

Survival Probability of a Random Walk Among a Poisson System of Moving Traps

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Abstract

We review some old and prove some new results on the survival probability of a random walk among a Poisson system of moving traps on \mathbb{Z}^d , which can also be interpreted as the solution of a parabolic Anderson model with a random time-dependent potential. We show that the annealed survival probability decays asymptotically as $e^{-\lambda_1 \sqrt{t}}$ for $d = 1$, as $e^{-\lambda_2 t / \log t}$ for $d = 2$, and as $e^{-\lambda_d t}$ for $d \geq 3$, where λ_1 and λ_2 can be identified explicitly. In addition, we show that the quenched survival probability decays asymptotically as $e^{-\tilde{\lambda}_d t}$, with $\tilde{\lambda}_d > 0$ for all $d \geq 1$. A key ingredient in bounding the annealed survival probability is what is known in the physics literature as the Pascal principle, which asserts that the annealed survival probability is maximized if the random walk stays at a fixed position. A corollary of independent interest is that the expected cardinality of the range of a continuous time symmetric random walk increases under perturbation by a deterministic path.

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1 Introduction

1.1 Model and results

Let $X := (X(t))_{t \geq 0}$ be a simple symmetric random walk on \mathbb{Z}^d with jump rate $\kappa \geq 0$, and let $(Y_j^y)_{1 \leq j \leq N_y, y \in \mathbb{Z}^d}$ be a collection of independent simple symmetric random walks on \mathbb{Z}^d with jump rate $\rho > 0$, where N_y is the number of walks that start at each $y \in \mathbb{Z}^d$ at time 0, $(N_y)_{y \in \mathbb{Z}^d}$ are i.i.d. Poisson distributed with mean $\nu > 0$, and $Y_j^y := (Y_j^y(t))_{t \geq 0}$ denotes the j -th walk starting at y at time 0. Let us denote the number of walks Y at position $x \in \mathbb{Z}^d$ at time $t \geq 0$ by

$$\xi(t, x) := \sum_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y} \delta_x(Y_j^y(t)). \quad (1.1)$$

It is easy to see that for each $t \geq 0$, $(\xi(t, x))_{x \in \mathbb{Z}^d}$ are i.i.d. Poisson distributed with mean ν , so that $(\xi(t, \cdot))_{t \geq 0}$ is a stationary process, and furthermore it is reversible in the sense that $(\xi(t, \cdot))_{0 \leq t \leq T}$ is equally distributed with $(\xi(T - t, \cdot))_{0 \leq t \leq T}$. We will interpret the collection of walks Y as traps, and

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at each time t , the walk X is killed with rate $\gamma\xi(t, X(t))$ for some parameter $\gamma > 0$. Conditional on the realization of the field of traps ξ , the probability that the walk X survives by time t is given by

$$Z_{t,\xi}^\gamma := \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) \, ds \right\} \right], \quad (1.2)$$

where \mathbb{E}_0^X denotes expectation with respect to X with $X(0) = 0$. We call this the *quenched survival probability*, which depends on the random medium ξ . When we furthermore average over ξ , which we denote by \mathbb{E}^ξ , we obtain the *annealed survival probability*

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \mathbb{E}^\xi \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) \, ds \right\} \right]. \quad (1.3)$$

We will study the long time behavior of the annealed and quenched survival probabilities, and in particular, identify their rate of decay and their dependence on the spatial dimension d and the parameters κ, ρ, ν and γ .

Here are our main results on the decay rate of the annealed and quenched survival probabilities.

Theorem 1.1 [Annealed survival probability] *Assume that $\gamma \in (0, \infty]$, $\kappa \geq 0$, $\rho > 0$ and $\nu > 0$, then*

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \begin{cases} \exp \left\{ -\nu \sqrt{\frac{8\rho t}{\pi}} (1 + o(1)) \right\}, & d = 1, \\ \exp \left\{ -\nu \pi \rho \frac{t}{\log t} (1 + o(1)) \right\}, & d = 2, \\ \exp \left\{ -\lambda_{d,\gamma,\kappa,\rho,\nu} t (1 + o(1)) \right\}, & d \geq 3, \end{cases} \quad (1.4)$$

where $\lambda_{d,\gamma,\kappa,\rho,\nu}$ depends on $d, \gamma, \kappa, \rho, \nu$, and is called the *annealed Lyapunov exponent*. Furthermore, $\lambda_{d,\gamma,\kappa,\rho,\nu} \geq \lambda_{d,\gamma,0,\rho,\nu} = \nu\gamma / (1 + \frac{\gamma G_d(0)}{\rho})$, where $G_d(0) := \int_0^\infty p_t(0) \, dt$ is the Green function of a simple symmetric random walk on \mathbb{Z}^d with jump rate 1 and transition kernel $p_t(\cdot)$.

