $(r,\eta) \in \mathbb{S}'_\theta$. Then define

$$\tilde{\mathbb{S}}_\theta := \left\{ (r,\mathcal{I},\mathcal{Z}) \subset \mathbb{L}_\theta : \max_{(x,i)\in\mathcal{I}} x = r \right\}.$$

From Proposition 1, note that if $w_0 = (r_0, \mathcal{I}(0), \mathcal{Z}(0)) \in \tilde{\mathbb{S}}_\theta$ then $w_t = (r_t, \mathcal{I}(t), \mathcal{Z}(t)) \in \tilde{\mathbb{S}}_\theta$ for $t \geq 0$. We now define the *labeled process* starting from w_0 as the triple $\{w_t : t \geq 0\} = \{(r_t, \mathcal{I}(t), \mathcal{Z}(t)) : t \geq 0\}$, with a law given by a probability measure $\mathbb{P}_w$ defined on the Skorohod space $D([0,\infty); \tilde{\mathbb{S}}_\theta)$. Throughout this paper, we will occasionally use the notation $\mathbb{P}_{r,eta}$ to denote any law $\mathbb{P}_w$ with an initial condition w compatible with r and the particle count η.

Using sub-additivity we have the following result (see also Lemma 3 of [5]),

Lemma 1. *Suppose that* $(r,1),\ldots,(r,a) \in \mathcal{I}(0)$*, all initially at* r*. Then* $\rho_k \leq \nu_k$.

Let us now define $\mathcal{R}(t)$ as the set of labels obtained after removing from $\mathcal{I}(t)$ all labels (x,i) with $x < r = \sup\{y : (y,i) \in \mathcal{I}(0)\}$. We define for $y \leq r_t$ the particle count

$$\zeta(t,y) := \sum_{(x,i)\in\mathcal{R}(t)} 1(Z_{x,i}(t) = y). \tag{4}$$

2.3. **Exponential density norm of particles.** Assume that the initial condition of our process is (r,η). Let us also fix two integers z_1, z_2, such that $z_1 < z_2 \leq r-1$

and follow the individual particles which originated at $z_1 < y \le z_2$:

$$\eta_{z_1,z_2}(t,y) := \sum_{(x,i):z_1<x\le z_2} 1(Z_{x,i}(t)=y),$$

We will also write $\eta_z(t,y)$ for $\eta_{-\infty,z}(t,y)$. We will use the notation,

$$m_{z_1,z_2}(t) := \sum_{x=z_1+1}^{z_2} \eta_{z_1,z_2}(x,t),$$

to denote the total number of such particles which are still in the same interval at time t.

For $\theta > 0$ and $t \ge 0$ define,

$$\phi_z(t,r,\eta) := \sum_{x\in\mathbb{Z}} e^{\theta(x-r_t)}\eta_z(t,x),$$

which we will call the *exponential density norm* of particles. Sometimes we will write $\phi_z(t)$ instead of $\phi_z(t,r,\eta)$.

3. The renewal structure

Let us now define the renewal structure that will be used to define the regeneration times. The exponential density norm of particles will be an important ingredient and will enable us to control the number of particles far from the front. Let us now fix some integer L satisfying

$$aL \ge M, \tag{5}$$

and real numbers θ, α_1 and α_2 satisfying

$$0 < 2\sinh 2\theta < \alpha_1 < \alpha_2 < \alpha = \lim_{t\to\infty} \tilde{r}_t^r/t. \tag{6}$$

Let us now consider the labeled process w_t with its natural filtration $\mathcal{F}_t$ with an initial condition $w_0 \in \tilde{\mathbb{S}}_\theta$ having particles with labels $(r,1),\dots,(r,a)$ at site r, and any allowable configuration of particles with labels to the left of r. Call $\eta(0)$ the initial particle count corresponding to w_0.

Define the stopping times,

$$W := \inf\{t \ge 0 : \phi_{r-L}(t,r,\eta(0)) \ge e^{\theta(\lfloor\alpha_1 t\rfloor-(r_t-r))}\},$$

and

$$V := \inf\{t\ge 0 : \max_{r-L<z<r}\max_{1\le i\le a} Z_{z,i}(t) > \lfloor\alpha_1 t\rfloor + r\}.$$

When $W=\infty$, none of the particles initially to the left of $r-L$ ever touches the line $\lfloor\alpha_1 t\rfloor + r$. Define,

$$U := \inf\{t\ge 0 : \tilde{r}_t^r - r < \lfloor\alpha_2 t\rfloor\}.$$

We then let

$$D := \min\{U,V,W\}.$$

We will also need to define $U\circ\theta_s, V\circ\theta_s$ and $W\circ\theta_s$ as the first times U,V or W happen starting from the initial condition w_s for $s\ge 0$, and $D\circ\theta_s := \min\{U\circ\theta_s, V\circ\theta_s, W\circ\theta_s\}$.

For each $y \in \mathbb{Z}$, let

$$T_y := \inf\{t \geq 0 : r_t \geq y\}.$$

Fix p such that

$$0 < pe^{\theta} < 1. \tag{7}$$

We will furthermore impose the following additional condition on L,

$$(a-1)e^{-L\theta} < p \tag{8}$$

Now define for $x \geq r$,

$$J_x := \inf\{j \geq 1 : \phi_{x+(j-1)L}(T_{x+jL}) \leq p \text{ and } m_{x+(j-1)L,x+jL}(T_{x+jL}) \geq aL/2\}. \tag{9}$$

Define the sequence of $\mathcal{F}_t$-stopping times, $\{S_k : k \geq 0\}$ and $\{D_k : k \geq 1\}$ as follows. Let $S_0 := 0$ and $R_0 = r$. Then define

$$S_1 := T_{R_0+J_{R_0}L} \qquad D_1 := D \circ \theta_{S_1} + S_1, \qquad R_1 := r_{D_1}$$

and for $k \geq 1$,

$$S_{k+1} := T_{R_k+J_{R_k}L} \qquad D_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}, \qquad R_{k+1} = r_{D_{k+1}}$$

Let

$$K := \inf\{k \geq 1 : S_k < \infty, D_k = \infty\},$$

and define the *regeneration time*

$$\kappa := S_K, \tag{10}$$

with the understanding that $\kappa = \infty$ on the event $\{k \geq 1 : S_k < \infty, D_k = \infty\} = \emptyset$. Note that κ is *not* a stopping time with respect to $\mathcal{F}_t$.

Define $\mathcal{G}$, the information up to time κ, defined as the completion with respect to $\mathbb{P}_w$ of the smallest σ-algebra containing all sets of the form $\{\kappa \leq t\} \cap A, A \in \mathcal{F}_t$.

Proposition 2. *For every initial condition $w \in \tilde{\mathbb{S}}_\theta$ with at least one particle at the rightmost visited site,*

$$\kappa < \infty, \qquad \mathbb{P}_w - \text{a.s.} \tag{11}$$

Furthermore, if $a\delta_0$ denotes a configuration with rightmost visited site 0 such that the number of particles at 0 is a and the number of particles at each site $x < 0$ is 0,

$$\mathbb{E}_{a\delta_0}\left[\kappa^2 | U = \infty\right] < \infty \qquad \textit{and} \qquad \mathbb{E}_{a\delta_0}\left[r_\kappa^2 | U = \infty\right] < \infty. \tag{12}$$

Proposition 2 will be proved in Subsection 5.6. Recall the definition (4) of ζ. The key observation is

Proposition 3. *Let A be a Borel subset of $D([0,\infty);\mathbb{S}'_\theta)$ and $w \in \tilde{\mathbb{S}}_\theta$. Then,*

$$\mathbb{P}_w[\tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A | \mathcal{G}] = \mathbb{P}_{a\delta_0}[\eta(\cdot) \in A | U = \infty].$$

Proof. We have to show that for any $B \in \mathcal{G}$,

$$\mathbb{P}_w[B, \{\tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A\}] = \mathbb{P}_w[B]\ \mathbb{P}_{a\delta_0}[\eta(\cdot) \in A \mid U = \infty]. \tag{13}$$

Now, using (11),

$$\begin{aligned} &\mathbb{P}_w[B, \{\tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A\}] = \mathbb{P}_w[\{\kappa < \infty\}, B, \{\tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A\}] \\ = &\sum_{k=1}^{\infty} \mathbb{P}_w\left[\{S_k < \infty, D_k = \infty\}, B, \tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A\right] \\ = &\sum_{k=1}^{\infty}\sum_{x\in\mathbb{Z}} \mathbb{P}_w[r_{S_k} = x, S_k < \infty, D_k = \infty, B, \tau_{-x}\zeta(S_k + \cdot) \in A]. \end{aligned} \tag{14}$$

From the definition of $\mathcal{G}$ there is an event $B_k \in \mathcal{F}_{S_k}$ such that $B_k = B$ on $\kappa = S_k$. Therefore, we can continue developing (14) to obtain,

$$\begin{aligned} = &\sum_{k=1}^{\infty}\sum_{x\in\mathbb{Z}} \mathbb{P}_w\left[r_{S_k} = x, S_k < \infty, D_k = \infty, B_k, \tau_{-x}\zeta(S_k + \cdot) \in A\right] \\ = &\sum_{k,x} \mathbb{E}_w\left[1(r_{S_k} = x, S_k < \infty, B_k)\mathbb{P}_w\left[D_k = \infty, \tau_{-x}\zeta(S_k + \cdot) \in A \mid \mathcal{F}_{S_k}\right]\right], \end{aligned} \tag{15}$$

where $\mathbb{E}_w$ is the expectation defined by $\mathbb{P}_w$. But on the events $S_k < \infty$ and $r_{S_k} = x$, we have that

$$\zeta_w(S_k + \cdot) = \eta_{a\delta_x}(\cdot) \tag{16}$$

when $U_k = V_k = W_k = \infty$, and that $\eta_{a\delta_x}(\cdot)$ is independent of the configuration of particles initially to the left of x. Here, $a\delta_x$, is the configuration with rightmost visited site x and with a particles at site x with labels $(x, 1), \dots, (x, a-1)$ and none elsewhere. Indeed, on the event $V_k = W_k = \infty$, the particles with initial positions z to the left of x, are never to the right of $\lfloor \alpha_1 t\rfloor + x$. And on the event $U_k = \infty$, the front r_t is always to the right of $\lfloor \alpha_2 t\rfloor + x$ and hence of $\lfloor \alpha_1 t\rfloor + x$. Therefore, there is no effect of the particles initially to the left of x on the front r_t, so that $\zeta_w(S_k + \cdot) = \eta_{a\delta_x}(\cdot)$. Then, (16) combined with the independence of U_k and $V_k \wedge W_k$ given $\mathcal{F}_{S_k}$, the translation invariance, and the strong Markov property imply that on the events $S_k < \infty$ and $r_{S_k} = x$,

$$\begin{aligned} &\mathbb{P}_w\left[U_k = \infty, V_k \wedge W_k = \infty, \tau_{-x}\zeta(S_k + \cdot) \in A \mid \mathcal{F}_{S_k}\right] \\ = &\ \mathbb{P}_w\left[U_k = \infty, \tau_{-x}\eta_{a\delta_x}(\cdot) \in A \mid \mathcal{F}_{S_k}\right]\mathbb{P}_w\left[V_k \wedge W_k = \infty \mid \mathcal{F}_{S_k}\right] \\ = &\ \mathbb{P}_{a\delta_0}[U = \infty, \eta(\cdot) \in A]\mathbb{P}_w[V_k \wedge W_k = \infty \mid \mathcal{F}_{S_k}]. \end{aligned} \tag{17}$$

Summarizing, we have,

$$\begin{aligned} &\mathbb{P}_w[B, \tau_{-r_\kappa}\zeta(\kappa + \cdot) \in A] \\ &= \mathbb{P}_{a\delta_0}[U = \infty, \eta(\cdot) \in A]\sum_{k,x} \mathbb{P}_w[V_k \wedge W_k = \infty, r_{S_k} = x, S_k < \infty, B_k]. \end{aligned} \tag{18}$$

Letting $A = \mathbb{S}'_\theta$ gives

$$\mathbb{P}_w[B] = \mathbb{P}_{a\delta_0}[U = \infty]\sum_{k,x} \mathbb{P}_w[V_k \wedge W_k = \infty, r_{S_k} = x, S_k < \infty, B_k]. \tag{19}$$

(18) and (19) together imply (13). ☐

Now define $\kappa_1 \le \kappa_2 \le \cdots$ by $\kappa_1 := \kappa$ and for $n \ge 1$

$$\kappa_{n+1} := \kappa_n + \kappa(w_{\kappa_n+\cdot}).$$

where $\kappa(w_{\kappa_n+\cdot})$ is the regeneration time starting from $w_{\kappa_n+\cdot}$ and we set $\kappa_{n+1} = \infty$ on $\kappa_n = \infty$ for $n \ge 1$. We will call κ_1 the *first regeneration time* and κ_n the *n-th regeneration time*.

