EXPONENTIAL ERGODICITY AND RALEIGH-SCHRÖDINGER SERIES FOR INFINITE DIMENSIONAL DIFFUSIONS

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ABSTRACT. We consider an infinite dimensional diffusion on $T^{\mathbb{Z}^d}$, where T is the circle, defined by an infinitesimal generator of the form $L = \sum_{i\in\mathbb{Z}^d}\left(\frac{a_i(\eta)}{2}\partial_i^2 + b_i(\eta)\partial_i\right)$, with $\eta \in T^{\mathbb{Z}^d}$, where the coefficients a_i, b_i are finite range, bounded with bounded second order partial derivatives and the ellipticity assumption $\inf_{i,\eta} a_i(\eta) > 0$ is satisfied. We prove that whenever ν is an invariant measure for this diffusion satisfying the logarithmic Sobolev inequality, then the dynamics is exponentially ergodic in the uniform norm, and hence ν is the unique invariant measure. As an application of this result, we prove that if $A = \sum_{i\in\mathbb{Z}^d} c_i(\eta)\partial_i$, and c_i satisfy the condition $\sum_{i\in\mathbb{Z}^d}\int c_i^2 d\nu < \infty$, then there is an $\epsilon_c > 0$, such that for every $\epsilon \in (-\epsilon_c, \epsilon_c)$, the infinite dimensional diffusion with generator $L_\epsilon = L + \epsilon A$, has an invariant measure ν_ϵ having a Radon-Nikodym derivative g_ϵ with respect to ν, which admits the analytic expansion $g_\epsilon = \sum_{k=0}^\infty \epsilon^k f_k$, where $f_k \in L_2[\nu]$ are defined through $f_0 = 1$, $\int f_k d\nu = 0$ and the recurrence equations $L^* f_{k+1} = A^* f_k$.

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1. INTRODUCTION.

Let T be the unit circle. Consider $\Omega := C([0,\infty); T^{\mathbf{Z}^d})$, the space of continuous functions from $[0,\infty)$ to $T^{\mathbf{Z}^d}$, with the topology of uniform convergence in compact subsets of $[0,\infty)$. Let S_t be the unique semi-group on the set $C(T^{\mathbf{Z}^d})$ of continuous real functions on $T^{\mathbf{Z}^d}$ endowed with the uniform norm $\|\cdot\|_\infty$, associated to the generator which is the closure on Ω of the operator (L, D_0) where $L := \sum_{i\in\mathbf{Z}^d}\left(\frac{1}{2}a_i(\eta)\partial_i^2 + b_i(\eta)\partial_i\right)$, with $\partial_i := \frac{\partial}{\partial\eta_i}$, and D_0 is the set of local functions with continuous second order partial derivatives. Here, $a : T^{\mathbf{Z}^d} \to [0,\infty)^{\mathbf{Z}^d}$, $b : T^{\mathbf{Z}^d} \to R^{\mathbf{Z}^d}$ are Borel-measurable functions which we call sets of *coefficients*, $\eta \in T^{\mathbf{Z}^d}$, and a_i, b_i and η_i are their i-th components. We say that the coefficients a and b are bounded if $sup_{i,\eta}\{a_i, |b_i|\} < \infty$ and of finite range $R \in \mathbf{Z}^+$ if for each $i \in \mathbf{Z}^d$, $a_i(\eta)$ and $b_i(\eta)$ depend only on coordinates η_j of η such that $|j - i| \le R$. We say that the coefficients a and b have bounded second order partial derivatives if $\sup_{i,j,k,\eta}\left\{\left|\frac{\partial^2 a_i}{\partial\eta_j\partial\eta_k}\right|, \left|\frac{\partial^2 b_i}{\partial\eta_j\partial\eta_k}\right|\right\} < \infty$. The operator (L, D_0) defined above, with coefficients a and b that are bounded, of finite range and with bounded second order partial derivatives, is closable. Its closure, which we will denote $(L, D(a,b))$, is an infinitesimal generator and defines a Markov semi-group S_t on the space $C(T^{\mathbf{Z}^d})$ (see [7] and [4]). Such a process will be called a *finite range infinite dimensional diffusion family with bounded coefficients a and b with bounded second order partial derivatives.* If

$$a := \inf_{i,\eta} a_i(\eta) > 0, \tag{1}$$

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we will say that this diffusion is *uniformly elliptic*. Given a probability measure ν defined on $T^{\mathbf{Z}^d}$, throughout the paper we will use the notation $\langle f\rangle_\nu := \int f d\nu$. We will call this measure an *invariant measure* for the infinite dimensional diffusion, if $\int f d\nu = \int S_t f d\nu$ for every $f \in C(T^{\mathbf{Z}^d})$ and $t > 0$.

Few general results exist providing sufficient conditions for the existence of a unique invariant measure, or for the exponential ergodicity of infinite dimensional diffusions, specially out of the subclass of reversible processes. Furthermore, given that in general it is difficult to explicitely describe the invariant measures, it is natural to wonder what is the breadth of, for example, the classical theory of analytic perturbations for the invariant measures. In this paper we address these issues, within the context of probability measures satisfying the logarithmic Sobolev inequality and not satisfying any kind of reversibility assumptions. We say that a probability measure ν on $T^{\mathbf{Z}^d}$ endowed with its Borel σ-field $\mathcal{B}$, satisfies the logarithmic Sobolev inequality with respect to the Laplacian operator if there is a constant $\gamma > 0$ such that for every function $f \in D_0$ it is true that

$$\left\langle f^2 \ln \frac{f}{\sqrt{\langle f^2\rangle_\nu]}} \right\rangle_\nu \leq \gamma \left\langle \sum_{i\in\mathbf{Z}^d} (\partial_i f)^2 \right\rangle_\nu . \tag{2}$$

The first result of this paper provides a sufficient condition for exponential ergodicity of infinite dimensional diffusions. Let us define for $f \in D_0$, the triple semi-norm

$$|||f||| := \sup_{i,j,\eta} \left| \frac{\partial^2 f}{\partial\eta_i \partial\eta_j} \right| .$$

Theorem 1. *Let S_t be the semi-group of a finite range uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}^d}$ with bounded coefficients a and b with bounded second order partial derivatives. Let ν be a probability measure on $T^{\mathbf{Z}^d}$ which is invariant with respect to S_t and which satisfies the logarithmic Sobolev inequality with respect to the Laplacian operator. Then, there exist positive constants γ and A such that for any function $f \in D_0$ we have,*

$$\sup_{\eta\in T^{\mathbf{Z}^d}} |S_t f(\eta) - \langle f\rangle_\nu| \leq A|||f|||e^{-\gamma t}.$$

Theorem 1 is an improvement of a result of Zegarlinski [11] for reversible processes, formulated here in terms of invariant measures instead of Gibbs measures. As a corollary of Theorem 1, we obtain the following considerable improvement of Theorem 1 of [9]. For every $t \geq 0$ and measure μ on $T^{\mathbf{Z}^d}$, we define μS_t as the unique measure such that $\int S_t f d\mu = \int f d(\mu S_t)$ for every continuous function f on $T^{\mathbf{Z}^d}$.