Note that in dimensions 1 and 2, the annealed survival probability decays sub-exponentially, and the pre-factor in front of the decay rate is surprisingly independent of $\gamma \in (0, \infty]$ and $\kappa \geq 0$. The key ingredient in the proof is what is known in the physics literature as the *Pascal principle*, which asserts that in (1.3), if we condition on the random walk trajectory X , then the annealed survival probability is maximized when $X \equiv 0$. We will extract a rigorous proof of this from the work of Moreau, Oshanin, Bénichou and Coppey [MOBC03, MOBC04]. As a corollary of the Pascal principle, we will also show in Corollary 2.1 that the expected cardinality of the range of a continuous time symmetric random walk increases under perturbation by a deterministic path.

In contrast to the annealed case, the quenched survival probability always decays exponentially.

Theorem 1.2 [Quenched survival probability] *Assume that $d \geq 1$, $\gamma > 0$, $\kappa \geq 0$, $\rho > 0$ and $\nu > 0$. Then there exists deterministic $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$ depending on $d, \gamma, \kappa, \rho, \nu$, called the *quenched Lyapunov exponent*, such that \mathbb{P}^ξ -a.s.,*

$$Z_{t,\xi}^\gamma = \exp \left\{ -\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu} t (1 + o(1)) \right\} \quad \text{as } t \rightarrow \infty. \quad (1.5)$$

Furthermore, $0 < \tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu} \leq \gamma\nu + \kappa$ for all $d \geq 1, \gamma > 0, \kappa \geq 0, \rho > 0$ and $\nu > 0$.

Remark. When $\gamma < 0$, $Z_{t,\xi}^\gamma$ can be interpreted as the expected number of branching random walks in the catalytic medium ξ . See Section 1.3 for more discussion on this model. As will be outlined out at the end of Section 4.1, (1.5) also holds in this case, and lies in the interval $[-\gamma\nu - \kappa, \infty)$.

In Proposition 3.2 below, we will also give an upper bound of the same order as in Theorem 1.1 for the survival probability $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$, where $(\xi(0, x))_{x \in \mathbb{Z}^d}$ is deterministic and satisfies some constraints. These constraints hold asymptotically a.s. for i.i.d. Poisson distributed $(\xi(0, x))_{x \in \mathbb{Z}^d}$. Therefore we call this a *semi-annealed* bound, which we will use in Section 3 to obtain sub-exponential bounds on the quenched survival probability in dimensions 1 and 2.

1.2 Relation to the parabolic Anderson model

The annealed and quenched survival probabilities $Z_{t,\xi}^\gamma$ and $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ are closely related to the solution of the parabolic Anderson model (PAM), namely, the solution of the following parabolic equation with random potential ξ :

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= \kappa \Delta u(t, x) - \gamma \xi(t, x) u(t, x), & x \in \mathbb{Z}^d, t \geq 0, \\ u(0, x) &= 1, \end{aligned} \quad (1.6)$$

where γ, κ and ξ are as before, and $\Delta f(x) = \frac{1}{2d} \sum_{\|y-x\|=1} (f(y) - f(x))$ is the discrete Laplacian on \mathbb{Z}^d , which is also the generator of a simple symmetric random walk on \mathbb{Z}^d with jump rate 1.

By the Feynman-Kac formula, the solution u admits the representation

$$u(t, 0) = \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(t-s, X(s)) ds \right\} \right], \quad (1.7)$$

which differs from $Z_{t,\xi}^\gamma$ in (1.2) by a time reversal in ξ . When we average $u(t, 0)$ over the random field ξ , by the reversibility of $(\xi(t, \cdot))_{0 \leq s \leq t}$, we have

$$\mathbb{E}^\xi[u(t, 0)] = \mathbb{E}^\xi \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(t-s, X(s)) ds \right\} \right] = \mathbb{E}_0^X \mathbb{E}^\xi \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} \right] = \mathbb{E}^\xi[Z_{t,\xi}^\gamma]. \quad (1.8)$$

Therefore Theorem 1.1 also applies to the annealed solution $\mathbb{E}^\xi[u(t, 0)]$. Despite the difference between $Z_{t,\xi}^\gamma$ and $u(t, 0)$ due to time reversal, Theorem 1.2 also holds with $u(t, 0)$ in place of $Z_{t,\xi}^\gamma$.

