

FRONT PROPAGATION IN AN EXCLUSION ONE-DIMENSIONAL REACTIVE DYNAMICS

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ABSTRACT. We consider an exclusion process representing a reactive dynamics of a pulled front on the integer lattice, describing the dynamics of first class X particles moving as a simple symmetric exclusion process, and static second class Y particles. When an X particle jumps to a site with a Y particle, their position is interchanged and the Y particle becomes an X one. Initially, there is an arbitrary configuration of X particles at sites $\dots, -1, 0$, and Y particles only at sites $1, 2, \dots$, with a product Bernoulli law of parameter $\rho, 0 < \rho < 1$. We prove a law of large numbers and a central limit theorem for the front defined by the right-most visited site of the X particles at time t . These results corroborate Monte-Carlo simulations performed in a similar context. We also prove that the law of the X particles as seen from the front converges to a unique invariant measure. The proofs use regeneration times: we present a direct way to define them within this context.

1. INTRODUCTION

Few mathematical results exist about microscopic models of pulled front propagation representing non-equilibrium pattern formation in chemical reactions, or physical or biological phenomena (see [10] and [11] for a review of the physical literature). One-dimensional interacting particle systems which are microscopic versions of the Fisher-Kolmogorov-Petrovsky-Piscunov equation, representing evolutionary phenomena in genetics, have been studied in [1] and [4]. More recently, in several works (see [7], [9] and [2]), systems of branching interacting random walks on the lattice were studied, which consider the spatial ordering of particles: two types of particles perform independent continuous time simple random walk movement: X particles which jump at rate D_X and Y particles at rate D_Y . Upon contact with an X particle, a Y particle becomes X . In [3] the one-dimensional case where $D_Y = 0$, representing the combustion of a propellant towards a stationary state, was analyzed. There, the particles are symmetric random walks with initially one Y particle at each site $1, 2, \dots$ and an arbitrary configuration of X particles at $-2, -1, 0$ with a finite l_1 norm with a certain exponential weight. If we call r_t the position of the right-most visited site at time t by an X particle, it was proved that a.s. $r_t/t \rightarrow v$, with $v > 0$ and that $\epsilon^{1/2}(r_{\epsilon^{-1}t} - \epsilon^{-1}vt)$ converges to a Brownian motion

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with non-degenerate variance. In [7], for the case of symmetric random walks with $D_X = D_Y$ a shape theorem was proved in arbitrary dimensions. In particular, in dimension $d = 1$ it was proved that if the initial configuration of all the particles is a product Poisson measure with a finite number of X particles, r_t satisfies a strong law of large numbers.

Mai, Sokolov, Kuzovkov and Blumen [8] performed Monte Carlo simulations for a variation of the above described model in which both the X and the Y particles perform symmetric simple exclusion. These numerical computations indicate that in the case $D_X = D_Y$, with an initial condition which is a product Bernoulli measure, the front has a ballistic movement with normal fluctuations.

In this paper we study a process where the X particles perform symmetric simple exclusion but where $D_Y = 0$. We prove a law of large numbers and a functional central limit theorem for the position of the foremost visited site and for the number of activated particles, giving an indication that corroborates the behavior observed in the numerical simulations of [8]. In the model we consider, there are two types of particles: the X particles which move as a symmetric simple exclusion process; the Y particles, which do not move. Initially there are no X particles at sites $x > 0$, while the configuration $\eta := \{\eta(0, x) : x \leq 0\}$ of X particles at sites $x \leq 0$, where $\eta(0, x)$ is the number of X particles at site x , is such that $\eta(0, 0) = 1$ but otherwise arbitrary. Initially there are no Y particles at $0, -1, -2, \dots$ while at sites $1, 2, \dots$, the Y particles are distributed according to a product Bernoulli distribution of parameter ρ . When an X particle jumps to a site where there is a Y particle, their position is interchanged and the Y particle becomes an X particle. Since the Y particles do not move, the dynamics can be defined in terms of the configuration of the X particles and the rightmost visited site at time $t \geq 0$, which we call r_t . We adopt the convention that $r_0 = 0$. The state space of the process is then

$$\Omega := \left\{ (r, p, \eta) : r \in \mathbb{Z}, p \in \mathbb{Z}, \eta \in \{0, 1\}^{\{\dots, r-1, r\}} \right\}.$$

Here p represents a counter for the number of Y particles which have been activated. The infinitesimal generator of the process is

$$\begin{aligned} Lf(r, p, \eta) := & \sum_{x, y \leq r, |x-y|=1} \eta(x)(1 - \eta(y))(f(r, p, \sigma_{x,y}\eta) - f(r, p, \eta)) \\ & + \rho \eta(r)(f(r+1, p+1, \eta + \delta_{r+1}) - f(r, p, \eta)) \\ (1) \quad & + (1 - \rho) \eta(r)(f(r+1, p, \eta) - f(r, p, \eta)), \end{aligned}$$

where δ_x denotes the configuration with one particle at x , while $\sigma_{x,y}\eta$ denotes the configuration obtained from η after flipping the values of $\eta(x)$ and $\eta(y)$.

We will use the notation $\eta(t, x)$ for the number of particles at time $t \geq 0$ at site x , p_t the value of the counter for the number of activated Y particles (so that $p_t - p_0$ is the actual number of activated particles) and r_t the position of the front at time $t \geq 0$. We will use the notation $\eta(t) := \{\eta(t, x) : x \leq r_t\}$ for the configuration of particles at time t and will call the process $\{(r_t, p_t, \eta(t)) : t \geq 0\}$ the *exclusion reactive process*.

Theorem 1. *Assume that initially $r = p = 0$, $\eta(0, 0) = 1$ and $\eta(0, x) \in \{0, 1\}$ is arbitrary for $x < 0$.*

- (i) *There exist a $v > 0$ and a $w > 0$ which do not depend on the initial condition $\{\eta(0, x) : x \leq 0\}$, such that a.s.*

$$\lim_{t \rightarrow \infty} \frac{r_t}{t} = v, \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{p_t}{t} = w.$$

- (ii) *There exist σ_1 and σ_2 , $0 < \sigma_1, \sigma_2 < \infty$, which do not depend on the initial condition $\{\eta(0, x) : x \leq 0\}$, such that*

$$\epsilon^{1/2}(r_{\epsilon^{-1}t} - \epsilon^{-1}vt), \quad \text{and} \quad \epsilon^{1/2}(p_{\epsilon^{-1}t} - \epsilon^{-1}wt), \quad t \geq 0,$$

converge in law as $\epsilon \rightarrow 0$ to Brownian motions with variances σ_1 and σ_2 , respectively.

Theorem 2. *Consider the process as seen from the front, $\tau_{-r_t}\eta(t)$. There exist exactly two invariant measures: One supported on the configuration with no particles, and another, μ_∞ . The domain of attraction of the first consists of exactly the configuration with no particles. Any nontrivial configuration in $\{0, 1\}^{\{\dots, -1, 0\}}$ is in the domain of the second; if we denote by μ_t the distribution of the process $\tau_{-r_t}\eta(t)$, then $\mu_t \rightarrow \mu_\infty$ in the sense of weak convergence of probability measures.*

As in [2], this model does not satisfy any obvious sub-additivity property which would give a direct proof of part (i) of theorem 1. But in a certain sense, the interaction given by the exclusion dynamics of this paper, is stronger than the independent random walk dynamics of [3], or the annihilating random walks of [2].

The proof of these results is based in regeneration time methods as discussed in [2] and [3]. Nevertheless, to reduce the tail estimates of these regenerations times to manageable expressions, we have mapped the exclusion reactive process to a zero-range reactive dynamics, with total jump rate 1 per site. This mapping and the definition of the regeneration times is given in section 2.