Corollary 1. *Consider a finite range uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}^d}$ with bounded coefficients a and b with bounded second order partial derivatives. Suppose that ν is an invariant measure which satisfies the logarithmic Sobolev inequality with respect to the Laplacian operator. Then, ν is unique and for every probability measure μ on $T^{\mathbf{Z}^d}$, one has that $\lim_{t\to\infty} \mu S_t = \nu$.*

Theorem 2, which we formulate below, establishes the existence of a Raleigh-Schrödinger series with a positive radius of convergence around invariant measures of uniformly elliptic diffusions with finite range and bounded coefficients, satisfying the logarithmic Sobolev inequality with respect to the Laplacian operator. To state Theorem 2, we need to introduce the following regularity condition on coefficients.

Condition (R). *We say that a set of coefficients $c = \{c_i : i \in \mathbb{Z}^d\}$ satisfies the regularity condition* **(R)***, if*

$$C_0 := \sqrt{\left\| \sum_i c_i^2 \right\|_\infty} < \infty.$$

Any operator of the form

$$A = \sum_{i \in \mathbb{Z}^d} c_i(\eta) \partial_i,$$

where c is a set of coefficients, will be called a *diagonal first order operator*. For each $p \geq 1$, we denote by $L_p[\nu]$ the Banach space of functions $f \in T^{\mathbb{Z}^d}$ with norm $\|f\|_{p,\nu} := \left(\int |f| d\nu)\right)^{1/p}$. We adopt the convention that $L_2[\nu]$ is the space of complex valued square integrable functions and given two functions $f, g \in L_2[\nu]$, we denote their inner product by $(f, g)_\nu = \int \bar{f} g d\nu$. It will be shown, that if L is the generator of an elliptic infinite dimensional diffusion and A a diagonal first order operator with coefficients satisfying condition (**R**) and having both operators coefficients which are bounded and of finite range with bounded second order partial derivatives, then the closure of the operators (L, D_0) and $(L+A, D_0)$, on $L_2[\nu]$ have the same domain if ν is an invariant measure for L satisfying the logarithmic Sobolev inequality with respect to the Laplacian operator. Furthermore, given an operator $(T, D(T))$ on $L_2[\nu]$, we denote by $(T^*, D(T^*))$ its adjoint. We then have that for every function $f \in L_2[\nu]$, the equation

$$L_0^* g = -A^* f, \tag{3}$$

has a solution $g \in L_2[\nu]$. Here, equation (3) should be interpreted in the weak sense as $(g, L_0\phi)_\nu = -(f, A\phi)_\nu$, for every $\phi \in D_0$. We will also show that 0 is a simple eigenvalue of L_0^*. This implies that the equation (3) has a unique solution g such that $\langle g \rangle_\nu = 0$. We will denote by $M_2[\nu]$ the set of probability measures on $T^{\mathbf{Z}^d}$ which have a Radon-Nikodym derivative with respect to ν which is square integrable in $L_2[\nu]$.

Theorem 2. *Let L_0 be the infinitesimal generator of a uniformly elliptic infinite dimensional diffusion with finite range, bounded coefficients a and b having bounded second order partial derivatives. Let A be a diagonal first order operator with finite range, bounded coefficients having second order partial derivatives and satisfying the regularity condition* (**R**)*. Assume that ν is an invariant measure for the infinite dimensional diffusion with generator L_0, which satisfies the logarithmic Sobolev inequality with respect to the Laplacian operator. Then, for each $\epsilon \in (-\epsilon_c, \epsilon_c)$, where $\epsilon_c := \frac{a}{C_0\sqrt{\gamma}}$, the diffusion with generator $L_\epsilon := L_0 + \epsilon A$ has a unique invariant measure ν_ϵ in $M_2[\nu]$ with a Radon-Nikodym derivative g_ϵ with respect to ν with the following expansion in $L_2[\nu]$,*

$$g_\epsilon = \sum_{k=0}^{\infty} \epsilon^k f_k,$$

where $\{f_k : k \geq 0\}$ is the unique sequence of functions in $L_2[\nu]$ defined by $f_0 := 1$, the conditions $\langle f_k \rangle_\nu = 0$, and the recurrence relations

$$L_0^* f_{k+1} = -A^* f_k.$$

Furthermore, there exists a constant C such that for every $k \geq 1$ one has that $\|f_k\|_{2,\nu} \leq C\epsilon_c^{-k}$.

Theorem 2 does not require the unperturbed generator to be reversible with respect to the invariant measure. In [6], within the context of systems which satisfy the Einstein relation, a similar expansion was derived for interacting particle systems, under the assumption that the unperturbed generator is reversible.

A crucial ingredient in the proofs of Theorems 1 and 2, is the observation that the Dirichlet form of the generator of a diffusion having an invariant measure that satisfies the logarithmic Sobolev inequality, coincides with the Dirichlet form of the symmetrization of the generator. For Theorem 1, this together with the spectral gap, implies an exponentially fast convergence to the equilibrium measure in the $L_2[\nu]$ norm. One can then get exponentially fast convergence in the supremum norm through the following three additional ingredients: comparisons between a truncated version of the dynamics and the full dynamics, Gross lemma and uniform norm estimates on the marginal distribution of the process which are obtained using Girsanov theorem. The proof of Theorem 2, uses the machinery of analytic perturbation theory for operators which have a relatively bounded perturbation on Banach spaces. The uniqueness of the perturbed invariant measure in $M_2[\nu]$ of Theorem 2 follows from Corollary 1.

In the next section of this paper, we derive some important consequences of the property that a diffusion has an invariant measure satisfying the logarithmic Sobolev inequality (and hence the spectral gap). In section 3, we use the results of section 2, to prove Theorem 1. Theorem 2 is proved in section 4. In section 5, Theorem 2 is illustrated within the context of interacting Brownian motions.