Theorem 1.3 [Quenched solution of PAM] *Let $d \geq 1$, $\gamma > 0$, $\kappa \geq 0$, $\rho > 0$, $\nu > 0$ and $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu} > 0$ be the same as in Theorem 1.2. Then \mathbb{P}^ξ -a.s.,*

$$u(t, 0) = \exp \left\{ -\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu} t(1 + o(1)) \right\} \quad \text{as } t \rightarrow \infty. \quad (1.9)$$

Remark. By Theorem 1.2 and the remark following it, for any $\gamma \in \mathbb{R}$, $t^{-1} \log u(t, 0)$ converges in probability to $-\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$ because $u(t, 0)$ is equally distributed with $Z_{t,\xi}^\gamma$. However, we were only able to strengthen this to almost sure convergence for the $\gamma > 0$ case, but not for $\gamma < 0$.

1.3 Review of related results

The study of trapping problems has a long history in the mathematics and physics literature. We review some models and results most relevant to our problem.

Immobile Traps Extensive studies have been carried out for the case of immobile traps, i.e., $\rho = 0$ and $\xi(t, \cdot) \equiv \xi(0, \cdot)$ for all $t \geq 0$. A continuum version is Brownian motion among Poissonian obstacles, where a ball of size 1 is placed and centered at each point of a mean density 1 homogeneous Poisson point process in \mathbb{R}^d , acting as traps or obstacles, and an independent Brownian motion starts at the origin and is killed at rate γ times the number of obstacles it is contained in. Using a large deviation principle for the Brownian motion occupation time measure, Donsker and Varadhan [DV75] showed that the annealed survival probability decays asymptotically as $\exp\{-C_{d,\gamma} t^{\frac{d}{d+2}}(1 + o(1))\}$. Using spectral techniques, Sznitman [S98] later developed a coarse graining method, known as the *method of enlargement of obstacles*, to show that the quenched survival probability decays asymptotically as $\exp\{-\tilde{C}_{d,\gamma} \frac{t}{(\log t)^{2/d}}(1 + o(1))\}$. Similar results have also been obtained for random walks among immobile Bernoulli traps (i.e. $\xi(0, x) \in \{0, 1\}$), see e.g. [DV79, B94, A94, A95]. Traps with a more general form of the trapping potential ξ have also been studied in the context of the parabolic Anderson model (see e.g. Biskup and König [BK01]), where alternative techniques to the method of enlargement of obstacles were developed and the order of sub-exponential decay of the survival probabilities may

vary depending on the distribution of ξ . Compared to our results in Theorems 1.1 and 1.2, we note that when the traps are moving, both the annealed and quenched survival probabilities decay faster than when the traps are immobile. The heuristic reason is that, the walk survives by finding large space-time regions void of traps, which are easily destroyed if the traps are moving. Another example is a Brownian motion among Poissonian obstacles where the obstacles move with a deterministic drift. It has been shown that the annealed and quenched survival probabilities decay exponentially if the drift is sufficiently large, see e.g. [S98, Thms. 5.4.7 and 5.4.9].

Mobile Traps The model we consider here has in fact been studied earlier by Redig in [R94], where he considered a trapping potential ξ generated by a reversible Markov process, such as a Poisson system of random walks, or the symmetric exclusion process in equilibrium. Using spectral techniques applied to the process of moving traps viewed from the random walk, he established an exponentially decaying upper bound for the annealed survival probability when the empirical distribution of the trapping potential, $\frac{1}{t} \int_0^t \xi(s, 0) ds$, satisfies a large deviation principle with scale t . This applies for instance to ξ generated from either a Poisson system of independent random walks or the symmetric exclusion process in equilibrium, in dimensions $d \geq 3$.

