LARGE DEVIATIONS OF THE FRONT IN A ONE DIMENSIONAL MODEL OF $X + Y \to 2X$

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Abstract. We investigate the probabilities of large deviations for the position of the front in a stochastic model of the reaction $X + Y \to 2X$ on the integer lattice in which Y particles do not move while X particles move as independent simple continuous time random walks of total jump rate 2. For a wide class of initial conditions, we prove that a large deviations principle holds and we show that the zero set of the rate function is the interval $[0, v]$, where v is the velocity of the front given by the law of large numbers. We also give more precise estimates for the rate of decay of the slowdown probabilities. Our results indicate a gapless property of the generator of the process as seen from the front, as it happens in the context of nonlinear diffusion equations describing the propagation of a pulled front into an unstable state.

1. Introduction

We consider a microscopic model of a one-dimensional reaction-diffusion equation, with a propagating front representing the passage from an unstable equilibrium to a stable one. It is defined as an interacting particle system on the integer lattice $\mathbb{Z}$ with two types of particles: X particles, that move as independent, continuous time, symmetric, simple random walks with total jump rate $D_X = 2$; and Y particles, which are inert and can be interpreted as random walks with total jump rate $D_Y = 0$. Initially, each site $x = 0, -1, -2, \ldots$ bears a certain number $\eta(x) \geq 0$ of X particles (with at least one site x such that $\eta(x) \geq 1$), while each site $x = 0, 1, \ldots$ bears a fixed number a of particles of type Y (with $1 \leq a < +\infty$). When a site $x = 1, 2, \ldots$ is visited by an X particle for the first time, all the Y particles located at site x are instantaneously turned into X particles, and start moving. The *front* at time t is defined as the rightmost site that has been visited by an X particle up to time t, and is denoted by r_t, with the convention $r_0 := 0$. This model can be interpreted as an infection process, where the X and Y particles represent ill and healthy individuals respectively. It can also be interpreted as a combustion reaction, where the X and Y particles correspond to heat units and reactive molecules respectively, modeling the combustion of a propellant into a stable stationary state. We will denote this model the $X + Y \to 2X$ *front propagation process* with jump rates D_X and D_Y. Within

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the physics literature, a number of studies have been done both numerically and analytically of this process for different values of D_X and D_Y and of corresponding variants where the infection of a Y particle by an X particle at the same site is not instantaneous, drawing analogies with continuous space time nonlinear reaction-diffusion equations having uniformly traveling wave solutions [19], [15, 16, 17], [23], [8]. A particular well-known example is the F-KPP equation studied by Fisher [10] and Kolmogorov, Petrovsky and Piscounov [13].

Mathematically not too much is known. For the case $D_Y = 0$, when $\sum_{x\leq 0} \exp(\theta x)\eta(x) < +\infty$ for a small enough $\theta > 0$, a law of large numbers with a deterministic speed $0 < v < +\infty$ not depending on the initial condition is satisfied (see [22] and [4]):

$$\lim_{t\to+\infty} t^{-1} r_t = v \qquad a.s. \tag{1}$$

In [4] it was proved that the fluctuations around this speed satisfy a functional central limit theorem and that the marginal law of the particle configuration as seen from the front converges to a unique invariant measure as $t \to \infty$. Furthermore, a multi-dimensional version of this process on the lattice $\mathbb{Z}^d$, with an initial configuration having one X particle at the origin and one Y particle at every other site was studied in [22], [1], proving an asymptotic shape theorem as $t \to \infty$ for the set of visited sites. A similar result was proved by Kesten and Sidoravicius [12] for the case $D_X = D_Y > 0$ with a product Poisson initial law. In particular, in dimension $d = 1$ they prove a law of large numbers for the front as in (1). For the case $D_X > D_Y > 0$, even the problem of proving a law of large numbers in dimension $d = 1$ remains open (see [11]).

Within a certain class of one-dimensional nonlinear diffusion equations having uniformly traveling wave solutions describing the passage from an unstable to a stable state, it has been observed that for certain initial conditions the velocity of the front at a given time has a rate of relaxation towards its asymptotic value which is algebraic (see [8], [19] and physics literature references therein). These are the so called *pulled* fronts, whose speed is determined by a region of the profile linearized about the unstable solution. For the F-KPP equation, Bramson [3] proved that the speed of the front at a given time is below its asymptotic value and that the convergence is algebraic. In general, the slow relaxation is due to a gapless property of a linear operator governing the convergence of the centered front profile towards the stationary state. A natural question is wether such a behavior can be observed in the $X + Y \to 2X$ front propagation type processes. Deviations from the law of large numbers of a larger size than those given by central limit theorem should shed some light on such a question: in particular it would be reasonable to expect a large deviations principle with a degenerate rate function, reflecting a slow convergence of the particle configuration as seen from equilibrium towards the unique invariant measure [4]. In this paper, we investigate for the case $D_Y = 0$ the large time asymptotics of the distribution of r_t/t,

$$\mathbb{P}\left[\frac{r_t}{t} \in \cdot\right].$$

Our main result is that a full large deviations principle holds, with a degenerate rate function on the interval $[0, v]$, when the initial condition satisfies the following growth condition:
Assumption (G). For all $\theta > 0$

$$\sum_{x \le 0} \exp(\theta x)\eta(x) < +\infty. \tag{2}$$

Theorem 1. Large Deviations Principle *There exists a rate function* $I : [0, +\infty) \to [0, +\infty)$ *such that, for every initial condition satisfying* **(G)***,*

$$\limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{P}\left[\frac{r_t}{t} \in C\right] \le - \inf_{b \in C} I(b), \qquad \text{for} \quad C \subset [0, +\infty) \quad \text{closed},$$

and

$$\liminf_{t \to +\infty} \frac{1}{t} \log \mathbb{P}\left[\frac{r_t}{t} \in G\right] \ge - \inf_{b \in G} I(b), \qquad \text{for} \quad G \subset [0, +\infty) \quad \text{open}.$$

Furthermore, I *is identically zero on* $[0, v]$*, positive, convex and increasing on* $(v, +\infty)$*.*

It is interesting to notice that the rate function I is independent of the initial conditions within the class **(G)**: the large deviations of the empirical distribution function of the process as seen from the front appear to exhibit a uniform behavior for such initial conditions. Furthermore, this result seems to be in agreement with the phenomenon of slow relaxation of the velocity in the so-called pulled reaction diffusion equations. In [8], a nonlinear diffusion equation of the form

$$\partial_t \phi = \partial_x^2 \phi + f(\phi) \tag{3}$$

is studied where f is a function chosen so that $\phi = 0$ is an unstable state and the equation develops pulled fronts. It is argued that for steep enough initial conditions, the velocity relaxes algebraically towards the asymptotic speed, providing an explicit expansion up to order $O(1/t^2)$. Such a non-exponential decay is explained by the fact that the linearization of (3) around the uniformly translating front, gives a linear equation for the perturbation governed by a gapless Schrödinger operator. The position of the front in the $X + Y \to 2X$ particle system can be decomposed as $r_t = \int_0^t Lg(\eta_s)ds + M_t$, where L is the generator of the centered dynamics, g is an explicit function and M_t is a martingale. The fact that under assumption **(G)** the zero set of the large deviations principle of Theorem 1 is the interval $[0, v]$ is an indication that the symmetrization of L is a gapless operator.

The second result of this paper gives more precise estimates for the probability of the slowdown deviations. Let

$$U(\eta) := \limsup_{x \to -\infty} \frac{1}{\log|x|} \log\left(\sum_{y=0}^{x} \eta(y)\right), \; u(\eta) := \liminf_{x \to -\infty} \frac{1}{\log|x|} \log\left(\sum_{y=0}^{x} \eta(y)\right),$$

and

$$s(\eta) := \min(1, U(\eta)).$$

For the statement of the following theorem we will write U, u, s instead of $U(\eta), u(\eta), s(\eta)$.

Theorem 2. Slowdown deviations estimates. *Let η be an initial condition satisfying* **(G)***. Then the following statements are satisfied.*

(a) *For all $0 \leq c < b < v$, as t goes to infinity,*

$$\mathbb{P}\left[c \leq \frac{r_t}{t} \leq b\right] \geq \exp\left(-t^{s/2+o(1)}\right). \tag{4}$$

(b) *In the special case where $\eta(x) \geq a$ for all $x \leq 0$, one has that, for every $0 \leq b < v$, as t goes to infinity,*

$$\mathbb{P}\left[\frac{r_t}{t} \leq b\right] \leq \exp\left(-t^{1/3+o(1)}\right). \tag{5}$$

(c) *When $u < +\infty$, as t goes to infinity,*

$$\exp\left(-t^{U/2+o(1)}\right) \leq \mathbb{P}\left[r_t = 0\right] \leq \exp\left(-t^{u/2+o(1)}\right). \tag{6}$$

One may notice that the slowdown probabilities considered in (4) and in (6) exhibit distinct behaviors when $u > 1$. Furthermore, the results contained in Theorems 1 and 2 should be compared with the case of the random walk in random environment with positive or zero drift [21, 20].

A natural question is whether it is possible to relax assumption **(G)** in Theorem 1. It appears that even if assumption **(G)** is but mildly violated, the slowdown behavior is not in accordance with that described by Theorem 1. Moreover, if assumption **(G)** is strongly violated, the law of large numbers with asymptotic velocity v breaks down, so that the speedup part of Theorem 1 cannot hold either.

Theorem 3. *The following properties hold:*

(i) *Assume there is a $\theta > 0$ such that*

$$\liminf_{x \to -\infty} \eta(x) \exp(\theta x) = +\infty.$$

Then there exists $b > 0$ such that

$$\limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{P}\left[\frac{r_t}{t} \leq b\right] < 0.$$

(ii) *There exists $\theta' > 0$ and $v' > v$ such that, when*

$$\liminf_{x \to -\infty} \eta(x) \exp(\theta' x) = +\infty,$$

then

$$\mathbb{P}\left[\liminf_{t \to +\infty} \frac{r_t}{t} \geq v'\right] = 1.$$

It is important to stress that the proof of Theorem 1 would not be much simplified if we considered initial conditions with only a finite number of particles. Indeed, condition **(G)** is an assumption which delimits sensible initial data. To prove Theorem 1 we first establish that for initial conditions consisting only of a single particle at the origin, for all $b \geq 0$, the limit

$$\lim_{t \to +\infty} t^{-1} \log \mathbb{P}(r_t \geq bt) \tag{7}$$

exists. The proof of this fact relies on a soft argument based on the sub-additivity property of the hitting times. On the other hand, it is not difficult to show that for b large enough the decay of $\mathbb{P}(r_t \geq bt)$ is exponentially fast. Nevertheless, showing this for b arbitrarily close to but larger than the speed v is a subtler problem. For example, it is not clear how the standard sub-additive arguments could help. Our main tool to tackle this problem is the regeneration structure of the process defined in [4]. To overcome the fact that the regeneration times and positions have only polynomial tails, we couple the original process with one where the X particles have a small bias to the right, so that they jump to the right with probability $1/2 + \epsilon$ for some small $\epsilon > 0$, and the position of the front in the biased process dominates that of the front in the original process. We then use the regeneration structure to study the biased model and how it relates to the original one as ϵ tends to zero. In particular, if v_ϵ is the speed of the biased front, we establish via uniform bounds on the moments of the regeneration times and positions that

$$\lim_{\epsilon \to 0^+} v_\epsilon = v.$$

Furthermore, we show that the regeneration times and positions of the biased model have exponentially decaying tails. Combining these arguments proves that the limit in (7) is positive for any $b > v$. We then establish that this limit exists and has the same value for all initial conditions satisfying **(G)** by exploiting a comparison argument.

To show that the rate function vanishes on $[0, v]$ (and more precisely (5)), we first consider initial conditions having a uniformly bounded number of particles per site. In this case it is essentially enough to observe that the probability that the front remains at zero up to time t is bounded from below by $(1/\sqrt{t})^{t^{1/2+o(1)}}$, since there are at most of the order of $t^{1/2+o(1)}$ random walks that yield a non-negligible contribution to this event. Similar estimates on hitting times of random walks are used to prove (6) and Theorem 3, while more refined arguments are needed to establish (4) for arbitrary initial conditions within the class **(G)**. On the other hand, the proof of the upper bound for the slowdown probabilities (5) in Theorem 2 is more involved, and relies on arguments using the sub-additivity property and the positive association of the hitting times, together with estimates on their tails and their correlations, refining an idea already used in [22] in a similar context.

The rest of the paper is organized as follows. In Section 2, we give a formal definition of the model and introduce its basic structural properties, including sub-additivity and monotonicity of hitting times. In Section 3, we explain how Theorem 1 is proved, building on results proved in other sections. Section 4 is devoted to the proof of the fact that speedup large deviations events have exponentially small probabilities. Section 5 contains our estimates on slowdown probabilities, with the proofs of Theorems 2 and 3. Several appendices contain proofs that are not included in the core of the paper.

2. Construction and basic properties

Throughout the sequel we will use the convention $\inf \emptyset = +\infty$.

2.1. **Construction of the process.**

For our purposes, we have to define on the same probability space not only the original model, but also models including random walks with an arbitrary bias defined through a parameter $0 \le \epsilon < 1/2$.

In the sequel, we assume that we have a reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ giving us access to an i.i.d. family of random variables

$$[(\tau_n(u,i), U_n(u,i));\, n \ge 1,\, u \in \mathbb{Z},\, 1 \le i \le a],$$

such that, for all (n,u,i), $\tau_n(u,i)$ has an exponential(2) distribution, and $U_n(u,i)$ has the uniform distribution on $[0,1]$, and $\tau_n(u,i)$ and $U_n(u,i)$ are independent.

For every $n \ge 1$, $(x,i) \in \mathbb{Z} \times \{1,\dots,a\}$ and $0 \le \epsilon < 1/2$, we let

$$\varepsilon_n(x,i,\epsilon) := 2(\mathbf{1}(U_n(x,i) \le 1/2 + \epsilon)) - 1.$$

Let $(Y^\epsilon_{x,i,t})_{t \ge 0}$ be the continuous-time random walk started at $Y^\epsilon_{x,i,0} := 0$, whose sequence of time steps is $(\tau_n(x,i))_{n \ge 1}$, and whose sequence of space increments is $(\varepsilon_n(x,i,\epsilon))_{n \ge 0}$.

A configuration of particles is a triple $w = (F,r,A)$, where $r \in \mathbb{Z}$, A is a non-empty subset of $\mathbb{Z} \times \{1,\dots,a\}$ such that $\max\{x;\, (x,i) \in A\} \le r$, and $F : A \to \{-\infty,\dots,r\}$ is a map. To every index $(x,i) \in A$ corresponds the position $F(x,i)$ of an X particle, and we say that (x,i) is the birthplace of the corresponding particle. We see that such a configuration carries more information than just the number of X particles at each site, since every X particle is labeled by its birthplace (x,i). Note that, to allow for various types of initial configurations, we do not require that the initial configuration of the model satisfies $F(x,i) = (x,i)$. In fact, any distribution of X particles on $\{\dots,-1,0\}$ with a finite number of particles at each site can be encoded by such a triple $w = (F,r,A)$.

For $w = (F,r,A)$ and $(x,i) \in A$, we use the notation $w(x,i)$ to denote the configuration (F',r',A') with $r' = r$, $A' = \{(x,i)\}$, and $F'(x,i) = F(x,i)$. For a configuration of particles $w = (F,r,A)$ and $q \in \{1,2,\dots,\}$, we define a configuration $w \oplus q = (F',r',A')$ by $A' := A \cup \{r+1,\dots,r+q\} \times \{1,\dots,a\}$, $r' := r+q$, $F' := F$ on A, and $F'(x,i) := x$ for $(x,i) \in \{r+1,\dots,r+q\} \times \{1,\dots,a\}$.

The following definitions list special kinds of configurations that are used in the sequel. For $u \in \mathbb{Z}$ and $1 \le i \le a$, let δ_u be defined by $A := \{(u,1)\}$, $r := u$ and $F(u,1) := u$; let $a\delta_u$ be defined by $A := \{u\} \times \{1,\dots,a\}$, $r := u$, $F(u,i) := u$ for every $1 \le i \le a$; let $\mathcal{I}_u$ be defined by $A := \{-\infty,u\} \times \{1,\dots,a\}$, $r := u$, $F(x,i) := x$ for every $(x,i) \in A$.

For $w=(F,r,A)$ and $\theta>0$, we let

$$f_\theta(w):=\sum_{(x,i)\in A}\exp(\theta(F(x,i)-r)).$$

and η_w be the map defined on $\{\dots,r-1,r\}$ so that $\eta_w(x)$ is the number of particles at site x of the configuration w. Hence

$$\eta_w(x):=\#\{(y,i)\in A;\ F(y,i)=x\}.$$

As a consequence, $f_\theta(w)=\sum_{x\le r}\eta_w(x)\exp(\theta(x-r))$. When $r=0$, we define for $x\le 0$

$$H_w(x):=\sum_{y=0}^{x}\eta_w(y). \tag{8}$$

Now, for every $\theta>0$, let

$$\mathbb{L}_\theta:=\{w=(F,r,A);\ f_\theta(w)<+\infty\}.$$

Observe that $\mathcal{I}_u$, δ_u and $a\delta_u$ belong to $\mathbb{L}_\theta$ for all $u\in\mathbb{Z}$ and $\theta>0$.

For $w=(F,r,A)$ and $(x,i)\in\mathbb{Z}\times\{1,\dots,a\}$, let $\chi_\theta(w,x,i):=\mathbf{1}((x,i)\in A)\exp(\theta(F(x,i)-r))$. We equip $\mathbb{L}_\theta$ with the metric d_θ defined as follows: for $w=(F,r,A)$ and $w'=(F',r',A')$,

$$d_\theta(w,w'):=|r-r'|+\sum_{(x,i)\in\mathbb{Z}\times\{1,\dots,a\}}|\chi_\theta(w,x,i)-\chi_\theta(w',x,i)|.$$

The metric space (L_θ,d_θ) is a Polish space. We let $\mathcal{D}(\mathbb{L}_\theta)$ denote the space of càdlàg functions from $[0,+\infty)$ to $\mathbb{L}_\theta$ equipped with the Skorohod topology and the corresponding Borel $\sigma-$field.

Now, for every $0\le\epsilon<1/2$, and every $w=(F,r,A)\in\mathbb{L}_\theta$, we define a collection of random variables $(X_t^\epsilon(w))_{t\ge0}=(F_t^\epsilon(w),r_t^\epsilon(w),A_t^\epsilon(w))_{t\ge0}$ which describes the time-evolution of the configuration of particles. In order to alleviate notations, the dependence of F_t^ϵ, r_t^ϵ, A_t^ϵ with respect to w will not explicitly mentioned in the sequel when there is no ambiguity. Moreover, we shall often not mention the dependence with respect to ϵ when $\epsilon=0$, and for example, use the notation r_t instead of r_t^0.

The definition is done through the following inductive procedure. Let $\sigma_0^\epsilon:=0$, $r_0^\epsilon:=r$, $A_0^\epsilon:=A$, and for every $t\ge0$ and $(x,i)\in A_0^\epsilon$, let $F_t^\epsilon(x,i):=F(x,i)+Y_{x,i,t}^\epsilon$. Assume that, for some $n\ge1$, we have already defined $\sigma_0^\epsilon\le\ldots\le\sigma_{n-1}^\epsilon$, A_t^ϵ and r_t^ϵ for every $0\le t\le\sigma_{n-1}^\epsilon$, and $F_t^\epsilon(x,i)$ for every $t\ge0$ and $(x,i)\in A^\epsilon_{\sigma^\epsilon_{n-1}}$. Let

$$\sigma_n^\epsilon:=\inf\left\{t>\sigma_{n-1}^\epsilon;\ \text{there is an }(x,i)\in A^\epsilon_{\sigma^\epsilon_{n-1}}\text{ such that }F_t^\epsilon(x,i)=r^\epsilon_{\sigma^\epsilon_{n-1}}+1\right\},$$

Now, for $\sigma_{n-1}^\epsilon<t<\sigma_n^\epsilon$, let $r_t^\epsilon:=r^\epsilon_{\sigma^\epsilon_{n-1}}$, $A_t^\epsilon:=A^\epsilon_{\sigma_{n-1}}$, and let $r^\epsilon_{\sigma^\epsilon_n}:=r^\epsilon_{\sigma^\epsilon_{n-1}}+1$ and $A^\epsilon_{\sigma^\epsilon_n}:=A^\epsilon_{\sigma^\epsilon_n}\cup\{(r^\epsilon_{\sigma^\epsilon_n},i);\ 1\le i\le a\}$. Then, for $x=r^\epsilon_{\sigma^\epsilon_n}$, $i\in\{1,\dots,a\}$, and $t\ge\sigma_n^\epsilon$, let $F_t^\epsilon(x,i):=x+Y_{x,i,t-\sigma_n^\epsilon}$. We shall see that $\sup_n\sigma_n^\epsilon=+\infty$ a.s.

From the results in [4], (where only the case $\epsilon=0$ is treated, but it is immediate to adapt them to the present setting), the following results hold. For any $0\le\epsilon<1/2$ and $w\in\mathbb{L}_\theta$, almost surely with respect to $\mathbb{P}$:

- for every $n \geq 1$, $\sigma_{n-1}^{\epsilon} < \sigma_n^{\epsilon} < +\infty$, and there is a unique $(x,i) \in A_{\sigma_{n-1}^{\epsilon}}$ such that $F_{\sigma_n^{\epsilon}}^{\epsilon}(x,i) = r_{\sigma_n^{\epsilon}}^{\epsilon}$;
- $\lim_{n\to+\infty} \sigma_n^{\epsilon} = +\infty$;
- for all $t \geq 0$, $X_t^{\epsilon}(w) \in \mathbb{L}_\theta$;
- the map $t \mapsto X_t^{\epsilon}(w)$ belongs to $\mathcal{D}(\mathbb{L}_\theta)$.

For any $0 \leq \epsilon < 1/2$, $\theta > 0$, and $w = (F,r,A) \in \mathbb{L}_\theta$, let $\mathbb{Q}_w^{\epsilon,\theta}$ denote the probability distribution of the random process $(X_t^{\epsilon}(w))_{t\geq 0}$, viewed as a random element of $\mathcal{D}(\mathbb{L}_\theta)$. Again, as in [4],

Proposition 1. *For any $0 \leq \epsilon < 1/2$ and $\theta > 0$, the family of probability measures $(\mathbb{Q}_w^{\epsilon,\theta})_{w\in\mathbb{L}_\theta}$ defines a strong Markov process on $\mathbb{L}_\theta$.*

In the sequel, we use $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$ of random variables defined on $(\Omega, \mathcal{F})$. The notation $\mathbb{E}_w^{\epsilon,\theta}$ is used to denote the expectation with respect to $\mathbb{Q}_w^{\epsilon,\theta}$ of random variables defined on $\mathcal{D}(\mathbb{L}_\theta)$ equipped with its Borel $\sigma-$field.

2.2. **Properties of hitting times.** For $w = (F,r,A) \in \mathbb{L}_\theta$, and $u \geq r$, we define the first time that the front touches site u, given that the initial condition was w,

$$T_w^{\epsilon}(u) := \inf\{t > 0;\, r_t^{\epsilon} = u\}.$$

For all $u, v \in \mathbb{Z}$ such that $u < v$, $1 \leq i \leq a$, and $0 \leq \epsilon < 1/2$, let

$$\mathbb{A}^{\epsilon}(u,i,v) := \inf\left\{\sum_{k=1}^{m} \tau_k(u,i);\, u + \sum_{k=1}^{m} \varepsilon_k(u,i,\epsilon) = v,\, m \geq 1\right\}. \tag{9}$$

This represents the first time that the random walk born at (u,i) hits site v (assuming that the walk starts at u at time zero).

Proposition 2. *Let $w = (F,r,A) \in \mathbb{L}_\theta$.*

(i) *For all $u > r$ and $0 \leq \epsilon < 1/2$, $\mathbb{P}-$a.s.*

$$T_w^{\epsilon}(u) = \inf \sum_{j=1}^{L-1} \mathbb{A}^{\epsilon}(x_j, i_j, x_{j+1}),$$

where the infimum is taken over all finite sequences with $L \geq 2$, $x_1, \ldots, x_L \in \mathbb{Z}$ and $i_1, \ldots, i_{L-1}$ such that $x_1 = F(y_1, i_1)$ for some $(y_1, i_1) \in A$, $r < x_2 < \cdots < x_{L-1} < u$, $x_L = u$, $i_2, \ldots, i_{L-1} \in \{1, \ldots, a\}$.

(ii) *For all $u > r$ and $0 \leq \epsilon < 1/2$, the following identity holds $\mathbb{P}-$a.s.*

$$T_w^{\epsilon}(u) = \inf_{(x,i)\in A} T_{w(x,i)}^{\epsilon}(u)$$

(iii) *For all $r < u < v$ and $0 \leq \epsilon < 1/2$, the following sub-additivity property holds $\mathbb{P}-$a.s.*

$$T_w^{\epsilon}(v) \leq T_w^{\epsilon}(u) + T_{w\oplus(u-r)}^{\epsilon}(v).$$

(iv) *For any $0 \leq \epsilon_1 \leq \epsilon_2 < 1/2$, and all $u > r$, $\mathbb{P}-$almost surely, $T_w^{\epsilon_1}(u) \geq T_w^{\epsilon_2}(u)$.*

Proof. The proof of (i) is quite similar to that in [22], and so is the proof that (iii) is a consequence of (i). Then (ii) is a simple consequence of (i). As for (iv), this is an easy consequence of the characterization in (i) and of the fact that, for every $(x,i) \in \mathbb{Z} \times \{1,\dots,a\}$ and $n \geq 1$, $\varepsilon_n(x,i,\epsilon_1) \leq \varepsilon_n(x,i,\epsilon_2)$. □

An immediate consequence of (iv) in the above proposition is the following result.

Corollary 1. *For any $w \in L_\theta$, $0 \leq \epsilon_1 \leq \epsilon_2 < 1/2$, $\mathbb{P}-$almost surely, for all $t \geq 0$, $r_t^{\epsilon_1}(w) \leq r_t^{\epsilon_2}(w)$.*

3. Proof of the large deviations principle for $t^{-1}r_t$

Proposition 3. *There exists a convex function $J : (0,+\infty) \to [0,+\infty)$ such that, for all $b \in (0,+\infty)$,*

$$\lim_{n\to+\infty} n^{-1} \log \mathbb{P}(T^0_{\delta_0}(n) \leq bn) = -J(b).$$

Proof. For any $b > 0$, and all $n \geq 1$, it is easily checked that $\mathbb{P}(T^0_{\delta_0}(n) \leq bn) > 0$. Then let $u_n(b) := \log \mathbb{P}(T^0_{\delta_0}(n) \leq bn)$. Observe that, by subadditivity (part *(iii)* of Proposition 2), $T^0_{\delta_0}(n+m) \leq T^0_{\delta_0}(n) + T^0_{\delta_0 \oplus n}(n+m)$. Now, by part *(ii)* of Proposition 2, $T^0_{\delta_0\oplus n}(n+m) \leq T^0_{\delta_n}(n+m)$, since the infimum characterizing $T^0_{\delta_0\oplus n}(n+m)$ runs over a larger set than the infimum characterizing $T^0_{\delta_n}(n+m)$. As a consequence, $T^0_{\delta_0}(n+m) \leq T^0_{\delta_0}(n) + T^0_{\delta_n}(n+m)$. We deduce that, for all $m,n \geq 1$, and all $b,c > 0$,

$$\{T^0_{\delta_0}(n) \leq bn\} \cap \{T^0_{\delta_n}(n+m) \leq cm\} \subseteq \{T^0_{\delta_0}(n+m) \leq bn+cm\}. \tag{10}$$

Now, observe that $T^0_{\delta_0}(n)$ and $T^0_{\delta_n}(n+m)$ are independent random variables, since their definitions involve disjoint sets of independent random walks. As a consequence,

$$\mathbb{P}(\{T^0_{\delta_0}(n) \leq bn\} \cap \{T^0_{\delta_n}(n+m) \leq cm\}) = \mathbb{P}(T^0_{\delta_0}(n) \leq bn)\mathbb{P}(T^0_{\delta_n}(n+m) \leq cm). \tag{11}$$

¿From the above two relations (10), (11), and the fact that, by translation invariance of the model, $T^0_{\delta_0}(m)$ and $T^0_{\delta_n}(n+m)$ possess the same distribution, we deduce that, for all $m,n \geq 1$, and all $b,c > 0$,

$$u_{n+m}\left(\frac{bn+cm}{n+m}\right) \geq u_n(b) + u_m(c). \tag{12}$$

Applying Inequality (12) above with $c = b$, we deduce that the sequence $(u_n(b))_{n\geq1}$ is super-additive. Since $u_n(b) \leq 0$ for all $n \geq 1$, we deduce from the standard subadditive lemma that there exists a non-negative real number $J(b)$ such that $\lim_{n\to+\infty} n^{-1}u_n(b) = -J(b)$. Moreover, by definition, $b \mapsto u_n(b)$ is non-decreasing, and so $b \mapsto J(b)$ is non-increasing.