For each $n \ge 1$ we define the σ-algebra, $\mathcal{G}_n$, as the completion with respect to $\mathbb{P}_w$ of the smallest σ-algebra containing all sets of the form $\{\kappa_1 \le t_1\} \cap \cdots \cap \{\kappa_n \le t_n\} \cap A$, $A \in \mathcal{F}_{t_n}$. Now, noting that $\{\kappa_1 = \infty\}$ is a null event for $\mathbb{P}_w$ one can see that $\{U < \infty\} \cap \{\kappa_1 < \infty\} = \{\tilde{r}_U \le r_{\kappa_1}\} \cap \{\kappa_1 < \infty\} \in \mathcal{G}_1$ (see Lemma 5 of [5]). Hence, $\{U = \infty\} \in \mathcal{G}_1$. So we have the following general version of Proposition 3,

Proposition 4. *Let A be a Borel subset of $D([0,\infty); \mathbb{S}'_\theta)$ and $w \in \tilde{\mathbb{S}}_\theta$. Then,*

$$\mathbb{P}_w[\tau_{-r_{\kappa_n}}\zeta(\kappa_n + \cdot) \in A \mid \mathcal{G}_n] = \mathbb{P}_{a\delta_0}[\eta(\cdot) \in A \mid U = \infty].$$

We can now describe the full renewal structure.

Corollary 1. *Let $w \in \tilde{\mathbb{S}}_\theta$. (i) Under $\mathbb{P}_w$, $\kappa_1, \kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \ldots$ are independent, and $\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \ldots$ are identically distributed with law identical to that of κ_1 under $\mathbb{P}_{a\delta_0}[\cdot|U = \infty]$. (ii) Under $\mathbb{P}_w$, $r_{\cdot\wedge\kappa_1}, r_{(\kappa_1+\cdot)\wedge\kappa_2} - r_{\kappa_1}, r_{(\kappa_2+\cdot)\wedge\kappa_3} - r_{\kappa_2}, \ldots$ are independent, and $r_{(\kappa_1+\cdot)\wedge\kappa_2} - r_{\kappa_1}, r_{(\kappa_2+\cdot)\wedge\kappa_3} - r_{\kappa_2}, \ldots$ are identically distributed with law identical to that of r_{κ_1} under $\mathbb{P}_{a\delta_0}[\cdot|U = \infty]$.*

4. Limit theorems

We now use the renewal structure to prove the law of large numbers and the central limit theorem for r_t. Throughout, we will consider an an initial condition $(0, \eta) \in \mathbb{S}'_\theta$ such that $\eta(0, 0) \ge 1$.

4.1. **Law of Large Numbers.** We will prove that,

$$\lim_{t\to\infty} \frac{r_t}{t} = v := \frac{\mathbb{E}_{a\delta_0}[r_{\kappa_1}|U = \infty]}{\mathbb{E}_{a\delta_0}[\kappa_1|U = \infty]}. \tag{20}$$

Note that we have that $\kappa_1 < \infty$, $\mathbb{P}_{0,\eta}$-a.s. Hence, by Corollary 1 a.s.

$$\lim_{n\to\infty} \frac{\kappa_n}{n} = \mathbb{E}_{a\delta_0}[\kappa_1|U = \infty], \quad \text{and} \quad \lim_{n\to\infty} \frac{r_{\kappa_n}}{n} = \mathbb{E}_{a\delta_0}[r_{\kappa_1}|U = \infty]. \tag{21}$$

Now, for $t \ge 0$, define $n_t := \sup\{n \ge 0 : \kappa_n \le t\}$, with the convention $\kappa_0 = 0$. From (21) we see that a.s. $n_t < \infty$. Also, $\lim_{t\to\infty} r_{\kappa_{n_t}}/t = v$. The limit (20) now follows from the observation,

$$\lim_{t\to\infty} t^{-1}|r_t - r_{\kappa_{n_t}}| = 0,$$

which is a consequence of the inequality $|r_t - r_{\kappa_{n_t}}| \le |r_{\kappa_{n_t+1}} - r_{\kappa_{n_t}}|$ and the fact that $\lim_{t\to\infty} r_{\kappa_{n_t}}/t = v$ a.s.

4.2. **Central limit theorem.** Consider the quantity B_t^ϵ defined in (1) and

$$\Sigma_m := \sum_{j=1}^m R_j, \tag{22}$$

where $R_j := r_{\kappa_{j+1}} - r_{\kappa_j} - (\kappa_{j+1} - \kappa_j)v$. Now, for $0 \le t \le T < \infty$,

$$|B_t^\epsilon - \epsilon^{1/2}\Sigma_{n_{t/\epsilon}}| \le 2\epsilon^{1/2} \sup_{0\le n \le n_{\lfloor \epsilon^{-1}T\rfloor}} (r_{\kappa_{n+1}} - r_{\kappa_n}) + 2v\epsilon^{1/2} \sup_{0\le n \le n_{\lfloor \epsilon^{-1}T\rfloor}} (\kappa_{n+1} - \kappa_n). \tag{23}$$

On the other hand, from Corollary 2, we can conclude that for every $u > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}_{a\delta_0}[\epsilon^{1/2} \sup_{0\le n\le n_{\lfloor \epsilon^{-1}T\rfloor}} (\kappa_{n+1} - \kappa_n) > u] = 0.$$

Hence, in probability

$$\sup_{0\le n\le n_{\lfloor \epsilon^{-1}T\rfloor}} \epsilon^{1/2}(\kappa_{n+1} - \kappa_n) \to 0. \tag{24}$$

and

$$\sup_{0\le n\le n_{\lfloor \epsilon^{-1}T\rfloor}} \epsilon^{1/2}(r_{\kappa_{n+1}} - r_{\kappa_n}) \to 0.$$

This proves that $B_t^\epsilon - \epsilon^{1/2}\Sigma_{n_{\epsilon^{-1}t}}$ converges to 0 in probability, uniformly on compact sets of t. From Donsker's invariance principle, we know that $\sqrt{\epsilon}\Sigma_{\cdot/\epsilon}$ converges in law to a Brownian motion with variance $\mathbb{E}_{a\delta_0}[(r_{\kappa_1} - \kappa_1 v)^2 | U = \infty]$, where $\Sigma_s, s \ge 0$, now stands for the linear interpolation of $\Sigma_m, m \ge 0$. Using that $\lim_{t\to\infty} n_t/t = 1/\mathbb{E}_{a\delta_0}[\kappa_1 | U = \infty]$ we can conclude that as $\epsilon \to 0$, B_t^ϵ converges to a Brownian motion with variance,

$$\sigma^2 := \frac{\mathbb{E}_{a\delta_0}[(r_{\kappa_1} - \kappa_1 v)^2 | U = \infty]}{\mathbb{E}_{a\delta_0}[\kappa_1 | U = \infty]}. \tag{25}$$

4.3. **Non-degeneracy of the variance.** We will show that $\sigma^2 > 0$. It is enough to show that there exists some β, $0 < \beta < v$ such that,

$$\mathbb{P}_{a\delta_0}[r_{\kappa_1} = L, L\beta^{-1} \le \kappa_1 | U = \infty] > 0.$$

Now,

$$\mathbb{P}_{a\delta_0}[r_{\kappa_1} = L, L\beta^{-1} \le \kappa_1, U = \infty] \ge \mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U, D \circ \theta_{S_1} = \infty].$$

But the right hand side can be written as

$$\mathbb{E}_{a\delta_0}[1(L\beta^{-1} < S_1 < U)\mathbb{E}_{a\delta_0}[1(\min\{V\circ\theta_{S_1}, W\circ\theta_{S_1}\} = \infty)1(U\circ\theta_{S_1} = \infty) \mid \mathcal{F}_{S_1}]].$$

Now note that given $\mathcal{F}_{S_1}$, $U\circ\theta_{S_1}$, $V\circ\theta_{S_1}$ and $W\circ\theta_{S_1}$ are independent. Hence,

$$\mathbb{E}_{a\delta_0}[1(\min\{V\circ\theta_{S_1}, W\circ\theta_{S_1}\} = \infty)1(U\circ\theta_{S_1} = \infty) \mid \mathcal{F}_{S_1}] = \mathbb{P}_{a\delta_0}[V\circ\theta_{S_1} = \infty \mid \mathcal{F}_{S_1}]\mathbb{P}_{a\delta_0}[W\circ\theta_{S_1} = \infty \mid \mathcal{F}_{S_1}]\mathbb{P}_{a\delta_0}[U\circ\theta_{S_1} = \infty \mid \mathcal{F}_{S_1}]. \tag{26}$$

This implies that,

$$\mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U, D\circ\theta_{S_1} = \infty] \ge C\mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U],$$

for some constant $C > 0$. Now, we have to show that $\mathbb{P}_{a\delta_0}[L\beta^{-1} < S_1 < U] > 0$. Note that the event $\{L\beta^{-1} < S_1 < U\}$ contains the following event: one of the initial a particles at 0 jumps to site 1 at some time v_1, such that $\beta^{-1} < v_1 < 2\beta^{-1}$; the other $a-1$ particles initially at 0 stay at the same site during the time interval $[0, 2L\beta^{-1}]$; at time v_1, one of the a particles originating at site 1 jumps to site 2 at some time $v_2 + v_1$ such that $\beta^{-1} < v_2 < 2\beta^{-1}$; the other $a-1$ particles born at site 1 stay at the same site during the time interval $[0, 2L\beta^{-1}]$; in general, if k is such that $3 \le k \le L$, at time $v_k + v_{k-1} + \cdots + v_1$ one of the particles born at site k moves to site $k+1$, and $\beta^{-1} < v_k < 2\beta^{-1}$; all other $a-1$ particles born at site k stay at the same site during the time interval $[0, 2L\beta^{-1}]$. Note that $T_L = v_1 + \cdots + v_L$ and at this time we have $\phi_0(T_L) \le (a-1)e^{-L\theta}$. By (8) this quantity is smaller than p. It is easy to see that the above described event has positive probability.

4.4. **Ergodic theorem.** Let $\tilde{p}_t$ be the law at time t of the process as seen from the front

$$\tau_{-r_t}\eta(t) \in \tilde{\Omega} := \{0, 1, 2, \dots\}^{\mathbb{Z}_-}$$

under $\mathbb{P}_{0,\eta(0)}$. Note that $\tau_{-r_t}\eta(t)$ is itself a Markov process with infinitesimal generator

$$\tilde{\mathcal{L}} f(\eta) = \eta(0)[f(\tau_{-1}(\eta - \delta_0) + a\delta_0) - f(\eta)] + \sum_{\substack{x,y \le 0,\\ |x-y|=1}} \eta(x)[f(\eta - \delta_x + \delta_y) - f(\eta)]$$

Let f be a bounded continuous local function f on $\tilde{\Omega}$ Denote by $\ell(f)$ the smallest integer ℓ such that $f(\eta)$ does not depend on $\eta(x), x < -\ell$. The formula

$$\int_{\Omega_0} f d\tilde{p}_\infty = \frac{\mathbb{E}_{a\delta_0}[\int_{\kappa_N}^{\kappa_{N+1}} f(\tau_{-r_s}\eta(s))ds \mid U = \infty]}{\mathbb{E}_{a\delta_0}[\kappa_1 \mid U = \infty]}, \quad N(\alpha_2 - \alpha_1) > \ell(f) \tag{27}$$

defines a probability measure $\tilde{p}_\infty$ on $\tilde{\Omega}$. The righthand side of (27) does not depend on N provided that condition $N(\alpha_2 - \alpha_1) > \ell(f)$ holds. This shows that the family of probability measures defined on finite cylinders by this formula is consistent.