2. Symmetrization of the infinitesimal generator.

Here, we will show that a uniformly elliptic infinite dimensional diffusion with finite range and bounded coefficients having second order partial derivatives, has a Dirichlet form which looks like the Dirichlet form of its symmetrized generator. This in turn implies an exponential convergence result to equilibrium in the corresponding L_2 norm.

If ν is an invariant measure for an infinite dimensional diffusion with Markov semi-group S_t and generator $(L, D(a,b))$, then for every function $f \in C(T^{\mathbf{Z}^d})$ one has that,

$$||S_t f||_{2,\nu} \leq ||f||_{2,\nu}. \tag{4}$$

It is possible to continuously extend the Markov semi-group S_t as a Markov semi-group to $L_2[\nu]$. It is a standard fact to show that the infinitesimal generator of such an extension is the closure in $L_2[\nu]$ of the (L, D_0) (see for example [7]). We will denote it $(L, \bar{D}(a,b))$. In the sequel, we will make no notational distinction between the semi-group on $C(T^{\mathbf{Z}^d})$ or on $L_2[\nu]$. Given a subset $\Lambda \subset \mathbf{Z}^d$, let us denote by $\mathcal{F}_\Lambda$, the σ-algebra in $\mathcal{B}$ of sets generated by the coordinates $k \in \Lambda$. We will denote by ν_Λ the restriction of ν to the $\mathcal{F}_\Lambda$. Furthermore, for $r > 0$, we will define the box $\Lambda_r := [-r, r]^d \cap \mathbf{Z}^d$.

Proposition 1. *Consider a uniformly elliptic infinite dimensional diffusion with finite range, bounded coefficients a and b with bounded second order partial derivatives. Let Let $(L, \bar{D}(a,b))$ be its infinitesimal generator and ν be an invariant measure of such a diffusion. Then, the following are satisfied.*

(i) *For every $f \in \bar{D}(a,b)$*

$$(f, Lf)_\nu = -\sum_{i \in \mathbf{Z^d}} \int a_i (\partial_i f)^2 d\nu.$$

(ii) *Assume that the invariant measure ν satisfies the spectral gap with constant γ. Then, for every $f \in \bar{D}(a,b)$ we have*

$$\|f - \langle f\rangle_\nu\|_{2,\nu}^2 \le -\frac{\gamma}{a}(f, Lf)_\nu.$$

We continue with the following important corollary of part (i) of the above proposition.

Corollary 2. *Consider a uniformly elliptic infinite dimensional diffusion with finite range, bounded coefficients a and b with bounded second order partial derivatives. Let Let $(L, \bar{D}(a,b))$ be its infinitesimal generator and ν be an invariant measure of such a diffusion. Let $A = \sum c_i \partial_i$ be a diagonal first order operator such that the coefficients c are of finite range, bounded, with bounded second order partial derivatives and satisfy the regularity condition* **(R)**. *Assume that ν satisfies the logarithmic Sobolev inequality. Then, for every λ positive we have that for every $f \in D_0$ the following inequality is satisfied,*

$$||Af||_{2,\nu} \le \frac{C_0}{\sqrt{a}}\frac{1}{\lambda}\|Lf\|_{2,\nu} + \frac{C_0}{\sqrt{a}}\lambda||f||_{2,\nu}, \tag{5}$$

where

$$C_0 := \sqrt{\left\|\sum_{i\in\mathbb{Z}^d} c_i^2\right\|_\infty}.$$

In particular, the operator (A, D_0) is relatively bounded with respect to (L, D_0) in $L_2[\nu]$, with L-bound 0. Hence the closure of $(L + A, D_0)$ has the same domain $\bar{D}(a,b)$ as L.

Proof. Let $f \in D_0$. Note that

$$\begin{aligned} ||Af||_{2,\nu}^2 &= \int \left(\textstyle\sum_i c_i \partial_i f\right)^2 d\nu \le C_0^2 \textstyle\sum_i \int (\partial_i f)^2 d\nu \\ &\le \tfrac{C_0^2}{a}(f, Lf)_\nu \le \tfrac{C_0^2}{a}\|f\|_{2,\nu}\|Lf\|_{2,\nu}. \end{aligned}$$

Here, we have used part (i) of Proposition 1 in the second inequality. On the other hand, for every positive λ, a, b we have that $ab \le \lambda^{-2}b^2 + \lambda^2 a^2$ It follows that

$$||Af||_{2,\nu}^2 \le \frac{C_0^2}{a}\left(\lambda^{-2}\|Lf\|_{2,\nu}^2 + \lambda^2\|f\|_{2,\nu}^2\right).$$

which proves inequality (5). By taking λ arbitrarily large, we see that A has L-bound 0. Finally, by Theorem 1.1 of page 190 of Kato [5], we can see that the closure of the operators (L, D_0) and $(L + A, D_0)$ in $L_2[\nu]$ have the same domain. □

Corollary 2, in turn implies the following.

Corollary 3. *Let $(L, \bar{D}(a,b))$ be the infinitesimal generator of a uniformly elliptic infinite dimensional diffusion with finite range, bounded coefficients a and b, with bounded second order partial derivatives. Let ν be an invariant measure of such a diffusion satisfying the logarithmic Sobolev inequality with respect to the Laplacian operator. Let a be the lower bound defined in (1). Then, for every $f \in L_2(\nu)$ it is true that,*

$$||S_t f - \langle f\rangle_\nu||_{2,\nu} \le e^{-\frac{a}{\gamma}t}||f - \langle f\rangle_\nu||_{2,\nu}. \tag{6}$$

To prove the proposition, we will introduce a truncated version of the infinite dimensional diffusion. Let us fix a natural $n \geq 1$. Define, a truncated version of the finite range infinite dimensional diffusion with bounded coefficients a and b, through the truncated infinitesimal generator (L_n, D_2) which is the closure on $L_2[\nu]$ of (L_n, D_0), where

$$L_n = \sum_{i \in \Lambda_n} \left(\bar{a}_i \partial_i^2 + \bar{b}_i \partial_i \right), \tag{7}$$

with $\bar{a}_i(\eta) := \int a_i(\eta) d\bar{\nu}_n$ and $\bar{b}_i(\eta) := \int b_i(\eta) d\bar{\nu}_n$, and $\bar{\nu}_n$ is the conditional probability of ν given the σ-algebra $\mathcal{F}_{\Lambda_n}$ of the cylinders in Λ_n. We will call this diffusion process the *truncated version at scale n of the infinite dimensional diffusion process with bounded coefficients a and b* and denote the corresponding semi-group on $L_2[\nu]$ by $\{S_t^n : t \geq 0\}$. Given $\eta \in T^{\mathbf{Z}^d}$, we define η^n, called the *configuration η truncated at scale n* by $\eta^n(x) = \eta(x)$ whenever $|x| \leq n$ and $\eta^n(x) = 1$ otherwise.