To establish that J is convex, consider b,c, such that $0 < b < c$, $t \in (0,1)$, $k \geq 1$, and apply (12) with $n_k := \lceil kt \rceil$ and $m_k := \lfloor k(1-t) \rfloor$. For large enough k, $\frac{bn_k+cm_k}{n_k+m_k} \leq tb+(1-t)c$, so that $u_{n_k+m_k}(tb+(1-t)c) \geq u_{n_k}(b) + u_{m_k}(c)$. Taking the limit as k goes to infinity, we deduce that $J(tb+(1-t)c) \leq tJ(b) + (1-t)J(c)$. □

Proposition 4. *The function J defined in Proposition 3 is identically zero on $[v^{-1}, +\infty)$, positive and decreasing on $(0, v^{-1})$.*

The proof of the above proposition makes use of the following result, which is the main result of Section 4.

Proposition 5. *For any $c > v$,*

$$\limsup_{t \to +\infty} t^{-1} \log \mathbb{P}(r_t^0(\mathcal{I}_0) \geq ct) < 0.$$

Proof Proposition 4. For $n \geq 1$, (ii) of Proposition 2 implies that $T^0_{\mathcal{I}_0}(n) \leq T^0_{\delta_0}(n)$ $\mathbb{P}-$a.s. In view of the immediate identity $\{T^0_w(n) \leq bn\} = \{r^0_{bn}(w) \geq n\}$, we deduce that

$$\mathbb{P}(T^0_{\delta_0}(n) \leq bn) \leq \mathbb{P}(r^0_{bn}(\mathcal{I}_0) \geq n).$$

¿From Proposition 5, we deduce that J is positive on $(0, v^{-1})$. On the other hand, by the law of large numbers (1), we see that J must be identically 0 on $(v^{-1}, +\infty)$. The function J being convex on $(0, +\infty)$, it is also continuous, so that $J(v^{-1}) = 0$. Moreover, as we have already noted, J is non-increasing. These facts imply that J is decreasing on $(0, v^{-1})$. □

Let I be defined by $I(b) := bJ(b^{-1})$ for $b > 0$ and $I(0) := 0$. ¿From the previous results on J, it is easy to deduce the following.

Corollary 2. *The function I is identically zero on $[0, v]$, positive, increasing and convex on $(v, +\infty)$.*

Proof. Only the convexity of I is not totally obvious. Note that, since J is convex, $b \mapsto J(b^{-1})$ is convex on $(0, +\infty)$ as the composition of two convex functions. Then, since $b \mapsto J(b^{-1})$ is also increasing and positive, the convexity of $b \mapsto bJ(b^{-1})$ on $(0, +\infty)$ follows easily.

□

Proposition 6. *Assume that the initial condition w satisfies $r = 0$ and* **(G)**. *Then, for all $b > 0$,*

$$\lim_{n \to +\infty} n^{-1} \log \mathbb{P}(T^0_w(n) \leq bn) = -J(b),$$

where J is the function defined in Proposition 3.

The proof of the proposition makes use of the following lemma.

Lemma 1. *Let $w = (F, r, A) \in \mathbb{L}_\theta$. For all $t \geq 0$, and all $\gamma > 0$,*

$$\mathbb{P}\left(\sup_{(x,i) \in A} \sup_{0 \leq s \leq t} F^0_s(x, i) \geq r + \gamma t \right) \leq f_\theta(w) \exp\left[-g_\gamma(\theta) t\right],$$

where

$$g_\gamma(\theta) := \gamma\theta - 2(\cosh\theta - 1).$$

Proof. Let $G := \left\{\sup_{(x,i)\in A}\sup_{0\le s\le t} F_s^0(x,i) > r+\gamma t\right\}$. For all integers $K \le 0$, let $G_K := \bigcup_{(x,i)\in A;\, F(x,i)\ge K}\{\sup_{0\le s\le t} F_s^0(x,i) > r+\gamma t\}$. Clearly, $K_1 \le K_2$ implies that $G_{K_2} \subset G_{K_1}$, and $\bigcup_{K\le 0} G_K = G$, whence $\mathbb{P}(G) = \lim_{K\to-\infty}\mathbb{P}(G_K)$. Now observe that, for all K, the process $(M_{K,s})_{s\ge 0}$ defined by $M_{K,s} := \sum_{(x,i)\in A;\, F(x,i)\ge K}\exp\left(\theta\left(F_s^0(x,i)-r\right) - 2(\cosh\theta-1)s\right)$ is a càdlàg martingale. Then note that, for all K, $G_K \subset \{\sup_{0\le s\le t} M_{K,s} \ge \exp(g_\gamma(\theta)t\}$, then apply the martingale maximal inequality to deduce that,

$$\mathbb{P}\left(\sup_{0\le s\le t} M_{K,s} \ge \exp(g_\gamma(\theta)t)\right) \le \sum_{(x,i)\in A;\, F(x,i)\ge K}\exp\left[\theta\left(F(x,i)-r\right) - g_\gamma(\theta)t\right].$$

For all K, the r.h.s. of the above inequality is upper bounded by $\le f_\theta(w)\exp\left[-g_\gamma(\theta)t\right]$. The conclusion follows. □

Proof of Proposition 6. Consider $0 < b < v^{-1}$, and fix $\theta > 0$. Choose $\gamma > 0$ large enough so that

$$g_\gamma(\theta)b > J(b).$$

Denote by $w = (F, r, A)$ the initial condition, and consider the set $B_n := \{(x,i);\, F(x,i) \le -\lceil\gamma bn\rceil\}$. Let $m_n := \sum_{(x,i)\in B_n}\exp(\theta(F(x,i) - \lceil\gamma bn\rceil)$. Now let $\Xi_n := \inf\{s \ge 0;\, \exists (x,i) \in B_n,\, F_s^0(x,i) = 0\}$. We see that $\Xi_n \le bn$ implies that $\sup_{(x,i)\in B_n}\sup_{0\le s\le bn} F_s^0(x,i) \ge 0$. Thanks to Lemma 1 and translation invariance of the model, we deduce that

$$\mathbb{P}(\Xi_n \le bn) \le m_n\exp(-g_\gamma(\theta)bn). \tag{13}$$

¿From the fact that w satisfies **(G)**, we obtain that, for all $\varphi > 0$, $y \le 0$, $\#\{(x,i) \in A;\, F(x,i) = y\} \le f_\varphi(w)\exp(-\varphi y)$. As a consequence, whenever $\varphi < \theta$, we have that

$$m_n \le f_\varphi(w)(1-\exp(\varphi-\theta))^{-1}\exp(\varphi\lceil\gamma bn\rceil). \tag{14}$$

Now consider $(x,i) \in A\setminus B_n$, so that $F(x,i) > -\lceil\gamma bn\rceil$. By an easy coupling argument, we see that, since $F(x,i) \le 0$,

$$\mathbb{P}(T_{w(x,i)} \le bn) \le \mathbb{P}(T_{\delta_0} \le bn). \tag{15}$$

Moreover, according to **(G)**,

$$\#A\setminus B_n \le f_\varphi(w)\exp(\varphi\lceil\gamma bn\rceil). \tag{16}$$

Now, by *(ii)* of Proposition 2,

$$\{\Xi_n > bn\}\cap\{T_w(n) \le bn\} \subset \left\{\inf_{(x,i)\in A\setminus B_n} T_{w(x,i)} \le bn\right\}.$$

We deduce from (13), (14), (15), (16) and the union bound that

$$\mathbb{P}(T_w(n) \le bn) \le f_\varphi(w)e^{\varphi\lceil\gamma bn\rceil}\left[(1-\exp(\varphi-\theta))^{-1}\exp(-g_\gamma(\theta)bn) + \mathbb{P}(T_{\delta_0} \le bn)\right]. \tag{17}$$

Now, according to Proposition 3,

$$\lim_{n\to+\infty} n^{-1}\log\mathbb{P}(T^0_{\delta_0}(n) \le bn) = -J(b).$$

Since we have chosen γ so that $g_\gamma(\theta)b > J(b)$, we deduce from (17) that

$$\limsup_{n\to+\infty} n^{-1}\log \mathbb{P}(T_w^0(n) \le bn) \le -J(b) + \varphi\gamma b.$$

Since $\varphi > 0$ is arbitrary, we deduce that

$$\limsup_{n\to+\infty} n^{-1}\log \mathbb{P}(T_w^0(n) \le bn) \le -J(b). \tag{18}$$

On the other hand, consider a given $(x, i) \in A$. Clearly

$$\mathbb{P}(T_w(n) \le bn) \ge \mathbb{P}(T_{w(x,i)}(n) \le bn).$$

Now consider $\tilde{\tau} = \inf\{s \ge 0;\ F_s^0(x, i) = 0\}$. Clearly, $\tilde{\tau}$ is a.s. finite, and, conditional upon $\tilde{\tau}$, $T_{w(x,i)}(n) - \tilde{\tau}$ has the (unconditional) distribution of $T_{\delta_0}(n)$. Choosing any M such that $\mathbb{P}(\{\tilde{\tau} \le M\}) > 0$, one has that $\mathbb{P}(T_{w(x,i)}(n) \le bn) \ge \mathbb{P}(\{\tilde{\tau} \le M\})\mathbb{P}(T_{\delta_0}(n) \le bn - M)$. Taking an arbitrary $c > b$, we deduce that

$$\liminf_{n\to+\infty} n^{-1}\log \mathbb{P}(T_w^0(n) \le bn) \ge -J(c).$$

By continuity of J, we conclude that

$$\liminf_{n\to+\infty} n^{-1}\log \mathbb{P}(T_w^0(n) \le bn) \ge -J(b).$$

The above inequality, together with (18) concludes the proof. □

Proof of Theorem 1. Consider a non-empty closed subset $F \subset [0, +\infty)$, and let $b := \inf F$. Assume that $b \le v$. We have that $\inf_F I = 0$, so the upper bound of the LDP for F is always satisfied. Assume now that $b > v$. One has that $\mathbb{P}(t^{-1}r_t^0(w) \in F) \le \mathbb{P}(r_t^0(w) \ge \lceil tb\rceil) = \mathbb{P}(T_w^0(\lceil tb\rceil) \le t)$. Proposition 6 entails that $\lim_{t\to+\infty} t^{-1}\log\mathbb{P}(T_w^0(\lceil tb\rceil \le t) \le -I(b)$, so that the upper bound of the LDP holds for F since I is non-decreasing.

Consider now an open set $G \subset (v, +\infty)$. For every $b \in G$, there exists an interval $[b, c) \subset G$. By the large deviations upper bound, we know that $\limsup_{t\to+\infty} t^{-1}\log\mathbb{P}(r_t^0(w) \ge bt) \le -I(b)$ and that $\limsup_{t\to+\infty} t^{-1}\log\mathbb{P}(r_t^0(w) \ge ct) \le -I(c)$. By strict monotonicity of I on $(v, +\infty)$, we have that $I(b) < I(c)$, so we can conclude that $\liminf_{t\to+\infty} t^{-1}\log\mathbb{P}(bt \le r_t^0(w) < ct) \ge -I(b)$. As a consequence, $\liminf_{t\to\infty} \frac{1}{t}\mathbb{P}(t^{-1}r_t^0(w) \in G) \ge -I(b)$. Since this holds for an arbitrary $b \in G$, the lower bound of the LDP for G follows.

Consider now a non-empty open set $G \subset [0, +\infty)$ such that $G \cap [0, v] \ne \emptyset$. Then $\inf_G I = 0$. On the other hand, there is a non-empty interval of the form $[c, b) \subset G \cap [0, v]$. In Section 5, we prove that, under Assumption **(G)**,

$$\liminf_{t\to\infty} t^{-1}\log\mathbb{P}\left[c \le \frac{r_t^0(w)}{t} \le b\right] = 0. \tag{19}$$

Applying Inequality (19), we see that $\liminf t^{-1}\log\mathbb{P}(t^{-1}r_t^0 \in G) = 0$, so that the lower bound of the LDP holds. □

4. Speedup probabilities

The main result in this section is Proposition 5:

$$\text{for any } b > v, \ \limsup_{t \to +\infty} t^{-1} \log \mathbb{P}(r_t^0(\mathcal{I}_0) \geq bt) < 0. \tag{20}$$

For the sake of readability, the reference to the initial condition $\mathcal{I}_0$ is often dropped in this section, so that r_t^ϵ should be read as $r_t^\epsilon(\mathcal{I}_0)$.

Our strategy for proving Proposition 5 is to exploit the renewal structure already used in [4] to prove the CLT. However, this renewal structure leads to random variables (renewal time, and displacement of the front at a renewal time) whose tails have polynomial decay (see Appendix 7, and asymptotic exponential bounds such as (20) cannot be derived from such random variables. Whether it is possible to modify the definition of the renewal structure so as to obtain random variables enjoying an exponential decay of the tails, as required for a direct proof of Proposition 5 is unclear and instead we make use of a different idea. Indeed, we apply the renewal structure defined in [4] to a perturbation of the original model, one in which the random walks have a small bias to the right. Again, a law of large numbers holds:

Proposition 7. *For all small enough $\epsilon \geq 0$, there exists $0 < v_\epsilon < +\infty$ such that*

$$\lim_{t \to \infty} t^{-1} r_t^\epsilon = v_\epsilon, \ \mathbb{P} - a.s. \text{ and in } L^1(\mathbb{P}).$$

The interest of introducing a bias to the right is that, reworking the estimates of [4] in this context, we can show that for any small value of the bias parameter $\epsilon > 0$, exponential decay of the tail of the renewal times holds, so that the following result can be proved.

Proposition 8. *There exists $\epsilon_0 > 0$ such that, for any $0 \leq \epsilon \leq \epsilon_0$, for any $b > v_\epsilon$,*

$$\limsup_{t \to +\infty} t^{-1} \log \mathbb{P}(r_t^\epsilon \geq bt) < 0.$$

On the other hand, it is shown in Corollary 1 above that, as expected, biasing the random walks to the right cannot decrease the position of the front, so that at each time t, a comparison holds between the position of the front in the original model and in the model with a bias. We deduce that

Proposition 9. *For any $0 \leq \epsilon < 1/2$ and $t \geq 0$, and all $x \in \{1, 2, \ldots\}$,*

$$\mathbb{P}(r_t^0 \geq x) \leq \mathbb{P}(r_t^\epsilon \geq x).$$

As a consequence, we can prove that (20) holds for all b such that there exists an $0 \leq \epsilon \leq \epsilon_0$ for which $v_\epsilon < b$. Noting that v_ϵ is a non-decreasing function of ϵ, we see that the following result would make our strategy work for all $b > v$:

Proposition 10.

$$\lim_{\epsilon \to 0+} v_\epsilon = v. \tag{21}$$

It is indeed natural to expect such a continuity property to hold, but proving it seems to require substantial work.

Indeed, write

$$v_\epsilon = \lim_{t\to+\infty} t^{-1}\mathbb{E}(r_t^\epsilon). \quad (22)$$

$$v = \lim_{t\to+\infty} t^{-1}\mathbb{E}(r_t^0).$$

For fixed t, it is possible (using the dominated convergence theorem) to prove that

$$\lim_{\epsilon\to 0+} \mathbb{E}(r_t^\epsilon) = \mathbb{E}(r_t^0). \quad (23)$$

Hence, to prove Identity (21), it is enough to prove that

$$\lim_{\epsilon\to 0+}\lim_{t\to+\infty} t^{-1}\mathbb{E}(r_t^\epsilon) = \lim_{t\to+\infty}\lim_{\epsilon\to 0+} t^{-1}\mathbb{E}(r_t^\epsilon).$$

Our strategy for proving Proposition 10 is based on the observation that, if some sort of uniformity with respect to $\epsilon \in [0, \epsilon_0]$ is achieved in (22), then the limits with respect to $\epsilon \to 0+$ and to $t \to +\infty$ in (22)-(23) can be exchanged. Reworking the estimates in [4] to obtain uniform upper bounds (with respect to $0 \le \epsilon \le \epsilon_0$) for the second moments of the random variables (renewal time, and displacement of the front at a renewal time) defined by the renewal structure, we can prove that the required uniformity in (22) holds.

4.1. **Some random variables on $\mathcal{D}(\mathbb{L}_\theta)$.** It will be convenient in the sequel to work with random variables defined on the canonical space of trajectories $\mathcal{D}(\mathbb{L}_\theta)$ rather than on $(\Omega, \mathcal{F}, \mathbb{P})$. We use the ˆ sign in order to make apparent the distinction between random variables defined on Ω and their counterparts. on $\mathcal{D}(\mathbb{L}_\theta)$.

On $\mathcal{D}(\mathbb{L}_\theta)$, we define the following random variables. Let $w. = (w_t)_{t\ge 0} = (\hat{F}_t, \hat{r}_t, \hat{A}_t)_{t\ge 0} \in \mathcal{D}(\mathbb{L}_\theta)$. The random process $(\hat{r}_t)_{t\ge 0}$ is defined through $(w_t)_{t\ge 0} = (\hat{F}_t, \hat{r}_t, \hat{A}_t)_{t\ge 0}$. Under the probability measure $\mathbb{Q}_w^{\epsilon,\theta}$ the process $(\hat{F}_t, \hat{r}_t, \hat{A}_t)_{t\ge 0}$ has the same law as $(F_t^\epsilon, r_t^\epsilon, A_t^\epsilon)_{t\ge 0}$.

For all $s \ge 0$, let $Z_{s,x,i}(w.) := x$ if $(x, i) \notin A_s$, and $Z_{s,x,i}(w.) = \hat{F}_s(x, i)$ otherwise. For $y \in \mathbb{Z}$, let $\hat{T}(y) := \inf\{s \ge 0;\ \hat{r}_s = y\}$ if $y \ge \hat{r}_0 + 1$, and let $\hat{T}(y) := 0$ otherwise. Let also $G_{s,x,i}(w.) := Z_{\hat{T}(x)+s,x,i}$. With respect to $\mathbb{Q}_w^{\epsilon,\theta}$, the processes $(G_{s,x,i})_{s\ge 0}$ form a family independent nearest-neighbor random walks on $\mathbb{Z}$ with jump rate 2 and step distribution $(1/2+\epsilon)\delta_{+1} + (1/2-\epsilon)\delta_{-1}$.

For $z \in \mathbb{Z}$, and $w = (F, r, A) \in \mathbb{L}_\theta$, define $\phi_z(w)$ by

$$\phi_z(w) := \sum_{(x,i)\in A\cap\{\dots,z-1,z\}\times\{1,\dots,a\}} \exp(\theta(F(x,i) - r)),$$

and for $z_1 < z_2 \in \mathbb{Z}$, let

$$m_{z_1,z_2}(w) := \sum_{(x,i)\in A\cap\{z_1+1,\dots,z_2\}\times\{1,\dots,a\}} \mathbf{1}(z_1 + 1 \le F(x,i) \le z_2).$$

We use the notation θ_s to denote the canonical time-shift on $\mathcal{D}(\mathbb{L}_\theta)$ and the notation ϖ_y to denote the truncated space-shift on $\mathcal{D}(\mathbb{L}_\theta)$ defined by $\varpi_y(F, r, A) = (F', r', A')$, with $A' = \{(x-y, i);\ (x,i) \in A,\ x \ge y\}$, $F'(x) := F(x+y)$, $r' := r - y$. In words, this corresponds to removing all the particles that are born at the left of y, and then shifting all birth positions by y. We denote by $(\mathcal{F}_t^{\epsilon,\theta})_{t\ge 0}$ the usual augmentation of the natural filtration on $\mathcal{D}(\mathbb{L}_\theta)$ with respect to the Markov family $(\mathbb{Q}_w^{\epsilon,\theta})_{w\in\mathbb{L}_\theta}$.

4.2. **An elementary speedup estimate.** The following lemma is stated in [4] in the case $\epsilon = 0$, and its adaptation to the more general case $0 \leq \epsilon < 1/2$ is straightforward.

Lemma 2. *Let* $\lambda(\epsilon,\theta) := 2(\cosh\theta - 1) + 4\epsilon\sinh\theta + a(1+2\epsilon)\exp\theta$ *and* $c_\gamma(\epsilon,\theta) := \gamma\theta - \lambda(\epsilon,\theta)$. *For all* $0 \leq \epsilon < 1/2$, $w \in \mathbb{L}_\theta$, *and* $t \geq 0$,

$$\mathbb{Q}_w^{\epsilon,\theta}(\hat{r}_t - \hat{r}_0 \geq \gamma t) \leq \phi_{\hat{r}_0}(w)\exp(-c_\gamma(\epsilon,\theta)t).$$

4.3. **Definition of the renewal structure.** We follow the definition of the renewal structure in [4]. Consider a parameter

$$M := 4(a+9). \tag{24}$$

Let $\nu_0 := 0$ and ν_1 be the first time one of the random walks $\{(G_{s,r_0,i})_{s\geq 0};\ 1 \leq i \leq a\}$, hits the site $\hat{r}_0+1$ (the random walks $(G_{s,x,i})$ are defined in section 4.1). Next, define ν_2 as the first time one of the random walks $\{(G_{s,z,i})_{s\geq 0};\ \hat{r}_0 \leq z \leq \hat{r}_0+1,\ 1 \leq i \leq a\}$ hits the site $\hat{r}_0 + 2$. In general, for $k \geq 2$, we define ν_k as the first time one of the random walks $\{(G_{s,z,i})_{s\geq 0};\ \hat{r}_0 \vee (\hat{r}_0 + k - M) \leq z \leq \hat{r}_0 + k - 1,\ 1 \leq i \leq a\}$, hits the site $\hat{r}_0 + k$. For $n \in \mathbb{N}$, let

$$\tilde{r}_t := \hat{r}_0 + n, \qquad \text{if} \qquad \sum_{k=0}^{n} \nu_k \leq t < \sum_{k=0}^{n+1} \nu_k.$$

The following proposition (see Lemma 1 from [4]), shows that the so-called auxiliary front $\tilde{r}_t$ can be used to estimate the position of the front $\hat{r}_t$.

Proposition 11. *For every* $0 \leq \epsilon < 1/2$, $\theta > 0$ *and* $w \in \mathbb{L}_\theta$, *the following holds* $\mathbb{Q}_w^{\epsilon,\theta}$*−almost surely:*

$$\textit{for every } t \geq 0,\ \tilde{r}_t \leq \hat{r}_t.$$

Now, observe that for any $w = (F, r, A)$ such that $r \times \{1, \ldots, a\} \subset A$ and $F(r,i) = r$ for all $1 \leq i \leq a$, with respect to $\mathbb{Q}_w^{\theta,\epsilon}$, for each $1 \leq j \leq M-1$, the random variables $(\nu_i)_{i\geq 1}$ are a.s. finite, and that the random variables $\{\nu_{Mk+j} : k \geq 1\}$ are i.i.d. and have finite expectation since $M \geq 3$. We deduce that a.s. (see also [5]),

$$\lim_{t\to\infty} \tilde{r}_t/t =: \alpha(\epsilon) > 0.$$

First note that $\alpha(\epsilon)$ does not depend on θ nor on w since the distribution of the random walks $(G_{s,x,i})_{s\geq 0}$ with respect to $\mathbb{Q}_w^{\theta,\epsilon}$ does not. Moreover, $\alpha(\epsilon)$ is a non-decreasing function of ϵ by an immediate coupling argument.

Now consider $\epsilon_0 < 1/2, \theta > 0, \alpha_1, \alpha_2 > 0$ such that

$$\begin{cases} 0 < \alpha_1 < \alpha_2 < \alpha(0), \\ \theta^{-1}(2(\cosh\theta - 1) + 4\epsilon_0\sinh\theta) < \alpha_1, \\ 4\epsilon_0 < \alpha_1. \end{cases} \tag{25}$$

In the sequel, we always assume that $0 \leq \epsilon \leq \epsilon_0$.

Let us define the following random variables on $\mathcal{D}(\mathbb{L}_\theta)$:

$$\begin{cases} U(w.) := \inf\{t \geq 0;\, \tilde{r}_t - \hat{r}_0 < \lfloor \alpha_2 t \rfloor\}, \\ V(w.) := \inf\{t \geq 0;\, \max_{\hat{r}_0 - L+1 \leq z \leq \hat{r}_0 - 1} Z_{t,x,i} > \lfloor \alpha_1 t \rfloor + \hat{r}_0\}, \\ W(w.) := \inf\{t \geq 0;\, \phi_{\hat{r}_0 - L}(w_t) \geq e^{\theta(\lfloor \alpha_1 t \rfloor - (\hat{r}_t - \hat{r}_0))}\}. \end{cases}$$

Note that, for all ϵ, U, V, W are stopping times with respect to $(\mathcal{F}_t^{\epsilon,\theta})_{t\geq 0}$, and that they are mutually independent with respect to $\mathbb{Q}_w^{\theta,\epsilon}$.

Let

$$D := \min(U, V, W).$$

Now let $p > 0$ be such that

$$p \exp(\theta) < 1,$$

and L such that

$$L^{1/4} \geq M + 1 \quad \text{and} \quad a \exp(-L\theta)(1 - \exp(-\theta))^{-1} < p. \tag{26}$$

For $x \in \mathbb{Z}$, let

$$J_x(w.) := \inf\{j \geq 1;\, \phi_{x+(j-1)L}(w_{\hat{T}(x+jL)}) \leq p,\, m_{x+jL-L^{1/4},x+jL}(w_{\hat{T}(x+jL)}) \geq a\lfloor L^{1/4} \rfloor/2\}.$$

Let $S_0 := 0$ and $R_0 := \hat{r}_0$. Then define for $k \geq 0$,

$$S_{k+1} := \hat{T}(R_k + J_{R_k} L), \qquad D_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}, \qquad R_{k+1} = r_{D_{k+1}}$$

$$K := \inf\{k \geq 1 : S_k < \infty, D_k = \infty\},$$

and define the *regeneration time*

$$\kappa := S_K,$$

Note that κ is *not* a stopping time with respect to $(\mathcal{F}_t^{\epsilon,\theta})_{t\geq 0}$. Define $\mathcal{G}^{\epsilon,\theta}$, the information up to time κ, as the smallest σ-algebra containing all sets of the form $\{\kappa \leq t\} \cap A$, $A \in \mathcal{F}_t^{\epsilon,\theta}$, $t \geq 0$.