Theorem 3. *We have $\tilde{p}_t \to \tilde{p}_\infty$ weakly as $t \to \infty$, and $\tilde{p}_\infty$ is invariant for $\tilde{L}$.*

Proof. Let f be bounded and continuous on $\tilde{\Omega}$. To prove convergence, first note that the last term in the decomposition

$$\int_{\Omega_0} f d\tilde{p}_t = \mathbb{E}_{0,\eta(0)}[\kappa_{N+1} \le t, f(\tau_{-r_t}\eta(t))] + \mathbb{E}_{0,\eta(0)}[\kappa_{N+1} > t, f(\tau_{-r_t}\eta(t))]$$

vanishes as $t \to \infty$. Also,

$$\begin{aligned}
&\mathbb{E}_{0,\eta(0)}[\kappa_{N+1} \le t, f(\tau_{-r_t}\eta(t))] \\
&= \textstyle\sum_{k\ge1, x\in\mathbb{Z}} \mathbb{E}_{0,\eta(0)}[\kappa_{N+k} \le t < \kappa_{N+k+1}, r_{\kappa_k} = x, f(\tau_{-r_t}\eta(t))] \\
&= \textstyle\sum_{k\ge1, x\in\mathbb{Z}} \mathbb{E}_{0,\eta(0)}\left[r_{\kappa_k} = x,\ \mathbb{E}_{\eta(0)}[\kappa_{N+k} \le t < \kappa_{N+k+1}, f(\tau_{-r_t}\eta(t)) | \mathcal{G}_k]\right] \\
&= \textstyle\sum_{k\ge1, x\in\mathbb{Z}} \mathbb{E}_{0,\eta(0)}\left[r_{\kappa_k} = x, \mathbb{E}_{0,\eta(0)}\left[\kappa_{N+k} \le t < \kappa_{N+k+1}, f\left(\tau_{-r_t}\zeta^{(k)}(t - \kappa_k)\right) | \mathcal{G}_k\right]\right]
\end{aligned} \tag{28}$$

where $\zeta^{(k)}$ is a short notation for $\zeta(\kappa_k+\cdot)$. Note that we have used that $N(\alpha_2-\alpha_1) > \ell(f)$. By Proposition 4, this quantity is equal to

$$\sum_{k\geq 1, x\in\mathbb{Z}} \int_0^t \mathbb{P}_{0,\eta(0)}[r_s = x, \kappa_k \in ds]$$
$$\times \mathbb{E}_{a\delta_0}\left[\kappa_N \leq t-s < \kappa_{N+1}, f\left(\tau_{-r_{t-s}}\eta(t-s)\right) \mid U=\infty\right]$$
$$\overset{u=t-s}{=} \sum_{k\geq 1, x\in\mathbb{Z}} \int_0^t \mathbb{P}_{0,\eta(0)}[r_u = x, t-\kappa_k \in du]$$
$$\times \mathbb{E}_{a\delta_0}\left[\kappa_N \leq u < \kappa_{N+1}, f\left(\tau_{-r_u}\eta(u)\right) \mid U=\infty\right]$$
$$= \int_0^t \mathcal{N}_t(du) F_f(u) \qquad (29)$$

where

$$\mathcal{N}_t([0,u]) = \sum_{k\geq 1} \mathbb{P}_{0,\eta(0)}[\kappa_k \in [t-u,t]]$$

and

$$F_f(u) = \mathbb{E}_{a\delta_0}\left[\kappa_N \leq u < \kappa_{N+1}, f\left(\tau_{-r_u}\eta(u)\right) \mid U=\infty\right].$$

We will use the following renewal theorem (Theorem 6.2 in [15]). To state the theorem we say that a random walk

$$S_n = S_0 + X_1 + \cdots + X_n, \qquad n = 0, 1, 2, \ldots$$

i.e. $X_1, X_2, \ldots$ are i.i.d. and independent of S_0, is a renewal process if S_0 is non-negative and X_k are strictly positive. We say it has *spread out* step-lengths if there exists an $r \geq 1$ and a nonnegative measurable function m such that $\int_{\mathbb{R}} m(x)dx > 0$ and

$$P(X_1 + \cdots + X_r \in A) \geq \int_A m(x)dx,$$

for all Borel sets B.

Theorem 4. *(Renewal theorem). Let S be a renewal process with spread out step lengths and $E[X_1] < \infty$. For Borel sets B, let*

$$N(B) = \sum_{k=0}^{\infty} \mathbf{1}_{\{S_k \in B\}}.$$

Then for each $h \in [0,\infty)$,

$$E[N(t+B)] \to |B|/E[X_1]$$

uniformly over Borel sets $B \subset [0,h]$. Here $|B|$ is the Lebesgue measure of B.

One can check the spread-out assumption in Theorem 4 as follows: With T_L the time of L-th jump for the particle with label $(0,1)$ first jumps, A the event that all these L jumps are to the right, B the event that no other particle moves between times 0 and 1, we have for $0 < s < t < 1$,

$$\begin{aligned}\mathbb{P}_{0,\eta(0)}[\kappa_2 - \kappa_1 \in (s,t]] &= \mathbb{P}_{a\delta_0}[\kappa_1 \in (s,t] \mid U = \infty] \\ &\geq \frac{\mathbb{P}_{a\delta_0}[T_L \in (s,t], A, B, U \circ \theta_1 = \infty, V \circ \theta_1 = \infty]}{\mathbb{P}_{a\delta_0}[U = \infty]} \\ &= C \int_s^t f_L(u)du, \end{aligned} \tag{30}$$

with f_L the L-fold convolution of the exponential density with rate 2 and C is a constant that we can check using independence satisfies $C > 0$. This shows that $\kappa_2 - \kappa_1$ is spread-out.

Hence from the renewal theorem,

$$\mathcal{N}_t(B) \to |B| / \mathbb{E}_{a\delta_0}[\kappa_1 \mid U = \infty] \quad \text{as } t \to \infty \tag{31}$$

uniformly over Borel sets B in any finite interval.

Since $F_f(u)$ is bounded and measurable, we have from (29)

$$\int_{\tilde{\Omega}} f d\tilde{p}_t \to \int_{\Omega_0} f d\tilde{p}_\infty$$

Because the process is Feller (Proposition 1), any limit measure is invariant. □

5. Expectations and variances of the regeneration times

5.1. Bounds on W.

Lemma 2. *Let $\{X_t : t \geq 0\}$ be a simple symmetric continuous time rate 2 random walk on $\mathbb{Z}$, such that $X_0 = x$. Let $M_t := x + \sup_{0 \leq s \leq t} |X_s - x|$. Then, for $t \geq 0$,*

$$E\left[e^{\theta M_t}\right] \leq 3 e^{\theta x + 2(\cosh\theta - 1)t},$$

where E is the expectation defined by the law of the random walk.

Proof. The reflection principle tells us that for every integer $n \geq 0$, $P[M_t \geq n] = 2P[X_t > n] + P[X_t = n]$. Hence we have, $P[M_t = n] \leq P[X_t = n] + 2P[X_t = n+1]$. Therefore, $E[e^{\theta M_t}] \leq E[e^{\theta X_t}] + 2e^{-\theta} E[e^{\theta X_t}] \leq 3E[e^{\theta X_t}]$. Finally remark that $E[e^{\theta X_t}] = e^{\theta x + 2(\cosh\theta - 1)t}$.

□

We will in several occasions consider the random process,

$$M_{x,i}(t) := Z_{x,i}(0) + \sup_{0 \leq s \leq t} |Z_{x,i}(s) - Z_{x,i}(0)|,$$

defined for each $(x,i) \in \mathcal{I}$. Furthermore, we will need to define for each initial condition $(r_0, \mathcal{I}(0), \mathcal{Z}(0))$ compatible with a particle count $\eta(0)$ and each $z \leq r_0 - 1$ the quantity,

$$\psi_z(t, r_0, \eta(0)) := \sum_{(x,i) \in \mathcal{I}(0), x \leq z} e^{\theta(M_{x,i}(t) - r_t)}.$$

Usually we will drop the argument, writing $\psi(t)$ instead of $\psi(t, r_0, \eta(0))$. Let us also note that since,

$$\phi_z(t, r_0, \eta(0)) := \sum_{(x,i)\in\mathcal{I}(0), x\le z} e^{\theta(Z_{x,i}(t)-r_t)}.$$

it is true that,

$$\phi_z(t) \le \psi_z(t), \tag{32}$$

for every $t \ge 0$ and $z \le r_0 - 1$. Due to condition (6), and the intermediate value theorem it is true that,

$$\mu := \theta\alpha_1 - 2(\cosh\theta - 1) > 0. \tag{33}$$

This enables us to obtain the following exponential bound.

Lemma 3. *For all initial conditions (r, η) such that $\phi_{r-L}(0, r, \eta) < \infty$ and $t \ge 0$ we have that,*

$$\mathbb{P}_{r,\eta}[t < W < \infty] \le C\phi_{r-L}(0, r, \eta)\exp\{-\mu t\},$$

where $C = 3e^\theta e^{2(\cosh\theta-1)}\frac{e^\mu}{1-e^{-\mu}}$.

Proof. Without loss of generality we assume $r = 0$. Let us first note that,

$$\mathbb{P}_w[t < W < \infty] \le \mathbb{P}_w\left[\cup_{s\ge t}\left\{\phi_{-L}(s) \ge e^{\theta(\lfloor\alpha_1 s\rfloor - r_s)}\right\}\right].$$

From inequality (32) and the fact that $M_{x,i}(t)$ is nondecreasing in t, it follows using Lemma 2 that,

$$\begin{aligned}
\mathbb{P}_w[t < W < \infty] &\le \sum_{n=[t]}^{\infty}\mathbb{P}_w\left[\sum_{(x,i)\in\mathcal{I}(0),x\le -L} e^{\theta M_{x,i}(n+1)} \ge e^{\theta\lfloor\alpha_1 n\rfloor}\right] \\
&\le 3\sum_{n=[t]}^{\infty} e^{2(\cosh\theta-1)(n+1)-\theta\lfloor\alpha_1 n\rfloor}\sum_{(x,i)\in\mathcal{I}(0),x\le -L} e^{\theta Z_{x,i}(0)} \\
&\le 3\phi_{-L}(0)\sum_{n=[t]}^{\infty} e^{2(\cosh\theta-1)(n+1)-\theta\lfloor\alpha_1 n\rfloor},
\end{aligned} \tag{34}$$

Summing up the last expression over n we finish the proof of the Lemma. □

Define for $t \ge 0$, and $z \le r$,

$$N_z(t) := e^{\theta r_t - 2(\cosh\theta-1)t}\phi_z(t).$$

Lemma 4. *Consider an initial condition $(0, \eta)$ and an integer z such that $z \le 0$ and $\phi_z(0) < \infty$. Then, $\{N_z(t) : t \ge 0\}$ is an $\mathcal{F}_t$-martingale.*

Proof. Let us remark that,

$$N_z(t) = \sum_{(x,i)\in\mathcal{I}(0),x\le z} e^{\theta Z_{x,i}(t)-2(\cosh\theta-1)t}.$$

Now, each one of the terms in the above sum is an $\mathcal{F}_t$-martingale. Furthermore, since $\phi_z(0) < \infty$, the martingales $\sum_{(x,i)\in\mathcal{I}(0),-n\le x\le z} e^{\theta Z_{x,i}(t)-2(\cosh\theta-1)t}$, converge in $L^1(\mathbb{P}_w)$ norm to $N_z(t)$ as $n\to\infty$. Thus, $\{N_z(t): t\ge 0\}$ is an $\mathcal{F}_t$-martingale. □

Lemma 5. *There is a $\delta > 0$ such that for all initial conditions w with particle count η, initial position of the front $r = 0$, such that $\phi_{-L}(0,0,\eta) \le p$,*

$$\mathbb{P}_w[W<\infty] < 1-\delta.$$

Proof. By inequality (33), note that,

$$\mathbb{P}_w[W<\infty] \le \mathbb{E}_w\left[e^{(\theta\alpha_1-2(\cosh\theta-1))W}1(W<\infty)\right]. \tag{35}$$

Now, from the definition of the exponential density norm and of the stopping time W, the a.s. right-continuity of the trajectories of the random walks, and Fatou's Lemma, it follows that $e^{\theta(\lfloor\alpha_1 W\rfloor - r_W)} \le \phi_{-L}(W)$. Hence, from inequality (35) we conclude that $\mathbb{P}_w[W<\infty]$ is bounded by,

$$e^\theta\mathbb{E}_w\left[e^{\theta r_W-2(\cosh\theta-1)W}\phi_{-L}(W)1(W<\infty)\right] = e^\theta\mathbb{E}_w[N_{-L}(W)1(W<\infty)]. \tag{36}$$

Now, note that $\mathbb{E}[N_{-L}(W)1(W<n)] \le \mathbb{E}_w[N_{-L}(n\wedge W)]$. Thus, by the optional stopping theorem and Fatou's Lemma,

$$\mathbb{E}_w[N_{-L}(W)1(W<\infty)] \le \lim_{n\to\infty}\mathbb{E}_w[N_{-L}(n\wedge W)] = N_{-L}(0) \le p.$$

This and the condition $pe^\theta < 1$, shows that $\mathbb{P}_w[W<\infty] < 1$. □

5.2. **Bounds on V.**

Lemma 6. *There is a C, $0 < C < \infty$, such that for all initial conditions w and all $t \ge 0$*

$$\mathbb{P}_w[t<V<\infty] \le C\exp\{-tC\}.$$

Proof. Without loss of generality we assume that initially $r = 0$. Note that the probability $P_w[t<V<\infty]$ is bounded by the probability that one of the random walks born at a site between $-L$ and -1 is at the right of $\lfloor\alpha_1 s\rfloor$ at some time $s \ge t$. Now this probability is bounded by the worst case in which initially all these random walks, aL, are at site 0. But this has probability,

$$aLP[t<\tau<\infty],$$

where $\tau := \inf\{t\ge 0 : X_t > \lfloor\alpha_1 t\rfloor\}$, $\{X_t : t\ge 0\}$ is a continuous time simple symmetric random walk on $\mathbb{Z}$, of total jump rate 2, starting from 0, and P is its law. It is easy to prove that this probability is bounded by $C \exp\{-Ct\}$ for some constant $C < \infty$ (for example, see Lemma 8 of [5]). □

Lemma 7. *There is a $\delta > 0$ such that for all initial conditions w,*

$$\mathbb{P}_w [V < \infty] < 1 - \delta.$$

Proof. Without loss of generality we can assume that $r = 0$. Note that the probability $\mathbb{P}_w[V < \infty]$ is upper bounded by the probability that a random walk within a group of aL independent ones all initially at site $x = 0$, at some time $t \geq 0$ is at the right of $\lfloor \alpha_1 t \rfloor$. But this probability is $1 - \gamma^{aL}$, where γ is the probability that a single random walk starting form $x = 0$ never is at the right of the curve $\{\lfloor \alpha_1 t \rfloor : t \geq 0\}$. By Lemma 8 of [5] we know that $\gamma < 1$. □

5.3. **Bounds on U.** The following two lemmas can be proved observing that at each instant of time $t \geq \nu_j$, with $j \geq M + 1$, the auxiliary process has at least $M \geq 20$ particles behind the front (see also [5]).