For each natural $n \geq 1$, consider the restriction $\nu_n := \nu_{\Lambda_n}$ of the invariant measure ν to the box Λ_n. Note that ν_n is the invariant measure of the diffusion process on T^{Λ_n} with infinitesimal generator (7), which is finite dimensional. Therefore it has a density v_n which by the compactness of T^{Λ_n} satisfies $\inf_\eta v_n(\eta) > 0$. Let $\{v_{n,i} : i \in \mathbf{Z}^d\}$ be the conditional probability densities of ν_n given $\mathcal{F}_{\Lambda_n - \{i\}}$. These are also smooth and satisfy $\inf_{\eta,i} v_{n,i}(\eta) > 0$. It is understood that both v_n and $v_{n,i}$ are functions only of the coordinates of η in Λ_n. We can therefore write the truncated generator (7) as

$$L_n = \sum_{i \in \Lambda_n} \left(\frac{1}{v_{n,i}} \partial_i (v_{n,i} \bar{a}_i \partial_i) - \frac{1}{v_{n,i}} \partial_i (v_{n,i} \bar{a}_i) \partial_i + \bar{b}_i \partial_i \right).$$

Consider the operator (S_n, D_0), where

$$S_n = \sum_{i \in \Lambda_n} \frac{1}{v_{n,i}} \partial_i (v_{n,i} \bar{a}_i \partial_i).$$

By an argument similar to the proof of Corollary 2, we can show that on $L_2[\nu]$, the operators (L_n, D_0) and (S_n, D_0) are closable and their closures have the same domain D_2. Now note that (S_n, D_0) is reversible with respect to the measure ν. It follows that the operator (S_n, D_2) is self-adjoint in $L_2[\nu]$ [7] and that for every $f, g \in D_2$,

$$\int g S_n f d\nu = - \sum_{i \in \Lambda_n} \int \bar{a}_i \partial_i g \partial_i f d\nu. \tag{8}$$

On the other hand, the fact that ν is an invariant measure implies that for every function $f \in D_0$ depending only on coordinates in the box Λ_n,

$$\sum_{i \in \Lambda_n} \int f \left(\frac{1}{v_{n,i}} \partial_i^2 (v_{n,i} \bar{a}_i) - \frac{1}{v_{n,i}} \partial_i (v_{n,i} \bar{b}_i) \right) d\nu = 0.$$

This implies that ν-a.s. (and also a.s. with respect to the Lebesgue measure on T^{Λ_n}),

$$\sum_{i \in \Lambda_n} \left(\frac{1}{v_{n,i}} \partial_i^2 (v_{n,i} \bar{a}_i) - \frac{1}{v_{n,i}} \partial_i (v_{n,i} \bar{b}_i) \right) = 0.$$

Hence, for every $f \in D_2$ it is true that,

$$\int f L_n f d\nu = \int f S_n f d\nu. \tag{9}$$

The proof of Proposition 1 now follows from the equalities (8), (9) after taking the limit when $n \to \infty$.

Let us now prove Corollary 3. Note that since ν satisfies the logarithmic Sobolev inequality with respect to the Laplacian operator (2) it necessarily satisfies the spectral gap inequality for every $f \in D_0$ (see for example [2]),

$$||f||_{2,\nu}^2 \leq \gamma \sum_{i \in \mathbf{Z}^d} \int (\partial_i f)^2 d\nu,$$

and hence for $f \in D_2$ depending only on the coordinates on Λ_n, using the uniform ellipticity condition of the coefficients $\{a_i : i \in \mathbf{Z}^d\}$ inherited by $\{\bar{a}_i : i \in \mathbf{Z}^d\}$ with the same lower bound, we get,

$$||f||_{2,\nu}^2 \leq -\frac{\gamma}{a} \int f S_n f \nu.$$

It now follows that for every $f \in L_2[\nu]$ depending only on coordinates on Λ_n, and $t > 0$

$$\begin{aligned} \frac{d}{dt}||S_t^n f - \langle f \rangle_\nu||_{2,\nu}^2 &= 2 \int (S_t^n f - \langle f \rangle_\nu) L_n (S_t^n f - \langle f \rangle_\nu) d\nu \\ &= 2 \int (S_t^n f - \langle f \rangle_\nu) S_n (S_t^n f - \langle f \rangle_\nu) d\nu \\ &\leq -2\frac{a}{\gamma}||S_t^n f - \langle f \rangle_\nu||_{2,\nu}^2. \end{aligned}$$

Thus,

$$||S_t^n f - \langle f \rangle_\nu||_{2,\nu} \leq e^{-\frac{a}{\gamma}t}||f - \langle f \rangle_\nu||_{2,\nu}. \tag{10}$$

Finally, since for $f \in D_0$ we have $\lim_{n\to\infty} ||L_n f - Lf||_2 = 0$, by Theorem I.2.12 of Trotter-Kurtz in Liggett [7], we can conclude that for every $t \geq 0$, $\lim_{n\to\infty} ||S_t^n f - S_t f||_2 = 0$. Therefore, taking the limit when $n \to \infty$ in (10) finishes the proof of (6) for functions $f \in L_2(\nu)$ depending only on a finite number of coordinates. Using the fact that this set of functions is dense in $L_2(\nu)$ we finish the proof of Corollary 3.

3. Proof of Theorem 1.

The basis of the proof of Theorem 1 is Corollary 3 of the previous section, a truncation estimate, Gross lemma and a uniform estimate on marginal distributions. Throughout, γ will denote the constant appearing in the logarithmic Sobolev inequality (2). Furthermore, we will adopt the convention that given any sequence $\{y_n\}$, and positive real number x, $y_x := y_{\lfloor x \rfloor}$.