4.4. **Properties of the renewal structure.** Throughout this section, we assume that $\theta, \alpha_1, \alpha_2, \epsilon_0$ satisfy the assumptions listed in part 4.3.

Proposition 12. *The following properties hold:*

(i) *There exist $0 < C, L^* < +\infty$ not depending on ϵ (but possibly depending on the choice of $\theta, \alpha_1, \alpha_2, \epsilon_0$) such that, for $L := L^*$, and all $0 \leq \epsilon \leq \epsilon_0$,*

$$\mathbb{E}_{\mathcal{I}_0}^{\epsilon,\theta}(\kappa^2) \leq C,\, \mathbb{E}_{a\delta_0}^{\epsilon,\theta}(\kappa^2 | U = +\infty) \leq C,$$

$$\mathbb{E}_{\mathcal{I}_0}^{\epsilon,\theta}((\hat{r}_\kappa)^2) \leq C \quad \text{and} \quad \mathbb{E}_{a\delta_0}^{\epsilon,\theta}((\hat{r}_\kappa)^2 | U = +\infty) \leq C.$$

(ii) *For all $0 < \epsilon \leq \epsilon_0$, there exist $0 < C(\epsilon), L(\epsilon), t(\epsilon) < +\infty$ such that, for $L := L(\epsilon)$,*

$$\mathbb{E}_{\mathcal{I}_0}^{\epsilon,\theta}(\exp(t(\epsilon)\kappa)) \leq C(\epsilon),\, \mathbb{E}_{a\delta_0}^{\epsilon,\theta}(\exp(t(\epsilon)\kappa) | U = +\infty) \leq C(\epsilon),$$

$$\mathbb{E}_{\mathcal{I}_0}^{\epsilon,\theta}(\exp(t(\epsilon)\hat{r}_\kappa)) \leq C(\epsilon) \quad \text{and} \quad \mathbb{E}_{a\delta_0}^{\epsilon,\theta}(\exp(t(\epsilon)\hat{r}_\kappa) | U = +\infty) \leq C(\epsilon).$$

Proposition 12 provides the key estimates needed for the proof of the main results in this section. Most of the technical work needed to prove it consists in a reworking of the estimates in [4], either proving that, for each positive value of the bias parameter ϵ, exponential estimates can be obtained instead of the polynomial ones derived in [4], or that the polynomial estimates already obtained in [4] can be made uniform with respect to $0 \leq \epsilon \leq \epsilon_0$. The proofs go along the lines of [4], and are deferred to Appendix 9. In the sequel, we always assume that either $L := L^*$ or $L := L(\epsilon)$. As a consequence of Proposition 12 we see that for all $0 \leq \epsilon \leq \epsilon_0$, $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(0 < \kappa < +\infty) = 1$ and $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(0 < \kappa < +\infty | U = +\infty) = 1$.

As in [4], the following propositions and corollary can be proved.

Proposition 13. *Let $0 \leq \epsilon \leq \epsilon_0$. If $w = \mathcal{I}_0$ or $w = a\delta_0$, then for any Borel subset Γ of $\mathcal{D}(\mathbb{L}_\theta)$,*

$$\mathbb{Q}^{\epsilon,\theta}_{w}(\varpi_{\hat{r}_\kappa}(w_{\kappa+t})_{t\geq 0} \in \Gamma | \mathcal{G}^{\epsilon,\theta}) = \mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\Gamma | U = +\infty) \qquad \mathbb{Q}^{\epsilon,\theta}_{w} - a.s.$$

Define $\kappa_1 := \kappa$ and for $i \geq 1$, $\kappa_{i+1} := \kappa_i + \kappa \circ \theta_{\kappa_i}$. Now, for all $i \geq 1$, define $\mathcal{G}^{\epsilon,\theta}_i$ as the smallest σ-algebra containing all sets of the form $\{\kappa_i \leq t\} \cap A$, $A \in \mathcal{F}^{\epsilon,\theta}_t$, $t \geq 0$.The following general version of Proposition 13 holds.

Proposition 14. *Let $0 \leq \epsilon \leq \epsilon_0$ and $i \geq 1$. If $w = \mathcal{I}_0$ and $w = a\delta_0$ then for any Borel subset Γ of $\mathcal{D}(\mathbb{L}_\theta)$,*

$$\mathbb{Q}^{\epsilon,\theta}_{w}(\varpi_{\hat{r}_{\kappa_i}}(w_{\kappa_i+t})_{t\geq 0} \in \Gamma | \mathcal{G}^{\epsilon,\theta}_i) = \mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\Gamma | U = +\infty) \qquad \mathbb{Q}^{\epsilon,\theta}_{w} - a.s.$$

Corollary 3. *The following properties hold:*

(i) *Under $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}$, $\kappa_1, \kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \dots$ are independent, and $\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \dots$ are identically distributed with law identical to that of κ under $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\cdot | U = +\infty)$.*

(ii) *Under $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}$, $\hat{r}_{\kappa_1}, \hat{r}_{\kappa_2} - \hat{r}_{\kappa_1}, \hat{r}_{\kappa_3} - \hat{r}_{\kappa_2}, \dots$ are independent, and $\hat{r}_{\kappa_2} - \hat{r}_{\kappa_1}, \hat{r}_{\kappa_3} - \hat{r}_{\kappa_2}, \dots$ are identically distributed with law identical to that of r_κ under $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\cdot | U = +\infty)$.*

We now give the proofs of Propositions 7, 8 and 10.

Proof of Proposition 7. First, note that the $\mathbb{P}-$a.s. convergence stated in Proposition 7 follows from the integrability of renewal times by a standard argument. To prove that the convergence also takes place in $L^1(\mathbb{P})$, we note that, from Lemma 2 above, it stems that $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_t) < +\infty$ for all t and that the family of random variables $(t^{-1}\hat{r}_t)_{t\geq 1}$ is uniformly integrable with respect to $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}$. The convergence in $L^1(\mathbb{P})$ then follows from the $\mathbb{P}-$a.s. convergence.

□

Proof of Proposition 8. Fix $0 < \epsilon \leq \epsilon_0$, and let $L := L(\epsilon)$. For all $t \geq 0$, define $a(t) := \sup\{n \geq 1; \kappa_n \leq t\}$, with the convention that $\sup \emptyset = 0$. From Corollary 3 and Proposition 12, we deduce that, $a(t) < +\infty$ a.s. for all $t \geq 0$ and that $\lim_{t\to+\infty} a(t) = +\infty$ a.s. Using the fact that the map $t \mapsto \hat{r}_t$ is non-decreasing, we have that $\hat{r}_t \leq \hat{r}_{\kappa_{a(t)+1}}$. Now observe that, for any $0 < \epsilon \leq \epsilon_0$, any $b > v_\epsilon$, and any $0 < c < +\infty$, by the union bound,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_t \geq bt) \leq \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(a(t) \geq \lfloor ct \rfloor) + \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{\kappa_{\lfloor ct \rfloor+1}} \geq bt).$$

Note that $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(a(t) \geq \lfloor ct \rfloor) \leq \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa_{\lfloor ct \rfloor} \leq t)$, and observe that, by a standard large deviations bound for the i.i.d non-negative sequence $(\kappa_{i+1}-\kappa_i)_{i\geq 1}$ and Proposition 12 for κ_1, whenever $c^{-1} < \mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty)$, $\limsup_{t\to+\infty} t^{-1} \log \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa_{\lfloor ct \rfloor} \leq t) < 0$. On the other hand, writing $\hat{r}_{\kappa_{\lfloor ct \rfloor+1}} = \hat{r}_{\kappa_1} + \sum_{i=1}^{\lfloor ct \rfloor+1}(\hat{r}_{\kappa_{i+1}} - \hat{r}_{\kappa_i})$, and using Proposition 12, together with a standard large deviations argument (see e.g. [6]), we have that, as soon as $b/c > \mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_\kappa | U = +\infty)$, $\limsup_{t\to+\infty} t^{-1} \log \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{\kappa_{\lfloor ct \rfloor+1}} \geq bt) < 0$.

Note that we can deduce from the renewal structure that

$$v_\epsilon = \frac{\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_\kappa | U = +\infty)}{\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty)}. \tag{27}$$

As a consequence, if $b > v_\epsilon$, we see that we can choose a $c > 0$ such that $c^{-1} < \mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty)$ and $b/c > \mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_\kappa | U = +\infty)$. □

Lemma 3. *There exists $0 < c < +\infty$ such that, for all $0 \leq \epsilon \leq \epsilon_0$,*

$$\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty) \geq c.$$

Proof. Use the fact that, by definition, $\kappa \geq \hat{T}(1)$, so that $\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty) \geq \mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\hat{T}(1)\mathbf{1}(U = +\infty))$. Now, by coupling, $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(U = +\infty) \geq \mathbb{Q}^{0,\theta}_{a\delta_0}(U = +\infty)$ for all $0 \leq \epsilon \leq \epsilon_0$. By coupling again, for all $u > 0$, $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\hat{T}(1) \geq u) \geq \mathbb{Q}^{\epsilon_0,\theta}_{a\delta_0}(\hat{T}(1) \geq u)$. Now, since $\mathbb{Q}^{\epsilon_0,\theta}_{a\delta_0}(\hat{T}(1) > 0) = 1$, we can find $u > 0$ small enough so that $\mathbb{Q}^{\epsilon_0,\theta}_{a\delta_0}(\hat{T}(1) \geq u) \geq 1 - (1/2)\mathbb{Q}^{0,\theta}_{a\delta_0}(U = +\infty)$. Putting the previous inequalities together, we see that, for all $0 \leq \epsilon \leq \epsilon_0$, $\mathbb{Q}^{0,\theta}_{a\delta_0}(\hat{T}(1) \geq u, U = +\infty) \geq (1/2)\mathbb{Q}^{0,\theta}_{a\delta_0}(U = +\infty)$. The conclusion follows. □

The following proposition contains the uniform convergence estimate that is required for the proof of Proposition 10. Broadly speaking, the idea is to control the convergence speed with second moment estimates on the renewal structure, so that uniform estimates on these moments yield uniform estimates on the convergence speed.

Proposition 15. *For all $\zeta > 0$, there exists $t_\zeta \geq 0$ such that, for all $t \geq t_\zeta$ and all $0 \leq \epsilon \leq \epsilon_0$,*

$$v_\epsilon \leq \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) + \zeta.$$

Proof. Let $0 < \lambda < 1$ be given, and let

$$m(t,\epsilon) := \left\lfloor (1-\lambda)t \left(\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty)\right)^{-1} \right\rfloor.$$

In the rest of the proof, we write m instead of $m(t,\epsilon)$ for the sake of readability. Note that, in view of Proposition 12, for all $0 \le \epsilon \le \epsilon_0$, $\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty) \le C^{1/2}$, so that $m \ge 1$ as soon as $t \ge C^{1/2}(1-\lambda)^{-1}$, which does not depend on ϵ.

We now re-use the random variables $a(t)$ defined in the proof of Proposition 8 above. Using the fact that $t \mapsto \hat{r}_t$ is non-decreasing, we see that $\hat{r}_t \ge \hat{r}_{\kappa_{a(t)}}$. Moreover, $\hat{r}_{\kappa_{a(t)}} \ge \hat{r}_{\kappa_{a(t)}} \mathbf{1}(a(t) \ge m)$, and $\hat{r}_{\kappa_{a(t)}} \mathbf{1}(a(t) \ge m) \ge \hat{r}_{\kappa_m} \mathbf{1}(a(t) \ge m)$ when $m \ge 1$. Taking expectations, we deduce that, when $m \ge 1$,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) \ge \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_{\kappa_m}) - \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_{\kappa_m}\mathbf{1}(a(t) < m)). \tag{28}$$

Consider the first term in the r.h.s. of (28) above, and observe that

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{\kappa_m}) = \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{\kappa}) + (m-1)\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_{\kappa} | U = +\infty).$$

¿From Proposition 12, $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_\kappa) \le C^{1/2}$ for all $0 \le \epsilon \le \epsilon_0$. Moreover, from Identity (27), $\left(\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_\kappa | U = +\infty)\right)\left(\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa | U = +\infty)\right)^{-1} = v_\epsilon$. We easily deduce that, as t goes to infinity, uniformly with respect to $0 \le \epsilon \le \epsilon_0$,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{\kappa_m}) = (1-\lambda)tv_\epsilon + O(1). \tag{29}$$

Consider now the second term in the r.h.s. of (28). By Schwarz's inequality,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_{\kappa_m}\mathbf{1}(a(t) < m)) \le \left(\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}\left[(t^{-1}\hat{r}_{\kappa_m})^2\right]\right)^{1/2} \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(a(t) < m)^{1/2}. \tag{30}$$

¿From Proposition 12 and Corollary 3, it is easily checked that

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}\left[(\hat{r}_{\kappa_m})^2\right] \le Cm^2. \tag{31}$$

On the other hand, one has that $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(a(t) < m) \le \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa_m \ge t)$. ¿From Proposition 12 and Corollary 3, the variance of κ_m with respect to $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}$ is bounded above by Cm, so that we can use the Bienaymé-Chebyshev's inequality to prove that, whenever $t > \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa_m)$,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(a(t) < m) \le Cm(t - \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa_m))^{-2}. \tag{32}$$

Now, using Proposition 12 as in the proof of (29) above, we can easily prove that, as t goes to infinity, uniformly with respect to $0 \le \epsilon \le \epsilon_0$,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa_m) = (1-\lambda)t + O(1).$$

Putting the above identity together with (32), (31) and (30), we deduce that, as t goes to infinity, uniformly with respect to $0 \le \epsilon \le \epsilon_0$,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_{\kappa_m}\mathbf{1}(a(t) < m)) \le Cm^{3/2}(\lambda t^2 + O(t))^{-1}.$$

In view of Lemma 3, we have that $m \le c^{-1}t$ for all $0 \le \epsilon \le \epsilon_0$, so we can conclude that, uniformly with respect to $0 \le \epsilon \le \epsilon_0$,

$$\lim_{t \to +\infty} \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_{\kappa_m}\mathbf{1}(a(t) < m)) = 0. \tag{33}$$

Plugging (29) and (33) in (28), we finally deduce that, as t goes to infinity, uniformly with respect to $0 \leq \epsilon \leq \epsilon_0$,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) \geq (1-\lambda)v_\epsilon + o(1).$$

The conclusion of the Proposition follows by noting that, since $v_\epsilon \leq v_{\epsilon_0}$, $(1-\lambda)v_\epsilon \geq v_\epsilon - \lambda v_{\epsilon_0}$.

□

Lemma 4. *For all $t \geq 0$,*

$$\lim_{\epsilon\to 0+} \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}^\epsilon_t) = \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}^0_t).$$

Proof. Consider a given $t \geq 0$. By Proposition 17 in Appendix 8, with $\mathbb{P}$ probability one, we can find a (random) $K \leq 0$ such that $\sup\{F^{\epsilon_0}_s(x,i);\, 0 \leq s \leq t,\, x < K,\, 1 \leq i \leq a\} \leq 0$, so that $\sup\{F^{\epsilon}_s(x,i);\, 0 \leq s \leq t,\, x < K,\, 1 \leq i \leq a\} \leq 0$ for all $0 \leq \epsilon \leq \epsilon_0$. As a consequence, for all $0 \leq \epsilon \leq \epsilon_0$, with probability one, $r^\epsilon_t(\mathcal{I}_0) = r^\epsilon_t(w(K))_{s\geq 0})$, where $w(K)$ is the configuration defined by $A = \{K,\dots,0\} \times \{1,\dots,a\}$, $r = 0$ and $F(x,i) = x$ for all $(x,i) \in A$.

Since, for every $0 \leq \epsilon \leq \epsilon_0$, with probability one $r^\epsilon_t \leq r^{\epsilon_0}_t$, we see that the value of r^ϵ_t is entirely determined by the trajectories up to time t of the random walks born at sites (x,i) with $K \leq x \leq r^{\epsilon_0}_t$. With probability one again, we are dealing with a finite number of random walks, and a finite number of steps. We now see that, for all ϵ small enough, these trajectories are identical to what they are for $\epsilon = 0$, so that $r^\epsilon_t = r^0_t$. Since $0 \leq r^\epsilon_t \leq r^{\epsilon_0}_t$ and $r^{\epsilon_0}_t$ is integrable w.r.t. $\mathbb{P}$, we can use the dominated convergence theorem to deduce the conclusion. □

Proof of Proposition 10. Let $\zeta > 0$, and, following Proposition 15, consider a t_ζ such that, for all $t \geq t_\zeta$ and all $0 \leq \epsilon \leq \epsilon_0$,

$$v_\epsilon \leq \mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) + \zeta.$$

Consider now, thanks to Proposition 7, a $t \geq t_\zeta$ such that $\mathbb{E}^{0,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) \leq v + \zeta$. Now, thanks to Lemma 4, we know that, for all ϵ small enough,

$$\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) \leq \mathbb{E}^{0,\theta}_{\mathcal{I}_0}(t^{-1}\hat{r}_t) + \zeta.$$

Putting together the above inequalities, we deduce that, for all ϵ small enough, $v_\epsilon \leq v + 3\zeta$. Since $v_\epsilon \geq v$, the conclusion follows.

□

Now Proposition 5 follows from Proposition 8, Proposition 9 and Proposition 10, as explained in the beginning of this section.

5. Slowdown large deviations

For $x \geq 1$ and $t \geq 0$, let $(\zeta_t)_{t\geq 0}$ denote a continuous time simple symmetric random walk starting form 0 of total jump rate 2. Let

$$\bar{G}_t(x) := P\left(\sup_{s\in[0,t)} \zeta_s < x\right),\; G_t(x) := P\left(\zeta_t \geq x\right).$$

In the sequel we will use the fact that for fixed $t\geq 0$, $G_t(\cdot)$ is non-decreasing and $\bar{G}_t(\cdot)$ is non-increasing, and that, thanks to the reflection principle,

$$1-\bar{G}_t(x)=2G_t(x)-P\left(\zeta_t=x\right). \tag{34}$$

5.1. **Proof of Theorem 2 (a) and (c).** We start with the proof of Theorem 2 (c). The fact that $r_t=0$ means that no particle in the initial configuration hits 1 before time t. Both the upper and lower bounds can then be understood heuristically as follows. Since we consider simple symmetric random walks, for large t, the constraint of not hitting 1 before time t has a cost only for particles within a distance of order $t^{1/2}$ of the origin. Now these particles perform independent random walks, and their number has an order of magnitude lying between $t^{u(\eta_w)/2}$ and $t^{U(\eta_w)/2}$.

We start with the lower bound. When $U=+\infty$, the inequality holds trivially, so we assume in the sequel that $U<+\infty$. The event $t^{-1}r_t(w)=0$, implies that none of the random walks corresponding to particles in the initial condition w hit 1 before time t. By independence of the random walks, the corresponding probability equals

$$\prod_{x=0}^{-\infty}\bar{G}_t(-x+1)^{\eta_w(x)}.$$

Now let $b_1>0$ be such that $1-2s\geq\exp\left(-4s\right)$ for all $0\leq s\leq b_1$. From (34), we see that for any $t\geq 0$ and $y\leq 0$, $\bar{G}_t(-y+1)\geq 1-2G_t(-y+1)$. By the central limit theorem, we can find t_0 and $K>0$ such that, for all $t\geq t_0$ and $y\leq -Kt^{1/2}$, $G_t(-y+1)\leq b_1$. Let $k_t:=\lceil Kt^{1/2}\rceil$. Then, for all $t\geq t_0$

$$\prod_{x=-k_t}^{-\infty}\bar{G}_t(-x+1)^{\eta_w(x)}\geq\exp\left(-4\sum_{x=-k_t}^{-\infty}\eta_w(x)G_t(-x+1)\right).$$

Now, by definition of G_t,

$$\begin{aligned}\sum_{x=0}^{-\infty}\eta_w(x)G_t(-x+1)=&\quad E\left(\textstyle\sum_{x=0}^{-\infty}\eta_w(x)\mathbf{1}(\zeta_t\geq -x+1)\right)\\ =E\left[\mathbf{1}(\zeta_t\geq 1)\left(\sum_{x=0}^{-\zeta_t+1}\eta_w(x)\right)\right]=&\quad E\left[\mathbf{1}(\zeta_t\geq 1)(H_w(-\zeta_t+1))\right].\end{aligned}$$

By assumption, $H_w(x)\leq |x|^{U+o(1)}$. Hölder's inequality yields that

$$E\left[\mathbf{1}(\zeta_t\geq 1)(H_w(-\zeta_t+1))\right]\leq t^{U/2+o(1)}.$$

We deduce that for all $t\geq t_0$

$$\prod_{x=-k_t}^{-\infty}\bar{G}_t(x)^{\eta_w(x)}\geq\exp(-t^{U/2+o(1)}). \tag{35}$$

Now, for $-k_t<y\leq 0$, observe that $\bar{G}_t(-y+1)\geq\bar{G}_t(1)$. As a consequence,

$$\prod_{x=0}^{-k_t+1}\bar{G}_t(-x+1)^{\eta_w(x)}\geq\bar{G}_t(1)^{H_w(-k_t+1)}.$$

But there exists $c_4 > 0$, such that, for large enough t, $\bar{G}_t(1) \geq c_4 t^{-1/2}$. Using again the fact that $H_w(x) \leq |x|^{U+o(1)}$, it is easy to deduce that $\bar{G}_t(1)^{H_w(-k_t+1)} \geq \exp(-t^{U/2+o(1)})$, whence

$$\prod_{x=0}^{-k_t+1} \bar{G}_t(-x+1)^{\eta_w(x)} \geq \exp(-t^{U/2+o(1)}). \tag{36}$$

From (35) and (36), we deduce that

$$\mathbb{P}(t^{-1} r_t^0(w) \leq 0) \geq \exp(-t^{U/2+o(1)}).$$

Now, let us prove the upper bound when $u < +\infty$. Using an argument similar to the one used in the proof of the lower bound above, we easily obtain that

$$\mathbb{P}(t^{-1} r_t(w) = 0) \leq \exp\left(-E\left[\mathbf{1}(\zeta_t \geq 1)(H_w(-\zeta_t+1))\right]\right).$$

It is then easy to deduce that

$$E\left[\mathbf{1}(\zeta_t \geq 1)(H_w(-\zeta_t+1))\right] \geq t^{u/2+o(1)},$$

and the upper bound follows.

We now turn to the proof of Theorem 2 (a). The idea of the proof when $s(\eta) = 1/2$ is to combine the following two arguments. First, for $b > 0$, it costs nothing to prevent all the particles in the initial condition from hitting $\lfloor bt \rfloor$ up to time t. Intuitively, this result comes from the fact that hitting $\lfloor bt \rfloor$ before time t has an exponential cost for any particle in the initial condition within distance $O(t)$ of the origin, and, due to **(G)**, there is a subexponentially large number of such particles.

Second, in the worst case where all the particles attached to sites $1 \leq x \leq bt$ are turned into X particles instantaneously at time zero, the cost of preventing all these particles from hitting bt up to time t is of order $\exp(-t^{1/2+o(1)})$, due to the lower bound in (6) proved above, The actual proof is in fact more complex since we want to consider probabilities of the form $\mathbb{P}(ct \leq r_t \leq bt)$, and not only $\mathbb{P}(r_t \leq bt)$, and deal also with the case $s(\eta) < 1/2$.

We state two lemmas before giving the proof.

Lemma 5. *Consider an initial condition $w = (F, 0, A)$ satisfying assumption* **(G)***. Then, for all $b > 0$, and all $\varphi > 0$,*

$$\mathbb{P}\left[\max_{(x,i)\in A} \sup_{0\leq s\leq t} F_s(x,i) \geq bt\right] \leq f_\varphi(w) \exp\left[t(\cosh(2\varphi)-1)\right] G_t(\lfloor bt \rfloor)^{1/2}.$$

Proof. The probability we are looking at is the probability that at least one of the random walks corresponding to particles in w exceeds bt before time t. By the union bound, this probability is smaller than

$$\sum_{x=0}^{-\infty} \eta_w(x)(1-\bar{G}_t(-x+\lfloor bt \rfloor)) \leq \sum_{x=0}^{-\infty} 2\eta_w(x) G_t(-x+\lfloor bt \rfloor)).$$

Now observe that by definition of G_t,

$$\sum_{x=0}^{-\infty} \eta_w(x) G_t(-x+\lfloor bt \rfloor) = E\left(\sum_{x=0}^{-\infty} \eta_w(x) \mathbf{1}(\zeta_t \geq -x + \lfloor bt \rfloor) \right)$$

$$= E\left[\mathbf{1}(\zeta_t \geq \lfloor bt \rfloor) \left(\sum_{x=0}^{-\zeta_t + \lfloor bt \rfloor} \eta_w(x) \right) \right] = E\left[\mathbf{1}(\zeta_t \geq \lfloor bt \rfloor)(H_w(-\zeta_t + \lfloor bt \rfloor)) \right].$$

¿From assumption **(G)**, we deduce that, for all $\varphi > 0$, $H_w(x) \leq f_\varphi(w)\exp(-\varphi x)$ for all $x \leq 0$. As a consequence, when $\zeta_t \geq \lfloor bt \rfloor$, $H_w(-\zeta_t + \lfloor bt \rfloor) \leq H_w(-\zeta_t) \leq f_\varphi(w)\exp(\varphi \zeta_t)$. Applying Schwarz's inequality, we see that

$$E\left[\mathbf{1}(\zeta_t \geq \lfloor bt \rfloor)(H_w(-\zeta_t + \lfloor bt \rfloor)) \right] \leq \mathbb{P}\left(\zeta_t \geq \lfloor bt \rfloor \right)^{1/2} f_\varphi(w) E\left[\exp(2\varphi \zeta_t) \right]^{1/2}.$$

Now note that $E\left[\exp(2\varphi\zeta_t)\right] = \exp[2(\cosh(2\varphi) - 1)t]$. □

Lemma 6. *Consider an initial condition $w = (F, 0, A)$ satisfying assumption* **(G)**. *Then, for all $\varphi > 0$*

$$\mathbb{E}\left[\sum_{(x,i) \in A_t} \exp\left(\varphi(F_t(x,i) - r_t) \right) \right] \leq \exp[2(\cosh(\varphi) - 1)t] f_\varphi(w) + a\mathbb{E}(r_t).$$

Proof. Write $\sum_{(x,i)\in A_t} = \sum_{(x,i)\in A} + \sum_{(x,i)\in A_t \setminus A}$. For $(x,i) \in A$, observe that $\exp\left(\varphi(F_t(x,i) - r_t)\right) \leq \exp\left(\varphi F_t(x,i)\right)$ and that $\mathbb{E}\left[\exp(\varphi F_t(x,i))\right] = \exp[\varphi x + 2(\cosh(\varphi) - 1)t]$. As a consequence,

$$\mathbb{E}\left[\sum_{(x,i) \in A} \exp\left(\varphi(F_t(x,i) - r_t) \right) \right] \leq \exp[2(\cosh(\varphi) - 1)t] f_\varphi(w). \tag{37}$$

On the other hand, observe that $A_t \setminus A = \{1, \ldots, r_t\} \times \{1, \ldots, a\}$. Since it is always true that $F_t(x,i) \leq r_t$,

$$\sum_{(x,i) \in A_t \setminus A} \exp\left(\varphi(F_t(x,i) - r_t) \right) \leq a r_t. \tag{38}$$