Lemma 8. *There is a constant C, $0 < C < \infty$, such that for all initial conditions w with particles $(r, 1), \ldots, (r, a)$ at the rightmost site r, and all $t > 0$*

$$\mathbb{P}_w [t < U < \infty] \leq Ct^{-M/2}.$$

Lemma 9. *There is a $\delta > 0$ such that for all initial conditions w with particles $(r, 1), \ldots, (r, a)$ at the rightmost site r,*

$$\mathbb{P}_w [U < \infty] < 1 - \delta.$$

5.4. **Bounds on D.** The following lemma is elementary.

Lemma 10. *There is a constant C, $0 < C < \infty$, such that for every $t > 0$*

$$\mathbb{P}_{a\delta_0}[\nu_1 > t] \leq Ct^{-a/2},$$

while for every $j \geq M + 1$ and $t > 0$

$$\mathbb{P}_{a\delta_0}[\nu_j > t] \leq Ct^{-M/2},$$

so that $\mathbb{E}_{a\delta_0}[\nu_j^{M/2}] < \infty$.

From here we obtain the following estimate.

Lemma 11. *Let β be such that $0 < \beta < \alpha$. Then there is a constant C, $0 < C < \infty$, such that the following statements are true.*

a) *Assume that η has at least a particles at 0. Then,*

$$\mathbb{P}_{0,\eta} [T_n > n/\beta] \leq Cn^{-a/2}.$$

b) *Assume that η is such that $m_{-L,0}(0) \geq aL/2$. Then,*

$$\mathbb{P}_{0,\eta} [T_n > n/\beta] \leq Cn^{-M/4}.$$

c) *Assume that η is a configuration with at least one particle. Then, for all $k \geq M$ we have,*

$$\mathbb{P}_{0,\eta}\left[T_{n+k} - T_k > n/\beta\right] \leq Cn^{-M/4}.$$

Proof. Let us prove part (a). First remark that $\mathbb{P}_{0,\eta}[T_n > n/\beta] \leq \mathbb{P}_{a\delta_0}[T_n > n/\beta]$. Now, $T_n = \sum_{i=1}^{n} \rho_i$. Hence, by Lemma 1 we have $T_n \leq \sum_{j=1}^{n} \nu_j$. Therefore, $\mathbb{P}_{a\delta_0}\left[T_n > n/\beta\right] \leq \mathbb{P}_{a\delta_0}\left[\sum_{i=1}^{n} \nu_i > n/\beta\right]$.

Choose now β' such that $\beta < \beta' < \alpha$. Then since $1/\beta = (1/\beta - 1/\beta') + 1/\beta'$ and ν_1 is stochastically larger than ν_j for $j \geq 2$, we have for $n \geq M+1$,

$$\mathbb{P}_{a\delta_0}\left[T_n > n/\beta\right] \leq M\mathbb{P}_{a\delta_0}\left[\nu_1 > \frac{n}{M}\left(\frac{1}{\beta} - \frac{1}{\beta'}\right)\right] + \mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n} \nu_i > \frac{1}{\beta'}\right]. \quad (37)$$

But, $\mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n} \nu_i > \frac{1}{\beta'}\right] \leq \mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n} \gamma_i > c\right]$, where $\gamma_j := \nu_j - 1/\alpha$ and $c := \frac{1}{\beta'} - \frac{1}{\alpha} > 0$. On the other hand, for each $0 \leq i < l$, $l = \lfloor (M+1)/(a+1) \rfloor$, the random variables $\{\gamma_{kl+i} : k \geq 1\}$ are independent. Thus,

$$\mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{i=M+1}^{n} \nu_i > \frac{1}{\beta'}\right] \leq \sum_{i=0}^{l-1} \mathbb{P}_{a\delta_0}\left[\frac{1}{n}\sum_{k:(M+1-i)/l \leq k \leq n} \gamma_{kl+i} > (a+1)c/M\right].$$

Now, for $q \geq 2$, if $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean zero, and if $E[|X_i|^q] < \infty$, then $E[|\sum_{i=1}^{n} X_i|^q] \leq Cn^{q/2}$ for some $C < \infty$ (see item 16, page 60 of [10]). Hence, since by Lemma 10 we have $\mathbb{E}_{a\delta_0}[\gamma_j^{M/2}] < \infty$, it follows that the last expression of the above display is bounded by, $Cn^{-M/4}$, for some other constant $C < \infty$. Finally observe that $M \geq 2(a+1)$, and use again Lemma 10 to bound the first term of inequality (37) to finish the proof. The proofs of parts (b) and (c) are similar using the inequality (5) satisfied by the parameters M and L. □

Let us now obtain the estimates for the stopping time D. From lemmas 3, 6 and 8 we obtain

Corollary 2. *There is a constant $C = C(p)$, $0 < C < \infty$, such that for all initial conditions w with $\phi_{-L}(0, w) \leq p$, and with particles $(r, 1)\ldots,(r, a)$ at the rightmost visited site r, and for all $t > 0$,*

$$\mathbb{P}_w\left[t < D < \infty\right] \leq Ct^{-M/2}.$$

We also have the following lemma.

Lemma 12. *There is a $\delta > 0$ such that, for all initial conditions w with particle count η and initial position of the front $r = 0$ such that $\phi_y(0, 0, \eta) \leq p$ and with particles $(0, 1), \ldots, (0, a)$ at 0,*

$$\mathbb{P}_w\left[D<\infty\right]<1-\delta.$$

Proof. Since W, V and U are independent,

$$\mathbb{P}_w\left[D<\infty\right]=1-\mathbb{P}_w\left[W=\infty\right]\mathbb{P}_w\left[V=\infty\right]\mathbb{P}_w\left[U=\infty\right]. \tag{38}$$

Applying lemmas 5, 7 and 9, we end up the proof. □

We finish this subsection with three lemmas and a corollary which will be subsequently used to obtain estimates for the stopping time S. The following lemma will be proved in Section 6.

Lemma 13. *There are constants C and γ_0, $0<C<\infty$ and $\gamma_0>0$, such that for all $w\in\tilde{\mathbb{S}}_\theta$, $\gamma\geq\gamma_0$ and $t\geq 0$,*

$$\mathbb{P}_w[r_t\geq\gamma t]\leq\phi_0(0,w)e^{-Ct}. \tag{39}$$

Lemma 14. *There is a constant $C=C(p)$, $0<C<\infty$, such that for all initial conditions w such that $\phi_{-L}(0,w)\leq p$, and $t>0$,*

$$\mathbb{P}_w\left[r_D>t,D<\infty\right]\leq Ct^{-M/2}.$$

Proof. Note that,

$$\begin{aligned}\mathbb{P}_w\left[r_D>\gamma t,D<\infty\right]&\leq\mathbb{P}_w\left[r_D>\gamma t,D\leq t\right]+\mathbb{P}_w\left[t<D<\infty\right]\\&\leq\mathbb{P}_w\left[r_t>\gamma t\right]+\mathbb{P}_w\left[t<D<\infty\right].\end{aligned} \tag{40}$$

The statement now follows from (39) of Lemma 13, the fact that $\phi_0(0,w)\leq\phi_{-L}(0,w)+aL$ and Corollary 2.

□

Lemma 15. *Consider an initial condition w with rightmost visited site $r=0$, at least a particles at 0 and such that $\phi_{-L}(0,w)\leq p$. Then, $\mathbb{P}_w$-a.s. on the event $\{D<\infty\}$ we have,*

$$\phi_{-L}(D)\leq e^{\theta}.$$

Proof. First note that by the assumption $\phi_{-L}(0,w)\leq p<1$, necessarily we have $D>0$. Now, by definition of U, note that whenever $t\leq U<\infty$, we have $\tilde{r}_t\geq\lfloor\alpha_2 t\rfloor$. By Lemma 1 we have $r_t\geq\tilde{r}_t$. It follows that $r_t\geq\lfloor\alpha_2 t\rfloor$. Therefore, if $t\leq U<\infty$, we have

$$\lfloor\alpha_1 t\rfloor-r_t\leq-(\lfloor\alpha_2 t\rfloor-\lfloor\alpha_1 t\rfloor)\leq 0. \tag{41}$$

Therefore, if $D=U$, inequality (41) shows that $\lfloor\alpha_1 D\rfloor-r_D\leq 0$. Hence, since in this case with probability one $D<W$, it follows that,

$$\phi_{-L}(D)\leq e^{\theta(\lfloor\alpha_1 D\rfloor-r_D)}\leq 1. \tag{42}$$

Similarly, if $D = V$, since $D < U$ and $D < W$ happen with probability one, inequality (42) still holds a.s. On the other hand, if $W < \infty$ we have,

$$\phi_{-L}(W) \leq e^{\theta} e^{\theta(\alpha_1 W - r_W)}, \tag{43}$$

since in the worst case scenario, at time W all particles jump one step to the right. Hence, if $D = W$, since with probability one we have $D < U$, by inequality (41), the exponent in the right hand side of (43) is non-positive, so that $\phi_{-L}(W) \leq e^{\theta}$. □

Corollary 3. *There is a constant C, $0 < C < \infty$, such that for all initial condition w with rightmost visited site $r = 0$, such that $\phi_{-L}(0, w) \leq p$ and at least the particles with labels $(0,1), \dots, (0,a)$ at 0,*

$$\mathbb{E}_w\left[\phi_{r_D}(D), D < \infty\right] < C.$$

Proof. Placing ourselves in the worst case scenario were all particles born between sites $-L$ and r_D are at site r_D at time D, we see that $\phi_{r_D}(D) \leq \phi_{-L}(D) + a(L + r_D)$. Hence, by Lemma 15, $\phi_{r_D}(D) \leq e^{\theta} + a(L + r_D)$. Lemma 14 together with the fact that $M \geq 3$ finishes the proof.

□

5.5. **Bounds on S.** We will now perform some key estimates which will let us obtain fast enough decay estimates for the tail probabilities of J in Lemma 20.