In [2] and [3], the regeneration times were defined following [12], in terms of two alternating sequences of stopping times. One of these sequences defines the first times at which activated particles behind the leading one branch. This definition is performed using a space-time line which decouples the behavior of the old particles behind the front with respect to the behavior of the front itself. This is the approach used in this paper to define the regeneration times in section 2. However, we consider important to have an understanding from a more fundamental point of view about alternative ways which could be used to define regeneration times within the context of interacting particle systems representing reactive fronts. For this reason, in section 6, we present a definition of regeneration times for the exclusion reactive process, which is not done in terms of sequences of stopping times and which does not require a space-time line decoupling the dynamics at the left from the dynamics at the right of the front. The approach we present is in the spirit of Kesten [6] within the context of Random Walks in Random Environments.

In section 3, theorems 1 and 2 are proved assuming that the regeneration times and the corresponding position of the front have finite second moments. In section 4, these estimates are performed. The non-degeneracy of the variances σ_1 and σ_2 of Theorem 1 is proved in section 5.

2. ZERO-RANGE REACTIVE PROCESS AND REGENERATION TIMES

The first step in the proof of theorems 1 and 2, will be to couple the exclusion reactive process defined by (1), with a zero range process where particles branch at the right-most visited site. Then, it will be enough to prove a law of large numbers and a functional central limit theorem for the right-most visited site of this new process, together with the convergence towards an invariant measure for the law of this process as seen from its front. In the first subsection we will define the zero-range reactive process. In subsection 2.2, we will construct a version of the zero-range reactive process where particles are labeled. In subsection 2.3, we construct an auxiliary process, which gives lower bounds for the position of the front. Then, in subsection 2.4, we will define the regeneration times for the labeled version of the zero-range reactive process following some of the methods introduced in [6].

2.1. One dimensional zero-range reactive process. Consider a configuration (r, p, η) of the stochastic combustion process with exclusion. Let $q := r - p$. We define $\zeta(q)$ as the number of particles between the site $r + 1$ and the rightmost empty site. In other words, $\zeta(q) := r - x_1$, where $x_1 := \sup\{x \leq 0 : \eta(x) = 0\}$. Next define $x_2 := \sup\{x < x_1 : \eta(x) = 0\}$, the position of the second right-most empty site, and $\zeta(q - 1) := x_1 - x_2 - 1$, the number of particles between x_1 and x_2 . In general, for $n > 2$, we define $x_n := \sup\{x < x_{n-1} : \eta(x) = 0\}$, while $\zeta(q - n + 1) := x_{n-1} - x_n - 1$. Let $\zeta(t) := \{\zeta(x, t) : x \leq q\}$ and $q_t := r_t - p_t$ with initial condition $(q_0, \zeta_0) = (q, \zeta)$. It is easy to check that the stochastic process $\{(q_t, p_t, \zeta(t)), t \geq 0\}$ follows the dynamics of a one-dimensional reactive process with a zero-range dynamics with infinitesimal generator

$$\begin{aligned}
 Lf(q, p, \zeta) &= \sum_{x, y \leq q: |x-y|=1} 1(\eta(x) > 0)(f(q, p, \zeta^{x,y}) - f(q, p, \zeta)) \\
 &\quad + (1 - \rho)1(\eta(q) > 0)(f(q + 1, p, \zeta - \delta_q + \delta_{q+1}) - f(q, p, \zeta)) \\
 (2) \quad &\quad + \rho 1(\eta(q) > 0)(f(q, p + 1, \zeta + \delta_q) - f(q, p, \zeta)).
 \end{aligned}$$

This is a zero-range process with total jump rate 2 at those sites strictly to the left of the rightmost visited site q , with jump rate 1 to the left and $1 - \rho$ to the right at site q , and with branching at rate $\rho 1(\zeta(q) > 0)$ at site q . Note that p_t represents the number of times a branching has occurred. We will call the triple $\{(q_t, p_t, \zeta(t)) : t \geq 0\}$ the *zero-range reactive process*.

2.2. Labeled process. We will make an explicit construction of the zero-range reactive process where each particle carries a label $z \in \mathbb{Z}$, representing the priority it has. The movement of a given particle will not be affected by particles with smaller labels. The construction will be performed in terms of a stochastic process $\{\mathcal{Y}(t), q_t) : t \geq 0\}$, where at time $t \geq 0$, the first component $\mathcal{Y}(t)$ represents the

positions on \mathbb{Z} of a random number of particles, and q_t the position of the rightmost visited site. Thus, the state space of this process is $\mathbb{S} := \mathbb{T} \times \mathbb{Z}$, where $\mathbb{T} := \cup_{A \subset \mathbb{Z}} \mathbb{Z}^A$. A typical element of \mathbb{T} will be denoted by $\{y_x : x \in A\}$, where A is the set of labels. We will furthermore use the notation $p := \sup\{x : x \in A\}$ and p_t , when the corresponding set A_t is time dependent.

Let us fix an initial condition $(\{y_x : x \in A_0\}, q_0) \in \mathbb{S}$. Now associate to each $x \in A_0 \cup \{z \in \mathbb{Z} : z > p_0\}$ a discrete time simple symmetric random walk X_x starting from 0, and a sequence $\{\tau_x^{(i)} : i \geq 1\}$ of i.i.d. rate 2 exponential random variables, which will represent the potential jump times of an associated continuous time random walk. Let us also choose another sequence $\{v_i : i \geq 1\}$ of i.i.d. Bernoulli random variables of parameter ρ . We choose all these random variables independent of each other.

Let us first define the dynamics of our process for an initial condition $(\{y_x : x \in A_0\}, q_0)$ such that $A_0 \subset \mathbb{Z}$ is finite, and such that $q_0 = \sup\{y_x : x \in A_0\}$. We will associate to each discrete time random walk $X_x, x \in A_0$, a continuous time random walk Y_x , such that $Y_x(0) = y_x$. Let n_1 be the cardinality of the set $\{y_x : x \in A_0\}$. We identify at each site in this set, the particle with the smallest label: let us call them x_1, \dots, x_{n_1} . Let x_* be the label within the group x_1, \dots, x_{n_1} where the minimum $\tau_1 := \min\{\tau_{x_1}^{(1)}, \dots, \tau_{x_{n_1}}^{(1)}\}$ is achieved. If $y_{x_*} < q_0$, we let the continuous time random walk Y_{x_*} jump at time τ_1 according to X_{x_*} , while the other random walks do not move. Thus, the first change in the process $\{(\{Y_x(t) : x \in A_t\}, q_t) : t \geq 0\}$ occurs at time τ_1 when $Y_{x_*}(\tau_1) = y_{x_*} + X_{x_*}(1)$. If $y_{x_*} = q_0$, and if the discrete time random walk X_{x_*} jumps to the left at time 1, we let Y_{x_*} jump to the left at time τ_1 . Thus $Y_{x_*}(\tau_1) = y_{x_*} + X_{x_*}(1)$. If $y_{x_*} = q_0$, but the random walk X_{x_*} jumps to the right at time 1, we let Y_{x_*} jump to the right at time τ_1 only if the Bernoulli random variable $v_1 = 0$, case in which $Y_{x_*}(\tau_1) = y_{x_*} + X_{x_*}(1)$ and $q_{\tau_1} = q_0 + 1$, while if $v_1 = 1$, a random walk Y_{p_0+1} , which will follow the trajectory of the random walk $X_{p_0+1} + q_0$, is created at time τ_1 while the remaining random walks do not move, so that $A_{\tau_1} = A_0 \cup \{p_0 + 1\}$. This defines the dynamics of the process $\{(\mathcal{Y}(t), q_t); t \geq 0\}$ in the time interval $[0, \tau_1]$, with $\mathcal{Y}(t) := \{Y_x(t) : x \in A_t\}$.