The result follows from putting together (37) and (38). □

Proof of Theorem 2. Let $\alpha, \delta \in (0,1)$ be such that $c < v(1-\alpha) < b$, $c < (1-\alpha)(1-\delta)v < (1-\alpha)(1+\delta)v < b$, and define $\gamma := b - (1-\alpha)(1+\delta)v$. For each $t > 0$, define the events

$$\begin{aligned} B_t &:= \left\{ v(1-\alpha)(1-\delta)t \leq r_{(1-\alpha)t} \leq v(1-\alpha)(1+\delta)t \right\}, \\ C_t &:= \left\{ \max_{(x,i)\in A_{t(1-\alpha)},\, t(1-\alpha)\leq s \leq t} F_s(x,i) \leq r_{t(1-\alpha)} + \gamma t \right\}, \\ D_t &:= \left\{ \max_{r_{(1-\alpha)t} < x \leq bt,\, 1 \leq i \leq a,\, 0 \leq s \leq \alpha t} x + Y_{x,i,s} \leq bt \right\}. \end{aligned}$$

Observe that

$$B_t \cap C_t \cap D_t \subset \{ct \leq r_t \leq bt\}. \tag{39}$$

Indeed, thanks to the choice of δ, B_t implies that $r_{(1-\alpha)t} \geq ct$, so that $r_t \geq ct$. On the other hand, since $r_{(1-\alpha)t} < bt$ on B_t, the event $B_t \cap \{r_t > bt\}$ implies that either a particle born before time $t(1-\alpha)$ at a position $x \leq r_{t(1-\alpha)}$, or a particle born between time $(1-\alpha)t$ and t at a position $r_{(1-\alpha)t} < x < bt$, exceeds bt at a time between $t(1-\alpha)$ and t. The former possibility is ruled out by $B_t \cap C_t$, since on $B_t \cap C_t$, $r_t \leq r_{(1-\alpha)t} + \gamma t \leq bt$. The latter possibility is ruled out by D_t. Now define

$$l(t) := \exp[2(\cosh(\varphi) - 1)(1-\alpha)t] f_\varphi(w) + a\mathbb{E}(r_{(1-\alpha)t}),$$

and

$$H_t := \left\{ \sum_{(x,i) \in A_{(1-\alpha)t}} \exp\left(\varphi(F_{(1-\alpha)t}(x,i) - r_{(1-\alpha)t})\right) \leq 2l(t) \right\}.$$

By Lemma 6 and Markov's inequality, for all $t \geq 0$, $\mathbb{P}\left(H_t\right) \geq 1/2$. Moreover, by the law of large numbers (1), $\lim_{t\to+\infty} \mathbb{P}(B_t) = 1$. We deduce that there exists a t_0 such that, for all $t \geq t_0$, $\mathbb{P}(B_t \cap H_t) \geq 1/4$. Let us call $\mathcal{F}_t$ the $\sigma-$algebra generated by the history of the particle system up to time t. Observe that B_t and H_t belong to $\mathcal{F}_{(1-\alpha)t}$, and by Lemma 5, on H_t,

$$\mathbb{P}(C_t^c | \mathcal{F}_{(1-\alpha)t}) \leq 2l(t) \exp\left[\alpha t(\cosh(2\varphi) - 1)\right] G_{\alpha t}(\lfloor \gamma t \rfloor)^{1/2}.$$

We deduce that

$$\mathbb{P}(B_t \cap H_t \cap C_t^c) \leq 2l(t) \exp\left[\alpha t(\cosh(2\varphi) - 1)\right] G_{\alpha t}(\lfloor \gamma t \rfloor)^{1/2}. \tag{40}$$

Moreover,

$$\mathbb{P}(D_t | \mathcal{F}_{(1-\alpha)t}) \geq \mathbb{P}(r_{\alpha t}(\mathcal{I}_0) = 0),$$

so that

$$\mathbb{P}(B_t \cap H_t \cap D_t) \geq (1/4)\mathbb{P}(r_{\alpha t}(\mathcal{I}_0) = 0) \geq \exp(-t^{1/2+o(1)}), \tag{41}$$

where the last inequality is due to the lower bound in (6). By standard large deviations bounds for the simple random walk, there exists $\zeta(\alpha,\gamma) > 0$ depending only on γ and α such that, as t goes to infinity, $\liminf_{t\to+\infty} t^{-1} \log G_{\alpha t}(\lfloor \gamma t \rfloor) = -\zeta(\alpha,\gamma)$. Furthermore, $\lim_{t\to+\infty} t^{-1} \log(2l(t) \exp\left[\alpha t(\cosh(2\varphi) - 1)\right]) = \xi(\alpha,\varphi)$, where $\xi(\alpha,\varphi) := \alpha(\cosh(2\varphi) - 1) + 2(\cosh(\varphi) - 1)(1-\alpha)$. We see that, choosing φ small enough, $\xi(\alpha,\varphi) < \zeta(\alpha,\gamma)/2$. For such a φ, (40) and (41) show that $\mathbb{P}(B_t \cap H_t \cap C_t^c) = o(\mathbb{P}(B_t \cap H_t \cap D_t))$, so that $\mathbb{P}(B_t \cap H_t \cap D_t \cap C_t) \geq \exp(-t^{1/2+o(1)})$. It then follows from (39) that $\mathbb{P}(ct \leq r_t \leq bt) \geq \exp(-t^{1/2+o(1)})$, so we are done when $s(\eta) = 1/2$.

Now, let us choose an $(x,i) \in A$ for the initial condition $w = (F, 0, A)$. Define also $\tau = \inf\{s \geq 0;\ F_s(x,i) = 0\}$. Let

$$\begin{aligned}
K_t &:= \{(1-(1-\alpha)(1+\delta))t \leq \tau \leq (1-(1-\alpha)(1-\delta))t\},\\
L_t &:= \{ct \leq r_{(1-\alpha)(1-\delta)t+\tau}(w(x,i)) \leq r_{(1-\alpha)(1+\delta)t+\tau}(w(x,i)) \leq bt\},\\
L'_t &:= \{ct \leq r_{(1-\alpha)(1-\delta)t}(\delta_0) \leq r_{(1-\alpha)(1+\delta)t}(\delta_0) \leq bt\},\\
M_t &:= \{\text{ for all } (y,j) \in A \setminus \{(x,i)\} \text{ and all } 0 \leq s \leq t,\ F_s(y,j) \leq 0\}.
\end{aligned}$$

Observe that, on M_t, $r_t(w) = r_t(w(x,i))$. Moreover, $K_t \cap L_t \subset \{ct \leq r_t(w(x,i)) \leq bt\}$. As a consequence,

$$M_t \cap K_t \cap L_t \subset \{ct \leq r_t(w) \leq bt\}. \tag{42}$$

But according to the lower bound of Theorem 2 (b), $\mathbb{P}(M_t) \geq \exp(-t^{U(\eta_w)/2+o(1)})$. On the other hand, conditional upon τ, $r_{s+\tau}(w(x,i))$ has the (unconditional) distribution of $r_s(\delta_0)$, for all $s \geq 0$. As a consequence, $\mathbb{P}(K_t \cap L_t) = \mathbb{P}(K_t)\mathbb{P}(L'_t)$, and, by the law of large numbers (1), $\lim_{t\to+\infty} \mathbb{P}(L'_t) = 1$. Moreover, it is easily seen from elementary estimates on hitting times by a simple symmetric continuous time random walk that $\liminf_{t\to+\infty} t^{-1/2}\mathbb{P}(K_t) > 0$. Finally, M_t being defined in terms of random walks that do not enter the definition of K_t and L_t, we deduce that M_t is independent from $K_t \cap L_t$. We finally deduce that $\mathbb{P}(M_t \cap K_t \cap L_t) \geq \exp(-t^{U(\eta_w)/2+o(1)})$, and the result follows from (42).

□

5.2. **Proof of Theorem 3.** As for the upper bound in (6), we easily obtain that

$$\mathbb{P}(t^{-1}r_t(w) \leq bt) \leq \exp\left(-E\left[\mathbf{1}(\zeta_t \geq \lceil bt\rceil)H_w(-\zeta_t + \lceil bt\rceil)\right]\right).$$

It is easily checked that, for small enough $b > 0$,

$$\liminf_{t\to+\infty} t^{-1}\log E\left[\mathbf{1}(\zeta_t \geq \lceil bt\rceil)\exp\left(\theta(\zeta_t - \lceil bt\rceil)\right)\right] > 0. \tag{43}$$

This proves (i). We now prove (ii). Again, it is easily checked that, for all $b > 0$, there exists $\theta > 0$ such that (43) holds. Choosing $b > v$, the result follows.

5.3. **Proof of Theorem 2 (b).** Note that by coupling, it is enough to prove the result with an initial condition consisting of exactly a particles per site $x \leq 0$, that is, with $w = \mathcal{I}_0$. Hence, we will establish that for all $0 < b < v$, and all $\alpha > 0$,

$$\liminf_{t\to\infty}(\log t)^{-1}\left|\log \mathbb{P}\left[\frac{r^0_t(\mathcal{I}_0)}{t} \leq b\right]\right| \geq 1/3.$$

Using the fact that $\mathbb{P}(T^0_{\mathcal{I}_0}(\lfloor bt\rfloor) \geq t) \leq \mathbb{P}(r^0_t(\mathcal{I}_0) \leq bt) \leq \mathbb{P}(T^0_{\mathcal{I}_0}(\lceil bt\rceil) \geq t)$, it is easy to see that (5) is equivalent to the following result.

Proposition 16. *For every $c > v^{-1}$, as n goes to infinity,*

$$\mathbb{P}(T^0_{\mathcal{I}_0}(n) \geq cn) \leq \exp\left(-n^{1/3+o(1)}\right).$$

Our strategy for proving Proposition 16 can be sketched as follows. By subadditivity, for all $m \geq 1$

$$T^0_{\mathcal{I}_0}(n) \leq \sum_{j=0}^{\lfloor n/m \rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)),$$

so that

$$\mathbb{P}(T^0_{\mathcal{I}_0}(n) \geq cn) \leq \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m \rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)) \geq (mc)\lfloor n/m \rfloor\right). \tag{44}$$

Now, by translation invariance, for all $j \geq 0$, $T^0_{\mathcal{I}_{mj}}(m(j+1))$ and $T^0_{\mathcal{I}_0}(m)$ have the same distribution, and it can be shown that

$$\lim_{m \to +\infty} m^{-1}\mathbb{E}(T^0_{\mathcal{I}_0}(m)) = v^{-1}.$$

Hence, given $c > v^{-1}$ we can always find $m \geq 1$ such that $mc > \mathbb{E}(T^0_{\mathcal{I}_0}(m))$, so that the r.h.s. of (44) is the probability of a large deviation above the mean for the sum $\sum_{j=0}^{\lfloor n/m \rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1))$. We then seek to apply large deviations bounds for i.i.d. variables in order to estimate this probability. Of course, the random variables $\left\{T^0_{\mathcal{I}_{mj}}(m(j+1))\right\}_{j\geq 0}$ are *not* independent, but the dependency between $\left\{T^0_{\mathcal{I}_{mj}}(m(j+1))\right\}_{j\leq j_1}$ and $\left\{T^0_{\mathcal{I}_{mj}}(m(j+1))\right\}_{j\geq j_2}$ is weak when $j_2 - j_1$ is large. Indeed, for given j, $T^0_{\mathcal{I}_{mj}}(m(j+1))$ mostly depends on the behavior of the random walks born at sites close to mj. We implement this idea by using a technique already exploited in [22] in a similar context. Given $\ell \geq 1$, we define a family $\left\{T'_{\mathcal{I}_{mj}}(m(j+1))\right\}_{j\geq 0}$ of hitting times as follows: $T'_{\mathcal{I}_{mj}}(m(j+1))$ uses the same random walks as $T^0_{\mathcal{I}_{mj}}(m(j+1))$ for particles born at sites (x,i) with $mj - m\ell < x < m(j+1)$, but uses fresh independent random walks for particles born at sites (x,i) with $x \leq mj - m\ell$. We can then prove that the following properties hold:

(a) For all $j \geq 0$, the family $\left\{T'_{\mathcal{I}_{mj+pm(\ell+1)}}(mj + pm(\ell+1) + m)\right\}_{p\geq 0}$ is i.i.d.;
(b) when ℓ is large, the probability that $T'_{\mathcal{I}_{mj}}(m(j+1)) = T^0_{\mathcal{I}_{mj}}(m(j+1))$ is close to 1.

We can thus obtain estimates on the r.h.s. of (44) by estimating separately the probability that $T'_{\mathcal{I}_{mj}}(m(j+1)) = T^0_{\mathcal{I}_{mj}}(m(j+1))$ for all $0 \leq j \leq \lfloor m/n \rfloor$, and the probability that $\sum_{j=0}^{\lfloor n/m \rfloor} T'_{\mathcal{I}_{mj}}(m(j+1)) \geq (mc)\lfloor n/m \rfloor$. Now, thanks to property (a) above, this last sum can be split evenly into $\ell+1$ subsums of i.i.d. random variables distributed as $T^0_{\mathcal{I}_0}(m)$. Controlling the tail of $T^0_{\mathcal{I}_0}(m)$ then allows us to apply large deviation bounds for i.i.d. variables separately to each of these subsums. In fact, the proof of (5) is a bit more subtle, since it also makes use of a positive association property, but we do not go into the details here (see Remark 3 below).

5.4. **Proof of Proposition 16.** Observe that, since for all $u \in \mathbb{Z}$ and $k \geq 0$, $\mathcal{I}_u \oplus k = \mathcal{I}_{u+k}$, the subadditivity property (part (*iii*)) of Proposition 2 reads as:

$$\text{for all } n, m \geq 0,\ T^0_{\mathcal{I}_0}(n+m) \leq T^0_{\mathcal{I}_0}(n) + T^0_{\mathcal{I}_n}(m).$$

Now, let $c > v^{-1}$. Thanks to subadditivity, for all $m \geq 1$ we have that

$$\mathbb{P}(T^0_{\mathcal{I}_0}(n) \geq cn) \leq \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m \rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)) \geq cn\right).$$

In Steps 1 and 2 below, m and ℓ denote fixed positive integers, while α denotes a fixed real number $0 < \alpha < 1$. For the sake of readability, the dependence with respect to these numbers is usually not mentioned explicitly in the notations. Only in Step 3 have the values of m, ℓ and α to be specified.

5.4.1. *Step 1: Comparison with a sum of i.i.d. random variables.* Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that we have access to an i.i.d. family of random variables

$$\left[(\tau^j_k(u,i), U^j_k(u,i));\ j \geq 0,\ k \geq 1,\ u \in \mathbb{Z},\ 1 \leq i \leq a\right],$$

independent from

$$\left[(\tau_k(u,i), U_k(u,i));\ \ k \geq 1,\ u \in \mathbb{Z},\ 1 \leq i \leq a\right],$$

and such that, for all (j,k,u,i), $\tau^j_k(u,i)$ has an exponential(2) distribution, $U^j_k(u,i)$ has the uniform distribution on $(0,1)$, and $\tau^j_k(u,i)$ and $U^j_k(u,i)$ are independent.

Let

$$\varepsilon^j_n(x,i) := 2(\mathbf{1}(U^j_n(x,i) \leq 1/2)) - 1.$$

Now, for all $\ell \geq 1$ and $j \in \mathbb{Z}$, define, for all $u, v \in \mathbb{Z}$ such that $u < v$, and $1 \leq i \leq a$,

$$\mathbb{B}_j(u,i,v) := \inf\left\{\sum_{k=1}^{m} \tau^j_k(u,i);\ u + \sum_{k=1}^{m} \varepsilon^j_k(u,i) = v,\ m \geq 1\right\},$$

and let

$$\mathbb{C}_j(u,i,v) := \begin{cases} \mathbb{B}_j(u,i,v) \text{ if } u \leq mj - m\ell \\ \mathbb{A}^0(u,i,v) \text{ if } u > mj - m\ell \end{cases},$$

where $\mathbb{A}^0$ is defined in display (9) in Section 2. Then let

$$T'_{\mathcal{I}_{mj}}(m(j+1)) := \inf \sum_{g=1}^{L-1} \mathbb{C}_j(x_g, i_g, x_{g+1}),$$

where the infimum is taken over all finite sequences with $L \geq 2$, $x_1, \ldots, x_L \in \mathbb{Z}$ and $i_1, \ldots, i_{L-1}$ such that $x_1 \leq mj$, $mj < x_2 < \cdots < x_{L-1} < m(j+1)$, $x_L = m(j+1)$, $i_1, i_2, \ldots, i_{L-1} \in \{1, \ldots, a\}$. Clearly, $T'_{\mathcal{I}_{mj}}(m(j+1))$ and $T^0_{\mathcal{I}_{mj}}(m(j+1))$ have the same distribution. Moreover, we have the following lemma, whose proof is immediate.

Lemma 7. *For every $j \in \mathbb{Z}$, the family of random variables*

$$\left(T'_{\mathcal{I}mj+pm(\ell+1)}(mj + pm(\ell+1) + m)\right)_{p\in\mathbb{Z}}$$

is i.i.d.

We now study the event $\left\{T'_{\mathcal{I}_{mj}}(m(j+1)) = T^0_{\mathcal{I}_{mj}}(m(j+1))\right\}$. To this end, let

$$J_j := \inf \sum_{g=1}^{L-1} \mathbb{C}_j(x_g, i_g, x_{g+1}),\ K_j := \inf \sum_{g=1}^{L-1} \mathbb{A}^0_j(x_g, i_g, x_{g+1}),$$

where in both cases the infimum is taken over all finite sequences with $L \geq 2$, $x_1, \ldots, x_L \in \mathbb{Z}$ and $i_1, \ldots, i_{L-1}$ such that $x_1 \leq mj - m\ell$, $mj < x_2 < \cdots < x_{L-1} < m(j+1)$, $x_L = m(j+1)$, $i_1, i_2, \ldots, i_{L-1} \in \{1, \ldots, a\}$. Let also

$$L_j := \inf \sum_{g=1}^{L-1} \mathbb{A}^0_j(x_g, i_g, x_{g+1}),$$

where the infimum is taken over all finite sequences with $L \geq 2$, $x_1, \ldots, x_L \in \mathbb{Z}$ and $i_1, \ldots, i_{L-1}$ such that $mj - m\ell < x_1 \leq mj$, $mj < x_2 < \cdots < x_{L-1} < m(j+1)$, $x_L = m(j+1)$, $i_1, i_2, \ldots, i_{L-1} \in \{1, \ldots, a\}$.

Observe that, $T'_{\mathcal{I}_{mj}}(m(j+1)) = \min(J_j, L_j)$ and that $T'_{\mathcal{I}_{mj}}(m(j+1)) = \min(K_j, L_j)$. As a consequence,

$$\{\min(J_j, K_j) \geq L_j\} \subset \left\{T'_{\mathcal{I}_{mj}}(m(j+1)) = T^0_{\mathcal{I}_{mj}}(m(j+1))\right\}.$$

For $\alpha > 0$, we now define

$$D(j) := \left\{\min(J_j, K_j) < \alpha(m\ell)^2\right\},$$

and

$$F(j) := \{L_j \geq \alpha(m\ell)^2\},$$

so that

$$F(j)^c \cap D(j)^c \subset \left\{T'_{\mathcal{I}_{mj}}(m(j+1)) = T^0_{\mathcal{I}_{mj}}(m(j+1))\right\}. \tag{45}$$

Lemma 8. *There exist $\lambda_1(a)$ and $\lambda_2 > 0$, not depending on m, ℓ, α, such that*

$$\mathbb{P}(D(j)) \leq 4a\alpha(m\ell)^2 G_{\alpha(m\ell)^2}(m\ell)) + \lambda_1(a)\exp\left(-\lambda_2\alpha(m\ell)^2\right) =: \lambda.$$

Proof of Lemma 8. Consider the random walks born at sites (x, i) for $x \leq mj - \alpha(m\ell)^2$. By Lemma 1 choosing $\gamma = 1$ and $\theta > 0$ small enough so that $g_\gamma(\theta) > 0$, we obtain the existence of $\lambda_1(a) > 0$ and $\lambda_2 > 0$ such that the probability that any of the walks born at a site (x, i) with $x \leq mj - \alpha(m\ell)^2$ hits mj before time $\alpha(m\ell)^2$ is $\leq \lambda_1(a)\exp\left(-\lambda_2\alpha(m\ell)^2\right)$. On the other hand, for $mj - \alpha(m\ell)^2 < x \leq mj - m\ell$, the probability that a walk started at x hits mj before time $\alpha(m\ell)^2$ is less than the corresponding probability for the walk started at $mj - m\ell$, that is, $1 - \bar{G}_{\alpha(m\ell)^2}(m\ell)$. In turn, this probability is less than $2G_{\alpha(m\ell)^2}(m\ell)$. A union bound over all the corresponding events yields the result.

□

Lemma 9. *There exist $V_1(m), V_2(m) > 0$, not depending on ℓ, α, such that for all j,*

$$\mathbb{P}(F(j)) \leq V_1(m)\exp\left(-V_2(m)\alpha^{1/2}\ell\right).$$

Proof. By translation invariance, we can assume that $j=0$. Let $t=\alpha(m\ell)^2$. Since $F(0)$ implies that no random walk born at a site $-m\ell+1\leq x\leq 0$ hits 1 before time $\alpha(m\ell)^2$, one has that $\mathbb{P}(F(0)) = \prod_{-m\ell+1\leq x\leq 0}\bar{G}_t(1-x)^a$. Since $0\leq\alpha\leq 1$, we see that $t^{1/2}\leq m\ell$, so that $\mathbb{P}(F(0))\leq \prod_{-\lfloor t^{1/2}\rfloor+1\leq x\leq 0}\bar{G}_t(1-x)^a$. Using monotonicity of $\bar{G}_t$, we deduce that $\mathbb{P}(F(0))\leq \bar{G}_t(\lfloor t^{1/2}\rfloor)^{a\lfloor t^{1/2}\rfloor}$.

By the central limit theorem, $\lim_{t\to+\infty} G_t\left(\lfloor t^{1/2}\rfloor\right)>0$, so that, since $\bar{G}_t\leq 1-G_t$, $\limsup_{t\to+\infty}\bar{G}_t\left(\lfloor t^{1/2}\rfloor\right)<1$. As a consequence, we can find $\rho>0$, and $t_0\geq 0$ such that, for all $t\geq t_0$, $\bar{G}_t\left(\lfloor t^{1/2}\rfloor\right)\leq 1-\rho$. For $t\geq t_0$, we deduce that $P(F(0))\leq (1-\rho)^{a\lfloor t^{1/2}\rfloor}$. For $t\leq t_0$, we see that we can find a large enough V_1 such that $P(F(0))\leq V_1(1-\rho)^{a\lfloor t^{1/2}\rfloor}$, using only the trivial bound $\mathbb{P}(F(0))\leq 1$.

□

Lemma 10. *For all $t\geq 0$, the events $\left\{\sum_{j=0}^{\lfloor n/m\rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1))\geq cn\right\}$ and $\bigcup_{j=0}^{\lfloor n/m\rfloor} D(j)$ are negatively associated.*

Proof. For an integer $K\geq 1$, let

$$\mathbb{A}^0(u,i,v,K) := \inf\left\{\sum_{k=1}^m \tau_k(u,i);\ u+\sum_{k=1}^m \varepsilon_k(u,i,\epsilon)=v,\ 1\leq m\leq K\right\}.$$

Similarly, let

$$\mathbb{B}_j(u,i,v,K) := \inf\left\{\sum_{k=1}^m \tau^j_k(u,i);\ u+\sum_{k=1}^m \varepsilon^j_k(u,i)=v,\ 1\leq m\leq K\right\},$$

and let

$$\mathbb{C}_j(u,i,v,K) := \begin{cases}\mathbb{B}_j(u,i,v,K) \text{ if } u\leq mj-m\ell\\ \mathbb{A}^0(u,i,v,K) \text{ if } u>mj-m\ell\end{cases}.$$

Now let

$$T^\epsilon_w(u,K) := \inf\sum_{j=1}^{L-1}\mathbb{A}^\epsilon(x_j,i_j,x_{j+1},K)$$

where the infimum is taken over all finite sequences with $L\geq 2$, $x_1,\dots,x_L\in\mathbb{Z}$ and $i_1,\dots,i_{L-1}$ such that $(x_1,i_1)\in A$, $x_1\geq -K$, $r<x_2<\cdots<x_{L-1}<u$, $x_L=u$, $i_2,\dots,i_{L-1}\in\{1,\dots,a\}$.

Similarly, let

$$J_{j,K} := \inf\sum_{g=1}^{L-1}\mathbb{C}_j(x_g,i_g,x_{g+1},K),\ K_{j,K} := \inf\sum_{g=1}^{L-1}\mathbb{A}^0_j(x_g,i_g,x_{g+1},K),$$

where in both cases the infimum is taken over all finite sequences with $L\geq 2$, $x_1,\dots,x_L\in\mathbb{Z}$ and $i_1,\dots,i_{L-1}$ such that $-K\leq x_1\leq mj-m\ell$, $mj<x_2<\cdots< x_{L-1}<m(j+1)$, $x_L=m(j+1)$, $i_1,i_2,\dots,i_{L-1}\in\{1,\dots,a\}$.

Observe that $\mathbb{P}-$almost surely, for all $u < v$, the sequence $(T^0_{\mathcal{I}_u}(v, K))_{K\geq 1}$ is ultimately stationary, and that its limiting value is $T^0_{\mathcal{I}_u}(v)$. Similarly, $\mathbb{P}-$almost surely, the sequences $(J_{j,K})_{K\geq 1}$ and $(K_{j,K})_{K\geq 1}$ are ultimately stationary, and their respective limits are J_j and K_j.

Then let $S_{q,K} := \sum_{p=0}^{q} T^0_{\mathcal{I}pm(\ell+1)}(pm(\ell+1)+m, K)$ and

$$D(j, K) := \left\{\min(J_{j,K}, K_{j,K}) < \alpha(m\ell)^2\right\}.$$

Now let $g_1 := \mathbf{1}\,(S_q \geq t)$, $g_2 := \mathbf{1}\left(\bigcup_{p=0}^{q} D(p(\ell+1))\right)$, and $g_{1,K} := \mathbf{1}\,(S_{q,K} \geq t)$ and $g_{2,K} := \mathbf{1}\left(\bigcup_{p=0}^{q} D(p(\ell+1), K)\right)$.

Note that $(g_{1,K})_{K\geq 1}$ is a bounded sequence of random variables that is $\mathbb{P}-$a.s. ultimately stationary and converging to g_1 as K goes to infinity. The same holds for $(g_{2,K})_{K\geq 1}$ and g_2. Now, for every K, $g_{1,K}$ and $g_{2,K}$ are functions of a finite number of the random variables $(U_n(x,i), U^j_n(x,i), \tau^j(x,i), \tau(x,i);\, n \geq 1,\, x \in \mathbb{Z},\, 1 \leq i \leq a)$. Moreover, it is easy to check from the definitions that, with respect to these random variables, $g_{1,K}$ is non-increasing, while $g_{2,K}$ is non-decreasing. Since these random variables are independent, we deduce that $\mathbb{E}(-g_{1,K}g_{2,K}) \geq \mathbb{E}(-g_{1,K})\mathbb{E}(g_{2,K})$ (see e.g. [9]). Taking the limit as $K \to +\infty$, and using the dominated convergence theorem, we obtain the result.