Lemma 16. *There exists a constant C, $0 < C < \infty$, such that the following statements are satisfied.*

a) *For all initial conditions $w \in \tilde{\mathbb{S}}_\theta$ with at least a particles at the rightmost visited site, and all $n \geq 1$,*

$$\mathbb{P}_w[m_{0,n}(T_n) < an/2] \leq C \frac{1}{n^{a/2}}.$$

b) *For all initial conditions $w \in \tilde{\mathbb{S}}_\theta$ with at least a particles at the rightmost visited site, and all $n \geq 1$,*

$$\mathbb{P}_w[m_{r_D, r_D+n}(T_{r_D+n}) < an/2] \leq C \frac{1}{n^{aM/(4+2M)}}. \tag{44}$$

c) *For all nontrivial initial conditions $w \in \tilde{\mathbb{S}}_\theta$,*

$$\lim_{n \to \infty} \mathbb{P}_w[m_{-n,L}(T_L) < aL/2] = 0.$$

Proof of part (a). Choose $0 < \beta < \alpha$. Then,

$$\mathbb{P}_w\left[m_{0,n}(T_n) < \frac{an}{2}\right] \leq \mathbb{P}_w\left[m_{0,n}(T_n) < \frac{an}{2}, T_n \leq \frac{1}{\beta} n\right] + \mathbb{P}_w\left[T_n > \frac{1}{\beta} n\right]. \tag{45}$$

Note that the event $\{m_{0,n}(T_n) < an/2, T_n \leq n/\beta\}$ is contained in the event that at least one particle born at any of the sites $\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \dots, n$ hits some site $x \leq 0$ in a time shorter than or equal to n/β. Hence, we can conclude that,

$$\mathbb{P}_w\left[m_{0,n}(T_n) < \frac{an}{2}, T_{0,n} \le \frac{1}{\beta}n\right] \le a(n - \lfloor n/2 \rfloor)P[M'_{n/\beta} \ge n/2], \tag{46}$$

where P is the law of a simple symmetric rate 2 random walk $\{X_t : t \ge 0\}$ on $\mathbb{Z}$ starting from 0 and $M'_t := sup_{0\le s\le t} X_s$. Now, by the reflection principle, $P[M'_t \ge x] \le 2P[X_t \ge x]$. Hence, from inequality (46), we see that $\mathbb{P}_w\left[m_{0,n}(T_n) < an/2, T_{0,n} \le \frac{1}{\beta}n\right]$ is bounded by $a(n+1)P[X_{n/\beta} \ge n/2]$. But, for every $t \ge 0$ and positive integer x, $P[X_t \ge x] \le e^{-2tI(x/(2t))}$, where $I(u) = u\sinh^{-1} u - \sqrt{1+u^2} + 1$. Hence, $a(n+1)P[X_{n/\beta} \ge n/2] \le (a+1)(n+1)\exp\left\{-\frac{2n}{\beta}I(\beta/4)\right\}$. Finally, using the inequality $\mathbb{P}_w[T_n > n/\beta] \le \mathbb{P}_{a\delta_0}[T_n > n/\beta]$, part (a) of Lemma 11 to bound the second term of inequality (45) and using the fact that $(a+1)(n+1)\exp\left\{-\frac{2n}{\beta}I(\beta/4)\right\} \le C/n^{a/2}$ for n large enough, we conclude the proof.

Proof of part (b). By part (a) and Lemma 14, $\mathbb{P}_w[m_{r_D,r_D+n}(T_{r_D+n}) < an/2]$ is upper bounded by,

$$\begin{aligned}\textstyle\sum_{k:1\le k\le n} \mathbb{P}_w[m_{k,k+n}(T_{k+n}) < an/2] + \mathbb{P}_w[r_D > m, D < \infty]\\ \le Cm\frac{1}{n^{a/2}} + C\frac{1}{m^{M/2}},\end{aligned}$$

for some constant $C > 0$ and for every $m \ge 1$. Choosing $m = n^{\frac{a}{2+a}}$ we obtain (44).

Proof of part (c). Note that,

$$\mathbb{P}_w\left[m_{-n,L}(T_L) < \frac{aL}{2}\right] \le \mathbb{P}_w\left[m_{-n,L}(T_L) < \frac{aL}{2}, T_L \le n\right] + \mathbb{P}_w\left[T_L > n\right]. \tag{47}$$

Clearly $\lim_{n\to\infty} \mathbb{P}_w\left[T_L > n\right] = 0$. On the other hand, an argument similar to the one used to derive (46), shows that the first term of the righthand side of (47) is bounded by $aLP[M'_n \ge n]$, which tends to 0 as n tends to ∞. □

Throughout the sequel, to simplify notation, we will define on the event $\{D < \infty\}$ for each $n \ge 1$,

$$F_n := T_{r_D+Ln} - D.$$

Lemma 17. *For every $0 < \beta < \alpha$, there exists a constant $C < \infty$ depending only on β, such that for all initial conditions w with rightmost visited site r, with at least $aL/2$ particles at a distance strictly smaller than L to r, and such that $\phi_{r-L}(0, w) \le p$, and for all natural $n \ge 1$,*

$$\mathbb{P}_w\left[F_n > \frac{1}{\beta}Ln, D < \infty\right] \le C\frac{1}{(nL)^{M/4-1}}.$$

Proof. Without loss of generality we can assume that initially $r = 0$. Note that $\mathbb{P}_{0,\eta}\left[F_n > \frac{1}{\beta}Ln, D < \infty\right]$ is upper-bounded by,

$$\sum_{k:1\le k\le Ln} \mathbb{P}_w\left[F_n > \frac{1}{\beta}Ln, r_D = k, D < \infty\right] + \mathbb{P}_w\left[r_D > Ln, D < \infty\right]. \tag{48}$$

Now, on the event $\{D < \infty\}$ we have that $T_{r_D} \le D$ so that $F_n \le T_{r_D+Ln} - T_{r_D}$. Hence,

$$\mathbb{P}_w\left[F_n > \frac{1}{\beta}Ln, r_D = k, D < \infty\right] \le \mathbb{P}_w\left[T_{k+Ln} - T_k > \frac{1}{\beta}Ln\right].$$

Now, by part (c) Lemma 11, for all $k > M$ we have $\mathbb{P}_w\left[T_{k+Ln} - T_k > \frac{1}{\beta}Ln\right] \le \frac{C}{(nL)^{M/4}}$, for some constant $C < \infty$. On the other hand for $1 \le k \le M$, $\mathbb{P}_w\left[T_{k+Ln} - T_k > \frac{1}{\beta}Ln\right] \le \mathbb{P}_w\left[T_{M+Ln} > \frac{1}{\beta}Ln\right]$. Thus, by part (b) of Lemma 11, since the initial condition w has at least $aL/2$ particles to the right of $r = 0$ at a distance strictly smaller than L to the origin, we know that $\mathbb{P}_w\left[T_{M+Ln} > \frac{1}{\beta}Ln\right] \le \frac{C}{(nL)^{M/4}}$, for some other constant $C < \infty$. We therefore conclude that,

$$\sum_{k:1\le k\le \gamma Ln} \mathbb{P}_w\left[F_n > \frac{1}{\beta}Ln, r_D = k, D < \infty\right] \le C\frac{1}{(nL)^{M/4-1}}. \tag{49}$$

Using Lemma 14 to estimate the second term of display (48) and combining this with inequality (49) we finish the proof. □

Now we will be concerned with proving that given $D_{k-1} < \infty$, the stopping time S_k happens almost surely and has tails that decay fast enough.

Lemma 18. *Let $q \ge 1$ be an integer. Consider a sequence $\{a_k : k \ge 1\}$ of non-negative real numbers such that $\sum_{k=1}^\infty a_k < 1$ and $\sum_{k=1}^\infty k^q a_k < \infty$. Assume that $\{c_m : m \ge 1\}$ is a sequence such that,*

$$c_1 \le a_1, \tag{50}$$

and for every $m \ge 2$ we have that,

$$c_m \le a_m + \sum_{k=1}^{m-1} a_{m-k} c_k. \tag{51}$$

Then,

$$\sum_{k=1}^\infty k^q c_k < \infty.$$

Proof. We will use induction on $0 \le q' \le q$ to prove the lemma. We introduce the notation $A_{q'} := \sum_{k=1}^\infty k^{q'} a_k$ and $C_{q'} := \sum_{k=1}^\infty k^{q'} c_k$. Let us first show that if $A_0 < \infty$ then $C_0 < \infty$. Let $n \ge 2$ be a fixed natural. Summing up inequality (50) with inequalities (51) from $m = 2$ to $m = n$ we see that,

$$\sum_{k=1}^{n-1} c_k(1-\sum_{j=1}^{n-k} a_j)+c_n\leq A_0,$$

Taking the limit when $n\to\infty$ above and using Fatou's Lemma we conclude that,

$$\sum_{k=1}^{\infty} c_k\leq \frac{A_0}{1-A_0}<\infty.$$

Now assume that $C_{q'-1}<\infty$ for some $1\leq q'\leq q$. We will show that then $C_{q'}<\infty$. Summing up inequality (50) with inequalities (51), multiplied by $m^{q'}$, from $m=2$ to $m=n$ we see that,

$$\sum_{m=1}^{n} m^{q'}c_m\leq A_{q'}+\sum_{m=2}^{n}\sum_{k=1}^{m-1} m^{q'}a_{m-k}c_k. \tag{52}$$

Substituting the binomial expansion $m^{q'}=\sum_{i=0}^{q'}\binom{q'}{i}(m-k)^i k^{q'-i}$ on (52) and interchanging the order of the summations on m and on k, we conclude that,

$$\sum_{m=2}^{n-1} m^{q'}c_m\left(1-\sum_{j=1}^{n-m} a_j\right)+n^q c_n\leq A_{q'}+\sum_{i=1}^{q'}\binom{q'}{i}\sum_{k=1}^{n-1} k^{q-1}c_k\sum_{m=1}^{n-k} m^i a_m.$$

Taking the limit when $n\to\infty$ and using Fatou's Lemma, we get

$$C_{q'}\leq\frac{A_{q'}+\sum_{i=1}^{q'}\binom{q'}{i}C_{q'-i}A_i}{1-A_0}<\infty.$$

□

Lemma 19. *Consider an initial condition $w\in\tilde{\mathbb{S}}_\theta$ such that the rightmost visited site associated to w is $r=0$ and at least one particle.*

a) *For every $h>0,s>0$ and $n\geq 1$ we have*

$$\mathbb{P}_w\left[\psi_0(T_n)>h,T_n<s\right]\leq 3\frac{\psi_0(0,w)}{h}e^{2(\cosh\theta-1)s-\theta n}. \tag{53}$$

b) *For every $h>0,s>0$, $k\geq 1$ and $n\geq k$ we have*

$$\mathbb{P}_w\left[\psi_k(T_n)-\psi_{k-L}(T_n)>h,T_n-T_k<s|\,\mathcal{F}_{T_k}\right]\leq 3\frac{aL}{h}e^{2(\cosh\theta-1)s-\theta(n-k)}. \tag{54}$$

Proof. Note that of the event $T_n<s$, it is true that $e^{\theta M_{x,i}(T_n)}\leq e^{\theta M_{x,i}(s)}$. Therefore, since $\psi_0(T_n)=e^{-\theta n}\sum_{(x,i),x\leq 0}e^{\theta M_{x,i}(T_n)}$, we have

$$\psi_0(T_n)\leq e^{-\theta n}\sum_{(x,i),x\leq 0}e^{\theta M_{x,i}(s)}.$$

Now, by Lemma 2 we have that $\mathbb{E}_w\left[\sum_{(x,i),x\le 0} e^{\theta M_{x,i}(s)}\right] \le 3\psi_0(0,w)e^{2(\cosh\theta-1)s}$. Hence,

$$\mathbb{E}_w\left[\psi_0(T_n)\right] \le 3\psi_0(0,w)e^{2(\cosh\theta-1)s-\theta n}.$$

Using Tchebyshev's inequality we obtain (53). A similar argument, using the fact that $\psi_k(T_k)-\psi_{k-L}(T_k)\le aL$ proves (54). □

We end up this section with the following result providing a tail estimate for the law of J_{r_D} (with J_x for x integer, defined in (9)). An important idea in the proof is that essentially, the event that the exponential norm ϕ_{nL} is larger than p, is contained on the event that some of the exponential norms ϕ_{jL} are larger than $p/2^j$ for some $0\le j\le n-1$.