Let us now recursively define the process for arbitrary times. Assume that for some k , such that $k \geq 1$, n_k and τ_k have been defined and also the process in the time interval $[0, \tau_k]$. Let $p_k := \sup\{x : x \in A_{\tau_k}\}$. Call n_{k+1} the cardinality of the set $\{Y_x(\tau_k) : x \in A_{\tau_k}\}$, and identify at each site in this set the particle with smallest label: we call these labels $x_1, \dots, x_{n_{k+1}}$. Let x_* be the label where the minimum $\tau_{k+1} := \min\{\tau_{x_1}^{(k+1)}, \dots, \tau_{x_{n_{k+1}}}^{(k+1)}\}$ is achieved. Denote for $1 \leq i \leq n_k$ as N_{x_i} the total number of jumps performed up to time τ_k by the random walk Y_{x_i} . If $Y_{x_*}(\tau_k) < q_{\tau_k}$, we let the continuous time random walk Y_{x_*} jump at time τ_{k+1} according to X_{x_*} so that $Y_{x_*}(\tau_{k+1}) = y_{x_*} + X_{x_*}(N_{x_*} + 1)$, while the other random walks do not move. If $Y_{x_*}(\tau_k) = q_{\tau_k}$ and if the random walk X_{x_*} jumps to the left at the time $N_{x_*} + 1$, we let Y_{x_*} jump to the left at time τ_{k+1} . If $Y_{x_*}(\tau_k) = q_{\tau_k}$ but the random walk X_{x_*} jumps to the right at the time $N_{x_*} + 1$, we let Y_{x_*} jump to the right at time τ_{k+1} only if the Bernoulli random variable $v_{1+p_k-p_0} = 0$, while if $v_{1+p_k-p_0} = 1$ a random walk Y_{p_k+1} , following the discrete time trajectory of $q_{\tau_k} + X_{p_k+1}$, is created.

Let us now consider the case in which A_0 is not necessarily finite, but nevertheless $p_0 = \sup\{x : x \in A_0\} < \infty$. Consider the initial condition of positions of the particles $\mathcal{Y}(0) = \{y_x : x \in A_0\}$ and of the front q_0 . We use the following notations: given $A \subset \mathbb{Z}$ and $n \geq 1$, define $\mathcal{Y}^n(0) := \{y_x \in \mathcal{Y}(0) : y_x \leq q_0 - n\}$ and $A^n := \{x \in A : y_x \leq q_0 - n\}$. Since A_0^n is finite, we can define the process $\{(\mathcal{Y}^n(t), q_t^n) : t \geq 0\}$ as in the previous paragraphs.

Lemma 1. *There exists a set of full measure such that for every $t \geq 0$ the following statements are true.*

(i) *There is an n_0 such that if $n \geq n_0$,*

$$A_t^n = A_t^{n_0}.$$

(ii) *Let $p_t^n := \sup\{x : x \in A_t^n\}$. Then*

$$p_t := \lim_{n \rightarrow \infty} p_t^n,$$

exists.

(iii) *For every $x \in A_t := A_t^{n_0}$.*

$$Y_x(t) := \lim_{n \rightarrow \infty} Y_x^n(t),$$

exists.

(iv) *The limit*

$$q_t := \lim_{n \rightarrow \infty} q_t^n,$$

exists.

Proof. Let us prove part (i). Without loss of generality we assume that for each $n \geq 1$, $A_0^{n+1} \neq A_0^n$. Consider the event $E_n := \{A_t^m \neq A_t^n : \text{for some } m \geq n\}$. By the lemma of Borel-Cantelli it is enough to prove that

$$(3) \quad \sum_{n=1}^{\infty} P[E_n] < \infty.$$

Now the probability of the event E_n is upper bounded by the probability that in the time interval $[0, t]$, a Poisson process of rate 2 has performed at least n steps. Indeed, the event E_n is contained in the event that some particle initially at a distance larger than n from the front, is alone at the foremost visited site before time t . But this can happen only if at least n jumps were performed before time t . Thus,

$$P[E_n] \leq \sum_{k=n}^{\infty} e^{-2t} \frac{(2t)^k}{k!} \leq \frac{1}{n!},$$

which proves (3). Similar arguments can be used to prove parts (ii), (iii) and (iv). \square

Define now $\mathcal{Y}(t) := \{Y_x(t) : x \in A_t\}$. Then the triple $\{(\mathcal{Y}(t), q_t) : t \geq 0\}$, defines a probability measure \mathbb{P} on the Skorokhod space $D([0, \infty); \mathbb{S})$, which we will call the *labeled zero-range reactive process*. It turns out that the particle count

$$\zeta(t, x) = \sum_{x' \in A_t} 1(Y_{x'}(t) = x),$$

together with the pair p_t and q_t defined in lemma 1, satisfies the dynamics defined by the infinitesimal generator (2).

Consider an initial condition $w := (\{y_x : x \in A_0\}, q_0) \in \mathbb{S}$, such that $y_{p_0} = q_0$, where $p_0 = \sup\{x : x \in A\}$. Define $\delta_{p_0, q_0} \in \mathbb{S}$ as $\delta_{p_0, q_0} := (\{y_x : x \in B_0\}, q_0)$, where $B_0 := \{p_0\}$. This defines two coupled labeled zero-range reactive processes $\{(\{Y_x(t) : x \in A_t\}, q_t) : t \geq 0\}$ and $\{(\{Y'_x(t) : x \in B_t\}, q'_t) : t \geq 0\}$, with initial conditions w and δ_{p_0, q_0} respectively. The corresponding particle counts define two coupled zero range reactive processes with initial conditions corresponding to w and δ_{p_0, q_0} , which we will denote by ζ_w and $\zeta_{\delta_{p_0, q_0}}$ respectively. The corresponding rightmost visited sites of these processes will be denoted by q_t^w and $q_t^{\delta_{p_0, q_0}}$, whereas the counters of the number of activated particles by p_t^w and $p_t^{\delta_{p_0, q_0}}$. Furthermore we define

$$\xi(t, x) := \sum_{x' \in A_0 - \{p_0\}} 1(Y_{x'}(t) = x)$$

and

$$\zeta'_w(t, x) := \sum_{x' \in A'_t} 1(Y_{x'}(t) = x),$$

where $A'_t := (A_t - A_0) \cup \{p_0\}$. The processes $\{(\zeta_w(t), q_t^w) : t \geq 0\}$, $\{(\zeta_{\delta_{p_0, q_0}}(t), q_t^{\delta_{p_0, q_0}}) : t \geq 0\}$, $\{\zeta'_w(t) : t \geq 0\}$, and $\{\xi(t) : t \geq 0\}$ are then coupled.