Lemma 1. [Truncation estimate]. *For every $\delta > 0$ there exist constants $a > 0$ and $A > 0$ such that for all $f \in D_0$ the following statements are satisfied.*

(i) *For every $n \geq at$,*

$$\sup_{\eta \in T^{\mathbf{Z}^d}, u: 0 \leq u \leq t} |S_u f(\eta) - S_u f(\eta^n)| \leq A|||f|||e^{-\delta t}, \tag{11}$$

(ii) *For every $n \geq at$,*

$$\sup_{\eta \in T^{\mathbf{Z}^d}, u: 0 \leq u \leq t} |S_u f(\eta) - S_u^n f(\eta)| \leq A|||f|||e^{-\delta t}. \tag{12}$$

Lemma 2. **[Gross lemma]**. *Let $p(t)=1+e^{4t/\gamma}$. Then, for all $f\in L_2(\nu)$ and $t\ge 0$, it is true that*

$$||S_t f||_{p(t),\nu}\le ||f||_{2,\nu}. \tag{13}$$

Lemma 3. **[Uniform norm estimate on marginal distributions]**. *Let $\eta\in T^{\mathbf{Z}^d}$ and denote by δ_η the probability measure with a unique atom at η. Consider the evolution up to time 1 of this measure under the truncated version at scale n of the infinite dimensional diffusion, $\mu_n:=S_1^n\delta_\eta$. Denote by $g_{n,\eta}$, the Radon-Nikodym derivative of μ_n with respect to ν_{Λ_n}. Then, if $q(t):=1+e^{-4t/\gamma}$, for every $b>0$ we have that*

$$\lim_{t\to\infty}\sup_\eta ||g_{bt,\eta}||_{q(s),\nu}\le 1, \tag{14}$$

where s is given by the relation $t=1+s+s^2$.

Let us now show why do these four facts imply Theorem 1. Let $f\in D_0$. First, note that without loss of generality, we can assume that $\int f d\nu=0$. For $t\ge 1$, define $s\ge 0$ by the relation $t=1+s+s^2$. Remark that by parts (i) and (ii) of the truncation estimate with $\delta=1$, there exist constants $a>0$ and $A>0$ such that,

$$|S_t f(\eta)|\le\left|\int S_{s+s^2}f(\zeta)d\mu_{at}(\zeta^{at})\right|+A|||f|||e^{-t}, \tag{15}$$

where μ_{at} is the restriction to $T^{\Lambda_{at}}$ of the measure $S_1^{at}\delta_\eta$. Now

$$\left|\int S_{s+s^2}f(\zeta)d\mu_{at}\right|=\left|\int S_{s+s^2}f(\zeta)g_{at,\eta}d\nu_{\Lambda_{at}}\right|. \tag{16}$$

Since $\frac{1}{p(t)}+\frac{1}{q(t)}=1$, by Hölder's inequality, we see that the right-hand side of inequality (16) is upper-bounded by

$$||S_{s+s^2}f(\zeta)||_{p(s),\nu}||g_{at,\eta}||_{q(s),\nu}\le\left(||S_{s+s^2}f(\zeta)||_{p(s),\nu}+A|||f|||e^{-t}\right)||g_{at,\eta}||_{q(s),\nu}$$

$$\le\left(||S_{s^2}f(\zeta)||_{2,\nu}+A|||f|||e^{-t}\right)||g_{at,\eta}||_{q(s),\nu}\le\left(e^{-as^2/\gamma}||f||_{2,\nu}+A|||f|||e^{-t}\right)\sup_\eta ||g_{at,\eta}||_{q(s),\nu},$$

where in the first inequality we have used the truncation estimate of Lemma 1, in the second inequality Gross lemma (Lemma 2) and the contraction inequality (4), and in the last inequality the spectral gap estimate Corollary 3. Now, by Lemma 3, there is a $t_0>0$ such that $K_1(\nu):=\sup_{t\ge t_0}\sup_\eta ||g_{at,\eta}||_{q(s),\nu}<\infty$. Thus, by inequality (15), and the previous development, for $t\ge t_0$ we have that

$$|S_t f(\zeta)|\le K_2(\nu)\left(e^{-as^2/\gamma}||f||_{2,\nu}+A|||f|||e^{-t}\right),$$

where $K_2(\nu):=K_1(\nu)+1$. Observing that $||f||_{2,\nu}\le |||f|||$, and using the density of D_0 in $L_2(\nu)$ finishes the proof of Theorem 2.

Since they are standard results, the proofs of Lemmas 1 and 2 are omitted (the proof of Lemma 1 is analogous to that of Theorem 3 of [8] and the proof of Lemma 2 can be found in [3] and [11]). Hence, we now proceed to prove Lemma 3.

Proof of Lemma 3. We will follow the techniques of [3], [2] and [11]. Consider the semi-group $\{S_t^n : t \geq 0\}$ of the truncated version at scale n of the infinite dimensional diffusion process with coefficients a and b, with infinitesimal generator L_n as defined in (7). Let us call $P_{n,\eta}$ the law of such a process starting from $\eta \in T^{\mathbb{Z}^d}$ on the space $C([0,\infty); T^{\Lambda_n})$ endowed with its Borel σ-algebra, for $t \geq 0$, $P_{n,\eta,t}$ the restriction of such a law to $C([0,t); T^{\Lambda_n})$ and $\mathcal{F}_t$ the information up to time t of the process. For each $\eta \in T^{\Lambda_n}$ and $t > 0$, let $h_{n,\eta,t}$ be the Radon-Nikodym derivative of $S_t^n \delta_\eta$ with respect to the Lebesgue measure. Consider the finite dimensional diffusion on T^{Λ_n} defined by the infinitesimal generator

$$D_n = \sum_{i \in \Lambda_n} \partial_i (\bar{a}_i^n \partial_i).$$

Let us call $Q_{n,\eta}$ the law of this diffusion starting from $\eta \in T^{\Lambda_n}$ defined on $C([0,\infty); T^{\Lambda_n})$ with its Borel σ-algebra, and for $t \geq 0$, $Q_{n,\eta,t}$ its restriction to $\mathcal{F}_t$. Furthermore, call $d_{n,\eta,t}$ the Radon-Nikodym derivative of the law of this process at time t with respect to Lebesgue measure. Classical heat-kernel estimates give us that

$$e^{-C_1|\Lambda_n| \ln |\Lambda_n|} \leq ||d_{n,\eta,1}||_\infty \leq e^{C_1|\Lambda_n|}, \tag{17}$$

for some constant $C_1 > 0$. Indeed, the upper bound can be computed using ideas of Nash found in Fabes-Stroock [1] (see also [8]). The lower bound, with an explicit dependence of the constants in the dimension is proven in Lemma 2.6 and Lemma 2.7 of [1]. Now, recalling that v_n is the density of the measure ν_{Λ_n} and noting that $v_n = \int d_{n,\eta,1} d\nu$, we see that it is also true that

$$e^{-C_1|\Lambda_n| \ln |\Lambda_n|} \leq ||v_n||_\infty \leq e^{C_1|\Lambda_n|}. \tag{18}$$