Lemma 20. *Assume that M and L satisfy (2) and (5), θ,α_1,α_2 (6) and p satisfies (7). Then, there is a constant C, $0<C<\infty$, and an integer L_0 such that if $L\ge L_0$, the following statements are satisfied.*

a) *Consider an initial condition $w\in\tilde{\mathbb{S}}_\theta$ with rightmost visited site $r=0$, such that the number of live particles at 0 is a, and such that $m_{-L,0}(0)\ge aL/2$. Then, for every $t>0$,*

$$\mathbb{P}_w[J_{r_D}>t, D<\infty] < Ct^{3-M/4}.$$

b) *Consider an initial condition $w\in\tilde{\mathbb{S}}_\theta$ with rightmost visited site $r=0$. Then, for every $t>0$,*

$$\mathbb{P}_w[J_0>t, U=\infty] < Ct^{3-M/4}.$$

Furthermore, for every nontrivial initial condition $w\in\tilde{\mathbb{S}}_\theta$,

$$J_0<\infty, \qquad \mathbb{P}_w-\text{a.s.} \tag{55}$$

Proof of part (a). Call $F_i' := T_{r_D+iL} = F_i + D$ for $i\ge 1$. For $n=1,2,\ldots$,

$$\mathbb{P}[J>n, D<\infty] \le \mathbb{P}\left[B_n, D<\infty\right], \tag{56}$$

where we have dropped the subscripts on $\mathbb{P}_w$ and J_{r_D} and defined

$$B_n := \cap_{i=1}^n\left\{\psi_{r_D+(i-1)L}(F_i', w_D) > p\right\}\cup B_i',$$
$$B_i' := \left\{m_{r_D+(i-1)L, r_D+iL}(F_i') < aL/2\right\}.$$

We have used here that $\phi_z(t)\le\psi_z(t)$ (see (32)). By the strong Markov property, part (a) of Lemma 19, and translation invariance, we see that for $\lambda>0$ and $n\ge 1$,

$$\begin{aligned}&\mathbb{P}\left[\psi_{r_D}(F_n', w_D) > \lambda, F_n < \tfrac{nL}{\alpha_1}, D<\infty\right]\\ &= \mathbb{E}_w\left[\mathbb{P}_{\tau_{-r_D}w_D}\left[\psi_0(T_{nL})>\lambda, T_{nL}<\tfrac{nL}{\alpha_1}\right]1(D<\infty)\right]\\ &\le \frac{C_1}{\lambda}e^{-\frac{nL}{\alpha_1}(\alpha_1\theta-2(\cosh\theta-1))}\end{aligned} \tag{57}$$

where $C_1 := 3\sup_w \mathbb{E}_w[\phi_{r_D}(D), D < \infty] < \infty$ by Corollary 3, the supremum being taken over all $w \in \tilde{\mathbb{S}}_\theta$ satisfying the conditions described, and we have used the fact that $\psi_{r_D}(D, r_D, \eta(D)) = \phi_{r_D}(D)$. Using (57) for $n = 1$, with $\lambda = p$, and Lemma 17 we see that,

$$\mathbb{P}[\psi_{r_D}(T_{r_D+L}) > p, D < \infty] \le \frac{C_1}{p} e^{-\frac{L}{\alpha_1}(\alpha_1\theta - 2(\cosh\theta - 1))} + \frac{C}{L^{M/4-1}}, \tag{58}$$

for some constant $C > 0$. Therefore, from (58) and part (b) of Lemma 16, we have that,

$$\begin{aligned}\mathbb{P}[B_1, D < \infty] &\le \mathbb{P}[\psi_{r_D}(T_{r_D+L}) > p, D < \infty] + \mathbb{P}[m_{r_D, r_D+L}(T_{r_D+L}) < aL/2] \\ &\le \tfrac{C_1}{p} e^{-\frac{L}{\alpha_1}(\alpha_1\theta - 2(\cosh\theta-1))} + \tfrac{C}{L^{M/4-1}} + \tfrac{C}{L^{aM/(4+2M)}},\end{aligned} \tag{59}$$

for some constant $C > 0$. Let us examine now the terms with $n \ge 2$ in (56). Note that in this case, $\psi_{r_D+(n-1)L} = \psi_{r_D} + \sum_{k=1}^{n-1}\Delta_k$ where

$$\Delta_k := \psi_{r_D+kL} - \psi_{r_D+(k-1)L}.$$

Since $\frac{1}{2^{n-1}} + \sum_{k=1}^{n-1}\frac{1}{2^{n-k}} = 1$, we have,

$$\left\{\psi_{r_D+(n-1)L} > p\right\} \subset \left\{\psi_{r_D} > p/2^{n-1}\right\} \cup \left[\cup_{k=1}^{n-1}\left\{\Delta_k > p/2^{n-k}\right\}\right]. \tag{60}$$

Let

$$A_0^n := \left\{\psi_{r_D}(F_n') > p/2^{n-1}\right\}, \quad A_k^n := \left\{\Delta_k(F_n') > p/2^{n-k}\right\},$$

for $1 \le k \le n-1$. From (60), for $n \ge 2$,

$$B_n \subset B_{n-1} \cap \left(B_n' \cup A_0^n \cup A_1^n \cup \cdots \cup A_{n-1}^n\right).$$

So for $n \ge 2$,

$$\mathbb{P}[B_n, D < \infty] \quad \le \quad \sum_{k=0}^{n-1}\mathbb{P}[A_k^n, B_{n-1}, D < \infty] + \mathbb{P}\left[B_n', B_{n-1}, D < \infty\right]. \tag{61}$$

By the strong Markov property, we have for any $\lambda \in \mathbb{R}$ and $1 \le k \le n-1$,

$$\mathbb{P}\left[F_n' - F_k' \ge \lambda,\ D < \infty \middle| \mathcal{F}_{F_{k-1}'}\right] \le \mathbb{P}_{w_{F_{k-1}'}}\left[T_{(n-k+1)L} - T_L \ge \lambda\right]. \tag{62}$$

Hence, by part (c) of Lemma 11 and the fact that $\alpha_1 < \alpha$, we have

$$\mathbb{P}\left[F_n' - F_k' \ge (n-k)L/\alpha_1 \middle| \mathcal{F}_{F_{k-1}'}\right] \le C((n-k)L)^{-M/4}.$$

By the strong Markov property again, and part (b) of Lemma 19,

$$\begin{aligned}&\mathbb{P}\left[\Delta_k(F_n') > p/2^{n-k}, F_n' - F_k' < (n-k)L/\alpha_1 \middle| \mathcal{F}_{F_{k-1}'}\right] \\ &\qquad \le 3\tfrac{aL}{p}2^{n-k}e^{-\frac{(n-k)L}{\alpha_1}(\alpha_1\theta - 2(\cosh\theta-1))}.\end{aligned} \tag{63}$$

Therefore, for $n \geq 2$ and $1 \leq k \leq n-1$,

$$\begin{aligned}\mathbb{P}\left[A_k^n|\mathcal{F}_{F'_{k-1}}\right] &\leq \mathbb{P}\left[\Delta_k(F'_n) > \tfrac{p}{2^{n-k}}, F'_n - F'_k < \tfrac{1}{\alpha_1}(n-k)L\,\middle|\,\mathcal{F}_{F'_{k-1}}\right] \\ &\quad + \mathbb{P}\left[F'_n - F'_k \geq \tfrac{1}{\alpha_1}(n-k)L\,\middle|\,\mathcal{F}_{F'_{k-1}}\right] \\ &\leq 3\tfrac{aL}{p}2^{n-k}e^{-\frac{(n-k)L}{\alpha_1}(\alpha_1\theta - 2(\cosh\theta - 1))} + \tfrac{C}{((n-k)L)^{M/4}} \qquad (64)\end{aligned}$$

From inequality (57) with $\lambda = 2^n/p$, Lemma 17 and the assumption that initially $m_{-L,0}(0) \geq aL/2$, we then obtain that for $n \geq 2$,

$$\mathbb{P}\left[A_0^n, D < \infty\right] \leq C_1 \frac{2^n}{p} e^{-\frac{nL}{\alpha_1}(\alpha_1\theta - 2(\cosh\theta - 1))} + \frac{C}{(nL)^{M/4-1}}.$$

Now, for $n \geq 2$, by part (a) of Lemma 16, the strong Markov property, and the fact that there are a particles at the rightmost visited site at time F'_{n-1},

$$\mathbb{P}\left[B'_n\middle|\,\mathcal{F}_{F'_{n-1}}\right] \leq \frac{C}{L^{a/2}}. \qquad (65)$$

Define a sequence

$$a_1 := 3\frac{C}{L^{aM/(4+2M)}}, \qquad (66)$$

$$a_n := \frac{4C}{((n-1)L)^{M/4-1}} \qquad \text{for } n \geq 2. \qquad (67)$$

Now note that there is a $L_0 \geq C_1$, such that if $L \geq L_0$, for $n = 1, 2, \ldots$, we have that

$$(3aL + C_1)\frac{2^n}{p} e^{-\frac{nL}{\alpha_1}(\alpha_1\theta - 2(\cosh\theta - 1))} \leq C(nL)^{1-M/4} \leq a_n/4. \qquad (68)$$

(which is possible by inequality (33)) and that,

$$\sum_{n=1}^{\infty} a_n < 1.$$

Let us now define $c_n := \mathbb{P}\left[B_n, D < \infty\right]$ for $n \geq 1$. We want to prove that the sequence $\{c_n : n \geq 1\}$ satisfies

$$c_1 \leq a_1 \qquad (69)$$

$$c_n \leq a_n + \sum_{k=1}^{n-1} a_{n-k}c_k \qquad n \geq 2. \qquad (70)$$

From (58), (68) and the fact that $1/L^{M/4-1} \leq 1/L^{aM/(4+2M)}$ (which follows from (2)) note that (69) is satisfied. Now note that by inequality (65), whenever $L \geq L_0$, for $n \geq 2$ we have that

$$\mathbb{P}[B'_n, B_{n-1}, D < \infty] \leq \frac{C}{L^{a/2}}\mathbb{P}[B_{n-1}, D < \infty] \leq a_1\mathbb{P}[B_{n-1}, D < \infty]. \qquad (71)$$

Inequality (65) and condition (68) imply $\mathbb{P}\left[A_0^n, D<\infty\right] \leq a_n/2$ and $\mathbb{P}\left[A_1^n, D<\infty\right] \leq a_n/2$. Hence, for $n \geq 2$,

$$\mathbb{P}[A_0^n, B_{n-1}, D<\infty] + \mathbb{P}[A_1^n, B_{n-1}, D<\infty] \leq a_n. \tag{72}$$

Similarly for $2 \leq k \leq n-2$ we have $\mathbb{P}\left[A_k^n | \mathcal{F}_{F'_{k-1}}\right] \leq a_{n-k+1}$. Thus, since $B_{n-1} \subset B_{k-1}$ for $2 \leq k \leq n-2$,

$$\mathbb{P}[A_k^n, B_{n-1}, D<\infty] \leq \mathbb{P}[A_k^n, B_{k-1}, D<\infty] \leq a_{n-k+1}\mathbb{P}[B_{k-1}, D<\infty]. \tag{73}$$

Also, by inequality (64) and condition (68) for $n \geq 2$, we have $\mathbb{P}\left[A_{n-1}^n | \mathcal{F}_{F'_{n-2}}\right] \leq \frac{a_2}{2}$. Thus, since $B_{n-1} \subset B_{n-2}$, for $n \geq 3$,

$$\mathbb{P}[A_{n-1}^n, B_{n-1}, D<\infty] \leq \mathbb{P}[A_{n-1}^n, B_{n-2}, D<\infty] \leq a_2\mathbb{P}[B_{n-2}, D<\infty]. \tag{74}$$

For $n=2$, (70) now follows after substituting estimates (71) and (72) in inequality (61), for $n=3$ after substituting (71), (72) and (74) in inequality (61) while for $n \geq 4$ it follows after substituting (71), (72), (73) and (74) in inequality (61).

But remark that,

$$\sum_{n=1}^{\infty} n^{M/4-3} a_n < \infty. \tag{75}$$

Hence, by Lemma 18, (69), (70) and inequality (56) we conclude that,

$$\sum_{n=1}^{\infty} n^{M/4-3}\mathbb{P}\left[J_{r_D} > n, D<\infty\right] < \infty.$$

This implies that $\limsup_{n\to\infty} n^{M/4-3}\mathbb{P}\left[J_{r_D} > n, D<\infty\right] = 0$. Thus, there exists a constant $C > 0$, such that for every $n \geq 1$ it is true that $\mathbb{P}\left[J_{r_D} > n, D<\infty\right] \leq C/n^{M/4-3}$. This together with the monotonicity in t of the expression $\mathbb{P}\left[J_{r_D} > t, D<\infty\right]$, finishes the proof.

Proof of part (b). This time, in analogy with (56), note that for $n = 1, 2, \ldots$,

$$\mathbb{P}[J_0 > n, U=\infty] \leq \mathbb{P}\left[B_n, U=\infty\right], \tag{76}$$

where again we have dropped the subscript on $\mathbb{P}_w$ but now

$$B_n := \cap_{i=1}^n \left\{\psi_{(i-1)L}(T_{iL}, w) > p\right\} \cup B'_i,$$
$$B'_i := \left\{m_{(i-1)L,iL}(T_{iL}) < aL/2\right\}.$$

An analysis similar to that of part (a) proves (69), (70) and (75) with $c_n := \mathbb{P}\left[B_n, U=\infty\right]$ for $n \geq 1$ and $\{a_n : n \geq 1\}$ as in (66) and (67). Part (b) now follows by Lemma 18 as in part (a).

Proof of (55). Note that $\{J_0=\infty\} \subset \{J_L=\infty\}$. Now, for every $n \geq 1$,

$$\mathbb{P}_w[J_L=\infty] \leq \mathbb{P}_w[J_L=\infty, m_{-n,L}(T_L) \geq aL/2] + \mathbb{P}_w[m_{-n,L}(T_L) < aL/2].$$

Now, following the proof of part (a), it is possible to show that $\mathbb{P}_w[J_L > t, m_{-n,L}(T_L) \geq aL/2] \leq Ct^{3-M/4}$, for some constant $C > 0$. Hence, for every $n \geq 1$,

$$\mathbb{P}_w[J_L = \infty] \leq \mathbb{P}_w[m_{-n,L}(T_L) < aL/2].$$

Taking the limit as $n \to \infty$ and using part (c) of Lemma 16 we finish the proof. □

Corollary 4. *For every nontrivial initial condition $w \in \tilde{\mathbb{S}}_\theta$, it is true that,*

$$S_1 < \infty \qquad \mathbb{P}_w - \text{a.s.}, \tag{77}$$

and for every $k \geq 2$,

$$\mathbb{P}_w[D_{k-1} < \infty, S_k < \infty] = \mathbb{P}_w[D_{k-1} < \infty]. \tag{78}$$

Proof. Assertion (77) is a consequence of (55) of Lemma 20 and the fact that $\{S_1 < \infty\} = \{J_0 < \infty\}$. Similarly, assertion (78) follows directly from part (a) of Lemma 20 and the fact that $\{D_{k-1} < \infty, S_k < \infty\} = \{D_{k-1} < \infty, J_{r_{D_{k-1}}} < \infty\}$.