Let V be the first time that some particle with a label smaller than p_0 is at the right-most visited site while no particle with label larger than or equal to p_0 is at the right-most visited site

$$V' := \inf \left\{ t \geq 0 : \xi(t, q_t) > 0, \zeta_{\delta_{p_0, q_0}}(t, q_t) = 0 \right\}.$$

Lemma 2. *Let $A \subset \mathbf{Z}$, $w := (\{y_x : x \in A_0\}, q_0) \in \mathbb{S}$ and $p_0 = \sup\{x : x \in A_0\}$. Assume that $y_{p_0} = q_0$. Consider the corresponding coupled process $\{(\zeta_w(t), q_t^w), (\zeta_{\delta_{p_0, q_0}}(t), q_t^{\delta_{p_0, q_0}}), \xi(t)); t \geq 0\}$. Then*

$$\begin{aligned} q_t^{\delta_{p_0, q_0}} &= q_t^w, & t < V', \\ p_t^{\delta_{p_0, q_0}} &= p_t^w, & t < V' \end{aligned}$$

and

$$\zeta_{\delta_{p_0, q_0}}(t) = \zeta'_w(t), \quad t < V'.$$

Proof. It is enough to observe that before time V' , none of the particles with labels smaller than p_0 affect the dynamics of the process $\{(\{Y_x(t) : x \in A_t\}, q_t) : t \geq 0\}$. \square

2.3. Auxiliary process. Let us now define a process which will be helpful to obtain estimates for the law of some stopping times used to define the regeneration times. Let $n \geq 2$ be a fixed natural number. Now let $\{Z_x : x \geq 0\}$ be a set of independent continuous times simple symmetric random walks of rate $2/n$, such that $Z_x(0) = x$. Define ν_1 as the first time the random walk Z_0 hits the site 1,

$$\nu_1 := \inf\{t \geq 0 : Z_0(t) = 1\}.$$

For m such that $2 \leq m \leq n$, define ν_m as the first time that any of the random walks Z_0, Z_1, \dots, Z_{m-1} hits site m ,

$$\nu_m := \inf\{t \geq 0 : \sup_{0 \leq i \leq m-1} Z_i(t) = m\}.$$

And for $m > n$, define

$$\nu_m := \inf\{t \geq 0 : \sup_{m-n \leq i \leq m-1} Z_i(t) = m\}.$$

Now define for $t \geq 0$,

$$\tilde{q}_t := m \quad \text{for} \quad \sum_{i=0}^m \nu_i \leq t < \sum_{i=0}^{m+1} \nu_i,$$

with the convention that $\nu_0 = 0$. In the sequel we will call $\{\tilde{q}_t : t \geq 0\}$, the *auxiliary process* with n particles.

Lemma 3. *Whenever $n \geq 3$, there is an $\alpha > 0$ such that*

$$\liminf_{t \rightarrow \infty} \frac{q_t^{\delta_{0,0}}}{t} \geq \alpha \quad a.s.$$

Proof. For each natural n , we can construct a process having the same law as $\{q_t^{\delta_{0,0}} : t \geq 0\}$: the last n activated particles have priority over the ones activated previously; nevertheless, if at a given time there are m particles from this group (of n particles), each one jumps at a rate $2/m$. We can couple this construction with the auxiliary process with n particles $\{\tilde{q}_t : t \geq 0\}$ in such a way that $q_t^{\delta_{0,0}} \geq \tilde{q}_t$ (for the details of such a coupling within a similar context see [2]). Now, it is easy to check that

$$\lim_{t \rightarrow \infty} \frac{\tilde{q}_t}{t} =: \alpha \quad a.s.$$

□

2.4. Regeneration times. Let us consider an initial condition $(\{y_x : x \in A_0\}, q_0)$ such that $y_{p_0} = q_0$. Let also α_1 and α_2 be such that $0 < 2\alpha_1 < \alpha_2 < \alpha$. Define

$$T := \inf\{t \geq 0 : Z_{p_0, q_0}(t) = q_0 - 1\},$$

which is the first time that the leading particle at q_0 jumps backwards. Now let

$$\bar{q}_t^{\delta_{p_0, q_0}} = Z_{p_0, q_0}(t) \quad \text{for } 0 \leq t < T$$

and $\bar{q}_t^{\delta_{p_0, q_0}} = q_t^{\delta_{p_0, q_0}}$ for $t \geq T$. Define

$$U := \inf\{t \geq 0 : \bar{q}_t^{\delta_{p_0, q_0}} - q_0 < \lfloor \alpha_2 t \rfloor \text{ or } \zeta_{\delta_{p_0, q_0}}(t, 0) = 0\},$$

$$V := \inf\{t \geq 0 : \xi(t, \lfloor \alpha_1 t \rfloor + q_0) > 0\}$$

and

$$D := \min\{U, V\}.$$

The definition of U in terms of $\bar{q}_t^{\delta_{p_0, q_0}}$ instead of $q_t^{\delta_{p_0, q_0}}$ is necessary because we have to avoid the possibility that before time D some particle in the configuration ξ branches before the front $q_t^{\delta_{p_0, q_0}}$ has moved. Furthermore, the fact that $2\alpha_1 < \alpha_2$ guarantees that for times $t > 1/\alpha_1$ the function $\lfloor \alpha_2 t \rfloor$ is never equal to $\lfloor \alpha_1 t \rfloor$. Note that U , V and D are stopping times with respect to the natural filtration $\{\mathcal{F}_t : t \geq 0\}$ of the labeled zero-range reactive process.

Let us also define the first time U and V happen after time $s \geq 0$,

$$U \circ \theta_s := \inf\{t \geq 0 : \bar{q}_t^{\delta_{p_s, q_s}} - q_s < \lfloor \alpha_2 t \rfloor\},$$

$$V \circ \theta_s := \inf\{t \geq 0 : \xi_{w_s}(t, \lfloor \alpha_1 t \rfloor + q_s) > 0\},$$

and $D \circ \theta_s := \min\{U \circ \theta_s, V \circ \theta_s\}$, where $w_s := (\{Y_x(s) : x \in A_s\}, q_s)$.

For each $y \in \mathbb{N}$, define the \mathcal{F}_t -stopping time

$$N_y := \inf\{t \geq 0 : p_t = p_0 + y\}.$$

We now define sequences of stopping times $\{S_k : k \geq 0\}$ and $\{D_k : k \geq 1\}$ as follows. First let $S_0 := 0$ and $R_0 := 0$. Then define for $k \geq 0$,

$$S_{k+1} := N_{R_{k+1}}, \quad D_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}, \quad R_{k+1} := p_{D_{k+1}},$$

where we adopt the convention that $p_{D_k} = \infty$ and $S_{k+1} = \infty$ in the event $D_k = \infty$. We similarly define $U_k := U \circ \theta_{S_k} + S_k$ and $V_k := V \circ \theta_{S_k} + S_k$ for $k \geq 1$. Let now

$$K := \inf\{k \geq 1 : S_k < \infty, D_k = \infty\},$$

and define the *regeneration time*,

$$\kappa := S_K.$$

As in [2], κ is not a stopping time. Define \mathcal{G} , the information up to time κ , as the completion of the σ -algebra generated by events of the form $\{t \leq \kappa\} \cap A$, with $A \in \mathcal{F}_t$.

The following proposition will be proved in section 4. We call an element $w \in \mathbb{S}$ *nontrivial*, if it corresponds to a configuration with at least one particle behind the front or at it.

Proposition 1. *For every non-trivial initial condition $w \in \mathbb{S}$,*

$$(4) \quad \kappa < \infty, \quad \mathbb{P}_w - a.s.$$

Furthermore,

$$(5) \quad \mathbb{E}_{\delta_{0,0}}[\kappa^2 | U = \infty] < \infty \quad \text{and} \quad \mathbb{E}_{\delta_{0,0}}[r_\kappa^2 | U = \infty] < \infty.$$

Proposition 2. *Let F be a Borel subset of $D([0, \infty); \Omega)$. Then, for every nontrivial $w \in \mathbb{S}$,*

$$\mathbb{P}_w[(q_{\kappa+\cdot} - q_\kappa, p_{\kappa+\cdot} - p_\kappa, \tau_{-q_\kappa, -p_\kappa} \zeta'_{w_\kappa}(\kappa + \cdot)) \in F | \mathcal{G}] = \mathbb{P}_{\delta_{0,0}}[(q, p, \zeta(\cdot)) \in F | U = \infty].$$