□

Now consider the following inclusion.

$$\left\{\sum_{j=0}^{\lfloor n/m\rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)) \geq cn\right\} \subseteq X \cup Y \cup Z, \tag{46}$$

where

$$X := \left\{\sum_{j=0}^{\lfloor n/m\rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)) \geq cn\right\} \cap \bigcup_{j=0}^{\lfloor n/m\rfloor} D(j),$$

$$Y := \left\{\sum_{j=0}^{\lfloor n/m\rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)) \geq cn\right\} \cap \bigcap_{j=0}^{\lfloor n/m\rfloor} \left(D(j)^c \cap F(j)^c\right),$$

$$Z := \bigcup_{j=0}^{\lfloor n/m\rfloor} F(j).$$

Let

$$f(n,c) := \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m\rfloor} T^0_{\mathcal{I}_{mj}}(m(j+1)) \geq cn\right).$$

Then $f(n,c) \leq \mathbb{P}(X) + \mathbb{P}(Y) + \mathbb{P}(Z)$. Now, according to Lemmas 10, 8 we see that

$$\mathbb{P}(X) \leq f(n,c) \times (\lfloor n/m\rfloor + 1)\lambda.$$

¿From (45), we see that

$$\mathbb{P}(Y)\le \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m\rfloor} T'_{\mathcal{I}_{mj}}(m(j+1))\ge cn\right).$$

¿From Lemma 9, we see that,

$$\mathbb{P}(Z)\le (\lfloor n/m\rfloor+1)V_1(m)\exp\left(-V_2(m)\alpha^{1/2}\ell\right).$$

This leads to the following bound.

$$\delta(n)f(n,c)\le \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m\rfloor} T'_{\mathcal{I}_{mj}}(m(j+1))\ge cn\right)+(\lfloor n/m\rfloor+1)V_1(m)\exp\left(-V_2(m)\alpha^{1/2}m\ell\right). \tag{47}$$

where $\delta(n):=1-(\lfloor n/m\rfloor+1)\lambda$.

Using the independence properties of the random variables $T'_{\mathcal{I}_{mj}}(m(j+1))$ (Lemma 7), and the union bound, we see that the following inequality holds

$$\mathbb{P}\left(\sum_{j=0}^{\lfloor n/m\rfloor} T'_{\mathcal{I}_{mj}}(m(j+1))\ge cn\right)\le (\ell+1)\mathfrak{F}_m^{\otimes k(n)}\left([cn(\ell+1)^{-1},+\infty)\right),$$

where $\mathfrak{F}_m^{\otimes k}$ denotes the distribution of the sum of k independent copies of $T^0_{\mathcal{I}_0}(m)$, and where $k(n):=1+\lfloor\frac{n/m-1}{\ell+1}\rfloor$.

5.4.2. *Step 2: Large deviations estimates for i.i.d. random variables.* We start with a general bound on the tail of $T^0_w(0,m)$.

Lemma 11. *There exist $A_m, c_m>0$ such that, for all $w=(F,0,A)$ and $t\ge 0$*

$$\mathbb{P}(T^0_w(0,m)\ge t)\le A_m\exp\left(-c_m H_w(-\lfloor t^{1/2}\rfloor)\right).$$

Proof. Observe that the event $T^0_w(m)\ge t$ implies that none of the random walks born at sites $F(x,i)$, $(x,i)\in A$ has hit m before time t. As a consequence, $\mathbb{P}(T^0_w(m)\ge t)\le\prod_{x=0}^{-\lfloor t^{1/2}\rfloor}\bar{G}_t(-x+m)^{\eta_w(x)}$. Using monotonicity of $\bar{G}_t$, we deduce that $\mathbb{P}(T^0_w(m)\ge t)\le G_t(m+\lfloor t^{1/2}\rfloor)^{H_w(-\lfloor t^{1/2}\rfloor)}$. Re-using the notations of the proof of Lemma 9, we see that, for all $t\ge t_0$, $\mathbb{P}(T^0_w(m)\ge t)\le(1-\rho)^{H_w(-\lfloor t^{1/2}\rfloor)}$. Now, for $t\le t_0$, we can find A_m such that, using only the trivial bounds $H_w(-\lfloor t^{1/2}\rfloor)\ge 0$ and $\mathbb{P}(T^0_w(m)\ge t)\le 1$, $\mathbb{P}(T^0_w(m)\ge t)\le A_m(1-\rho)^{H_w(-\lfloor t^{1/2}\rfloor)}$ for all $0\le t\le t_0$. □

Remark 1. *The lower bound (4) shows that the upper bound of Lemma 11 yields the right order of magnitude for the tail of $T_w(m)$, at least when $U(\eta_w)<2$.*

The probabilities of large deviations for $\mathfrak{F}_m^{\otimes k}$ are described by the following lemma, whose proof is deferred to Appendix 6.

Lemma 12. *Let $(R_j)_{j\geq 1}$ be a sequence of i.i.d. non-negative random variables with common distribution μ. Let $M := \int x d\mu(x)$. Assume that there exist $A, c > 0$ such that for every $x \geq 0$*

$$\mu([x,+\infty)) \leq A \exp\left(-cx^{1/2}\right). \tag{48}$$

Then $M < +\infty$ and for all $f > M$, there exists $h(f) > 0$ and $n(f)$ such that if $n \geq n(f)$

$$P\left(n^{-1}(R_1 + \cdots + R_n) \geq f\right) \leq \exp\left(-h(f)n^{1/2}\right).$$

5.4.3. *Step 3: Conclusion.* Lemma 12 above can be applied to probabilities of large deviations of the form $\mathfrak{F}_m^{\otimes k}([bk,+\infty))$, where $b > \mathbb{E}(T_{\mathcal{I}_0}(m))$, and our goal is to control probabilities of the form $\mathfrak{F}_m^{\otimes k(n)}\left([cn(\ell+1)^{-1},+\infty)\right)$. It is easily checked that

$$cn(\ell+1)^{-1} \geq k(n)cm\left(1+\tfrac{m(\ell+1)}{n}\right)^{-1}. \tag{49}$$

Now observe that Kingman's subadditive ergodic theorem (see e.g. [14]) can be applied to the sequence of random variables $(T^0_{\mathcal{I}_u}(v))_{u\leq v}$. Indeed, these variables are non-negative, integrable (Lemma 11), and satisfy the required distributional translation invariance properties. We deduce that

$$\lim_{m\to+\infty} m^{-1}\mathbb{E}(T_{\mathcal{I}_0}(m)) = v^{-1}. \tag{50}$$

As a consequence, for all $c > v^{-1}$, we can find $m \geq 1$ large enough so that

$$cm > \mathbb{E}(T_{\mathcal{I}_0}(m)).$$

In the sequel, we assume that m is chosen such that this inequality holds. Now let us choose $\ell := \ell_n = n^{1/3}$. Taking into account Lemmas 11, 12, and (49), we see that, as n goes to infinity, there exists a constant $h_1 > 0$ such that

$$\mathfrak{F}_m^{\otimes k(n)}\left([cn(\ell_n+1)^{-1},+\infty)\right) = O\left(\exp(-h_1 n^{1/3})\right). \tag{51}$$

Now, for $0 < \zeta < 1/2$, let us choose $\alpha := \alpha_n = n^{-\zeta}$, and consider Inequality (47). With our definitions, $\alpha_n^{1/2}(m\ell_n) = mn^{1/3-\zeta/2}$ while $m\ell_n = mn^{1/3}$. As a consequence, a moderate deviations bound for the simple random walk (see e.g. [6]) yields that $G_{\alpha_n(m\ell_n)^2}(m\ell_n+1) = O\left(\exp(-h_2 n^{\zeta})\right)$ for some constant $h_2 > 0$, whence the fact that $\delta(n) = 1 + o(1)$. Using (51), we see that Inequality (47) entails that, for large n,

$$f(n,c) \leq O\left(\exp\left(-h_3 n^{1/3-\zeta/2}\right)\right).$$

Since ζ can be taken arbitrarily small, the conclusion of Proposition 16 follows.

Remark 2. *In view of (4) and (5), we see that our upper and lower bounds on slowdown probabilities do not match. One may wonder whether it is possible to improve upon either of these bounds so as to find the exact order of magnitude of slowdown large deviations probabilities. What we can prove (the details are not given here) is that the* $\exp\left(-n^{1/3+o(1)}\right)$ *bound in Proposition 16 gives the best order of magnitude that can be reached by following our proof strategy based on subadditivity. Indeed, despite the fact that each $T^0_{\mathcal{I}_{mj}}(m(j+1))$ has a tail decaying roughly as*

$\exp\left(-t^{1/2}\right)$, *so that the probabilities of large deviations above the mean would be of order* $\exp\left(-n^{-1/2}\right)$ *if these random variables were independent, the positive dependence between these variables makes such large deviations much more likely, with probabilities of order* $\exp\left(-n^{1/3}\right)$.

Remark 3. *One may wonder whether the use of association (see Lemma 10) is really needed in the proof. Indeed, a simpler approach would be to bound the probability of the event X in (46) above by* $\mathbb{P}\left(\bigcup_{j=0}^{\lfloor n/m \rfloor} D(j)\right)$. *By properly choosing α_n and ℓ_n, we could make this probability of the order of* $\exp\left(-n^{1/3+o(1)}\right)$, *compared to the* $\exp\left(-h_2 n^{-\zeta}\right)$ *obtained in the proof of Proposition 16. However, such a choice interferes with the other bounds used in the proof, (making α_n smaller increases the probability of $F(j)$. The best order of magnitude we could obtain with that simpler method is* $\exp\left(-n^{2/7+o(1)}\right)$.

6. Appendix: large deviations of i.i.d. random variables with $\exp(-t^{1/2})$ tails

Neither the result stated in Lemma 12 nor the idea of its proof are new, but we failed in finding a reference providing both a statement suited to our purposes and a short proof, so we chose to give a detailed exposition.

We refer to the paper [18] for a review of results concerning large deviations of random variables with subexponential tails, and to Theorem 4.1 in [2] for an example of a result from which Lemma 12 may be derived. See also the recent preprint [7].

Proof of Lemma 12. Let A and c be as in the statement of the lemma. And let G be defined by $G(x) := \mu([x,+\infty))$.

Let A_n be the following event: $A_n := \bigcap_{1\leq i\leq n}\{R_i \leq n\}$. By the union bound, $P(A_n^c) \leq n\mu([n,+\infty))$, so that, by Assumption (48) above and Lemma 13 below,

$$P(A_n^c) = O\left[n \exp\left(-(c/2)n^{1/2}\right)\right]. \tag{52}$$

We now apply the Cramér bound for i.i.d. random variables possessing finite exponential moments (see e.g. [6]) to the i.i.d. bounded random variables $R_{i,n}$ defined by $R_{i,n} := \min\left(R_i, n\right)$. For every $\lambda > 0$, the following inequality holds.

$$P\left(n^{-1}(R_{1,n}+\cdots+R_{n,n}) \geq f\right) \leq \exp\left[-n\lambda f\right]\left[E\exp\left(\lambda R_{1,n}\right)\right]^n. \tag{53}$$

Let $\lambda_n := (c/3)n^{-1/2}$ and $K_n := n^{1/4}$. By definition $E\exp\left(\lambda_n R_{1,n}\right) = \int_{[0,n)}\exp(\lambda_n x)d\mu(x) + \exp(\lambda_n n)\mu([n,+\infty))$. Let us split the above integral into $\int_{[0,n)} = \int_{[0,K_n)} + \int_{[K_n,n)}$. Fix a real number $\alpha > 0$. Since $\lambda_n K_n$ goes to zero as n goes to infinity, we have, for all large enough n (depending on α), an inequality of the following form: for every $x \in [0,K_n)$, $\exp(\lambda_n x) \leq 1 + (1+\alpha)\lambda_n x$. Taking the integral in this inequality, we obtain that, for all large enough n,

$$\int_{[0,K_n)} \exp(\lambda_n x)d\mu(x) \leq \mu([0,K_n)) + (1+\alpha)\lambda_n \int_{[0,K_n)} x d\mu(x).$$

Since α is arbitrary in the above argument, we see that

$$\int_{[0,K_n)} \exp(\lambda_n x) d\mu(x) \leq \mu([0,K_n)) + (1+o(1))\lambda_n \int_{[0,K_n)} x d\mu(x). \quad (54)$$

By definition, $M = \int_{[0,K_n)} x d\mu(x) + \int_{[K_n,+\infty)} x d\mu(x)$. Integration by parts yields that $\int_{[K_n,+\infty)} x d\mu(x) = -[xG(x)]_{K_n}^{+\infty} + \int_{[K_n,+\infty)} G(x)dx$. Assumption (48) above says that $G(x) \leq A\exp(-cx^{1/2})$. As a consequence, $-[xG(x)]_{K_n}^{+\infty} \leq AK_n \exp\left(-cK_n^{1/2}\right)$. Moreover, Lemma 13 yields that $\int_{[K_n,+\infty)} G(x)dx = O\left[\exp\left(-(c/2)K_n^{1/2}\right)\right]$.

Putting the above estimates together, and using the definitions of λ_n and K_n, the above estimates clearly imply that $\int_{[K_n,+\infty)} x d\mu(x) = o(\lambda_n)$. Similarly, $\mu([K_n,+\infty)) = o(\lambda_n)$. As a consequence, Inequality (54) above yields that

$$\int_{[0,K_n)} \exp(\lambda_n x) d\mu(x) \leq 1 + (1+o(1))M\lambda_n.$$

We now study $\int_{[K_n,n)} \exp(\lambda_n x) d\mu(x)$. Integration by parts says that $\int_{[K_n,n)} \exp(\lambda_n x) d\mu(x) = -\left[\exp(\lambda_n x) G(x)\right]_{K_n}^n + \int_{K_n}^n \lambda_n \exp(\lambda_n x) G(x)dx$. Observe that, with our definitions of λ_n and K_n, for every $0 \leq x \leq n$, $\lambda_n x \leq (c/3)x^{1/2}$. As a consequence, $\exp(\lambda_n x) G(x) \leq A\exp\left(-(2c/3)x^{1/2}\right)$. This estimate, together with Lemma (13), yields that, as n goes to infinity, $\int_{K_n}^n \exp(\lambda_n x) G(x)dx = o(1)$. Similarly, $\left[\exp(\lambda_n x) G(x)\right]_{K_n}^n = o(\lambda_n)$. As a consequence, as n goes to infinity, $\int_{[K_n,n)} \exp(\lambda_n x) d\mu(x) = o(\lambda_n)$. Similarly, $\exp(\lambda_n n)\mu([n,+\infty)) = o(\lambda_n)$.

Finally, we obtain the following estimate: $E\exp(\lambda_n R_{1,n}) = 1 + \lambda_n m(1+o(1))$. As n goes to infinity, an expansion yields that $\left[E\exp(\lambda_n R_{1,n})\right]^n = \exp(nM\lambda_n(1+o(1)))$. ¿From Cramér's inequality (53), we obtain that

$$P\left(n^{-1}(R_{1,n} + \cdots + R_{n,n}) \geq f\right) \leq \exp\left(-n\lambda_n(f-M)(1+o(1))\right). \quad (55)$$

Now, on the event A_n, $R_i = R_{i,n}$ for all $1 \leq i \leq n$.

As a consequence, $P\left(n^{-1}(R_1 + \cdots + R_n) \geq f\right) \leq P\left(n^{-1}(R_{1,n} + \cdots + R_{n,n}) \geq f\right) + P(A_n^c)$.

The statement of the Lemma now follows from the bound (52) on $P(A_n^c)$ and the large deviations bound (55) for $R_{1,n} + \cdots + R_{n,n}$.

□

Lemma 13. *For every $\nu > 0$, as $x \to +\infty$,*

$$\int_x^{+\infty} \exp\left(-\nu u^{1/2}\right) du = O\left[\exp\left(-(\nu/2)x^{1/2}\right)\right].$$

Proof of Lemma 13. Observe that there exists $d_1 > 0$ such that, for every $u \geq 1$, $u^{1/2}\exp\left(-(\nu/2)u^{1/2}\right) \leq d_1$. As a consequence, $\exp\left(-\nu u^{1/2}\right) \leq d_1 u^{-1/2}\exp\left(-(\nu/2)u^{1/2}\right)$, so that

$$\int_x^{+\infty} \exp\left(-\nu u^{1/2}\right) du \leq d_1 \int_x^{+\infty} u^{-1/2}\exp\left(-(\nu/2)u^{1/2}\right) du.$$

The r.h.s. of the above inequality is then equal to $d_1(4/\nu)\exp\left(-(\nu/2)x^{1/2}\right)$.

□

7. Appendix: Polynomial tail of renewal variables when $\epsilon = 0$

That κ may have a polynomial tail is not necessarily an obstacle for proving an exponential bound for speedup large deviations, as long as $\hat{r}_\kappa$ has an exponential tail. However, as we now prove, both κ and $\hat{r}_\kappa$ have a polynomial tail under $\mathbb{Q}_w^{0,\theta}$ when w satisfies **(G)**.

Let w be such that $r \times \{1,\ldots,a\} \subset A$, $F(r,i) = r$ for all $1 \leq i \leq a$ and $\phi_{r-L}(w) \leq p$. Let $A_t := \{U \geq t\}$, so that $A_t = \{\tilde{r}_u - \hat{r}_0 \geq \lfloor \alpha_2 u \rfloor$ for all $u < t\}$. Let $B_t := \{\tilde{r}_t - \hat{r}_0 \leq \lfloor 2\alpha(0)t \rfloor\}$. Now choose $K > 0$ such that $K\alpha_2 > 2\alpha(0)$ and consider the event C_t that the (at most) aM random walks involved in the definition of $\nu_{\tilde{r}_t+1}$ remain below their position at time t during the time interval $[t, Kt]$. On $B_t \cap C_t$, $\tilde{r}_{Kt} - \hat{r}_0 \leq 2\alpha(0)t$, so that, with our choice of K, for t large enough (non-random), $B_t \cap C_t \subset \{U \leq Kt\}$.

Now, we know that $\mathbb{Q}_w^{0,\theta}(A_t) \geq \mathbb{Q}_w^{0,\theta}(U = +\infty) \geq \delta_2 > 0$. Moreover, it is easily checked that, by the law of large numbers, $\lim_{t\to+\infty} \mathbb{Q}_w^{0,\theta}(B_t^c) = 0$ uniformly with respect to all w such that $F(r,i) = r$ for all $1 \leq i \leq a$. As a consequence, for large enough t (not depending on w), $\mathbb{Q}_w^{0,\theta}(A_t \cap B_t) \geq \delta_2/2$. Moreover, conditional on $A_t \cap B_t$, C_t has a probability larger that $ct^{-aM/2}$ for some $c > 0$. As a consequence, there exists $d > 0$ such that, for large enough t (not depending on w), $\mathbb{Q}_w^{0,\theta}(A_t \cap B_t \cap C_t) \geq dt^{-aM/2}$, so that $\mathbb{Q}_w^{0,\theta}(t \leq U \leq Kt) \geq dt^{-aM/2}$. Since U, V and W are independent and $\mathbb{Q}_w^{0,\theta}(V = +\infty) \geq \delta_1 > 0$ and $\mathbb{Q}_w^{0,\theta}(W = +\infty) \geq \delta_3 > 0$, we deduce that $\mathbb{Q}_w^{0,\theta}(t \leq U < +\infty,\, V = +\infty,\, W = +\infty) > d\delta_1\delta_3 t^{-aM/2}$. Then observe that, on the event $\{t \leq U < +\infty,\, V = +\infty,\, W = +\infty\}$, one has that $D \geq t$ and $\hat{r}_D \geq \hat{r}_t \geq \lfloor \alpha_2 t \rfloor$. Since $\kappa \geq D \circ \theta_{S_1} + S_1$ and $\hat{r}_\kappa \geq \hat{r}_{D\circ\theta_{S_1}+S_1}$, this ends the proof.

8. Appendix: negligibility of remote particles

Proposition 17. *For any $w \in \mathbb{L}_\theta$, $0 \leq \epsilon < 1/2$, and any $t \geq 0$, with $\mathbb{P}$ probability one,*

$$\lim_{K\to-\infty} \sup_{0\leq s\leq t} \sum_{(x,i)\in A;\, x\leq r+K} \exp(\theta(F_s^\epsilon(x,i) - r)) = 0.$$

Proof. For all x, i, t, let $C_{x,i,t} := \exp(\theta(F_t^\epsilon(x,i) - r))$ and $\gamma := [2(\cosh\theta - 1) + 4\epsilon\sinh\theta]$. Let also

$$H_{K,k}(s) := \sum_{(x,i)\in A;\, r+K+k\leq x\leq r+K} C_{x,i,s} \text{ and } H_{K,-\infty}(s) := \sum_{(x,i)\in A;\, x\leq r+K} C_{x,i,s}.$$

Since for every (x,i), $(C_{x,i,s}\exp(-\gamma s))_{s\geq 0}$ is a càdlàg martingale, so is $(H_{K,k}(t)\exp(-\gamma t))_{t\geq 0}$, and we have the following inequality, valid for all $\lambda > 0$:

$$\mathbb{P}\left(\sup_{0\leq s\leq t} H_{K,k}(s)\exp(-\gamma s) > \lambda\right) \leq \lambda^{-1}\mathbb{E}\left(H_{K,k}(0)\right).$$

Now $\mathbb{E}\left(H_{K,k}(0)\right) = \sum_{(x,i)\in A;\, r+K+k\le x\le r+K} \exp(\theta(F^\epsilon(x,i)-r))$. We deduce that

$$\mathbb{P}\left(\sup_{0\le s\le t} H_{K,k}(s) > \lambda\right) \le \lambda^{-1}\exp(\gamma t) \sum_{(x,i)\in A;\, r+K+k\le x\le r+K} \exp(\theta(F^\epsilon(x,i)-r)). \tag{56}$$

Now observe that, for every s, the sequence $(H_{K,k}(s))_{k=0,-1,\cdots}$ is non-decreasing since we are summing non-negative terms. As a consequence, $\mathbb{P}\left(\sup_{0\le s\le t} H_{K,-\infty}(s) > \lambda\right)$ equals $\mathbb{P}\left(\bigcup_{k=0}^{-\infty} \sup_{0\le s\le t} H_{K,k}(s) > \lambda\right)$, which is the probability of the union of a non-decreasing sequence of events, and so is equal to $\lim_{k\to-\infty} \mathbb{P}\left(\sup_{0\le s\le t} H_{K,k}(s) > \lambda\right)$. As a consequence, by (56),

$$\mathbb{P}\left(\sup_{0\le s\le t} H_{K,-\infty}(s) > \lambda\right) \le \lambda^{-1}\exp(\gamma t) \sum_{(x,i)\in A;\, x\le r+K} \exp(\theta(F^\epsilon(x,i)-r)). \tag{57}$$

Now observe that, for every s, the sequence $(\sum_{(x,i)\in A;\, x\le r+K} C_{x,i,s})_{K=0,-1,\cdots}$ is non-increasing, since we are summing non-negative terms. As a consequence, $\lim_{K\to-\infty}\sup_{0\le s\le t} H_{K,-\infty}(s)$ exists, and $\mathbb{P}\left(\lim_{K\to-\infty}\sup_{0\le s\le t} H_{K,-\infty}(s) > \lambda\right)$ equals $\mathbb{P}\left(\bigcap_{K\le 0}\sup_{0\le s\le t} H_{K,-\infty}(s) > \lambda\right)$, which is the probability of the intersection of a non-increasing sequence of events, and so is equal to $\lim_{K\to-\infty}\mathbb{P}(\sup_{0\le s\le t} H_{K,-\infty}(s)) > \lambda)$. ¿From Inequality (57), we see that this last expression equals zero. ☐

9. Appendix: estimates on the renewal structure

In the sequel every constant C_i or δ_i appearing in the estimates is assumed to depend on $a, \theta, \epsilon_0, \alpha_1, \alpha_2, p, L, \epsilon$, unless there is a special mention that dependence with respect to some of these parameters is absent. The notation $(\xi_s^\epsilon)_{s\ge 0}$ stands for a nearest-neighbor random walk on $\mathbb{Z}$ with jump rate 2 and step distribution $(1/2+\epsilon)\delta_{+1} + (1/2-\epsilon)\delta_{-1}$, started at zero. The probability measure governing $(\xi_s^\epsilon)_{s\ge 0}$ is denoted by P. We use the shorthand $M' := M/4-1$, which is an integer number according to (24). We also use the notation

$$\mathbb{L}'_\theta := \{w = (F,r,A) \in \mathbb{L}_\theta;\ r\times\{1,\dots,a\}\subset A,\ F(r,i) = r \text{ for all } 1\le i\le a\}.$$

For every $x\in\mathbb{Z}$, let $M_{t,x,i} := \sup_{0\le s\le t} Z_{s,x,i}$. Let also, for $z\in\mathbb{Z}$,

$$\psi_{z,t} := \sum_{(x,i);\, x\le z,\, (x,i)\in A_t} \exp(\theta(M_{t,x,i} - \hat{r}_t)). \tag{58}$$

Let $\mu_\epsilon := \theta\alpha_1 - 2(\cosh\theta - 1) - 4\epsilon\sinh\theta$. Now, for all $0\le\epsilon\le\epsilon_0$, $\mu_\epsilon \ge \mu_{\epsilon_0}$, and, according to (25), $\mu_\epsilon \ge \mu_{\epsilon_0} > 0$ for all $0\le\epsilon\le\epsilon_0$.

Lemma 14. *There exists $C_1 < +\infty$ not depending on ϵ or L such that, for all $0\le\epsilon\le\epsilon_0$ and all $w=(F,r,A)\in\mathbb{L}_\theta$,*

$$\mathbb{Q}_w^{\epsilon,\theta}(t < W < +\infty) \le C_1\phi_{r-L}(w)\exp(-\mu_\epsilon t).$$

Proof. Without loss of generality we assume $r=0$. Let us first note that

$$\mathbb{Q}_w^{\epsilon,\theta}\left[t < W < \infty\right] \le \mathbb{Q}_w^{\epsilon,\theta}\left[\cup_{s\ge t}\left\{\phi_{-L}(w_s) \ge e^{\theta(\lfloor\alpha_1 s\rfloor - \hat{r}_s)}\right\}\right].$$

By the fact that $s\mapsto M_{s,x,i}$ is nondecreasing, and the union bound, we deduce that

$$\mathbb{Q}_w^{\epsilon,\theta}\left[t<W<\infty\right]\leq\sum_{n=\lfloor t\rfloor}^{+\infty}\mathbb{Q}_w^{\epsilon,\theta}\left[\sum_{(x,i)\in A\cap\{\dots,-(L-1),-L\}\times\{1,\dots,a\}}e^{\theta M_{n+1,x,i}}\geq e^{\theta\lfloor\alpha_1 n\rfloor}\right].$$

Using the Markov inequality, we obtain that

$$\mathbb{Q}_w^{\epsilon,\theta}\left[t<W<\infty\right]\leq\sum_{n=\lfloor t\rfloor}^{+\infty}\exp(-\theta\lfloor\alpha_1 n\rfloor)\sum_{(x,i)\in A\cap\{\dots,-(L-1),-L\}\times\{1,\dots,a\}}\mathbb{E}_w^{\epsilon,\theta}\left(e^{\theta M_{n+1,x,i}}\right).\tag{59}$$

For $(x,i)\in A$, write $(Z_{s,x,i})_s$ as the independent sum of a symmetric nearest neighbor random walk on $\mathbb{Z}$ with rate $2-4\epsilon$ and a Poisson process with rate 4ϵ. Using the reflection principle to treat the symmetric part, and the fact that the Poisson process part is non-decreasing, we deduce that

$$\mathbb{E}_w^{\epsilon,\theta}\left(e^{\theta M_{s,x,i}}\right)\leq 2\exp(\theta F(x,i))\exp\left(s\left[2(\cosh\theta-1)+4\epsilon\sinh\theta\right]\right).$$

Plugging the last identity into (59) and summing, we finish the proof of the Lemma.