□

5.6. **Variance bounds for the regeneration times and positions.** In this subsection we will prove Proposition 2. Let us first prove assertion (11) of Proposition 2. By Corollary 4, note that for every $k \geq 1$,

$$\mathbb{P}_w[\kappa = \infty] \leq \mathbb{P}_w[D_k < \infty].$$

But by the strong Markov property and Lemma 12, the righthand side of the above inequality is bounded by $(1-\delta)^k$. It follows that,

$$\mathbb{P}_w[\kappa = \infty] \leq (1-\delta)^k,$$

for every $k \geq 1$. Taking the limit when k tends to infinity concludes the proof of (11) of Proposition 2.

To prove (12) we will need the following lemma.

Lemma 21. *For every $\epsilon > 0$, there is a constant C, $0 < C < \infty$, such that*

$$\mathbb{P}_{a\delta_0}\left[\kappa > t \middle| U = \infty\right] \leq Ct^{-M/4+3+\epsilon}. \tag{79}$$

proof. Without loss of generality we will assume that initially,

$$r_0 = 0. \tag{80}$$

By the fact that $\kappa < \infty$, a.s., we can write,

$$\mathbb{P}\left[\kappa > t \middle| U = \infty\right] = \sum_{k=1}^{\infty} \mathbb{P}\left[S_k > t, K = k \middle| U = \infty\right],$$

where we have dropped the subscript $a\delta_0$ in $\mathbb{P}_{a\delta_0}$. Applying recursively the strong Markov property to the stopping times $\{S_j : j \geq 1\}$ we see that for every $k \geq 1$,

$$\mathbb{P}\left[S_k > t, K = k | U = \infty \right] \leq (1-\delta)^{k-1},$$

where $\delta > 0$ is given by Lemma 12. Let $0 < \beta < 1/2$. For any $l > 0$ we therefore have,

$$\mathbb{P}\left[\kappa > t | U = \infty \right] \leq \sum_{k=1}^{l} \mathbb{P}\left[t < S_k < \infty | U = \infty \right] + \delta^{-1}(1-\delta)^l. \tag{81}$$

Let $0 < \gamma < 1$ and consider the event,

$$A_k := \{ r_{D_1} - r_{S_1} < t^\gamma, r_{D_2} - r_{S_2} < t^\gamma, \ldots, r_{D_{k-1}} - r_{S_{k-1}} < t^\gamma, S_k < \infty \}$$

On A_k we have, $r_{S_k} \leq kt^\gamma + L \sum_{j=0}^{k-1} J_{r_{D_j}}$, where we adopt the convention $D_0 := 0$, so that $r_{D_0} = 0$ by (80). Since $\tilde{r}_t \leq r_t$, if $U = \infty$, then $r_t \geq \lfloor \alpha_2 t \rfloor$ for all $t > 0$. Therefore, on $A_k \cap \{U = \infty\}$,

$$\lfloor \alpha_2 S_k \rfloor \leq kt^\gamma + L \sum_{j=0}^{k-1} J_{r_{D_j}}.$$

Now define, the event

$$B_k := \{ J_{r_{D_0}} < t^\gamma, J_{r_{D_1}} < t^\gamma, \ldots, J_{r_{D_{k-1}}} < t^\gamma, S_k < \infty \}. \tag{82}$$

Then on $A_k \cap B_k \cap \{U = \infty\}$ we have,

$$\lfloor \alpha_2 S_k \rfloor \leq kt^\gamma (1 + L).$$

Hence for $t > (lt^\gamma(1+L)+1)/\alpha_2$ and $k \leq l$,

$$\mathbb{P}[t < S_k < \infty, A_k, B_k | U = \infty] = 0$$

and therefore,

$$\mathbb{P}[t < S_k < \infty | U = \infty] \leq \mathbb{P}[A_k^c, S_k < \infty | U = \infty] + \mathbb{P}[B_k^c, S_k < \infty | U = \infty]. \tag{83}$$

Using part (a) of Lemma 20 to bound the probability of the event $\{J_{r_0} \geq t^\gamma\} = \{J_0 \geq t^\gamma\}$ and part (b) to bound the probability of the events $\{J_{r_{D_j}} \geq t^\gamma\}$, for $1 \leq j \leq k-1$, we can see that the second term of the righthand side of inequality (83) is bounded by $Ckt^{-\gamma(M/4-3)}$. On the other hand, by Lemma 14, the first term is bounded by $Ckt^{-\gamma M/2}$. Choosing $l = C_1 \log t$ with $C_1 = (M/4-3)\left(\log(1-\delta)^{-1}\right)^{-1}$ and γ close enough to 1 we obtain (79). □

Proof of (12) of Proposition 2. The assertion for κ of (12) follows from Lemma 21 noting that $M \geq 21$ (by condition (2)) and that for r_κ from Lemma 21 and (39).

6. Construction and Feller property

Throughout, $\theta > 0$ is arbitrary, P is the joint law of the independent random walk used to define the process for finite initial conditions and E the corresponding expectation. By our construction note that r_t^ℓ is increasing in ℓ and hence we can define

$$r_t := \lim_{\ell\to\infty} r_t^\ell.$$

We will see that for every $t \geq 0$, a.s. $r_t < \infty$.

Consider $f_\theta(\eta^\ell)$ where

$$f_\theta(\eta) = \sum_x \eta(x)e^{\theta x}.$$

We compute

$$\mathcal{L}f_\theta(\eta^\ell) = \sum_x \eta^\ell(x)e^{\theta x}\left[e^\theta + e^{-\theta} - 2 + ((a+1)e^\theta - 1)1(x=r)\right].$$

Hence if we let $\lambda_{1,\theta} = e^\theta + e^{-\theta} - 2$ and $\lambda_{2,\theta} = (a+1)e^\theta + e^{-\theta} - 2$ then

$$\lambda_{1,\theta} f_\theta \leq \mathcal{L}f_\theta \leq \lambda_{2,\theta} f_\theta.$$

In particular,

$$E[f_\theta(\eta^\ell(t)) \mid \mathcal{F}_0] \leq e^{\lambda_{2,\theta} t} f_\theta(\eta^\ell(0)). \tag{84}$$

In addition, $f_\theta(\eta^\ell(t))$ is a nonnegative sub-martingale and therefore by Doob's inequality,

$$P(\sup_{0\leq s\leq t} f_\theta(\eta^\ell(s)) \geq e^{\gamma\theta t} \mid \mathcal{F}_0) \leq e^{-\gamma\theta t} E[f_\theta(\eta^\ell(t)) \mid \mathcal{F}_0]. \tag{85}$$

Since r_t^ℓ is the rightmost site which has been occupied up to time t we have $\sup_{0\leq s\leq t} f_\theta(\eta^\ell(s)) \geq e^{\theta r_t^\ell}$. Hence from (84) and (85) we have

$$P(r_t^\ell \geq \gamma t \mid \mathcal{F}_0) \leq e^{-c_{\gamma,\theta} t} f_\theta(\eta^\ell(0)) \tag{86}$$

where $c_{\gamma,\theta} = \gamma\theta - \lambda_{2,\theta}$. This proves that for each ℓ and $t \geq 0$, a.s. $r_t^\ell < \infty$ and hence $\lim_{n\to\infty} \tau_n = \infty$. Also, taking the limit when $\ell \to \infty$ in (86), we obtain,

$$P(r_t \geq \gamma t \mid \mathcal{F}_0) \leq e^{-c_{\gamma,\theta} t} f_\theta(\eta(0)) \tag{87}$$

This proves Lemma 13 of Section 5.4. Furthermore, if $(r,\eta) \in \mathbb{S}'_\theta$ then $f_\theta(\eta) < \infty$ so we have $r_t < \infty$ a.s.

Choose now γ large enough so that we have $c_{\gamma,\theta} > 0$. Define for each $y = 1, 2, \ldots$,

$$T_y := \inf\{t \geq 0 : r_t = y\}. \tag{88}$$

We have $r_{T_y\wedge t} = \lim_{\ell\to\infty} r^\ell_{T_y\wedge t}$. Let ℓ_k be the smallest natural number such that $r_{T_k\wedge t} = r^\ell_{T_k\wedge t}$ for all $\ell \geq \ell_k$. Then if $\bar\ell = \max\{\ell_1, \ldots, \ell_{r_t}\}$, the front $r_{\cdot\wedge t}$ generated by the initial condition η^ℓ up to time t, does not depend on ℓ if $\ell \geq \bar\ell$. This means that particles that are initially at any site $x \leq r - \bar\ell$, never visit any site to the right of the front before time t. Using attractiveness, we can then conclude that the sequence $\eta^\ell(s)$ is increasing for $s \leq t$ and $\ell \geq \bar\ell$. Therefore,

$$\eta(t) := \lim_{\ell\to\infty} \eta^\ell(t),$$

exists almost surely. Taking the limit when $\ell \to \infty$ in (84) and using Fatou's Lemma we see that,

$$E[f_\theta(\eta(t)) \mid \mathcal{F}_0] \le e^{\lambda_{2,\theta} t} f_\theta(\eta(0)). \tag{89}$$

Noting that r_t is increasing, this shows that $(r_t, \eta(t))$ stays in $\mathbb{S}'_\theta$. Hence we have shown that $(r_t, \eta(t))$ is a Markov process on $\mathbb{S}'_\theta$.

We next want to show that it satisfies the Feller property. We need some more preliminary estimates. Note that

$$\begin{aligned} &\mathcal{L} f_\theta^2 - 2 f_\theta \mathcal{L} f_\theta \\ &= \sum_x \eta(x) e^{2\theta x} \left[(e^\theta - 1)^2 + (e^{-\theta} - 1)^2 + ((a^2 - 1) e^{2\theta} - 2(a-1) e^\theta) \right] \\ &\le \lambda_{3,\theta} f_{2\theta} \end{aligned} \tag{90}$$

for some $\lambda_{3,\theta} < \infty$. Hence if $M_\theta(t) = f_\theta(\eta^\ell(t)) - \int_0^t \mathcal{L} f_\theta(\eta^\ell(s)) ds$ then $\langle M_\theta(t) \rangle = \int_0^t \left(\mathcal{L} f_\theta^2 - 2 f_\theta \mathcal{L} f_\theta \right) (\eta^\ell(s)) ds$, and then,

$$E[M_\theta^2(t) \mid \mathcal{F}_0] \le 2\lambda_{3,\theta} \int_0^t E[f_{2\theta}(\eta^\ell(s)) \mid \mathcal{F}_0] ds \le 2\lambda_{3,\theta} \lambda_{2,2\theta}^{-1} e^{\lambda_{2,2\theta} t} f_\theta(\eta^\ell(0)).$$

Here we used in the last inequality that $f_{2\theta}(\eta^\ell(0)) \le f_\theta(\eta^\ell(0))$, since there are no particles initially to the right of the origin. In particular, by Chebyshev's inequality,

$$P(f_\theta(\eta^\ell(t)) < e^{\lambda_{1,\theta} t} f_\theta(\eta^\ell(0)) - A \mid \mathcal{F}_0) \le A^{-2} e^{\lambda_{4,\theta} t} f_\theta(\eta^\ell(0)) \tag{91}$$

for some $\lambda_{4,\theta} < \infty$. We can pass to the limit to obtain,

$$P(f_\theta(\eta(t)) < e^{\lambda_{1,\theta} t} f_\theta(\eta(0)) - A \mid \mathcal{F}_0) \le A^{-2} e^{\lambda_{4,\theta} t} f_\theta(\eta(0)). \tag{92}$$

This proves that we have a well-defined Markov process starting from any initial data in $\mathbb{S}'_\theta$. Next we show the process satisfies the Feller property. We start by identifying the compact sets of $\mathbb{S}'_\theta$.