To prove Lemma 3, let us first write

$$||g_{bt,\eta}||_{q(s),\nu}^{q(s)} = \int \left(\frac{h_{bt,\eta,1}}{d_{bt,\eta,1}} \right)^{q(s)} \left(\frac{d_{bt,\eta,1}}{v_{bt}} \right)^{q(s)} d\nu.$$

Therefore, using the bounds (17) and (18), we see that there is a constant $C_2 > 0$ such that

$$||g_{bt,\eta}||_{q(s),\nu}^{q(s)} \leq \left(e^{C_2|\Lambda_{bt}| \ln |\Lambda_{bt}|} \right)^{q(s)-1} \int \left(\frac{h_{bt,\eta,1}}{d_{bt,\eta,1}} \right)^{q(s)} \frac{d_{bt,\eta,1}}{v_{bt}} d\nu. \tag{19}$$

Note that

$$\frac{h_{bt,\eta,1}}{d_{bt,\eta,1}} = E_{Q_{bt,\eta,1}} \left[\frac{dP_{bt,\eta,1}}{dQ_{bt,\eta,1}} \middle| \mathcal{F}_{=1} \right], \tag{20}$$

where for each $t \geq 0$, $\mathcal{F}_{=t}$ is the σ-algebra of events at time t. Then, by Jensen's inequality and the identity (20),

$$\begin{aligned} &\int \left(\tfrac{h_{bt,\eta,1}}{d_{bt,\eta,1}} \right)^{q(s)} \tfrac{d_{bt,\eta,1}}{v_{bt}} d\nu = \int \left(E_{Q_{bt,\eta,1}} \left[\tfrac{dP_{bt,\eta,1}}{dQ_{bt,\eta,1}} \middle| \mathcal{F}_{=1} \right] \right)^{q(s)} \tfrac{d_{bt,\eta,1}}{v_{bt}} d\nu \\ &\leq \int E_{Q_{bt,\eta,1}} \left[\left(\tfrac{dP_{bt,\eta,1}}{dQ_{bt,\eta,1}} \right)^{q(s)} \middle| \mathcal{F}_{=1} \right] \tfrac{d_{bt,\eta,1}}{v_{bt}} d\nu = E_{Q_{bt,\eta,1}} \left[\left(\tfrac{dP_{bt,\eta,1}}{dQ_{bt,\eta,1}} \right)^{q(s)} \right]. \end{aligned} \tag{21}$$

Now, by the Girsanov theorem, for every natural n and $t \geq 0$,

$$\frac{dP_{n,\eta,t}}{dQ_{n,\eta,t}} = \exp \left(\sum_{i \in \Lambda_n} \left(\int_0^t \frac{1}{a_i^n} \left(b_i^n - \frac{\partial a_i^n}{\partial \eta_i} \right) d\eta_i(s) - \int_0^t \frac{1}{2a_i^n} \left(b_i^n - \frac{\partial a_i^n}{\partial \eta_i} \right)^2 ds \right) \right).$$

Therefore, by the uniform ellipticity assumption and the boundedness of the coefficients and its derivatives, we know that there is a constant $C_3 > 0$ such that

$$E_{Q_{bt,\eta,1}}\left[\left(\frac{dP_{bt,\eta,1}}{dQ_{bt,\eta,1}}\right)^{q(s)}\right] \le \exp\left\{C_3(q(s)^2 - q(s))|\Lambda_{bt}|\right\} \le \exp\left\{2C_3 e^{-4s/\gamma}|\Lambda_{bt}|\right\}. \qquad (22)$$

Combining (22) with (21) and then with (19), we obtain see that there is a constant $C_4 > 0$ such that

$$\sup_\eta ||g_{bt,\eta}||_{q(s),\nu} \le \exp\left\{C_4 e^{-4s/\gamma}|\Lambda_{bt}| \ln |\Lambda_{bt}|\right\}. \qquad (23)$$

Since $\lim_{t\to\infty} e^{-4s/\gamma}|\Lambda_{bt}| \ln |\Lambda_{bt}| = 1$, taking the limit when t tends to ∞ in inequality (23), we obtain Lemma 3.

□

4. Proof of Theorem 2.

We need to recall some basic notions (see for example Kato [5]) which will be used throughout this section. Given a closed operator T defined on $L_2[\nu]$, we will denote its spectrum by $\Sigma(T)$. We will denote by $R(z,T)$ the resolvent operator for every $z \notin \Sigma(T)$. We say that a simple closed curve Γ in the complex plane $\mathbb{C}$ *separates* the spectrum $\Sigma(T)$, if there exist subsets of $\mathbb{C}$, Σ_1 and Σ_2 such that $\Sigma(T) = \Sigma_1 \cup \Sigma_2$, Σ_1 is in the exterior of Γ and Σ_2 is in the interior of Γ. Furthermore, we say that two subspaces M_1 and M_2 of $L_2[\nu]$ form a *decomposition associated to* Σ_1 *and* Σ_2 if $L_2[\nu] = M_1 \oplus M_2$, the spectrum of T_{M_1} is Σ_1 and the spectrum of T_{M_2} is Σ_2. Here $T_{M_1} := P_1 T$ and $T_{M_2} := P_2 T$, where P_1 is the projection of $L_2[\nu]$ onto M_1 along M_2 and P_2 the projection of $L_2[\nu]$ onto M_2 along M_2.

Firstly, we derive the following proposition giving some basic information about the effect of diagonal first order operators with coefficients satisfying condition **(R)** on the infinitesimal generator of infinite dimensional diffusions.

Proposition 2. *Consider a uniformly elliptic infinite dimensional diffusion with finite range, bounded coefficients a and b with bounded second order partial derivatives. Let $(L_0, \bar{D}(a,b))$ be its infinitesimal generator. Let ν be an invariant measure for this diffusion which satisfies the logarithmic Sobolev inequality. Let A be a diagonal first order perturbation with coefficients c satisfying the regularity condition* **(R)**, *of finite range, bounded with bounded second order partial derivatives. For each real ϵ, define*

$$L_\epsilon := L_0 + \epsilon A.$$

Then the following statements are satisfied.