□

Define for $t\geq 0$, and $z\leq\hat{r}_0$,

$$N_{t,z}(w.):=e^{\theta\hat{r}_t-[2(\cosh\theta-1)-4\epsilon\sinh\theta]t}\phi_z(w_t).\tag{60}$$

Lemma 15. *For all $0\leq\epsilon\leq\epsilon_0$, and all $w=(F,r,A)\in\mathbb{L}_\theta$, the family $(N_{t,z})_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t^{\epsilon,\theta})_{t\geq 0}$-martingale with respect to $\mathbb{Q}_w^{\epsilon,\theta}$,.*

Proof. Let us remark that,

$$N_{t,z}=\sum_{(x,i)\in A,x\leq z}e^{\theta Z_{t,x,i}-[2(\cosh\theta-1)-4\epsilon\sinh\theta]t}.$$

Now, each one of the terms in the above sum is an $(\mathcal{F}_t^{\epsilon,\theta})_{t\geq 0}$-martingale Furthermore, since $\phi_z(0)<+\infty$, the martingales $\sum_{(x,i)\in A,-n\leq x\leq z}e^{\theta Z_{t,x,i}-[2(\cosh\theta-1)-4\epsilon\sinh\theta]t}$, converge in $L^1(\mathbb{Q}_w^{\epsilon,\theta})$ to $N_{t,z}$ as $n\to\infty$. Thus, $(N_{t,z})_{t\geq 0}$ is an $(\mathcal{F}_t^{\epsilon,\theta})_{t\geq 0}$-martingale. That the paths are càdlàg is an easy consequence of $(w_s)_{s\geq 0}$ being càdlàg.

□

Lemma 16. *For every $0\leq\epsilon\leq\epsilon_0$ and $w=(F,r,A)\in\mathbb{L}_\theta$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[W<\infty\right]\leq\exp(\theta)\phi_{r-L}(w).$$

Proof. See [4]. □

Lemma 17. *There exist $0<C_2,C_3<+\infty$ not depending on ϵ or L such that, for all $0\leq\epsilon\leq\epsilon_0$, $w=(F,r,A)\in\mathbb{L}_\theta$ and $t\geq 0$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[t<V<\infty\right]\leq LC_2\exp(-C_3 t).$$

Proof. Without loss of generality, assume that $r = 0$. Then $\mathbb{Q}_w^{\epsilon,\theta}(t < V < +\infty)$ is bounded above by the probability that one of the random walks born at a site between $-L+1$ and -1 is at the right of $\lfloor \alpha_1 s \rfloor$ at some time $s \geq t$. By coupling, we see that the worst case is when all the walks start at zero, in which case, by the union bound, the probability is less than aL times the probability for a single random walk started at zero to exceed $\lfloor \alpha_1 s \rfloor$ at some time $s \geq t$. Let $\tau := \inf\{s \geq t; \xi_s^\epsilon \geq \lfloor \alpha_1 s \rfloor\}$.

Using the fact that $(\exp(\theta \xi_s^\epsilon - [2(\cosh\theta - 1) - 4\epsilon \sinh\theta]s))_{s\geq 0}$ is a martingale, and applying Doob's stopping theorem, we obtain the bound $P(\tau < +\infty) \leq \exp(\theta)\exp(-\mu_\epsilon t)$. The result follows. □

Lemma 18. *There exists $\delta_1 > 0$ not depending on ϵ such that, for all $0 \leq \epsilon \leq \epsilon_0$ and $w = (F, r, A) \in \mathbb{L}_\theta$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[V < \infty\right] \leq 1 - \delta_1.$$

Proof. Without loss of generality we can assume that $r = 0$. Note that the probability $\mathbb{Q}_w^{\epsilon,\theta}[V < \infty]$ is upper bounded by the probability that a random walk within a group of aL independent ones all initially at site $x = 0$, at some time $t \geq 0$ is at the right of $\lfloor \alpha_1 t \rfloor$. But this probability is $1 - f(\epsilon)^{aL}$, where $f(\epsilon) := P(\text{for all } s \geq 0, \xi_s^\epsilon \leq \lfloor \alpha_1 s \rfloor)$. By coupling, observe that f is a non-increasing function of ϵ. For $\epsilon = \epsilon_0$, the asymptotic speed of the walk is $4\epsilon_0$. Since, from (25) $\alpha_1 > 4\epsilon_0$, an easy consequence of the law of large numbers is that $f(\epsilon_0) > 0$. This ends the proof. □

Lemma 19. *There exists $0 < C_4 < +\infty$ not depending on ϵ or L such that for all $\epsilon \leq \epsilon_0$ and $w = (F, r, A) \in \mathbb{L}'_\theta$, and all $t > 0$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[t < U < \infty\right] \leq C_4 t^{-M'}.$$

Proof. The proof given in [5] for $\epsilon = 0$ is based on tail estimates on the random variables $(\nu_k)_{k\geq 0}$. By coupling, for all $0 \leq \epsilon < 1/2$, and every $s \geq 0$, $\mathbb{Q}_w^{\epsilon,\theta}(\nu_k \geq s) \leq \mathbb{Q}_w^{0,\theta}(\nu_k \geq s)$. Thus, the estimate in [5] is in fact uniform over ϵ. □

Lemma 20. *There exists $0 < C_{45} < +\infty$ not depending on ϵ or L such that for all $\epsilon \leq \epsilon_0$ and all $t > 0$,*

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}\left[\cup_{s\geq t} \hat{r}_s < \lfloor \alpha_1 s \rfloor\right] \leq C_{45} t^{-M'}.$$

Proof. Since we start with the initial condition $\mathcal{I}_0$, we can define a modified auxiliary front $(\tilde{r}'_s)_{s\geq 0}$ by replacing the random variables $(\nu_k)_{k\geq 0}$ used in the definition of $(\tilde{r}_s)_{s\geq 0}$ by the random variables $(\nu'_k)_{k\geq 0}$ defined as follows. Let $\nu'_0 := 0$ and, for $k \geq 1$, ν'_k is the first time one of the random walks $\{(G_{s,z,i})_{s\geq 0}; (\hat{r}_0 + k - M) \leq z \leq \hat{r}_0 + k - 1, 1 \leq i \leq a\}$, hits the site $\hat{r}_0 + k$. With this definition, $\tilde{r}'_s \leq \hat{r}_s$ for all $s \geq 0$, and, for each $1 \leq j \leq M - 1$, the random variables $\{\nu'_{Mk+j} : k \geq 0\}$ are i.i.d. with finite moment of order $M/2$, whereas this is only true for $\{\nu_{Mk+j} : k \geq 1\}$. The argument of [5] used to prove Lemma 19 can then be easily adapted to prove the present result. Alternatively, one can invoke Lemma 38. □

Lemma 21. *For every* $0 < \epsilon < 1/2$, *there exist* $0 < C_5(\epsilon), C_6(\epsilon) < +\infty$ *not depending on* L *such that, for every* $w = (F, r, A) \in \mathbb{L}'_\theta$, *and every* $t > 0$,

$$\mathbb{Q}_w^{\epsilon,\theta}\left[t < U < \infty\right] \leq C_5(\epsilon)\exp(-C_6(\epsilon)t).$$

Proof. We observe that, for a given $\epsilon > 0$, ν_k has an exponentially decaying tail due to the positive bias of the random walks $(G_{s,x,i})_{s\geq 0}$. Using standard large deviations estimates rather than moment estimates in the proof of Lemma 19, we get the result. □

Using a similar argument, we can prove the following Lemma.

Lemma 22. *For all* $0 < \epsilon \leq \epsilon_0$, *there exists* $0 < C_{53}(\epsilon), C_{54}(\epsilon) < +\infty$ *not depending on* L *such that for all* $\epsilon \leq \epsilon_0$ *and all* $t > 0$,

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}\left[\cup_{s\geq t}\hat{r}_s < \lfloor \alpha_1 s\rfloor\right] \leq C_{53}(\epsilon)\exp(-C_{54}(\epsilon)t).$$

Lemma 23. *There exists* $\delta_2 > 0$ *not depending on* ϵ *such that, for all* $0 \leq \epsilon \leq \epsilon_0$, $w = (F, r, A) \in \mathbb{L}'_\theta$, *and* $t > 0$,

$$\mathbb{Q}_w^{\epsilon,\theta}\left[U < \infty\right] \leq 1 - \delta_2.$$

Proof. By coupling, we see that $\mathbb{Q}_w^{\epsilon,\theta}\left[U < \infty\right]$ is a non-increasing function of ϵ. Thus the estimate for $\epsilon = 0$ proved in [5] is enough. □

Lemma 24. *Let* β *be such that* $0 < \beta < \alpha(0)$. *Then there exists* $0 < C_7(\beta) < \infty$ *not depending on* ϵ *or* L *such that, for all* $0 \leq \epsilon \leq \epsilon_0$, *the following properties hold for all* $w = (F, r, A) \in \mathbb{L}_\theta$.

a) *If* $r = 0$ *and* $w \in \mathbb{L}'_\theta$, *and* $n \geq 1$,

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(n) > n/\beta\right] \leq C_7(\beta)n^{-a/2}.$$

b) *Assume that* $r = 0$, $m_{-\lfloor L^{1/4}\rfloor,0}(w) \geq a\lfloor L^{1/4}\rfloor/2$ *and* $n \geq 1$. *Then,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(n) > n/\beta\right] \leq (C_7(\beta)\lfloor L^{1/4}\rfloor n^{-1/2})^{a\lfloor L^{1/4}\rfloor/2} + C_7(\beta)n^{-M'}.$$

c) *Assume that* $r = 0$. *For all* $k \geq M$ *and* $n \geq 1$, *we have,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(n+k) - \hat{T}(k) > n/\beta\right] \leq C_7(\beta)n^{-M'}.$$

Proof. The proof given in [4] for $\epsilon = 0$ is based on tail estimates for the random variables $(\nu_k)_{k\geq 0}$ and for hitting times of symmetric random walks, so that, by coupling, the estimates proved in [4] are in fact uniform over ϵ. □

Lemma 25. *Let* β *be such that* $0 < \beta < \alpha(0)$. *Then there exists* $0 < C_{37}(\beta) < \infty$ *not depending on* ϵ *or* L *such that, for all* $0 \leq \epsilon \leq \epsilon_0$, *for all* $w = (F, r, A) \in \mathbb{L}_\theta$ *such that* $r = 0$ *and* $m_{-\lfloor L^{1/4}\rfloor,0}(w) \geq a\lfloor L^{1/4}\rfloor/2$, *for all* $n \geq 1$,

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(nL) > nL/\beta\right] \leq C_{37}(\beta)(nL^{1/2})^{-M'}.$$

Proof. Easy consequence of Lemma 24 b), using the first inequality in (26). □

Lemma 26. *For all $0 < \epsilon < 1/2$ and β such that $0 < \beta < \alpha(0)$, there exist $0 < C_8(\epsilon, \beta), C_9(\epsilon, \beta) < \infty$ not depending on L such that: for every $w = (F, r, A) \in \mathbb{L}'_\theta$, and $n \geq 1$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(n) > n/\beta\right] \leq C_8(\beta, \epsilon) \exp(-C_9(\beta, \epsilon)n).$$

Proof. Stems from the exponential decay of the tail of ν_k, as in Lemma 21. □

Corollary 4. *There exists $0 < C_{10}, C_{11} < \infty$ not depending on ϵ or L such that, for all $\epsilon \leq \epsilon_0$, all $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$, and all $t > 0$,*

$$\mathbb{Q}_w^{\epsilon,\theta}(t < D < \infty) \leq C_{10}(t^{-M'} + L \exp(-C_{11}t)).$$

Corollary 5. *For every $0 < \epsilon < 1/2$, there exist $0 < C_{12}(\beta, \epsilon), C_{13}(\beta, \epsilon) < \infty$ not depending on L such that, for all $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$, and for all $t > 0$,*

$$\mathbb{Q}_w^{\epsilon,\theta}(t < D < \infty) \leq LC_{12}(\beta, \epsilon) \exp(-C_{13}(\beta, \epsilon)t).$$

Corollary 6. *There exists $0 < \delta_3 < \infty$ such that, for all $0 \leq \epsilon \leq \epsilon_0$, and all $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$,*

$$\mathbb{Q}_w^{\epsilon,\theta}(D < \infty) \leq 1 - \delta_3.$$

Proof of the corollaries 4, 5 and 6. See [4]. □

Lemma 27. *There exists $0 < C_{14}, C_{15} < +\infty$ not depending on ϵ or L such that, for all $0 \leq \epsilon \leq \epsilon_0$, all $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$, and all $t > 0$,*

$$\mathbb{Q}_w^{\epsilon,\theta}(\hat{r}_D - r > t, D < +\infty) \leq C_{14}\left(t^{-M'} + L \exp(-C_{15}t)\right).$$

Lemma 28. *For every $0 < \epsilon \leq \epsilon_0$, there exists $0 < C_{16}(\epsilon), C_{17}(\epsilon) < +\infty$ not depending on L such that, for all $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$, and for all $t > 0$,*

$$\mathbb{Q}_w^{\epsilon,\theta}(\hat{r}_D - r > t, D < +\infty) \leq LC_{16}(\epsilon) \exp(-C_{17}(\epsilon)t).$$

Proof of Lemmas 27 and 28. Consider $\gamma_0 > 0$ large enough so that

$$c_{\gamma_0}(\epsilon_0, \theta) > 0. \tag{61}$$

Observe that then $c_{\gamma_0}(\epsilon, \theta) \geq c_{\gamma_0}(\epsilon_0, \theta)$ for all $0 \leq \epsilon \leq \epsilon_0$.

Observe that by the union bound and the fact that $(\hat{r}_s)_s$ is non-decreasing, $\mathbb{Q}_w^{\epsilon,\theta}(\hat{r}_D - r > t, D < +\infty) \leq \mathbb{Q}_w^{\epsilon,\theta}(\hat{r}_{t\gamma_0^{-1}} - r > t, D \leq t\gamma_0^{-1}) + \mathbb{Q}_w^{\epsilon,\theta}(t\gamma_0^{-1} < D < +\infty)$. Moreover, note that, by definition, $\phi_r(0) \leq \phi_{r-L}(0) + aL$. Then apply Lemma 2 and Corollaries 4 and 5. □

Lemma 29. *Consider $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$. Then, for all $0 \leq \epsilon \leq \epsilon_0$, $\mathbb{Q}_w^{\epsilon,\theta}$-a.s. on the event $\{D < \infty\}$ we have,*

$$\phi_{r-L}(D) \leq e^\theta.$$

Proof. See [4]. □

Corollary 7. *There exists $0 < C_{18} < +\infty$ not depending on ϵ or L, such that, for all $0 \le \epsilon \le \epsilon_0$, and all $w = (F, r, A) \in \mathbb{L}'_\theta$ satisfying $\phi_{r-L}(w) \le p$,*

$$\mathbb{E}^{\epsilon,\theta}_w[\phi_{\hat{r}_D}(D), D < \infty] \le C_{18} L.$$

Proof. See [4]. □

Lemma 30. *There is a constant $0 < C_{19} < +\infty$ not depending on ϵ or L, such that, for all $0 \le \epsilon \le \epsilon_0$, and all $w = (F, r, A) \in \mathbb{L}'_\theta$:*

a) $\mathbb{Q}^{\epsilon,\theta}_w \left(m_{r, r+\lfloor L^{1/4} \rfloor}(w_{\hat{T}(r+\lfloor L^{1/4}\rfloor)}) < a\lfloor L^{1/4}\rfloor/2 \right) \le C_{19} L^{-a/8}$;

b) $\mathbb{Q}^{\epsilon,\theta}_w \left(m_{\hat{r}_D + L - \lfloor L^{1/4}\rfloor, \hat{r}_D + L}(w_{\hat{r}_D + L}) < a\lfloor L^{1/4}\rfloor/2 \right) \le C_{19} L^{-aM'/8(M'+1)}$.

Proof. Without loss of generality, assume that $r = 0$. For the sake of readability, let $n := \lfloor L^{1/4} \rfloor$. We start with the proof of a).

Choose $4\epsilon_0 < \beta < \alpha(0)$. Then,

$$\mathbb{Q}^{\epsilon,\theta}_w \left[m_{0,n}(w_{\hat{T}(n)}) < \frac{an}{2} \right] \le \mathbb{Q}^{\epsilon,\theta}_w \left[m_{0,n}(w_{\hat{T}(n)}) < \frac{an}{2}, \hat{T}(n) \le \frac{n}{\beta} \right] + \mathbb{Q}^{\epsilon,\theta}_w \left[\hat{T}(n) > \frac{n}{\beta} \right]. \tag{62}$$

Note that the event $\{m_{0,n}(w_{\hat{T})(n)} < an/2, \hat{T}(n) \le n/\beta\}$ is contained in the event that at least one particle born at any of the sites $\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \dots, n$ hits some site $x \le 0$ in a time shorter than or equal to n/β. Hence, we can conclude that,

$$\mathbb{Q}^{\epsilon,\theta}_w \left[m_{0,n}(w_{\hat{T}(n)}) < \frac{an}{2}, \hat{T}(n) \le \frac{n}{\beta} \right] \le a(n + 1 - \lfloor n/2 \rfloor) P[\Lambda^\epsilon_{n/\beta} \le -\lfloor n/2 \rfloor], \tag{63}$$

where and $\Lambda^\epsilon_t := \inf_{0 \le s \le t} \xi^\epsilon_s$.

Noting that, by coupling, $P[\Lambda^\epsilon_{n/\beta} \le -n/2]$ is non-increasing as a function of ϵ, we can assume that $\epsilon = 0$.

Now, by the reflection principle, $P[\Lambda^0_{n/\beta} \le -n/2] \le 2P[\xi^0_{n/\beta} \le -n/2]$. Hence, from inequality (63), we see that $\mathbb{Q}^{\epsilon,\theta}_w \left[m_{0,n}(\hat{T}(n)) < an/2, \hat{T}(n) \le \frac{n}{\beta} \right]$ is bounded above by $a(n+1)P[\xi^0_{n/\beta} \le -n/2]$. By a standard large deviations argument, for every $t \ge 0$ and positive integer x, $P[\xi^0_t \ge x] \le e^{-tg(x/t)}$, where $g(u) > 0$ for all $u > 0$. Hence, $a(n+1)P[\xi^0_{n/\beta} \le -n/2] \le a(n+1) \exp\left\{ -\frac{n}{\beta} g(\beta/2) \right\}$. Finally, using part a) of Lemma 24 to bound the second term of inequality (62) and using the fact that $a(n+1)\exp\left\{ -\frac{n}{\beta} g(\beta/2) \right\} \le 1/n^{a/2}$ for n large enough, we conclude the proof of a).

Now for b), $\mathbb{P}_w[m_{\hat{r}_D + L - n, \hat{r}_D + L}(\hat{T}(\hat{r}_D + L))(w_{\hat{T}(\hat{r}_D + L)}) < an/2]$ is upper bounded by,

$$\textstyle\sum_{k: 1 \le k \le m} \mathbb{Q}^{\epsilon,\theta}_w[m_{k+L-n, k+L}(w_{\hat{T}(k+L)}) < an/2] + \mathbb{Q}^{\epsilon,\theta}_w[\hat{r}_D > m, D < \infty]$$

Letting $m := L^{a/(8(M'+1))}$, and using part a) and Lemma 27, we obtain the result. $\square$

Throughout the sequel, to simplify notation, we will define on the event $\{D < \infty\}$ for each $n \geq 1$,

$$F_n := \hat{T}(\hat{r}_D + nL) - D,\ F'_n := \hat{T}(\hat{r}_D + nL).$$

Lemma 31. *For every $0 < \beta < \alpha(0)$, there exist $0 < C_{20}(\beta), C_{21}(\beta) < \infty$ not depending on ϵ, L, such that for all $0 \leq \epsilon \leq \epsilon_0$, and all $w = (F, r, A) \in \mathbb{L}_\theta$ such that $m_{r-\lfloor L^{1/4}\rfloor, r}(w) \geq a\lfloor L^{1/4}\rfloor/2$, and $\phi_{r-L}(w) \leq p$, and for all natural $n \geq 1$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, D < \infty\right] \leq C_{20}(\beta)(nL^{1/2})^{-M'+1}.$$

Proof. Without loss of generality we can assume that initially $r = 0$. Note that $\mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, D < \infty\right]$ is upper-bounded by

$$\sum_{k:1\leq k\leq \lfloor L^{1/2}\rfloor n} \mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, \hat{r}_D = k, D < \infty\right] + \mathbb{Q}_w^{\epsilon,\theta}\left[\hat{r}_D > n\lfloor L^{1/2}\rfloor, D < \infty\right]. \quad (64)$$

Now, on the event $\{D < \infty\}$ we have that $\hat{T}(\hat{r}_D) \leq D$ so that $F_n \leq \hat{T}(\hat{r}_D + nL) - \hat{T}(\hat{r}_D)$. Hence,

$$\mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, \hat{r}_D = k, D < \infty\right] \leq \mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(k + nL) - \hat{T}(k) > \frac{nL}{\beta}\right].$$

Now, by part c) of Lemma 24, for all $k \geq M$ we have $\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(k+nL) - \hat{T}(k) > \frac{nL}{\beta}\right] \leq \frac{C_7(\beta)}{(nL)^{M'}}$. On the other hand for $1 \leq k \leq M-1$, $\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(k+nL) - \hat{T}(k) > \frac{nL}{\beta}\right] \leq \mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(M+nL) > \frac{nL}{\beta}\right]$.

Now let $\beta < \beta' < \alpha(0)$. Observe that, when $nL^{1/2} \geq M(\beta'/\beta - 1)^{-1}$, $(nL + M)/\beta' \leq nL/\beta$, so that $\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(M+nL) > \frac{nL}{\beta}\right] \leq \mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(M+nL) > \frac{nL+M}{\beta'}\right]$.

Thus, by Lemma 25, since $m_{r-\lfloor L^{1/4}\rfloor, r}(w) \geq a\lfloor L^{1/4}\rfloor/2$, we know that

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\hat{T}(M+nL) > \frac{nL+M}{\beta'}\right] \leq (C_{37}(\beta)(nL^{1/2})^{-M'}). \quad (65)$$

When $nL^{1/2} \leq M(\beta'/\beta - 1)^{-1}$, the same bound holds, with a possibly larger constant, using only the trivial inequality $\mathbb{Q}_w^{\epsilon,\theta}(\cdot) \leq 1$. Using Lemma 27 to estimate the second term of display (64), and combining with (65), we finish the proof. $\square$

Lemma 32. *For every $0 < \epsilon \leq \epsilon_0$ and $0 < \beta < \alpha(0)$, there exist $0 < C_{22}(\beta, \epsilon), C_{23}(\beta, \epsilon) < \infty$ not depending on L, such that for all $w = (F, r, A) \in \mathbb{L}'_\theta$ such that $\phi_{r-L}(w) \leq p$, for all natural $n \geq 1$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, D < \infty\right] \leq C_{22}(\beta, \epsilon)\exp(-C_{23}(\beta, \epsilon)nL).$$

Proof. Consider $\ell > 0$ such that $\beta(1+\ell) < \alpha(0)$.

As in the proof of the previous lemma, $\mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, D < \infty\right]$ is upper-bounded by

$$\sum_{k:1\le k\le \lfloor \ell nL\rfloor} \mathbb{Q}_w^{\epsilon,\theta}\left[F_n > \frac{nL}{\beta}, \hat r_D = k, D<\infty\right] + \mathbb{Q}_w^{\epsilon,\theta}\left[\hat r_D > \lfloor \ell nL\rfloor, D<\infty\right]. \qquad (66)$$

By Lemma 28, $\mathbb{Q}_w^{\epsilon,\theta}\left[\hat r_D > \lfloor \ell nL\rfloor, D<\infty\right] \le LC_{16}(\epsilon)\exp(-C_{17}(\epsilon)\lfloor \ell nL\rfloor)$. On the other hand, for $1 \le k \le \lfloor \ell nL\rfloor$, $\mathbb{Q}_w^{\epsilon,\theta}\left[\hat T(k+nL) - \hat T(k) > \frac{nL}{\beta}\right] \le \mathbb{Q}_w^{\epsilon,\theta}\left[\hat T(\lfloor nL(1+\ell)\rfloor) > \frac{nL}{\beta}\right]$. By Lemma 26, $\mathbb{Q}_w^{\epsilon,\theta}\left[\hat T(\lfloor nL(1+\ell)\rfloor) > \frac{nL}{\beta}\right] \le C_8(\beta(1+\ell),\epsilon)\exp(-C_9(\beta(1+\ell),\epsilon)\lfloor nL(1+\ell)\rfloor)$.

□

Lemma 33. *Consider $w=(F,r,A)\in\mathbb{L}_\theta$ such that $r=0$. Then for all $0\le\epsilon\le\epsilon_0$, the following properties hold.*

a) *For every $h>0, s>0$ and $n\ge 1$ we have*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\psi_{0,\hat T(n)} > h, \hat T(n) < s\right] \le 2\frac{\phi_0(w)}{h}\exp(s(2(\cosh\theta-1)+4\epsilon\sinh\theta)-\theta n). \qquad (67)$$

b) *For every $h>0, s>0$, $k\ge 1$ and $n\ge k$ we have a.s.*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\psi_{k,\hat T(n)} - \psi_{k-L,\hat T(n)} > h, \hat T(n) - \hat T(k) < s \,\middle|\, \mathcal{F}^{\epsilon,\theta}_{\hat T(k)}\right]$$
$$\le 2\frac{aL}{h}\exp(s(2(\cosh\theta-1)+4\epsilon\sinh\theta)-\theta(n-k)).$$

Proof. See [4]. □

Corollary 8. *There exists $0<C_{24}<+\infty$ not depending on ϵ or L such that, for all $w=(F,r,A)\in\mathbb{L}_\theta$, for all $0\le\epsilon\le\epsilon_0$, $\lambda>0$, $n\ge 1$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\psi_{\hat r_D, F'_n} > \lambda,\ F_n \le \alpha_1^{-1}nL,\ D<+\infty\right] \le \lambda^{-1}C_{24}L\exp(-\alpha_1^{-1}nL\mu_{\epsilon_0}).$$

Proof. See [4]. □

Corollary 9. *There exists $0<C_{25}<+\infty$ not depending on ϵ or L such that, for all $0\le\epsilon\le\epsilon_0$, and all $w=(F,r,A)\in\mathbb{L}'_\theta$ such that $m_{r-\lfloor L^{1/4}\rfloor,r}(w)\ge a\lfloor L^{1/4}\rfloor/2$,*

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\{\psi_{\hat r_D} > p, \hat T(\hat r_D+L)\}\cup\{m_{\hat r_D+L-\lfloor L^{1/4}\rfloor,\hat r_D+L}(w_{\hat r_D+L}) < a\lfloor L^{1/4}\rfloor/2\},\ D<+\infty\right]$$
$$\le C_{25}L^{-aM'/(8(M'+1))}.$$

Proof. See [4]. □

Lemma 34. *Let $q\ge 1$ be an integer. Consider two sequences $(a_k)_{k\ge 1}$ and $(c_k)_{k\ge 1}$ of non-negative real numbers such that $\sum_{k=1}^{\infty} a_k < 1$ and such that*

$$c_1 \le a_1, \qquad (68)$$

and for every $m \geq 2$ we have that,

$$c_m \leq a_m + \sum_{k=1}^{m-1} a_{m-k} c_k. \tag{69}$$

For all integers $q \geq 0$, let $A_q := \sum_{k=1}^{+\infty} a_k k^q$ and $C_q := \sum_{k=1}^{+\infty} c_k k^q$. For $t \geq 0$, let $\mathcal{A}(t) := \sum_{k=1}^{+\infty} a_k \exp(tk)$ and $\mathcal{C}(t) := \sum_{k=1}^{+\infty} c_k \exp(tk)$. The following properties hold:

a) *Assume that $q \geq 1$ is such that $A_q < +\infty$. Then $C_k < +\infty$ for all $1 \leq k \leq q$, and*
$$C_q \leq (1 - A_0)^{-1} \left(A_q + \sum_{k=1}^{q} \binom{q}{k} C_{q-k} A_k \right).$$
b) *Assume that $\mathcal{A}(t_0) < +\infty$ for some $t_0 > 0$. Then $\mathcal{A}(t) < 1$ for all small enough $t > 0$ and, for all such t,*
$$\mathcal{C}(t) \leq (1 - \mathcal{A}(t))^{-1} \mathcal{A}(t).$$

Proof. Part a) is proved in [4]. As for part b), observe that the power series $a(z) := \sum_{k=1}^{+\infty} a_k z^k$ has a convergence radius $\geq \exp(t_0)$. As a consequence, the map $t \mapsto a(\exp(t))$ is well-defined and continuous for $t \leq t_0$. For $t = 0$, $a(\exp(t)) = \sum_{k=1}^{+\infty} a_k < 1$ by assumption. By continuity, $a(\exp(t)) < 1$ for all $t > 0$ small enough.