Proof. It is enough to prove that for every $B \in \mathcal{G}$,

$$\begin{aligned} & \mathbb{P}_w[B, (q_{\kappa+\cdot} - q_\kappa, p_{\kappa+\cdot} - p_\kappa, \tau_{-q_\kappa, -p_\kappa} \zeta'_{w_\kappa}(\kappa + \cdot)) \in F] \\ &= \mathbb{P}[B] \mathbb{P}_{\delta_{0,0}}[(q, p, \zeta(\cdot)) \in F | U = \infty]. \end{aligned}$$

As in [2], this can be done using lemma 2 and observing that on the event on the event $S_k < \infty$, $q_{s_k} = x$ and $p_{s_k} = y$, we have that

$$\zeta'_{w_\kappa}(S_k + \cdot) = \zeta_{\delta_{x,y}}(\cdot),$$

whenever $U_k = V_k = \infty$. □

We can now define a sequence $\kappa_1 \leq \kappa_2 \leq \dots$, with $\kappa_1 := \kappa$ while for $n \geq 1$,

$$\kappa_{n+1} := \kappa_n + \kappa(w_{\kappa_n} + \cdot),$$

where $\kappa(w_{\kappa_n} + \cdot)$ is the regeneration time starting from w_{κ_n} and we set $\kappa_{n+1} = \infty$ on the event $\kappa_n = \infty$. We call κ_1 the *first regeneration time* and κ_n the *n-th regeneration time*. Now define for each $n \geq 1$, the σ -algebra \mathcal{G}_n as the completion with respect to \mathbb{P} of the smallest σ -algebra containing all sets of the form $\{\kappa_1 \leq t_1\} \cap \dots \cap \{\kappa_n \leq t_n\} \cap A$, $A \in \mathcal{F}_{t_n}$. As in [2], we then have the following generalization of proposition 2.

Proposition 3. *Let F be a Borel subset of $D([0, \infty); \Omega)$. Then, for every nontrivial $w \in \mathbb{S}$ and $n \geq 1$,*

$$\begin{aligned} & \mathbb{P}_w[(q_{\kappa_n+\cdot} - q_{\kappa_n}, p_{\kappa_n+\cdot} - p_{\kappa_n}, \tau_{-q_{\kappa_n}, -p_{\kappa_n}} \zeta'_{w_{\kappa_n}}(\kappa_n + \cdot)) \in F | \mathcal{G}_n] \\ &= \mathbb{P}_{\delta_{0,0}}[(q, p, \zeta(\cdot)) \in F | U = \infty]. \end{aligned}$$

Proposition 4. *Let $w \in \mathbb{S}$. (i) Under \mathbb{P}_w , $\kappa_1, \kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \dots$ are independent, and $\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \dots$ are identically distributed with law identical to that of κ_1 under $\mathbb{P}_{\delta_{0,0}}[\cdot|U = \infty]$. (ii) Under \mathbb{P}_w , $q_{\cdot \wedge \kappa_1}, q_{(\kappa_1 + \cdot) \wedge \kappa_2} - q_{\kappa_1}, q_{(\kappa_2 + \cdot) \wedge \kappa_3} - q_{\kappa_2}, \dots$ are independent, and $q_{(\kappa_1 + \cdot) \wedge \kappa_2} - q_{\kappa_1}, q_{(\kappa_2 + \cdot) \wedge \kappa_3} - q_{\kappa_2}, \dots$ are identically distributed with law identical to that of q_{κ_1} under $\mathbb{P}_{\delta_{0,0}}[\cdot|U = \infty]$. (iii) Under \mathbb{P}_w , $p_{\cdot \wedge \kappa_1}, p_{(\kappa_1 + \cdot) \wedge \kappa_2} - p_{\kappa_1}, p_{(\kappa_2 + \cdot) \wedge \kappa_3} - p_{\kappa_2}, \dots$ are independent, and $p_{(\kappa_1 + \cdot) \wedge \kappa_2} - p_{\kappa_1}, p_{(\kappa_2 + \cdot) \wedge \kappa_3} - p_{\kappa_2}, \dots$ are identically distributed with law identical to that of p_{κ_1} under $\mathbb{P}_{\delta_{0,0}}[\cdot|U = \infty]$.*

3. PROOF OF THEOREMS 1 AND 2

3.1. Proof of theorem 1. The proof of parts (i) and (ii) of theorem 1 follow now using standard arguments (see for example [2] or in the context of Random Walks in Random Environment [12]). Indeed, using Propositions 1 and 4 we first prove that, a.s.

$$\lim_{n \rightarrow \infty} \frac{q_{\kappa_n}}{\kappa_n} = \frac{\mathbb{E}_w[q_{\kappa_1}|U = \infty]}{\mathbb{E}_w[\kappa_1|U = \infty]} =: v_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{p_{\kappa_n}}{\kappa_n} = \frac{\mathbb{E}_w[p_{\kappa_1}|U = \infty]}{\mathbb{E}_w[\kappa_1|U = \infty]} =: v_2.$$

An interpolation argument then proves that a.s.

$$\lim_{t \rightarrow \infty} \frac{q_t}{t} = v_1 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{p_t}{t} = v_2.$$

Since $r_t = q_t + p_t$, this implies that a.s.

$$\lim_{t \rightarrow \infty} \frac{r_t}{t} = v_1 + v_2,$$

proving part (i) of theorem 1. We then define $P_j := p_{\kappa_{j+1}} - p_{\kappa_j} - (\kappa_{j+1} - \kappa_j)v_2$ and $R_j := p_{\kappa_{j+1}} + q_{\kappa_{j+1}} - p_{\kappa_j} - q_{\kappa_j} - (\kappa_{j+1} - \kappa_j)(v_1 + v_2)$, and show that

$$\Sigma_m := \sum_{j=1}^m P_j \quad \text{and} \quad \Sigma'_m := \sum_{j=1}^m R_j,$$

converge in law to Brownian motions with variances

$$\sigma_1 := \frac{\mathbb{E}_w[(p_{\kappa_1} - v_2 \kappa_1)^2|U = \infty]}{\mathbb{E}_w[\kappa_1|U = \infty]} \quad \text{and} \quad \sigma_2 := \frac{\mathbb{E}_w[(r_{\kappa_1} - (v_1 + v_2)\kappa_1)^2|U = \infty]}{\mathbb{E}_w[\kappa_1|U = \infty]}$$

respectively. An interpolation argument can then be used to obtain the full limits proving that

$$(6) \quad \epsilon^{1/2}(p_{\epsilon^{-1}t} - \epsilon^{-1}tv_2) \quad \text{and} \quad \epsilon^{1/2}(r_{\epsilon^{-1}t} - \epsilon^{-1}t(v_1 + v_2)), \quad t \geq 0,$$

converge in law to Brownian motions with variances σ_1 and σ_2 respectively. In section 5 we will prove that these variances are positive.

3.2. Proof of theorem 2. Let μ_t be the law at time t of the exclusion reactive process seen from the front $\tau_{-r_t}\eta_t \in \Omega_0 := \{0, 1\}^{\mathbb{Z}^-}$. This is a Markov process with infinitesimal generator

$$L_0 f(\eta) = \rho \eta(0)(f(\tau_{-1}\eta + \delta_0) - f(\eta)) + (1 - \rho)\eta(0)(f(\tau_{-1}\eta) - f(\eta)) \\ + \sum_{x,y \leq 0, |x-y|=1} \eta(x)(1 - \eta(y))(f(\sigma_{x,y}\eta) - f(\eta)).$$

For a local function f on Ω_0 , define $l(f)$ as the smallest integer l such that $f(\eta)$ does not depend on $\eta(x)$ if $x < -l$. Now define the probability measure μ_∞ on Ω_0 by the formula

$$\int_{\Omega_0} f d\mu_\infty = \frac{\mathbb{E}_{\delta_0}[\int_{\kappa_N}^{\kappa_{N+1}} f(\tau_{-r_s}\eta_s) ds | U = \infty]}{\mathbb{E}_{\delta_0}[\kappa_1 | U = \infty]},$$

for N such that $N(\alpha' - \alpha'') > l(f)$. This defines a consistent family of probability measures on cylinders.