(i) *For every real ϵ and λ positive we have that for every $f \in D_0$ the following inequality is satisfied,*

$$||\epsilon A f||_{2,\nu} \le |\epsilon| \frac{C_0}{\sqrt{a}} \frac{1}{\lambda} \|L_0 f\|_{2,\nu} + |\epsilon| \frac{C_0}{\sqrt{a}} \lambda ||f||_{2,\nu},$$

where $C_0 := \sqrt{\left\|\sum_{i\in\mathbb{Z}^d} c_i^2\right\|_\infty}$.

(ii) *For every real ϵ, the operator (L_ϵ, D_0) is closable, having the same domain $\bar{D}(a,b)$ as the closure of (L_0, D_0) on $L_2[\nu]$.*

(iii) 0 *is a simple eigenvalue of the operator* $(L_0, \bar{D}(a,b))$. *Furthermore, the intersection of the open disc centered at* 0 *of radius* $\gamma/(2a)$ *with the spectrum* $\Sigma(L_0)$ *of* L_0 *is* $\{0\}$*, so that* 0 *is an isolated eigenvalue.*

Proof. Proof of parts (i) *and* (ii)*.* These statements are a rephrasing of Corollary 2.

Proof of part (ii)*.* It is enough to prove that 0 is a simple eigenvalue of the adjoint operator $(L_0^*, \bar{D}(a,b)^*)$. Assume that $g \in \bar{D}(a,b)$ is a normalized function such that

$$\int (L_0^* g) f d\nu = 0, \tag{24}$$

for every $f \in \bar{D}(a,b)$. To prove that 0 is a simple eigenvalue of L_0^*, it is enough to show that this implies that ν-a.s. $g = 1$. But the left-hand side of (24) can be written as $\int g L_0 f d\nu = 0$. Since ν satisfies the logarithmic Sobolev inequality and L_0 is a uniformly elliptic diffusion with bounded, finite range coefficients with second order partial derivatives, by Corollary 1, ν is the unique invariant measure. It follows that necessarily ν-a.s. $g = 1$. Let us now show that 0 is an isolated eigenvalue. From part (ii) of Proposition 1, we see that for every complex z such that $0 < |z| < \gamma/a - |z|$ one has that for every $f \in D_2$,

$$\|(L_0 - z)f\|_{2,\nu} \geq m \|f\|_{2,\nu},$$

where $m := \min\{|z|, \gamma/a - |z|\}$. This shows that every z such that $0 < |z| < \gamma/(2a)$ is in the resolvent set of L_0, which proves the last statement of part (ii) of the proposition.

□

Let $\epsilon_0 > 0$. We say that a family $\{T(\epsilon) : \epsilon \in (-\epsilon_0, \epsilon_0)\}$ of bounded operators defined on $L_2[\nu]$, is *holomorphic* in ϵ if and only if each ϵ has a neighborhood in which $T(\epsilon)$ is bounded and $(f, T(\epsilon)g)_\nu$ is holomorphic for every f, g in a dense subset of $L_2[\nu]$ (see section VII.1 of Kato [5]).

Lemma 4. *Let* A, L_0, L_ϵ *and* ν *be as in Proposition 2. Consider the complex contour* $\Gamma := \{z \in \mathbb{C} : |z| = \gamma/(2a)\}$*. Let*

$$\epsilon_c := \frac{a}{C_0 \sqrt{\gamma}}.$$

Then the following are satisfied.

(i) *For every* $\epsilon \in (-\epsilon_c, \epsilon_c)$*, the contour* Γ *separates the spectrum* $\Sigma(L_\epsilon)$ *of* L_ϵ *into two parts:* $\Sigma_1 := \{0\}$ *and* $\Sigma_{2,\epsilon} := \Sigma(L_\epsilon) - \{0\}$*.*

(ii) *For every* $\epsilon \in (-\epsilon_c, \epsilon_c)$*, the decomposition associated to* Σ_1 *and* $\Sigma_{2,\epsilon}$*,* $L_2[\nu] = M_{1,\epsilon} \oplus M_{2,\epsilon}$*, is such that* $M_{1,\epsilon}$ *is isomorphic to* $M_{1,0}$ *and* $M_{2,\epsilon}$ *to* $M_{2,0}$*.*

(iii) *The projection* P_ϵ *of* $L_2[\nu]$ *onto* $M_{1,\epsilon}$ *along* $M_{2,\epsilon}$ *is holomorphic as a function of* ϵ *for* $\epsilon \in (-\epsilon_c, \epsilon_c)$*. and admits the following expansion with radius of convergence* ϵ_c

$$P_\epsilon = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} \epsilon^k \int_\Gamma R(z, L_0)(-A\ R(z, L_0))^k dz.$$

Proof. Part (iii) of Proposition 2 and Theorem 3.16, page 212 of Kato [5], imply that Γ separates $\Sigma(L_\epsilon)$ into two non-empty pieces Σ_1 and Σ_2. A second application of Theorem 3.16 of [5], proves part (ii) of the lemma. On the other hand, for every $\epsilon \in (-\epsilon_c, \epsilon_c)$, 0 is an eigenvalue of the operator L_ϵ. It follows that 0 is an eigenvalue of L_ϵ^*. Hence, part (ii) of this lemma, implies part (i). To prove part (iii), note that for $z \in \Gamma$, whenever ϵ is non-negative and is such that $\|\epsilon A\ R(z, L_0)\|_{2,\nu} < 1$, we have the following expansion (see Theorem 1.5, page 66 and chapter VIII of [5])

$$R(z, L_\epsilon) = R(z, L_0) \sum_{k=0}^{\infty} \epsilon^k (-A\ R(z, L_0))^k.$$

Observing that for $z \in \Gamma$,

$$\left\| \frac{L_0}{L_0 - z} \right\|_{2,\nu} \leq 2 \qquad \text{and} \qquad \|R(z, L_0)\|_{2,\nu} \leq \frac{\gamma}{2a},$$

we conclude that for every $\lambda > 0$,

$$\|\epsilon A\ R(z, L_0)\|_{2,\nu} \leq \frac{2\epsilon C_0}{\sqrt{a}} \frac{1}{\lambda} + \frac{\epsilon C_0 \gamma}{2a^{3/2}} \lambda.$$

Taking the infimum over $\lambda > 0$, we conclude that

$$\|\epsilon A\ R(z, L_0)\|_{2,\nu} \leq \epsilon \frac{C_0 \sqrt{\gamma}}{a},$$

which proves the analyticity of the resolvent operator $R(z, L_\epsilon)$ for $z \in \Gamma$ when $\epsilon \in (-\epsilon_c, \epsilon_c)$. Finally, the fact that the projection P_ϵ of $L_2[\nu]$ onto $M_{1,\epsilon}$ along $M_{2,\epsilon}$ can be expressed as

$$P_\epsilon = -\frac{1}{2\pi i} \int_\Gamma R(z, L_\epsilon) dz,$$

proves part (iii).