Summing (68) and (69), we see that, for all $m \geq 1$ and $t \geq 0$, $\sum_{i=1}^{m} c_i \exp(ti) \leq a_1 \exp(t) + \sum_{i=2}^{m} \left(a_i \exp(ti) + \sum_{k=1}^{i-1} a_{i-k} c_k \exp(ti) \right)$, so that $\sum_{i=1}^{m} c_i \exp(ti) \leq \sum_{i=1}^{m} a_i \exp(ti) + \sum_{k=1}^{m-1} c_k \exp(tk) \left(\sum_{i=k+1}^{m} a_{i-k} \exp(t(i-k)) \right)$. As a consequence, $\sum_{i=1}^{m-1} c_i \exp(ti) \leq \mathcal{A}(t) + \mathcal{A}(t) \sum_{k=1}^{m-1} c_k \exp(tk))$. When $\mathcal{A}(t) < 1$, we deduce that $\sum_{i=1}^{m-1} c_i \exp(ti) \leq (1 - \mathcal{A}(t))^{-1} \mathcal{A}(t)$. Letting m go to infinity, we conclude the proof. □

Lemma 35. *Let $(O, \mathcal{H}, \mathbb{T})$ be a probability space, and $(\mathcal{H}_n)_{n \geq 1}$ be a non-decreasing sequence of sub-$\sigma-$algebras of $\mathcal{H}$. Let $(B_n)_{n \geq 1}$, $(A_k^n)_{n \geq 2, 0 \leq k \leq n-1}$ and $(B'_n)_{n \geq 2}$ be sequences of events in $\mathcal{H}$ such that the following properties hold:*

(i) *for all $n \geq 1$, $B_n \in \mathcal{H}_n$*
(ii) *for all $n \geq 2$, $B_n \subset B_{n-1} \cap \left(B'_n \cup A_0^n \cup A_1^n \cup \cdots \cup A_{n-1}^n \right)$.*

Now assume that we have defined a sequence $(a_n)_{n \geq 1}$ of non-negative real numbers enjoying the following properties:

(1) *$\mathbb{T}(B_1) \leq a_1$;*
(2) *for all $n \geq 2$, $\mathbb{T}(B'_n | \mathcal{H}_{n-1}) \leq a_1$ a.s.;*
(3) *for all $n \geq 3$, $\mathbb{T}(A_{n-1}^n | \mathcal{H}_{n-2}) \leq a_2$ a.s.;*
(4) *for all $n \geq 2$, $\mathbb{T}(A_0^n) \leq a_n / 2$ a.s.;*
(5) *for all $n \geq 2$, $\mathbb{T}(A_1^n) \leq a_n / 2$ a.s.;*
(6) *for all $n \geq 4$ and all $2 \leq k \leq n-2$, $\mathbb{T}(A_k^n | \mathcal{H}_{k-1}) \leq a_{n-k+1}$ a.s.;*

then, letting $c_n := \mathbb{T}(B_n)$ for all $n \geq 1$, the inequalities (68) and (69) are satisfied by the two sequences $(a_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$.

Proof. First, observe that Inequality (68) is a mere consequence of assumption (1). Assume now that $n \geq 2$. By the union bound,

$$\mathbb{T}(B_n) \quad \leq \quad \sum_{k=0}^{n-1} \mathbb{T}(A_k^n, B_{n-1}) + \mathbb{T}\left(B'_n, B_{n-1}\right). \tag{70}$$

Now, since $B_{n-1} \in \mathcal{H}_{n-1}$, assumption (2) entails that $\mathbb{T}(B'_n, B_{n-1}) \leq a_1 \mathbb{T}(B_{n-1})$. On the other hand, (4) and (5) imply that $\mathbb{T}(A_0^n) + \mathbb{T}(A_1^n) \leq a_n$.

When $n = 2$, we deduce from (70) that $\mathbb{T}(B_n) \leq \mathbb{T}(A_0^n) + \mathbb{T}(A_1^n) + \mathbb{T}(B'_n, B_{n-1})$, so that $\mathbb{T}(B_n) \leq a_n + a_1 \mathbb{T}(B_{n-1})$, and so (69) is proved for $n = 2$.

Assume now that $n \geq 3$. Since by assumption $B_{n-1} \subset B_{n-2}$, $\mathbb{T}(A_{n-1}^n, B_{n-1}) \leq \mathbb{T}(A_{n-1}^n, B_{n-2})$. Now, thanks to assumption (3) and to the fact that $B_{n-2} \in \mathcal{H}_{n-2}$, $\mathbb{T}(A_{n-1}^n, B_{n-2}) \leq a_2 \mathbb{T}(B_{n-2})$.

For $n = 3$, we deduce from (70) that $\mathbb{T}(B_n) \leq \mathbb{T}(A_0^n) + \mathbb{T}(A_1^n) + \mathbb{T}(A_{n-1}^n, B_{n-1}) + \mathbb{T}(B'_n, B_{n-1})$, so that $\mathbb{T}(B_n) \leq a_n + a_2 \mathbb{T}(B_{n-2}) + a_1 \mathbb{T}(B_{n-1})$, and so (69) is proved for $n = 3$.

Assume now that $n \geq 4$. For $2 \leq k \leq n-2$, the fact that $B_{n-1} \subset B_{k-1}$ implies that $\mathbb{T}(A_k^n, B_{n-1}) \leq \mathbb{T}(A_k^n, B_{k-2})$. Since $B_{k-1} \in \mathcal{H}_{k-1}$, assumption (6) entails that $\mathbb{T}(A_k^n, B_{k-1}) \leq a_{n-k+1} \mathbb{T}(B_{k-1})$.

As a consequence, plugging the previous estimates into Inequality (70), we obtain that

$$\mathbb{T}(B_n) \leq a_n + a_2 \mathbb{T}(B_{n-2}) + a_1 \mathbb{T}(B_{n-1}) + \sum_{k=2}^{n-2} a_{n-k+1} \mathbb{T}(B_{k-1}),$$

which is exactly (69).

□

Lemma 36. *There exists* $0 < L_0 < +\infty$ *not depending on* ϵ *such that, for all* $L \geq L_0$ *there exists* $0 < C_{26} < +\infty$ *not depending on* ϵ*, such that for all* $0 \leq \epsilon \leq \epsilon_0$*, the following properties hold.*

a) *For all* $n \geq 1$, $\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(J_0 \geq n) \leq C_{26} n^{3-M'}$.

b) *For all* $w = (F, r, A) \in \mathbb{L}'_\theta$ *such that* $m_{r-\lfloor L^{1/4} \rfloor, r}(w) \geq a\lfloor L^{1/4} \rfloor/2$*, and* $\phi_{r-L}(w) \leq p$*, we have that, for all* $n \geq 1$, $\mathbb{Q}_w^{\epsilon,\theta}(J_{\hat{r}_D} \geq n, D < +\infty) \leq C_{26} n^{3-M'}$

c) *For all* $n \geq 1$, $\mathbb{Q}_{a\delta_0}^{\epsilon,\theta}(J_0 \geq n, U > \hat{T}_{nL}) \leq C_{26} n^{3-M'}$.

In the sequel, we use the notation $\mathcal{F}_t$ instead of $\mathcal{F}_t^{\epsilon,\theta}$ to alleviate notations.

Proof of part a). For all $n \geq 1$, let

$$B_n := \cap_{i=1}^n \left\{ \psi_{(i-1)L, \hat{T}(iL)} > p \right\} \cup B'_i,$$

$$B'_i := \left\{ m_{iL - \lfloor L^{1/4} \rfloor, iL}(w_{\hat{T}(iL)}) < a\lfloor L^{1/4} \rfloor/2 \right\}.$$

Since $\phi_z(w_t) \leq \psi_{z,t}$, the following inequality holds:

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(J_0 > n) \leq \mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(B_n). \tag{71}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, let

$$\Delta_k^n := \psi_{kL,\hat{T}(nL)} - \psi_{(k-1)L,\hat{T}(nL)},$$

and let

$$A_0^n := \left\{ \psi_{0,\hat{T}(nL)} > p/2^{n-1} \right\}, \quad A_k^n := \left\{ \Delta_k^n > p/2^{n-k} \right\}.$$

We now prove that the assumptions (i)-(ii) of Lemma 35 are satisfied, with $(O, \mathcal{H})$ being the space $\mathcal{D}(\mathbb{L}_\theta)$ equipped with the cylindrical σ-algebra, and probability $\mathbb{Q}_w^{\epsilon,\theta}$, and $\mathcal{H}_n := \mathcal{F}_{\hat{T}(nL)}$ for all $n \geq 1$.

Assumption (i) is immediate. Note that, for $n \geq 2$, $\psi_{(n-1)L,\hat{T}(nL)} = \psi_{0,\hat{T}(nL)} + \sum_{k=1}^{n-1} \Delta_k^n$. Since $\frac{1}{2^{n-1}} + \sum_{k=1}^{n-1} \frac{1}{2^{n-k}} = 1$, we have that

$$\left\{ \psi_{(n-1)L,\hat{T}(nL)} > p \right\} \subset \left\{ \psi_{0,\hat{T}(nL)} > p/2^{n-1} \right\} \cup \left[\cup_{k=1}^{n-1} \left\{ \Delta_k^n > p/2^{n-k} \right\} \right], \tag{72}$$

so that (ii) is established.

We now look for a sequence $(a_n)_{n \geq 1}$ such that assumptions (1)-(6) of Lemma 35 are satisfied. Assume that $n \geq 2$. By the strong Markov property and Lemma 24 c), using the fact that, by (26), $L \geq M$, we have for any $1 \leq k \leq n-1$, a.s.

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta} \left(\hat{T}(nL) - \hat{T}(kL) \geq (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}((k-1)L)} \right) \leq C_7(\alpha_1)((n-k)L)^{-M'}.$$

By the strong Markov property again, and Lemma 33 b), using the fact that $\mu_\epsilon \geq \mu_{\epsilon_0}$, we have that a.s.

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta} \left[\Delta_k^n > p/2^{n-k}, \hat{T}(nL) - \hat{T}(kL) \leq (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}((k-1)L)} \right] \\ \leq 2aLp^{-1}2^{n-k} \exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1).$$

We deduce that, for $n \geq 2$, and $1 \leq k \leq n-1$, a.s.

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(A_k^n | \mathcal{F}_{\hat{T}((k-1)L)}) \leq C_7(\alpha_1)((n-k)L)^{-M'} + 2aLp^{-1}2^{n-k} \exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1). \tag{73}$$

Similarly, using Lemma 25, which is possible since $m_{-\lfloor L^{1/4} \rfloor,0}(\mathcal{I}_0) \geq a\lfloor L^{1/4} \rfloor/2$, we have that

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta} \left(\hat{T}(nL) \geq nL/\alpha_1 \right) \leq C_{37}(\alpha_1)(nL^{1/2})^{-M'}.$$

On the other hand, by Lemma 33 a), we have that

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta} \left[\psi_{0,\hat{T}_{nL}} > p/2^n, \hat{T}(nL) \leq nL/\alpha_1 \right] \leq 2p^{-1}2^{n-1}\phi_0(\mathcal{I}_0) \exp(-\mu_{\epsilon_0} nL/\alpha_1).$$

We deduce that

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(A_0^n) \leq C_{37}(\alpha_1)(nL^{1/2})^{-M'} + 2p^{-1}2^{n-1}\phi_0(\mathcal{I}_0) \exp(-\mu_{\epsilon_0} nL/\alpha_1). \tag{74}$$

Now, for $n \geq 2$, by part a) of Lemma 30, the strong Markov property, the fact that $(n-1)L \leq nL - \lfloor L^{1/4} \rfloor$ and that there are at least a particles at the rightmost visited site at time $\hat{T}(nL - \lfloor L^{1/4} \rfloor)$, a.s.

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(B_n' | \mathcal{F}_{\hat{T}((n-1)L)}) \leq C_{19} L^{-a/8}. \tag{75}$$

Finally, observe that, by the union bound, $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(B_1)$ is upper bounded by $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\psi_{0,\hat{T}(L)} > p, \hat{T}(L) \leq L/\alpha_1) + \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{T}(L) > L/\alpha_1) + \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(m_{L-\lfloor L^{1/4}\rfloor,L}(w_{\hat{T}(L)}) < a\lfloor L^{1/4}\rfloor/2)$.

Thanks to Lemma 24 a), Lemma 33 a) and Lemma 30 a), we obtain that

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(B_1) \leq 2p^{-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}. \tag{76}$$

Now we see that, by Inequalities (75) and (76), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := 2p^{-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}.$$

Now, for $m \geq 2$, let

$$\begin{aligned} a_m := 2&\left[C_7(\alpha_1)((m-1)L)^{-M'} + 2aLp^{-1}2^{m-1}\exp(-\mu_{\epsilon_0}(m-1)L/\alpha_1)\right] \\ &+2\left[C_{37}(\alpha_1)(mL^{1/2})^{-M'} + 2p^{-1}2^{m-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}mL/\alpha_1)\right]. \end{aligned}$$

Inequalities (73) and (74) entail assumptions (3)-(4)-(5)-(6) of Lemma 35. Note that the sequence $(a_m)_{m\geq 1}$ depends on ϵ_0 but not on ϵ. Moreover, observe that, for large enough L (not depending on ϵ), $\sum_{m=1}^{+\infty} a_m m^{M'-3} < +\infty$. On the other hand, as L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part a) of Lemma 36 then follows from applying Lemma 34.

□

Proof of part b). We use exactly the same strategy as for part a).

For all $n \geq 1$, let

$$B_n := \cap_{i=1}^n \left\{\psi_{\hat{r}_D+(i-1)L,\hat{T}(\hat{r}_D+iL)} > p,\, D < +\infty\right\} \cup B'_i,$$

$$B'_i := \left\{m_{\hat{r}_D+iL-\lfloor L^{1/4}\rfloor,\hat{r}_D+iL}(w_{\hat{T}(\hat{r}_D+iL)}) < a\lfloor L^{1/4}\rfloor/2,\, D < +\infty\right\}.$$

Since $\phi_z(w_t) \leq \psi_{z,t}$, the following inequality holds:

$$\mathbb{Q}^{\epsilon,\theta}_w(J_{\hat{r}_D} > n,\, D < +\infty) \leq \mathbb{Q}^{\epsilon,\theta}_w(B_n). \tag{77}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, on $\{D < +\infty\}$, let

$$\Delta^n_k := \psi_{\hat{r}_D+kL,\hat{T}(\hat{r}_D+nL)} - \psi_{\hat{r}_D+(k-1)L,\hat{T}(\hat{r}_D+nL)},$$

and let

$$A^n_0 := \left\{\psi_{\hat{r}_D,\hat{T}(\hat{r}_D+nL)} > p/2^{n-1},\, D < +\infty\right\}, \quad A^n_k := \left\{\Delta^n_k > p/2^{n-k},\, D < +\infty\right\},$$

for $1 \leq k \leq n-1$.

We now prove that the assumptions (i)-(ii) of Lemma 35 are satisfied, with $(O,\mathcal{H},\mathbb{T})$ being the space $\mathcal{D}(\mathbb{L}_\theta)$ equipped with the cylindrical $\sigma-$algebra, and probability $\mathbb{Q}^{\epsilon,\theta}_w$, and $\mathcal{H}_n := \mathcal{F}_{\hat{T}(\hat{r}_D+nL)}$ for all $n \geq 1$.

Assumptions (i) is immediate. Note that, for $n \geq 2$, on $\{D < +\infty\}$ $\psi_{\hat{r}_D+(n-1)L,\hat{T}(\hat{r}_D+nL)} = \psi_{\hat{r}_D,\hat{T}(\hat{r}_D+nL)} + \sum_{k=1}^{n-1}\Delta_k^n$. Since $\frac{1}{2^{n-1}} + \sum_{k=1}^{n-1}\frac{1}{2^{n-k}} = 1$, we have that

$$\left\{\psi_{\hat{r}_D+(n-1)L,\hat{T}(\hat{r}_D+nL)} > p,\, D < +\infty\right\} \subset \left\{\psi_{\hat{r}_D,\hat{T}(\hat{r}_D+nL)} > p/2^{n-1},\, D < +\infty\right\} \cup \left[\cup_{k=1}^{n-1}\left\{\Delta_k^n > p/2^{n-k},\, D < +\infty\right\}\right],$$

so that (ii) is established.

We now look for a sequence $(a_n)_{n\geq 1}$ such that assumptions (1)-(6) of Lemma 35 are satisfied. Assume that $n \geq 2$. By the strong Markov property and Lemma 24 c), using the fact that, by (26), $L \geq M$, we have for any $1 \leq k \leq n-1$, on the event $\{D < +\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}\left(\hat{T}(\hat{r}_D + nL) - \hat{T}(\hat{r}_D + kL) \geq (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}(\hat{r}_D+(k-1)L)}\right) \leq C_7(\alpha_1)((n-k)L)^{-M'}.$$

By the strong Markov property again, and Lemma 33 b), using the fact that $\mu_\epsilon \geq \mu_{\epsilon_0}$, we have that, on $\{D < +\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\Delta_k^n > p/2^{n-k}, \hat{T}(\hat{r}_D + nL) - \hat{T}(\hat{r}_D + kL) \leq (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}(\hat{r}_D+(k-1)L)}\right] \leq 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1).$$

We deduce that, for $n \geq 2$, and $1 \leq k \leq n-1$, on $\{D < +\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}(A_k^n | \mathcal{F}_{\hat{T}(\hat{r}_D+(k-1)L)}) \leq C_7(\alpha_1)((n-k)L)^{-M'} + 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1). \tag{78}$$

Similarly, using Lemma 31, which is possible since $m_{-\lfloor L^{1/4}\rfloor,0}(w) \geq a\lfloor L^{1/4}\rfloor/2$ and $\phi_{r-L}(w) \leq p$, we have that

$$\mathbb{Q}_w^{\epsilon,\theta}\left(\hat{T}(\hat{r}_D + nL) - D \geq nL/\alpha_1,\, D < +\infty\right) \leq C_{20}(\alpha_1)(nL^{1/2})^{-M'+1}.$$

On the other hand, by Corollary 8, we have that

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\psi_{\hat{r}_D,\hat{T}_{\hat{r}_D}+nL} > p/2^{n-1}, \hat{T}(nL) - D \leq nL/\alpha_1\right] \leq p^{-1}2^{n-1}C_{24}L\exp(-\alpha^{-1}nL\mu_{\epsilon_0}).$$

We deduce that

$$\mathbb{Q}_w^{\epsilon,\theta}(A_0^n) \leq C_{20}(\alpha_1)(nL^{1/2})^{-M'+1} + p^{-1}2^{n-1}C_{24}L\exp(-\alpha_1^{-1}nL\mu_{\epsilon_0}). \tag{79}$$

Now, for $n \geq 2$, by part a) of Lemma 30, the strong Markov property, the fact that $(n-1)L \leq nL - \lfloor L^{1/4}\rfloor$, and that there are at least a particles at the rightmost visited site at time $\hat{T}(\hat{r}_D + nL - \lfloor L^{1/4}\rfloor)$, on $\{D < +\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}(B_n' | \mathcal{F}_{\hat{T}(\hat{r}_D+(n-1)L)}) \leq C_{19}L^{-a/8}. \tag{80}$$

Finally, observe that, by Corollary 9,

$$\mathbb{Q}_w^{\epsilon,\theta}(B_1) \leq C_{25}L^{-aM'/(8(M'+1))}. \tag{81}$$

Now we see that, by Inequalities (80) and (81), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := C_{19} L^{-a/8} + C_{25} L^{-aM'/(8(M'+1))}.$$

Now, for $m \geq 2$, let

$$a_m := 2\left[C_7(\alpha_1)((m-1)L)^{-M'} + 2aLp^{-1}2^{m-1}\exp(-\mu_{\epsilon_0}(m-1)L/\alpha_1)\right] \\ +2\left[C_{20}(\alpha_1)(mL^{1/2})^{-M'+1} + p^{-1}2^{m-1}C_{24}L\exp(-\alpha_1^{-1}mL\mu_{\epsilon_0})\right].$$

Inequalities (78) and (79) entail assumptions (3)-(4)-(5)-(6) of Lemma 35.

Note that the sequence $(a_m)_{m\geq 1}$ depends on ϵ_0 but not on ϵ. Moreover, observe that, for large enough L (not depending on ϵ), $\sum_{m=1}^{+\infty} a_m m^{M'-3} < +\infty$. On the other hand, as L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part b) of Lemma 36 then follows from applying Lemma 34. □

Proof of part c). For all $n \geq 1$, let

$$B_n := \cap_{i=1}^{n} \left\{\psi_{(i-1)L,\hat{T}(iL)} > p,\, U > \hat{T}(iL)\right\} \cup B'_i,$$

$$B'_i := \left\{m_{iL-\lfloor L^{1/4}\rfloor, iL}(w_{\hat{T}(iL)}) < a\lfloor L^{1/4}\rfloor/2,\, U > \hat{T}(iL)\right\}.$$

Since $\phi_z(w_t) \leq \psi_{z,t}$, the following inequality holds:

$$\mathbb{Q}_w^{\epsilon,\theta}(J_0 > n, U > \hat{T}_{nL}) \leq \mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(B_n). \tag{82}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, let

$$\Delta_k^n := \psi_{kL,\hat{T}(nL)} - \psi_{(k-1)L,\hat{T}(nL)},$$

and let

$$A_0^n := \left\{\psi_{0,\hat{T}(nL)} > p/2^{n-1},\, U > \hat{T}(nL)\right\}, \quad A_k^n := \left\{\Delta_k^n > p/2^{n-k},\, U > \hat{T}(nL)\right\},$$

for $1 \leq k \leq n-1$.

We now prove that the assumptions (i)-(ii) of Lemma 35 are satisfied, with $(O,\mathcal{H})$ being the space $\mathcal{D}(\mathbb{L}_\theta)$ equipped with the cylindrical $\sigma-$algebra and probability $\mathbb{Q}_{a\delta_0}^{\epsilon,\theta}$, and $\mathcal{H}_n := \mathcal{F}_{\hat{T}(nL)}$. Assumption (i) is immediate. Assumption (ii) is proved as in a).

We now look for a sequence $(a_n)_{n\geq 1}$ such that assumptions (1)-(6) of Lemma 35 are satisfied.

Assume that $n \geq 2$. Exactly as in part a), we can prove that, for $n \geq 2$, and $1 \leq k \leq n-1$, a.s.

$$\mathbb{Q}_{a\delta_0}^{\epsilon,\theta}(A_k^n|\mathcal{F}_{\hat{T}((k-1)L)}) \leq C_7(\alpha_1)((n-k)L)^{-M'} + 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1). \tag{83}$$

Now, note that, on A_0^n, one has $\hat{T}(nL) \leq (nL+1)/\alpha_2$ since $U > \hat{T}(nL)$, whence $\hat{T}(nL) \leq nL/\alpha_1$ when $L \geq \alpha_1/(\alpha_2-\alpha_1)$.

On the other hand, by Lemma 33 a), we have that

$$\mathbb{Q}_{a\delta_0}^{\epsilon,\theta}\left[\psi_{0,\hat{T}_{nL}} > p/2^{n-1}, \hat{T}(nL) \leq nL/\alpha_1\right] \leq 2p^{-1}2^{n-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}nL/\alpha_1).$$

We deduce that

$$\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(A^n_0) \leq 2p^{-1}2^{n-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}nL/\alpha_1). \tag{84}$$

Exactly as in a), a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(B'_n|\mathcal{F}_{\hat{T}((n-1)L)}) \leq C_{19}L^{-a/8}. \tag{85}$$

Finally, observe that, by the union bound, $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(B_1)$ is upper bounded by $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\psi_{0,\hat{T}(L)} > p, \hat{T}(L) \leq L/\alpha_1) + \mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\hat{T}(L) > L/\alpha_1) + \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(m_{L-\lfloor L^{1/4}\rfloor,L}(w_{\hat{T}(L)}) < a\lfloor L^{1/4}\rfloor/2)$.

Thanks to Lemma 33 a) and Lemma 30 a) and Lemma 24, we obtain that

$$\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(B_1) \leq 2p^{-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}. \tag{86}$$

Now we see that, by Inequalities (85) and (86), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := 2p^{-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}.$$

Now, for $m \geq 2$, let

$$a_m := 2\left[C_7(\alpha_1)((m-1)L)^{-M'} + 2aLp^{-1}2^{m-1}\exp(-\mu_{\epsilon_0}(m-1)L/\alpha_1)\right] \\ +2\left[2p^{-1}\phi_0(a\delta_0)2^{m-1}\exp(-\mu_{\epsilon_0}mL/\alpha_1)\right].$$

Inequalities (83) and (84) entail assumptions (3)-(4)-(5)-(6) of Lemma 35.

Note that the sequence $(a_m)_{m\geq 1}$ depends on ϵ_0 but not on ϵ. Moreover, observe that, for large enough L (not depending on ϵ), $\sum_{m=1}^{+\infty} a_m m^{M'-3} < +\infty$. On the other hand, as L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part c) of Lemma 36 then follows from applying Lemma 34.