Theorem 3. $\lim_{t \rightarrow \infty} \mu_t = \mu_\infty$ weakly and μ_∞ is invariant for the generator L_0 .

Proof. First note that for f local and w an arbitrary initial condition, $\lim_{t \rightarrow \infty} \mathbb{E}_w[\kappa_{N+1} > t, f(\tau_{-r_t}\eta(t))] = 0$. Therefore in the decomposition

$$\int_{\Omega_0} f d\mu_t = \mathbb{E}_w[\kappa_{N+1} \leq t, f(\tau_{-r_t}\eta(t))] + \mathbb{E}_w[\kappa_{N+1} > t, f(\tau_{-r_t}\eta(t))],$$

it is enough to examine the first term. Now, as in [2], by Proposition 3

$$(7) \quad \mathbb{E}_w[\kappa_{N+1} \leq t, f(\tau_{-r_t}\eta(t))] = \int_0^t \mathcal{N}_t(du) F_f(u),$$

where

$$\mathcal{N}_t([0, u]) := \sum_{k \geq 1} \mathbb{P}_w(\kappa_k \in [t - u, t])$$

and

$$F_f(u) := \mathbb{E}_{\delta_{0,0}}[\kappa_N \leq u < \kappa_{N+1}, f(\tau_{-r_t}\eta(u)) | U = \infty].$$

Also, as in [3], we can check the spread out assumption of the renewal theorem (theorem 6.2 of [13]) to show that

$$\lim_{t \rightarrow \infty} \mathcal{N}_t([0, u]) = \frac{u}{\mathbb{E}_{\delta_{0,0}}[\kappa_1 | U = \infty]},$$

uniformly on compacts. Finally, since $|F_f| \leq \|f\|_\infty |F_1|$ and $\int F_1 du < \infty$ we conclude that

$$\lim_{t \rightarrow \infty} \int_{\Omega_0} f d\mu_t = \int_{\Omega_0} f d\mu_\infty.$$

□

4. ESTIMATES FOR THE REGENERATION TIMES

Let us first state the following estimate for the stopping time U .

Lemma 4. *For every $p \geq 1$ there exists a constant $C = C(p)$ such that for every initial condition w*

$$\mathbb{P}_w[t \leq U < \infty] \leq Ct^{-p}.$$

Proof. The proof is based on a comparison with the auxiliary process defined in subsection 2.3, and large deviation estimates as in the proof of lemma 7 of [2]. \square

Let us define \tilde{V} as the first time that independent continuous time simple symmetric random walks created at rate 1 at the origin, reach the site $\lfloor \alpha t \rfloor$. We have the following estimate for the stopping time V .

Lemma 5. *There is a constant $C > 0$ such that for every initial condition w*

$$\mathbb{P}_w[t \leq V < \infty] \leq \exp\{-Ct\}.$$

Proof. Let us note using the labeled construction of the zero-range reactive process, that we can bound the probability of event $\{t \leq V < \infty\}$ by the probability of the event $\{t \leq \tilde{V} < \infty\}$. It is not difficult to check that there is a constant $C > 0$ such that, $P[t \leq \tilde{V} < \infty] \leq \exp\{-Ct\}$ (see [2]). \square

From Lemmas 4 and 5 we can now directly prove (see [2]),

Lemma 6. *For every $p \geq 1$, there is a constant $C = C(p)$ such that for every initial condition w*

$$\mathbb{P}_w[t \leq D < \infty] \leq Ct^{-p}.$$

We continue with two important lemmas. The first one is proved using the auxiliary process as in [2].

Lemma 7. *There is a $\delta_1 > 0$ such that for every initial condition w*

$$\mathbb{P}_w[U = \infty] \geq \delta_1.$$

The second lemma which follows can be proved using the inequality $\mathbb{P}_w[V = \infty] \geq \mathbb{P}_w[\tilde{V} = \infty]$.

Lemma 8. *There is a $\delta_2 > 0$ such that for every initial condition w*

$$\mathbb{P}_w[V = \infty] \geq \delta_2.$$

We will also need the following lemma.

Lemma 9. *There is a constant $C > 0$ such that for every initial condition w with $p = q = 0$ and $M > 1$,*

$$\mathbb{P}_w[q_t \geq Mt] \leq C \exp\{-Ct\} \quad \text{and} \quad \mathbb{P}_w[p_t \geq Mt] \leq C \exp\{-Ct\}.$$

Proof. It is enough to note that the rate at which the process $\{q_t : t \geq 0\}$ increases is always bounded by 1. Similarly for the process $\{p_t : t \geq 0\}$. \square

Let us now prove (4) of Proposition 1. Note that for every $k \geq 1$, $\mathbb{P}_w[\kappa = \infty] \leq \mathbb{P}_w[D_k < \infty]$. Taking the limit when $k \rightarrow \infty$ we obtain (4). Part (5) of Proposition 1 is a consequence of the following lemma.

Lemma 10. *There is a constant $C > 0$ such that for every $p \geq 1$ and $t \geq 0$,*

$$\mathbb{P}_{\delta_{0,0}}[\kappa > t | U = \infty] \leq Ct^{-p}.$$

Proof. Note that

$$\mathbb{P}_{\delta_{0,0}}[\kappa > t | U = \infty] = \sum_{k=1}^{\infty} \mathbb{P}_{\delta_{0,0}}[S_k > t, K = k | U = \infty].$$

From Lemmas 7 and 8, using the strong Markov property we deduce that there is a $\delta > 0$ such that for every $l \geq 1$,

$$(8) \quad \mathbb{P}_{\delta_{0,0}}[\kappa > t | U = \infty] \leq \sum_{k=1}^l \mathbb{P}_{\delta_{0,0}}[S_k > t, K = k | U = \infty] + \delta^{-1}(1 - \delta)^l,$$

Let $0 < \gamma < 1$ and consider the event

$$A_k := \{r_{D_1} - r_{S_1} < t^\gamma, \dots, r_{D_{k-1}} - r_{S_{k-1}} < t^\gamma\}.$$

On A_k we have $r_{S_k} \leq kt^\gamma$. Since $\bar{q}_t = q_t$ (because the initial condition is $\delta_{0,0}$), if $U = \infty$ then $r_t \geq q_t > \lfloor \alpha t \rfloor$ for all $t \geq 0$. We then have on $A_k \cap \{U = \infty\}$ that $\lfloor \alpha S_k \rfloor \leq kt^\gamma$. Hence, for $t > (lt^\gamma + 1)/\alpha$ and $k \leq l$,

$$\mathbb{P}_{\delta_{0,0}}[t < S_k < \infty, A_k | U = \infty] = 0$$

and

$$(9) \quad \begin{aligned} \mathbb{P}_{\delta_{0,0}}[t < S_k < \infty | U = \infty] &\leq \mathbb{P}_{\delta_{0,0}}[t < S_k < \infty, A_k^c | U = \infty] \\ &\leq C \sum_{i=1}^{k-1} \mathbb{P}_{\delta_{0,0}}[r_{D_i} - r_{S_i} \geq t^\gamma, S_k < \infty], \end{aligned}$$

where we have used Lemma 7. Now, using Lemmas 9 and 6, we can show that for every $p \geq 1$ there exists a constant C such that

$$\mathbb{P}_{\delta_{0,0}}[r_{D_i} - r_{S_i} \geq t^\gamma, S_k < \infty] \leq Ct^{-p}.$$