Lemma 22. *The compact sets K in $\mathbb{S}'_\theta$ are those which are closed, bounded in norm $\|(r,\eta)\| = d((r,\eta),(0,0))$, and have* uniform tails *i.e.*

$$\lim_{N\to\infty} \sup_{(r,\eta)\in K} \sum_{x \le r-N} e^{\theta(x-r)} \eta(x) = 0. \tag{93}$$

Proof. Suppose $(r,\eta)_i$, $i = 1, 2, \ldots$ are elements of such a K. Since K is bounded, we can find a weakly convergent subsequence. We have to show they converge in norm as well. Relabeling, we can call the weakly convergent subsequence $(r,\eta)_i \to (r,\eta)$. Also, note that $r_i = r$ for large enough i. So without loss of generality assume $r_i = r$. Let $\epsilon > 0$ and choose N_0 such that

$$\sup_i \textstyle\sum_{x<r-N_0} e^{\theta(x-r)} \eta_i(x) < \epsilon. \tag{94}$$

Now choose i_0 so that for $i \ge i_0$,

$$\textstyle\sum_{r-N_0 \le x \le r} e^{\theta(x-r)} |\eta_i(x) - \eta(x)| < \epsilon. \tag{95}$$

It follows that for $i \ge i_0$,

$$d((r,\eta)_i, (r,\eta)) < 3\epsilon.$$

Since K is closed $(r,\eta) \in K$ and such a set is compact. Suppose on the other hand that a subset K of $\mathbb{S}'_\theta$ is compact. Fix $\epsilon > 0$. Let

$$B_N = \{(r,\eta) : \sum_{x \le r-N} e^{\theta(x-r)}\eta(x) < \epsilon\}.$$

B_N are open sets whose union is $\mathbb{S}'_\theta$. So B_N are an open cover of K. Since K is compact, there is a finite sub-cover, and hence an N such that $K \subset B_N$. In other words, K has uniform tails. □

Lemma 23. *For each $t > 0$, $\epsilon > 0$ and $K \subset \mathbb{S}'_\theta$ compact there exists a compact $K_0 \subset \mathbb{S}'_\theta$ such that*

$$P((\eta(t), r_t) \in K \mid (\eta(0), r_0 = 0) \notin K_0) < \epsilon. \tag{96}$$

Proof. Fix $t > 0$, $\epsilon > 0$ and $K \subset \mathbb{S}'_\theta$ compact. By Lemma 22, K is of the form

$$K = \{(r,\eta) \mid \textstyle\sum_{x \le r-N(m)} e^{\theta(x-r)}\eta(x) \le B/m, \text{ for all } m = 1,2,\ldots\} \tag{97}$$

for some $B < \infty$ and some $N(m) \uparrow \infty$ as $m \to \infty$ with $N(1) = 0$. We have to find a B_0 and $N_0(\cdot)$ so that for each $m_0 = 1,2,\ldots$, if $\sum_{x \le -N_0(m_0)} e^{\theta x}\eta(0,x) > B_0/m_0$ then $P(\sum_{x \le r_t - N(m)} e^{\theta(x-r_t)}\eta(t,x) \le B/m \text{ for all } m = 1,2,\ldots\}) < \epsilon$.

Choose γ large enough such that

$$P(r_t > \gamma t) < \epsilon/2. \tag{98}$$

We start with $m_0 = 1$. Use (92) with $A^2 = f_\theta(\eta(0))e^{\lambda_{4,\theta}t}\epsilon/2$. We can find B_0 so that if $f_\theta(\eta(0)) > B_0$ then $e^{\lambda_{1,\theta}t}f_\theta(\eta(0)) - A > Be^{\gamma t}$ and hence from (92) and (98), if $\sum_{x \le 0} e^{\theta x}\eta(0,x) > B_0$,

$$P(\textstyle\sum_{x \le r_t} e^{\theta(x-r_t)}\eta(t,x) \le B) < \epsilon. \tag{99}$$

Next we consider the case $m_0 > 1$. It is not hard to see that for each N, there exists $A = A(t) < \infty$ such that

$$P(\textstyle\sum_{x \le -N} e^{\theta x}\eta(t,x) \le \frac{1}{2}\sum_{x \le -N-A} e^{\theta x}\eta(0,x)) < \epsilon/2. \tag{100}$$

Indeed, the left hand side of the event in (100) is only smaller if we suppress the branching. If we temporarily denote $x_i(0)$ the initial positions of particles to the left of $-N-A$ then we have continuous time random walks and the event is that $\sum_i e^{\theta x_i(t)} 1(x_i(t) \le -N) \le \frac{1}{2}\sum_i e^{\theta x_i(0)}$. We can assume that $\sum_{x \le 0} e^{\theta x}\eta(0,x) \le B_0$, for otherwise we have (99). Then it is clear that there exists an A such that $P(\cup_i\{x_i(t) > -N\}) < \epsilon/4$. Hence we only need to show that $P(\sum_i e^{\theta x_i(t)} \le \frac{1}{2}\sum_i e^{\theta x_i(0)}) < \epsilon/4$ which is easy to deduce from the fact that $e^{-\lambda_{1,\theta}t}\sum_i e^{\theta x_i(t)}$ are martingales. This proves (100).

From (98) and (100) with $N_0(m) = N(\lfloor BB_0^{-1}2me^{\gamma t}\rfloor + 1) + A$, we have that if $\sum_{x \le -N_0(m)} e^{\theta x}\eta(0,x) > B_0/m$ and $\sum_{x \le 0} e^{\theta x}\eta(0,x) \le B_0$ then

$$P(\textstyle\sum_{x \le r_t - N(m')} e^{\theta(x-r_t)}\eta(t,x) \le B/m') \le \epsilon, \tag{101}$$

for $m' = \lfloor BB_0^{-1}2me^{\gamma t}\rfloor + 1$. This completes the proof. □

Lemma 24. *For each $\epsilon > 0$ and $t > 0$ there exists $\delta > 0$ such that if (r,η) and (r,η') are any two configurations of particles on $\mathbb{S}'_\theta$ with $\sum_{x\leq r} e^{\theta x}|\eta(x)-\eta'(x)| < \delta$, there is stopping time τ and a coupling of two copies $(r_s,\eta(s))$ and $(r'_s,\eta'(s))$ of our Markov process with generator $\mathcal{L}$ for $0 \leq s \leq \tau$ satisfying*

(1) $P(\tau < t) < \epsilon$.
(2) $E[d((r_t,\eta(t)),(r'_t,\eta'(t)))\mathbf{1}_{\tau>t}] < \epsilon$.
(3) $P[r_0 = r'_0 = r, \eta(0) = \eta, \eta'(0) = \eta'] = 1$.

Here P is the coupling measure and E the corresponding expectation.

Proof. Consider the difference $\zeta = \eta - \eta'$. We have $\sum_{x\leq 0} e^{\theta x}|\zeta(x)| < \delta$ so choosing δ sufficiently small we have $\zeta(x) = 0$ for $x \in \{-L,\dots,0\}$ for some large L. We attempt to couple the two processes by moving the particles together whenever possible. Then positive and negative parts of ζ move as independent random walks of positive and negative type, the two types annihilating on contact and the coupling succeeds up to the first time τ when a particle of either type hits r_s. It is easy to choose δ small enough, and therefore L large enough, so that (1) is satisfied. To prove (2), note that up to time τ, $d((r_t,\eta(t)),(r'_t,\eta'(t))) = \sum_{x\leq r_t} e^{\theta(x-r_t)}|\zeta(t,x)|$ and we can get an easy upper bound by using $r_t \geq 0$ and letting $\bar\zeta(t)$ be the process obtained by starting with $|\zeta(0)|$ and using the same random walks, but dropping the signs and the annihilations. Then $\sum_x e^{\theta(x-r_t)}|\zeta(t,x)| \leq \sum_x e^{\theta x}\bar\zeta(t,x)$. (2) follows since $e^{-\lambda_{1,\theta}t}\sum_x e^{\theta x}\bar\zeta(t,x)$ is a martingale. □

Proposition 5. *Let $(r,\eta) \in \mathbb{S}'_\theta$ and $P_{r,\eta}$ the law of the process $\{(r_t,\eta(t) : t \geq 0\}$ with initial condition (r,η) under P. Then, $P_t g(r,\eta) = E_{r,\eta}[g(r_t,\eta(t))]$, $t \geq 0$ form a Feller semi-group on $\mathbb{S}'_\theta$, where $E_{r,\eta}$ is the expectation associated to $P_{r,\eta}$.*

Proof. Suppose that g is continuous and vanishes at infinity and let $\epsilon > 0$. In particular $|g| \leq B < \infty$. There is a compact K such that $|g(r,\eta)| < \epsilon/2$ for (r,η) in the complement of K. By Lemma 23 there is a compact K_0 such that $P((r_t,\eta(t)) \in K \mid (r_0,\eta(0)) \notin K_0) < \epsilon/2B$. So if $(r,\eta) \notin K_0$,

$$P_t g(r,\eta) = E_{r,\eta}[g(r_t,\eta(t))\mathbf{1}_{(r_t,\eta(t))\in K}] + E_{r,\eta}[g(r_t,\eta(t))\mathbf{1}_{(r_t,\eta(t))\notin K}] < \epsilon.$$

This proves that $P_t g$ vanishes at infinity as well.

Next we show that $P_t g$ is continuous. Since g is continuous and vanishes at infinity, it is uniformly continuous. So we can choose $\epsilon_0 > 0$ so that $d((r,\eta),(r',\eta')) < \epsilon_0$ implies $|g(r,\eta) - g(r',\eta')| < \epsilon/3$. By Lemma 24, there exists δ such that if $d((r,\eta),(r',\eta')) < \delta$, then there is a stopping time $\tau \geq 0$ such that $P(d((r_t,\eta(t)),(r'_t,\eta'(t))) > \epsilon_0, \tau > t) < \epsilon/(6B)$ and $P(\tau < t) \leq \epsilon/(3B)$. Hence $|E_{r,\eta}[g(r_t,\eta(t))] - E_{r',\eta'}[g(r'_t,\eta'(t))]| \leq E[|g(r_t,\eta(t)) - g(r'_t,\eta'(t))|\mathbf{1}_{\tau>t}] + 2BP(\tau < t) \leq \epsilon$. This proves that $P_t g$ is continuous as well. □

References

[1] Alves, O.; Machado, F.; Popov, S. (2002). *The shape theorem for the frog model*, Ann. Appl. Probab., **12**, no. 2, 533–546.

[2] Alves, O.; Machado, F.; Popov, S.; Ravishankar, K. (2001). *The shape theorem for the frog model with random initial configuration.* Markov Processes Relat. Fields **7** (4), 525-539.

[3] Bandyopadhyay, A.; Zeitouni, O. *Random Walk in Dynamic Markovian Random Environment*, math.PR/0509066.

[4] Bramson, M.; Calderoni, P.; De Masi, A.; Ferrari, P.; Lebowitz, J.; Schonmann, R. (1986). *Microscopic selection principle for a diffusion-reaction equation*, J. Statist. Phys. **45**, no. 5-6, 905–920.

[5] Comets, F.; Quastel, J.; Ramírez, A.F. *Fluctuations of the front in a stochastic combustion model*, to appear in Ann. Inst. H. Poincaré Probab. Statist.

[6] Kesten, H. (1977). *A renewal theorem for random walk in a random environment.* Proc. Symp. Pure Math. **31**, 67-77.

[7] Kesten, H.; Sidoravicius, V. (2005). *The spread of a rumor or infection in a moving population.* Ann. Probab. **33**, no. 6, 2402–2462.

[8] Mai, J.; Sokolov, I.M.; Kuzovkov, V.N.; Blumen, A. (1997). *Front form and velocity in a one-dimensional auto-catalytic $A+B \to 2A$ reaction*, Phys. Rev. E **56**, 4130-4134.

[9] Panja, D. (2004). *Effects of Fluctuations on Propagating Fronts*, Physics Reports **393**, 87-174.

[10] Petrov, V. (1975). *Sums of independent random variables*, Springer-Verlag.

[11] Ramírez, A. F.; Sidoravicius, V. (2004). *Asymptotic behavior of a stochastic combustion growth process.* J. Eur. Math. Soc. **6**, no. 3, 293–334.

[12] van Saarloos, W. (2003). *Front propagation into unstable states*, Phys. Rep. **386**, 29.

[13] Sznitman, A.S. (2000). *Slowdown estimates and central limit theorem for random walks in random environment*, J. Eur. Math. Soc. **2**, no. 2, 93-143.

[14] Sznitman, A.S.; Zerner, M. (1999). *A law of large numbers for random walks in random environment*, Ann. Probab., **27**, 4, 1851-1869.

[15] Thorisson, H.(2000). *Coupling, stationarity, and regeneration.* Probability and its Applications (New York). Springer-Verlag, New York.

(Francis Comets) Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 7- Denis Diderot, 2, Place Jussieu, F-75 251 Paris Cedex 05, France

E-mail address: `comets@math.jussieu.fr`

(Jeremy Quastel) Departments of Mathematics and Statistics, University of Toronto, 40 St. George Street, Toronto, Ontario M5S 1L2, Canada

E-mail address: `quastel@math.toronto.edu`

(Alejandro F. Ramírez) Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile

E-mail address: `aramirez@mat.puc.cl`