□

Lemma 37. *For every $\epsilon > 0$, there exists $L_1(\epsilon) < +\infty$ such that, for all $L \geq L_1(\epsilon)$, there exists $0 < C_{27}(\epsilon), C_{28}(\epsilon) < +\infty$ such that the following properties hold.*

a) *For all $n \geq 1$, $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(J_0 \geq n) \leq C_{27}(\epsilon)\exp(-C_{28}(\epsilon)n)$.*
b) *For all $w \in \mathbb{L}'_\theta$ such that $m_{r-\lfloor L^{1/4}\rfloor,r}(w) \geq a\lfloor L^{1/4}\rfloor/2$, and $\phi_{r-L}(w) \leq p$, we have that, for all $n \geq 1$, $\mathbb{Q}^{\epsilon,\theta}_{w}(J_{\hat{r}_D} \geq n, D < +\infty) \leq C_{27}(\epsilon)\exp(-C_{28}(\epsilon)n)$.*
c) *For all $n \geq 1$, $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(J_0 \geq n, U > \hat{T}_{nL}) \leq C_{27}(\epsilon)\exp(-C_{28}(\epsilon)n)$.*

Proof of part a). We use exactly the same definitions as in the proof of part a) of Lemma 36, except that we look for a different sequence $(a_n)_{n\geq 1}$ such that assumptions (1)-(6) of Lemma 35 are satisfied. Assume that $n \geq 2$. By the strong Markov property and Lemma 26, we have that, for any $1 \leq k \leq n-1$, a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left(\hat{T}(nL) - \hat{T}(kL) \geq (n-k)L/\alpha_1|\mathcal{F}_{\hat{T}((k-1)L)}\right) \leq C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(n-k)L).$$

As in the proof of Lemma 36, a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left[\Delta^n_k > p/2^{n-k}, \hat{T}(nL)-\hat{T}(kL) \le (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}((k-1)L)}\right] \le 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1).$$

We deduce that, for $n\ge 2$, and $1\le k\le n-1$, a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(A^n_k|\mathcal{F}_{\hat{T}((k-1)L)}) \le C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(n-k)L) + 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1). \quad (87)$$

By Lemma 26 again,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left(\hat{T}(nL)\ge nL/\alpha_1\right) \le C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)nL).$$

On the other hand, as in the proof of Lemma 36,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left[\psi_{0,\hat{T}_{nL}} > p/2^{n-1}, \hat{T}(nL)\le nL/\alpha_1\right] \le 2p^{-1}2^{n-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}nL/\alpha_1).$$

We deduce that

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(A^n_0) \le C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)nL) + 2p^{-1}2^{n-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}nL/\alpha_1). \quad (88)$$

Now, for $n\ge 2$, as in the proof of Lemma 36, a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(B'_n|\mathcal{F}_{\hat{T}((n-1)L)}) \le C_{19}L^{-a/8}. \quad (89)$$

Similarly,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(B_1) \le 2p^{-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}. \quad (90)$$

Now we see that, by Inequalities (89) and (90), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := 2p^{-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}.$$

Now, for $m\ge 2$, let

$$a_m := 2\left[C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(m-1)L) + 2aLp^{-1}2^{m-1}\exp(-\mu_{\epsilon_0}(m-1)L/\alpha_1)\right] + 2\left[C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)mL) + 2p^{-1}2^{m-1}\phi_0(\mathcal{I}_0)\exp(-\mu_{\epsilon_0}mL/\alpha_1)\right].$$

Inequalities (87) and (88) entail assumptions (3)-(4)-(5)-(6) of Lemma 35. Now observe that, for L large enough, $\sum_{n=1}^{+\infty} a_n\exp(tn) < +\infty$ for $t>0$ small enough. As L goes to infinity, $\sum_{n=1}^{+\infty} a_n$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\ge 1}$. Part a) then follows from applying Lemma 34.

□

Proof of part b). We re-use exactly the same definitions as in the proof of part b) of Lemma 36, except that we look for a different sequence $(a_n)_{n\ge 1}$ such that assumptions (1)-(6) of Lemma 35 are satisfied. Assume that $n\ge 2$. By the strong Markov property and Lemma 26, we have for any $1\le k\le n-1$, on $\{D<+\infty\}$ a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{w}\left(\hat{T}(\hat{r}_D+nL)-\hat{T}(\hat{r}_D+kL) \ge (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}(\hat{r}_D+(k-1)L)}\right) \le C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(n-k)L).$$

As in Lemma 36, we have that, on $\{D < +\infty\}$ a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\Delta_k^n > p/2^{n-k}, \hat{T}(\hat{r}_D + nL) - \hat{T}(\hat{r}_D + kL) \leq (n-k)L/\alpha_1 | \mathcal{F}_{\hat{T}(\hat{r}_D+(k-1)L)}\right] \leq 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1).$$

We deduce that, for $n \geq 2$, and $1 \leq k \leq n-1$, on $\{D < +\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}(A_k^n | \mathcal{F}_{\hat{T}(\hat{r}_D+(k-1)L)}) \leq C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(n-k)L) + 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1). \quad (91)$$

Similarly, using Lemma 32, which is possible since $\phi_{r-L}(w) \leq p$, we have that

$$\mathbb{Q}_w^{\epsilon,\theta}\left(\hat{T}(\hat{r}_D + nL) \geq nL/\alpha_1,\ D < +\infty\right) \leq C_{22}(\alpha_1,\epsilon)L\exp(-C_{23}(\alpha_1,\epsilon)nL).$$

As in the proof of Lemma 36, we have that

$$\mathbb{Q}_w^{\epsilon,\theta}\left[\psi_{\hat{r}_D,\hat{T}_{\hat{r}_D+nL}} > p/2^{n-1}, \hat{T}(nL) \leq nL/\alpha_1\right] \leq p^{-1}2^{n-1}C_{24}L\exp(-\alpha^{-1}nL\mu_{\epsilon_0}).$$

We deduce that

$$\mathbb{Q}_w^{\epsilon,\theta}(A_0^n) \leq C_{22}(\alpha_1,\epsilon)L\exp(-C_{23}(\alpha_1,\epsilon)nL) + p^{-1}2^{n-1}C_{24}L\exp(-\alpha_1^{-1}nL\mu_{\epsilon_0}). \quad (92)$$

Now, for $n \geq 2$, as in Lemma 36 a.s.

$$\mathbb{Q}_w^{\epsilon,\theta}(B'_n | \mathcal{F}_{\hat{T}(\hat{r}_D+(n-1)L)}) \leq C_{19}L^{-a/8}, \quad (93)$$

and

$$\mathbb{Q}_w^{\epsilon,\theta}(B_1) \leq C_{25}L^{-aM/(16(M+1))}. \quad (94)$$

Now we see that, by Inequalities (93) and (94), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := C_{19}L^{-a/8} + C_{25}L^{-aM/(16(M+1))}.$$

Now, for $m \geq 2$, let

$$a_m := 2\left[C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(m-1)L) + 2aLp^{-1}2^{m-1}\exp(-\mu_{\epsilon_0}(m-1)L/\alpha_1)\right] + 2\left[C_{22}(\alpha_1,\epsilon)L\exp(-C_{23}(\alpha_1,\epsilon)nL) + p^{-1}2^{m-1}C_{24}L\exp(-\alpha_1^{-1}mL\mu_{\epsilon_0})\right].$$

Inequalities (91) and (92) entail assumptions (3)-(4)-(5)-(6) of Lemma 35. Now observe that, for L large enough, $\sum_{n=1}^{+\infty} a_n \exp(tn) < +\infty$ for $t > 0$ small enough. As L goes to infinity, $\sum_{n=1}^{+\infty} a_n$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part b) then follows from applying Lemma 34. □

Proof of part c). We use exactly the same definitions as in the proof 36 c), except that we look for a different sequence $(a_n)_{n\geq 1}$ such that assumptions (1)-(6) of Lemma 35 are satisfied.

Assume that $n \geq 2$. Exactly as in the proof of part a) of the present lemma, we can prove that, for $n \geq 2$, and $1 \leq k \leq n-1$, a.s.

$$\mathbb{Q}_{\mathcal{I}_0}^{\epsilon,\theta}(A_k^n | \mathcal{F}_{\hat{T}((k-1)L)}) \leq C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(n-k)L) + 2aLp^{-1}2^{n-k}\exp(-\mu_{\epsilon_0}(n-k)L/\alpha_1). \quad (95)$$

As in the proof of Lemma 36 c),

$$\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(A_0^n) \leq 2p^{-1}2^{n-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}nL/\alpha_1). \tag{96}$$

Similarly, a.s.

$$\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(B'_n|\mathcal{F}_{\hat{T}((n-1)L)}) \leq C_{19}L^{-a/8}, \tag{97}$$

and

$$\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(B_1) \leq 2p^{-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}. \tag{98}$$

Now we see that, by Inequalities (97) and (98), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := 2p^{-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}L/\alpha_1) + C_7(\alpha_1)L^{-a/2} + C_{19}L^{-a/8}.$$

Now, for $m\geq 2$, let

$$a_m := 2\left[C_8(\alpha_1,\epsilon)\exp(-C_9(\alpha_1,\epsilon)(m-1)L) + 2aLp^{-1}2^{m-1}\exp(-\mu_{\epsilon_0}(m-1)L/\alpha_1)\right] \\ +2\left[2p^{-1}2^{m-1}\phi_0(a\delta_0)\exp(-\mu_{\epsilon_0}mL/\alpha_1)\right].$$

Inequalities (95) and (96) entail assumptions (3)-(4)-(5)-(6) of Lemma 35. Now observe that, for L large enough, $\sum_{n=1}^{+\infty} a_n\exp(tn) < +\infty$ for $t>0$ small enough. As L goes to infinity, $\sum_{n=1}^{+\infty} a_n$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part c) then follows from applying Lemma 34. □

Lemma 38. *Let $(Y_i)_{i\geq 1}$ be a sequence of random variables on a probability space $(O,\mathcal{H},\mathbb{T})$, and $(\mathcal{H}_i)_{i\geq 0}$ an non-decreasing sequence of sub-$\sigma-$algebras of $\mathcal{H}$ such that $\mathcal{H}_0=\{\emptyset,O\}$. Assume that the following properties hold:*

- *for all $i\geq 1$, Y_i is measurable with respect to $\mathcal{H}_i$;*
- *there exists an integer $q\geq 1$ and a constant $0<c_1(q)<+\infty$ such that a.s. $\mathbb{E}_{\mathbb{T}}(Y_i^{2q}|\mathcal{H}_{i-1})\leq c_1(q)$.*

Then there exists a constant $0<c_2(q)<+\infty$, depending only on q and $c_1(q)$, such that for all $t\geq 0$ and $n\geq 1$,

$$\mathbb{T}\left(\sup_{k\geq n} k^{-1}\left|Y_1+\cdots Y_k - \sum_{i=1}^k \mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})\right|\geq t\right)\leq c_2(q)n^{-q}t^{-2q}.$$

Proof. Observe that $\mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})$ exists and is finite for all i since $\mathbb{E}_{\mathbb{T}}(Y_i^{2q}|\mathcal{H}_{i-1})<+\infty$. Now let $Z_i := Y_i - \mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})$. Observe that, with our assumptions, $\mathbb{E}_{\mathbb{T}}(Z_i|\mathcal{H}_{i-1})=0$ a.s. Moreover, thanks e.g. to Jensen's inequality, $\mathbb{E}_{\mathbb{T}}(Z_i^{2q}|\mathcal{H}_{i-1})\leq c_3(q)$, where $c_3(q)$ depends only on q and $c_1(q)$.

We now prove by induction on ℓ that, for all $0\leq\ell\leq q$ there exists a constant $0<c_4(\ell)<+\infty$, depending only on ℓ, q and $c_1(q)$, such that, for all $n\geq 1$,

$$\mathbb{E}_{\mathbb{T}}((Z_1+\cdots+Z_n)^{2\ell})\leq c_4(\ell)n^{\ell}. \tag{99}$$

For $\ell=0$, the result is trivially true for all $n\geq 1$. Now consider $0\leq\ell\leq q-1$, assume that the result holds for ℓ, and let us prove that it holds for $\ell+1$. For all

$n \geq 1$,

$$\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_{n+1})^{2\ell+2}) = \sum_{k=0}^{2\ell+2} \binom{2\ell+2}{k} \mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell+2-k} Z_{n+1}^k).$$

With our assumptions, $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell+1} Z_{n+1}) = 0$. Now, by Jensen's inequality, $\mathbb{E}_{\mathbb{T}}(Z_{n+1}^2 | \mathcal{H}_n) \leq c_3(q+1)^{1/(q+1)}$ a.s. By our induction hypothesis, we see that $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell}) \leq c_4(\ell) n^\ell$, with $c_4(\ell)$ depending only on q, ℓ, and $c_1(q)$. As a consequence, $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell} Z_{n+1}^2) \leq c_4(\ell) c_3(q)^{1/(q+1)} n^q$. On the other hand, by Jensen's inequality, for $k \geq 3$, $\mathbb{E}_{\mathbb{T}} \left| (Z_1 + \cdots + Z_n)^{2\ell+2-k} \right| \leq \mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell})^{(2\ell+2-k)/2\ell} \leq (c_4(\ell) n^\ell)^{(2\ell+2-k)/2\ell}$. Similarly, $\mathbb{E}_{\mathbb{T}} \left(\left| Z_{n+1}^k \right| | \mathcal{H}_n \right) \leq c_3(q)^{k/2q}$ a.s., so that $\left| \mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell+2-k} Z_{n+1}^k) \right| \leq c_3(q)^{k/2q)} (c_4(\ell) n^\ell)^{(2\ell+2-k)/2\ell}$. Putting these estimates together, we obtain that

$$\begin{aligned} \mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_{n+1})^{2\ell+2}) - \mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell+2}) \leq \\ \binom{2\ell+2}{2} c_4(\ell) c_3(q)^{1/q} n^\ell \\ + \sum_{k=3}^{2\ell} \binom{2\ell+2}{2\ell+2-k} c_3(q)^{k/2q} (c_4(\ell) n^\ell)^{(2\ell+2-k)/2\ell}. \end{aligned}$$

Since the are only terms of order n^ℓ or less in the r.h.s. of the above inequality, summing, we easily deduce that $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell+2}) \leq c_4(\ell+1) n^{\ell+1}$ for all $n \geq 1$, with a constant $c_4(\ell+1)$ depending only on ℓ, q, and $c_1(q)$, so the induction step from q to $q+1$ is complete.

Now observe that the sequence $(M_k)_{k \geq 0}$ defined by $M_0 := 0$ and $M_k := k^{-1}(Z_1 + \cdots + Z_k)$ is a martingale with respect to $(\mathcal{H}_k)_{k \geq 0}$. As a consequence, using the maximal inequality for martingales and Inequality (99), we see that, for all integers $n \geq 1$ and $\ell \geq 0$,

$$\mathbb{T}\left(\sup_{2^\ell n \leq k \leq 2^{\ell+1} n} |M_k| \geq t \right) \leq c_4(q) \left(2^{\ell+1} n \right)^{-q} t^{-2q}.$$

By the union bound,

$$\mathbb{T}\left(\sup_{k \geq n} k^{-1} \left| Y_1 + \cdots Y_k - \sum_{i=1}^k \mathbb{E}_{\mathbb{T}}(Y_i | \mathcal{H}_{i-1}) \right| \geq t \right)$$

is bounded above by

$$\sum_{\ell=0}^{+\infty} \mathbb{T}\left(\sup_{2^\ell n \leq k \leq 2^{\ell+1} n} |M_k| \geq t \right)$$

and so by

$$\sum_{\ell=0}^{+\infty} c_4(q) \left(2^{\ell+1} n \right)^{-q} t^{-2q}.$$

The conclusion follows. □

Lemma 39. *Let $(Y_i)_{i\geq 1}$ be a sequence of non-negative integer-valued random variables on a probability space $(O,\mathcal{H},\mathbb{T})$, and $(\mathcal{H}_i)_{i\geq 0}$ an non-decreasing sequence of sub-$\sigma-$algebras of $\mathcal{H}$ such that $\mathcal{H}_0=\{\emptyset,O\}$. Assume that the following properties hold:*

- *for all $i\geq 1$, Y_i is measurable with respect to $\mathcal{H}_i$;*
- *there exists $0<c_1,c_2<+\infty$ such that for all $i\geq 1$ and $k\geq 0$, $\mathbb{T}(Y_i\geq t|\mathcal{H}_{i-1})\leq c_1\exp(-c_2k)$.*

Then there exists c_3 depending only on c_1,c_2 such that, for all $t>c_3$, there exist $0<c_5,c_6<+\infty$ such that, for all $1\leq n\leq m$, $\mathbb{T}(Y_1+\cdots+Y_n\geq mt)\leq c_5\exp(-c_6m)$.

Proof. For $0<\lambda<c_2$, one has a.s.

$$\begin{aligned}\mathbb{E}_{\mathbb{T}}(\exp(\lambda Y_i)|\mathcal{H}_{i-1}) &\leq 1+\sum_{k=1}^{+\infty}(e^{\lambda k}-e^{\lambda(k-1)})\mathbb{T}(Y_i\geq k|\mathcal{H}_{i-1})\\ &\leq 1+c_1(1-e^{-\lambda})\frac{e^{\lambda-c_2}}{1-e^{\lambda-c_2}}.\end{aligned}$$

Letting $j(\lambda):=c_1(1-e^{-\lambda})\frac{e^{\lambda-c_2}}{1-e^{\lambda-c_2}}$, we deduce that

$$\mathbb{E}_{\mathbb{T}}(\exp(\lambda(Y_1+\cdots+Y_m)))\leq(1+j(\lambda))^m.$$

Then, by Markov's inequality,

$$\mathbb{T}(Y_1+\cdots+Y_m\geq mt)\leq\exp(-m\lambda t)\mathbb{E}_{\mathbb{T}}(\exp(\lambda(Y_1+\cdots+Y_m)),$$

so that

$$\mathbb{T}(Y_1+\cdots+Y_m\geq mt)\leq\exp\left[-m\left(\lambda t+\log(1+j(\lambda))\right)\right]. \tag{100}$$

As λ goes to zero, we see that $j(\lambda)=c_3\lambda+o(\lambda)$, with $c_3:=\frac{e^{-c_2}}{1-e^{-c_2}}$. Choosing λ small enough in (100) yields the result when $n=m$. For $n\leq m$, observe that by assumption $Y_1+\cdots+Y_n\leq Y_1+\ldots+Y_m$.

□

Lemma 40. *For $L\geq L_0$, there exists $0<C_{74},C_{75}<+\infty$ such that, for all $0\leq\epsilon\leq\epsilon_0$, and all $k\geq 1$,*

a) $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{S_k}>kC_{75}+u,\,K>k)\leq C_{74}k^2u^{-4}$;
b) $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_{S_k}>kC_{75}+u,\,U=+\infty,\,K>k)\leq C_{74}k^2u^{-4}$.

Proof. Fix $L\geq L_0$. Observe that, for any $k\geq 1$, on $\{K>k\}$,

$$\hat{r}_{S_k}=\hat{r}_0+(\hat{r}_{S_1}-\hat{r}_0)+\sum_{j=1}^{k-1}\left(\hat{r}_{S_{j+1}}-\hat{r}_{D_j}+\hat{r}_{D_j}-\hat{r}_{S_j}\right)\mathbf{1}(K\geq j). \tag{101}$$

Observe that, for $w=w_{\hat{r}_{S_j}}$ with $1\leq j\leq K$, denoting $w=(F,r,A)$, the three conditions $w\in\mathbb{L}'_\theta$, $\phi_{r-L}(w)\leq p$, and $m_{r-\lfloor L^{1/4}\rfloor,r}(w)\geq a\lfloor L^{1/4}\rfloor/2$ are satisfied. As a consequence, by Lemma 27 and the strong Markov property, for all $1\leq j\leq k-1$, and all $t>0$, a.s. $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{D_j}-\hat{r}_{S_j}>t,\,K\geq j|\mathcal{F}_{S_j})\leq C_{14}\left(t^{-M'}+L\exp(-C_{15}t)\right)$.

Now letting, for $j \geq 1$, $Y_j := \left(\hat{r}_{D_j} - \hat{r}_{S_j}\right)\mathbf{1}(K \geq j)$, and $\mathcal{H}_j := \mathcal{F}_{S_{j+1}}$, we see that the assumptions of Lemma 38 are satisfied with $q = 2$, since $M' = a + 8$.

Thanks to the above observation on $w = w_{\hat{r}_{S_j}}$, and to the fact that, on $\{K \geq j\}$, $\hat{r}_{S_{j+1}} - \hat{r}_{D_j} = LJ_{\hat{r}_{D_j}}$, we see that, by Lemma 36 b) and the strong Markov property, for all $1 \leq j \leq k-1$, and all $t > 0$, a.s. $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{S_{j+1}} - \hat{r}_{D_j} > t,\, K \geq j|\mathcal{F}_{S_j}) \leq C_{26}(\lfloor L^{-1}t \rfloor)^{3-M'}$. Similarly, thanks to Lemma 36 a), one also has that, for all $t > 0$, a.s. $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{S_1} - \hat{r}_0 > t,\, K \geq j|\mathcal{F}_{S_j}) \leq C_{26}(\lfloor L^{-1}t \rfloor)^{3-M'}$.

Now letting $Y_1 := \hat{r}_{S_1} - \hat{r}_0$, and, for $j \geq 2$, $Y_j := \left(\hat{r}_{S_j} - \hat{r}_{D_{j-1}}\right)\mathbf{1}(K \geq j)$, and $\mathcal{H}_j := \mathcal{F}_{S_j}$, we see that the assumptions of Lemma 38 are again satisfied with $q = 2$. Applying Lemma 38, we deduce the existence of two constants C_{751}, C_{741} not depending on ϵ such that for all $k \geq 1$ and $u > 0$,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left(\sum_{j=1}^{k-1}\left(\hat{r}_{D_j} - \hat{r}_{S_j}\right)\mathbf{1}(K \geq j) > kC_{751} + u,\, K > k\right) \leq C_{741}k^2u^{-4},$$

and

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left(\hat{r}_{S_1} - \hat{r}_0 + \sum_{j=1}^{k-1}\left(\hat{r}_{S_{j+1}} - \hat{r}_{D_j}\right)\mathbf{1}(K \geq j) > kC_{751} + u,\, K > k\right) \leq C_{741}k^2u^{-4}.$$

Part a) of the lemma then follows from the two above inequalities, (101), and the union bound.

To prove part b), we note that, for all $k \geq 1$, on $\{K > k,\, U = +\infty\}$,

$$\hat{r}_{S_k} = \hat{r}_0 + (\hat{r}_{S_1} - \hat{r}_0)\mathbf{1}(U = +\infty) + \sum_{j=1}^{k-1}\left(\hat{r}_{S_{j+1}} - \hat{r}_{D_j} + \hat{r}_{D_j} - \hat{r}_{S_j}\right)\mathbf{1}(K \geq j). \quad (102)$$

We can use the same argument as in the proof of part a) to deal with $\sum_{j=1}^{k-1}\left(\hat{r}_{D_j} - \hat{r}_{S_j}\right)\mathbf{1}(K \geq j)$ and $\sum_{j=1}^{k-1}\left(\hat{r}_{S_{j+1}} - \hat{r}_{D_j}\right)\mathbf{1}(K \geq j)$. To deal with the remaining term $(\hat{r}_{S_1} - \hat{r}_0)\mathbf{1}(U = +\infty)$, observe that $\hat{r}_{S_1} - \hat{r}_0 = LJ_0$, and apply Lemma 36 c). □

Lemma 41. *For all $0 \leq \epsilon \leq \epsilon_0$, and $L \geq L_1(\epsilon)$, there exist $0 < C_{97}(\epsilon), C_{98}(\epsilon), C_{99}(\epsilon) < +\infty$ such that, for all $k \leq m$,*

a) $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{S_k} > mC_{97}(\epsilon),\, K > k) \leq C_{98}(\epsilon)\exp(-C_{99}(\epsilon)m)$; *and*
b) $\mathbb{Q}^{\epsilon,\theta}_{a\delta_0}(\hat{r}_{S_k} > mC_{97}(\epsilon),\, U = +\infty,\, K > k) \leq C_{98}(\epsilon)\exp(-C_{99}(\epsilon)m)$.

Proof. Adapt the proof of Lemma 40, using Lemma 39 instead of Lemma 38, and Lemma 37 instead of Lemma 36. □

Proposition 18. *For all $L \geq L_0$, there exists $0 < C_{29} < +\infty$ not depending on ϵ such that, for all $0 \leq \epsilon \leq \epsilon_0$,*

a) $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa^2) \leq C_{29}$, $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}((\hat{r}_\kappa)^2) \leq C_{29}$;
b) $\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\kappa^2|U = +\infty) \leq C_{29}$, $\mathbb{E}^{\epsilon,\theta}_{a\delta_0}((\hat{r}_\kappa)^2|U = +\infty) \leq C_{29}$.

Proof of Proposition 18. Observe that, for any integer $\ell \geq 1$,

$$\{\kappa > t\} \subset \{K > \ell\} \cup \bigcup_{k=1}^{\ell} \{K = k,\, S_k > t\},$$

whence

$$\{\kappa > t\} \subset \{K > \ell\} \cup \bigcup_{k=1}^{\ell} \{K = k,\, \hat{r}_{S_k} \geq \lfloor \alpha_1 t \rfloor\} \cup \{\cup_{s \geq t} \hat{r}_s < \lfloor \alpha_1 s \rfloor\}. \tag{103}$$

By the union bound,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa > t) \leq \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(K > \ell) + \sum_{k=1}^{\ell} \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{S_k} \geq \lfloor \alpha_1 t \rfloor, K = k) + \mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\cup_{s \geq t} \hat{r}_s < \lfloor \alpha_1 s \rfloor). \tag{104}$$

Now remember δ_3 defined in Corollary 6 and let $\ell := -4 \log\left((1-\delta_3)^{-1} \lceil t \rceil\right)$. By (26), $\phi_{r-L}(\mathcal{I}_0) \leq p$ so that $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(D < +\infty) \leq 1 - \delta_3$. Moreover, for all $j \geq 1$, on $K \geq j$, $\phi_{r-L}(w_{\hat{r}_{S_j}})$, so that, by the strong Markov property, we have a.s. $\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(D < +\infty | \mathcal{F}_{S_j}) \leq 1 - \delta_3$. We deduce that

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(K > \ell) \leq (1-\delta_3)^{\ell} \leq t^{-4}. \tag{105}$$

Now observe that, for large enough t (not depending on ϵ), $\lfloor \alpha_1 t \rfloor \geq \ell C_{75} + \alpha_1 t/2$. Using Lemma 40 a), we deduce that, for all $1 \leq k \leq \ell$,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}(\hat{r}_{S_k} > \lfloor \alpha_1 t \rfloor,\, K > k) \leq C_{74} k^2 (\alpha_1 t/2)^{-4}. \tag{106}$$

Finally, by Lemma 20,

$$\mathbb{Q}^{\epsilon,\theta}_{\mathcal{I}_0}\left[\cup_{s \geq t} \hat{r}_s < \lfloor \alpha_1 s \rfloor\right] \leq C_{45} t^{-M'}. \tag{107}$$

Plugging (105), (106) and (107) into (104), we deduce the conclusion of part a) regarding $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\kappa^2)$. The conclusion for $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}((\hat{r}_\kappa)^2)$ follows by an application of Lemma 2.

As for part b), observe that the estimate in (105) is still valid when $\mathcal{I}_0$ is replaced by $a\delta_0$. On the other hand, the estimate obtained in (106) follows from Lemma 40 b). Then, by definition, the event $U = +\infty$ rules out the event $\cup_{s \geq t} \hat{r}_s < \lfloor \alpha_1 s \rfloor$. Part b) is then proved exactly as part a), noting that, $Q_{a\delta_0}(U = +\infty) \geq 1 - \delta_2$. □

Proposition 19. *For all $0 < \epsilon \leq \epsilon_0$, and $L \geq L_1(\epsilon)$, there exists $0 < C_{30}(\epsilon), C_{31}(\epsilon) < +\infty$ such that*

- a) $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\exp(-C_{30}(\epsilon)\kappa)) \leq C_{31}(\epsilon)$, $\mathbb{E}^{\epsilon,\theta}_{\mathcal{I}_0}(\exp(-C_{30}(\epsilon)\hat{r}_\kappa) \leq C_{31}(\epsilon)$;
- b) $\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\exp(-C_{30}(\epsilon)\kappa | U = +\infty) \leq C_{31}(\epsilon)$, $\mathbb{E}^{\epsilon,\theta}_{a\delta_0}(\exp(-C_{30}(\epsilon)\hat{r}_\kappa | U = +\infty) \leq C_{31}(\epsilon)$.

Proof of Proposition 19. The proof is very similar to the proof of Proposition 18, but this time, we use $\ell := \lfloor (1/2) C_{97}(\epsilon)^{-1} \alpha_1 t \rfloor$, so that the r.h.s. of (105) now decays exponentially as $t \to +\infty$.

We then use Lemma 41 instead of Lemma 40, noting that, for large enough t, $\lfloor \alpha_1 t \rfloor \geq \ell C_{97}(\epsilon)$. Finally, we use Lemma 22 instead of Lemma 20, and the conclusion follows as in the proof of Proposition 18.

□

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