□

From Lemma 4, we have directly the following corollary regarding the adjoints L_0^* and A^* of L_0 and A in $L_2[\nu]$ respectively.

Corollary 4. *Let A, L_0, L_ϵ and ν be as in Proposition 2. Consider the complex contour $\Gamma := \{z \in \mathbb{C} : |z| = \gamma/(2a)\}$. Let*

$$\epsilon_c := \frac{a}{C_0 \sqrt{\gamma}}.$$

Then the following are satisfied.

(i) *For every $\epsilon \in (-\epsilon_c, \epsilon_c)$, the contour Γ separates the spectrum $\Sigma(L_\epsilon^*)$ of L_ϵ^* into two parts: $\Sigma_1^* := \{0\}$ and $\Sigma_{2,\epsilon}^* := \Sigma(L_\epsilon^*) - \{0\}$.*

(ii) *For every $\epsilon \in (-\epsilon_c, \epsilon_c)$, the decomposition associated to Σ_1^* and $\Sigma_{2,\epsilon}^*$, $L_2[\nu] = M_{1,\epsilon}^* \oplus M_{2,\epsilon}^*$, is such that $M_{1,\epsilon}^*$ is isomorphic to $M_{1,0}^*$ and $M_{2,\epsilon}^*$ to $M_{2,0}^*$.*

(iii) *The projection P_ϵ^* of $L_2[\nu]$ onto $M_{1,\epsilon}^*$ along $M_{2,\epsilon}^*$ is holomorphic as a function of ϵ for $\epsilon \in (-\epsilon_c, \epsilon_c)$. and admits the following expansion with radius of convergence ϵ_c*

$$P_\epsilon^* = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} \epsilon^k \int_{\bar{\Gamma}} (-R(z, L_0^*) A^*)^k R(z, L_0^*) dz.$$

Let us now prove Theorem 2. By parts (i) and (ii) of Corollary 4, we see that for each $\epsilon \in (-\epsilon_c, \epsilon_c)$ there exists a unique invariant measure ν_ϵ of the infinite dimensional diffusion with generator L_ϵ in $M_2[\nu]$. On the other hand, we know that $g := 1$ is an eigenfunction associated to the eigenvalue 0 of L_0 in $L_2[\nu]$. Let

$$g'_\epsilon := P_\epsilon^* g.$$

By part (iii) of Corollary 4, we know that g'_ϵ admits the expansion

$$g'_\epsilon = \sum_{k=0}^{\infty} \epsilon^k f'_k,$$

where $f'_0 := g$ and

$$f'_k := -\frac{1}{2\pi i} \int_{\bar{\Gamma}} (-R(z, L_0^*)A^*)^k R(z, L_0^*) g dz.$$

By parts (i) and (ii) of Corollary 4, necessarily $L_\epsilon^* g'_\epsilon = 0$. Hence, for every $f \in \bar{D}(a, b)$,

$$\sum_{k=0}^{\infty} \epsilon^k (f'_k, (L_0 + \epsilon A) f)_\nu = 0.$$

Matching equal powers of ϵ in the above equation, we conclude that for each $k \geq 0$, $h := f_{k+1}$ is solution of the equation

$$L_0^* h = -A^* f'_k. \tag{25}$$

Since the kernel $ker(L_0^*)$ of the operator L_0^* is one-dimensional and $A^* f'_k$ is orthogonal to $ker(L_0^*)$, it follows that the sequence of functions $f_0 := f'_0$ and $f_k := f'_k - \langle f'_k \rangle_\nu$, $k \geq 1$, is the only sequence satisfying (25) under the condition that the average of each term with respect to ν vanishes. But since $\langle g_\epsilon \rangle_\nu = 0$, we see that $\left\langle \sum_{k=1}^{\infty} \epsilon^k f'_k \right\rangle_\nu = 0$. It follows that $g_\epsilon := 1 + \sum_{k=1}^{\infty} \epsilon^k f_k$ is the Radon-Nikodym derivative of ν_ϵ with respect to ν. This finishes the proof of Theorem 2.

5. Interacting Brownian Motions.

Here we consider an illustration of Theorem 2 within the context of interacting Brownian motions. Let $\{V_i : i \in \mathbb{Z}\}$ be a set of smooth functions defined on $T^{\mathbb{Z}}$. We assume that they are bounded, of finite range R (in other words, for each $i \in \mathbb{Z}$, V_i is a function only of η_j for j such that $|j - i| \leq R$) and with bounded second order partial derivatives. Consider the infinitesimal generator

$$L_0 = \sum_{i \in \mathbb{Z}} \left(\partial_i^2 + (\partial_i V_i) \partial_i \right).$$

This can also we written as $L_0 = \sum_{i \in \mathbb{Z}} e^{-V_i} \partial_i (e^{V_i} \partial_i)$. It is a well known fact that this process has an invariant measure ν which satisfies the logarithmic Sobolev inequality. We wish to quantify the effect over the invariant measure of perturbing the above generator by the following operator:

$$A := \partial_0.$$

By Theorem 2, the diffusion with generator $L_\epsilon := L_0 + \epsilon A$, has an invariant measure with a Radon-Nikodym derivative g_ϵ with respect to ν, which admits the expansion

$$g_\epsilon = 1 + \sum_{k=1}^{\infty} \epsilon^k f_k,$$

with $L_0^* f_{k+1} = -A^* f_k$, $\langle f_k \rangle_\nu = 0$ for $k \geq 0$ and $f_0 := 1$. Hence, we see that f_1 satisfies the equation

$$\sum_{j \in \mathbb{Z}} e^{V_j} \partial_j (e^{-V_j} \partial_j f_1) = -\partial_0 e^{V_0}.$$

Then, writing $f_1 = -L_0^{-1}(\partial_0 e^{V_0})$, we have up to first order

$$g_\epsilon = 1 - \epsilon L_0^{-1}(\partial_0 e^{V_0}) + O(\epsilon^2),$$

where $\limsup_{\epsilon\to 0} \|O(\epsilon^2)\|_{2,\nu}/\epsilon^2 < \infty$.

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