Substituting this estimate back into (9) and then into (8), we conclude the proof. \square

5. NON-DEGENERACY

To prove the non-degeneracy of the limit of the first expression in display (6) we will prove that for some $\alpha < \beta < v_2$ it is true that

$$(10) \quad \mathbb{P}_{\delta_{0,0}}[p_{\kappa_1} = 1, \beta^{-1} < \kappa_1 | U = \infty] > 0.$$

Note that,

$$\mathbb{P}_{\delta_{0,0}}[p_{\kappa_1} = 1, \beta^{-1} < \kappa_1, U = \infty] \geq \mathbb{P}_{\delta_{0,0}}[\beta^{-1} < S_1 < U, D \circ \theta_{S_1} = \infty].$$

The lower bound of the above inequality can be written as

$$\mathbb{E}_{\delta_{0,0}}[1(\beta^{-1} < S_1 < U) \mathbb{E}_{\delta_{0,0}}[U \circ \theta_{S_1} = \infty, V \circ \theta_{S_1} = \infty | \mathcal{F}_{S_1}]].$$

Now

$$\begin{aligned} & \mathbb{E}_{\delta_{0,0}}[U \circ \theta_{S_1} = \infty, V \circ \theta_{S_1} = \infty | \mathcal{F}_{S_1}] = \\ & \mathbb{E}_{\delta_{0,0}}[U \circ \theta_{S_1} = \infty | \mathcal{F}_{S_1}] \mathbb{E}_{\delta_{0,0}}[V \circ \theta_{S_1} = \infty | \mathcal{F}_{S_1}] \geq \delta_1 \delta_2, \end{aligned}$$

where in the inequality we have used Lemmas 7 and 8. Thus,

$$\mathbb{P}_{\delta_{0,0}}[\beta^{-1} < S_1 < U, D \circ \theta_{S_1} = \infty] \geq \delta_1 \delta_2 \mathbb{P}_{\delta_{0,0}}[\beta^{-1} < S_1 < U].$$

Now, the probability of the right-hand term can be lower bounded by the probability that the particle initially at site 0 with label 0, performs a first jump at some time t towards the right and then at time $t + t'$ it branches at site 1 such that $\beta^{-1} < t + t' < \alpha^{-1}$. Clearly this probability is positive.

To prove the non-degeneracy of the second expression in display (6) this time we will show that for some $\alpha < \beta < v_1 + v_2$ it is true that

$$(11) \quad \mathbb{P}_{\delta_{0,0}}[r_{\kappa_1} = 1, \beta^{-1} < \kappa_1 | U = \infty] > 0.$$

But note that $\{r_{\kappa_1} = 1\} = \{p_{\kappa_1} = 1\}$. Therefore inequality (11) follows from (10).

6. AN ALTERNATIVE DEFINITION OF REGENERATION TIMES

6.1. Asymmetric simple random walk. Let us show through a simple example how to prove the independence of the increments of the regeneration times, without defining them through a sequence of stopping times. What we present here is essentially contained in the paper of Kesten [6].

Consider a discrete time asymmetric simple random walk $\{X_n : n \geq 0\}$, which at each site jumps to the right with probability q and to the left with probability p , with $q > p$. Define the random time

$$\kappa := \min\{n \geq 0 : \min_{k \geq n} X_k > \max_{j < n} X_j\},$$

with the convention that $\max_{j < 0} X_j = -1$. In words, κ is the first time the random walk visits a new site, without never afterward moving to the left of such a site.

Let us define the σ -algebra \mathcal{F}_κ of κ as the one generated by events of the form $\{\kappa = n\} \cap \{X_1 = x_1, \dots, X_n = x_n\}$ for $n \geq 0$ and x_1, \dots, x_n in \mathbb{Z}^d . If we call P the law of the random walk $\{X_n\}$, starting from 0, the following equality is satisfied for every subset $A \in \mathbb{Z}^{\mathbb{N}}$:

Lemma 11.

$$(12) \quad P[X_{\kappa+} - X_\kappa \in A | \mathcal{F}_\kappa] = P[X \in A | \kappa = 0].$$

Let us prove this equality using directly the Markov property. For convenience, we define $Y_n := \sup_{k \leq n} X_k$, for $n \geq 0$. Note first that

$$(13) \quad \{\kappa = n\} = \left\{ \inf_{m \geq n} X_m \geq X_n \right\} \cap \left\{ \sup_{1 \leq p \leq n-1} X_p = X_n - 1 \right\} \cap B_n,$$

where

$$B_n := \left\{ \forall k < n : \text{there exists a } j, \text{ such that } k < j < n \text{ and } X_j < Y_k \right. \\ \left. \text{or there exists a } j, \text{ such that } 0 \leq j < k \text{ and } Y_j = X_k \right\}.$$

Let us remark that $B_n \in \mathcal{F}_n$, where $\{\mathcal{F}_n : n \geq 0\}$ is the natural filtration of the random walk $\{X_n : n \geq 0\}$. Furthermore, the condition $q > p$, ensures that $P[\kappa = n] > 0$. Defining $C_n := \{\sup_{1 \leq p \leq n-1} X_p = X_n - 1\} \cap B_n \in \mathcal{F}_n$, we now have

$$\begin{aligned} & P[X_{\kappa+} - X_\kappa \in A | \kappa = n, X_1 = x_1, \dots, X_n = x_n] \\ = & \frac{P[X_{\kappa+} - X_\kappa \in A, \inf_{m \geq n} X_m \geq X_n, C_n, X_1 = x_1, \dots, X_n = x_n]}{P[\inf_{m \geq n} X_m \geq X_n, C_n, X_1 = x_1, \dots, X_n = x_n]}. \end{aligned}$$

On the other hand, by the fact that $C_n \in \mathcal{F}_n$, the Markov property, and translation invariance, we have that

$$\begin{aligned} & P \left[X_{\kappa+} - X_\kappa \in A, \inf_{m \geq n} X_m \geq X_n, C_n, X_1 = x_1, \dots, X_n = x_n \right] \\ = & E \left[\mathbf{1}_{\{C_n, X_1 = x_1, \dots, X_n = x_n\}} P \left[X_{n+} - X_n \in A, \inf_{m \geq n} X_m \geq X_n \middle| \mathcal{F}_n \right] \right] \\ (14) \quad = & P[C_n, X_1 = x_1, \dots, X_n = x_n] P[X \in A, \inf_{m \geq 0} X_m \geq 0]. \end{aligned}$$

Choosing A as the whole space in the previous development, we conclude that

$$\begin{aligned} & P \left[\inf_{m \geq n} X_m \geq X_n, C_n, X_1 = x_1, \dots, X_n = x_n \right] \\ = & P[C_n, X_1 = x_1, \dots, X_n = x_n] P[\inf_{m \geq n} X_m \geq 0], \end{aligned}$$

which combined with (14) completes the proof of (12).

6.2. Exclusion reactive process. Let us now show how the approach presented in the context of a symmetric simple random walk can be implemented for the exclusion reactive process to define a version of the regeneration times. To define the regeneration times we will consider the holes (empty sites) of the process as second class particles.

We will construct the exclusion reactive process associating to each bond connecting two nearest neighbor sites independent Poisson processes each one of rate 1. Let us assume that the initial condition is of the form $(0, \eta(0))$, where $\eta(0)$ is any nontrivial configuration of particles. If at a given time the rightmost visited site is r and $x + 1 \leq r$, whenever the Poisson clock connecting sites x and $x + 1$ rings, the state at sites x and $x + 1$ is interchanged: if the state of the process was η , this is changed to $\sigma_{x,x+1}\eta$. In this case the front stays at r . If at a given time the rightmost visited site r is occupied and the Poisson clock connecting sites r and $r + 1$ rings, with probability ρ a particle is created at site $r + 1$ and with probability $1 - \rho$ the particle at r jumps to site $r + 1$. In both cases the front advances one step to $r + 1$. Define τ_1 as the first time the front advances one step creating a new particle by

$$\tau_1 := \inf\{t \geq 0 : r_t > 0 \text{ and } \eta(r_t - 1, t) = \eta(r_t, t) = 1\}.$$

In general for $n \geq 2$ define recursively τ_n as the first time after time τ_{n-1} that the front advances one step creating a new particle by

$$\tau_n := \inf\{t \geq \tau_{n-1} : r_t > r_{\tau_{n-1}} \text{ and } \eta(r_t - 1, t) = \eta(r_t, t) = 1\}.$$

For each $n \geq 1$, at the time τ_n we will fill up the holes defined by the configuration $\eta(\tau_n)$ with particles: the original particles in this configuration are first class with respect to the new ones, which we will call *holes*. After time τ_n holes can never activate a new particle.

Let us now adopt the convention that whenever the Poisson clock of a bond corresponding to two occupied sites rings, the corresponding particles or holes are interchanged. We can then label each particle and each hole, following its trajectory. We will assign the label 0 to the particles initially at r or to the left of r . We assign the label 1 to the particle activated at time τ_1 . In general for $n \geq 2$, we assign the label n to the particle activated at time τ_n . Similarly, we assign the label 0 to all the holes initially at r or to the left of r , and to the new holes created before time τ_1 . We assign the label 1 to the holes created after time τ_1 but before time τ_2 , and for $n \geq 2$ we assign the label n to the holes created after time τ_n but before time τ_{n+1} . Let us call Υ the state space of this process consisting of the ordered pairs $(r, \boldsymbol{\eta})$, where r is an integer representing the position of the front and $\boldsymbol{\eta}$ is a configuration of labeled particles and holes at sites $\dots, r - 1, r$. Let us call Q the law of this process in the corresponding Skorohod space. We now define for each $n \geq 1$, the stopping time D_n as the first time after time τ_n that some of the particles or holes with labels strictly smaller than n (thus, excluding the foremost particle, created at time τ_n) is at the front

$$D_n := \inf\{t \geq 0 : \tilde{\eta}(\tau_n + t, r_{\tau_n + t}) = 1\}.$$

Here, at time $t \geq 0$, $\tilde{\eta}(\tau_n + t)$ is the particle-hole count of those particles or holes in $\eta(\tau_n + t)$ with labels strictly smaller than n . Now define the first regeneration time

$$\kappa_1 := \inf\{\tau_m : m \geq 1, D_m = \infty\}.$$

We then define recursively for $n \geq 2$,

$$\kappa_n := \inf\{\tau_m : \tau_m > \kappa_{n-1}, D_m = \infty\}.$$

Let us call $\delta_0 \in \Upsilon$ any initial condition with rightmost visited site $r = 0$, one particle with label 0 at 0 and none elsewhere, and one hole at each site $x < 0$ each one with label -1 . Given this initial condition, we define D_0 as the first time that one of the holes with label -1 is at the front. Call \mathcal{G}_1 the information up to time κ_1 . We have the following proposition corresponding to Proposition 2. We will in general call η the particle count corresponding to a state $\boldsymbol{\eta}$ and $\Omega' := \{(r, \eta) : r \in \mathbb{Z}, \eta \in \{0, 1\}^{\{\dots, r-1, r\}}\}$.

Proposition 5. *Let F be a Borel subset of $D([0, \infty); \Omega')$. Then, for every nontrivial $\boldsymbol{\eta} \in \Upsilon$, which has only particles with label 0 and holes with label 0,*

$$Q_{\boldsymbol{\eta}}[(r_{\kappa_1+} - r_{\kappa_1}, \tau_{-r_{\kappa_1}} \eta'(\kappa_1 + \cdot)) \in F | \mathcal{G}_1] = Q_{\delta_0}[(r, \eta(\cdot)) \in F | D_0 = \infty],$$

where η' is the particle count of those particles created at time κ_1 or after.

Proof. Note that the events $A \cap \{\kappa_1 = \tau_n\}$ with $A \in \mathcal{F}_{\tau_n}$ for some $n \geq 1$, generate the σ -algebra \mathcal{G}_1 . Therefore, it is enough to prove that

$$(15) \quad \begin{aligned} Q_{\boldsymbol{\eta}}[(r_{\kappa_1+} - r_{\kappa_1}, \tau_{-r_{\kappa_1}} \eta'(\kappa_1 + \cdot)) \in F, A, \kappa_1 = \tau_n] \\ = Q_{\delta_0}[(r, \eta(\cdot)) \in F | D_0 = \infty] Q_{\delta_0}[A, \kappa_1 = \tau_n], \end{aligned}$$

for each natural n and $A \in \mathcal{F}_{\tau_n}$. Now, note that $\{\kappa_1 = \tau_n\} = \{D_n = \infty\} \cap \{D_1 < \infty\} \dots \cap \{D_{n-1} < \infty\}$. But note that for every $1 \leq j < n$, $\{D_n = \infty\} \cap \{D_j < \infty\} = \{D_n = \infty\} \cap \{D_j < \tau_n\}$. Therefore

$$\{\kappa_1 = \tau_n\} = \{D_n = \infty\} \cap \{D_1 < \tau_n\} \dots \cap \{D_{n-1} < \tau_n\}.$$

Defining $B := \{D_1 < \tau_n\} \dots \cap \{D_{n-1} < \tau_n\}$, it follows that the right-hand side of (15) equals

$$\begin{aligned} & Q_{\boldsymbol{\eta}}[(r_{\kappa_1+} - r_{\kappa_1}, \tau_{-r_{\kappa_1}} \eta'(\kappa_1 + \cdot)) \in F, A, \kappa_1 = \tau_n] \\ &= Q_{\boldsymbol{\eta}}[(r_{\tau_n+} - r_{\tau_n}, \tau_{-r_{\tau_n}} \eta'(\tau_n + \cdot)) \in F, A, B, D_n = \infty] \\ &= Q_{\boldsymbol{\eta}}[A, B, Q_{\boldsymbol{\eta}}[(r_{\tau_n+} - r_{\tau_n}, \tau_{-r_{\tau_n}} \eta'(\tau_n + \cdot)) \in F, D_n = \infty | \mathcal{F}_{\tau_n}]]. \end{aligned}$$

Now, by translation invariance and the strong Markov property, $Q_{\boldsymbol{\eta}}[(r_{\tau_n+} - r_{\tau_n}, \tau_{-r_{\tau_n}} \eta'(\tau_n + \cdot)) \in F, D_n = \infty | \mathcal{F}_{\tau_n}] = Q_{\delta_0}[(r, \eta(\cdot)) \in F, D_0 = \infty]$. Hence,

$$(16) \quad \begin{aligned} & Q_{\boldsymbol{\eta}}[(r_{\kappa_1+} - r_{\kappa_1}, \tau_{-r_{\kappa_1}} \eta'(\kappa_1 + \cdot)) \in F, A, \kappa_1 = \tau_n] \\ &= Q_{\boldsymbol{\eta}}[A, D_1 < \tau_n, \dots, D_{n-1} < \tau_n] Q_{\delta_0}[(r, \eta(\cdot)) \in F, D_0 = \infty]. \end{aligned}$$

Choosing $F = \Omega'$ in the above equality we see that, $Q_{\eta}[A, \kappa_1 = \tau_n] = Q_{\eta}[A, D_1 < \tau_n, \dots, D_{n-1} < \tau_n]Q_{\delta_0}[D_0 = \infty]$. Substituting this back into (16) we conclude the proof of the proposition. \square

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