Annihilating Two-type Random Walks In [BL91], Bramson and Lebowitz studied a model from chemical physics, where there are two types of particles, A s and B s, both starting initially with an i.i.d. Poisson distribution on \mathbb{Z}^d with density $\rho_A(0)$ resp. $\rho_B(0)$. All particles perform independent simple symmetric random walk with jump rate 1, particles of the same type do not interact, and when two particles of opposite types meet, they annihilate each other. This system models a chemical reaction $A + B \rightarrow \text{inert}$. It was shown in [BL91] that when $\rho_A(0) = \rho_B(0) > 0$, then $\rho_A(t)$ and $\rho_B(t)$ (the densities of the A and B particles at time t) decay with the order $t^{-d/4}$ in dimensions $1 \leq d \leq 4$, and decay with the order t^{-1} in $d \geq 4$. When $\rho_A(0) > \rho_B(0) > 0$, it was shown that $\rho_A(t) \rightarrow \rho_A(0) - \rho_B(0)$ as $t \rightarrow \infty$, and $-\log \rho_B(t)$ increases with the order \sqrt{t} in $d = 1$, $t/\log t$ in $d = 2$, and t in $d \geq 3$, which is the same as in Theorem 1.1. Heuristically, as $\rho_B(t) \rightarrow 0$ and $\rho_A(t) \rightarrow \rho_A(0) - \rho_B(0) > 0$, we can effectively model the B particles as uncorrelated single random walks among a Poisson field of moving traps with density $\rho_A(0) - \rho_B(0)$. In light of Theorem 1.1, it is natural to conjecture that $\rho_B(t)$ decays exactly as prescribed in Theorem 1.1 with $\nu = \rho_A(0) - \rho_B(0)$ and $\gamma = \infty$, whence we obtain not only the logarithmic order of decay as in [BL91], but also the constant pre-factor. However we will not address this issue here.

Random Walk Among Moving Catalysts Instead of considering ξ as a field of moving traps, we may consider it as a field of moving catalysts for a system of branching random walks which we call reactants. At time 0, a single reactant starts at the origin which undergoes branching. Independently, each reactant performs simple symmetric random walk on \mathbb{Z}^d with jump rate κ , and undergoes binary branching with rate $|\gamma|\xi(t, x)$ when the reactant is at position x at time t . This model was studied by Kesten and Sidoravicius in [KS03], and in the setting of the parabolic Anderson model, studied by Gärtner and den Hollander in [GH06]. For the catalytic model, γ is negative in (1.2), (1.3), (1.7) and (1.8), and $Z_{t,\xi}^\gamma$ and $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ now represent the quenched, resp. annealed, expected number of reactants at time t . It was shown in [GH06] that $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ grows double exponentially fast (i.e., $t^{-1} \log \log \mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ tends to a positive limit as $t \rightarrow \infty$) for all $\gamma < 0$ in dimensions $d = 1$ and 2 . In $d \geq 3$, there exists a critical $\gamma_{c,d} < 0$ such that $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ grows double exponentially for $\gamma < \gamma_{c,d}$, and grows exponentially (i.e., $t^{-1} \log \mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ tends to a positive limit as $t \rightarrow \infty$) for all $\gamma \in (\gamma_{c,d}, 0)$. In the quenched case, however, it was shown in [KS03] that $Z_{t,\xi}^\gamma$ only exhibits exponential growth (with $\log Z_{t,\xi}^\gamma$ shown to be of order t) regardless of the dimension $d \geq 1$ and the strength of interaction $\gamma < 0$. Such dimension dependence bears similarities with our results for the trap model in Theorems 1.1 and 1.2.

Directed Polymer in a Random Medium We used $Z_{t,\xi}^\gamma$ to denote the survival probability, because $Z_{t,\xi}^\gamma$ and $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ are in fact the quenched resp. annealed partition functions of a directed polymer model in a random time-dependent potential ξ at inverse temperature γ . The directed polymer is modeled

by $(X(s))_{0 \leq s \leq t}$. In the polymer measure, a trajectory $(X(s))_{0 \leq s \leq t}$ is re-weighted by the survival probability of a random walk following that trajectory in the environment ξ . Namely, we define a change of measure on $(X(s))_{0 \leq s \leq t}$ with density $e^{-\gamma \int_0^t \xi(s, X(s)) ds} / Z_{t, \xi}^\gamma$ in the quenched model, and with density $\mathbb{E}^\xi [e^{-\gamma \int_0^t \xi(s, X(s)) ds}] / \mathbb{E}^\xi [Z_{t, \xi}^\gamma]$ in the annealed model. Qualitatively, the polymer measure favors trajectories which seek out space-time regions void of traps. However, a more quantitative geometric characterization as was carried out for the case of immobile traps (see e.g. [S98]) is still lacking.

For readers interested in more background on the problem of a Brownian motion (or random walk) in time-independent potential, we refer to the book by Sznitman [S98] on Brownian motion among Poissonian obstacles, and the survey by Gärtner and König [GK05] on the parabolic Anderson model. For readers interested in more recent studies of a random walk in time-dependent catalytic environments, we refer to the survey by Gärtner, den Hollander and Maillard [GHM09]. For readers interested in more recent studies of the trapping problem in the physics literature, we refer to the papers of Moreau, Oshanin, Bénichou and Coppey [MOBC03, MOBC04] and the references therein.

1.4 Outline

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 on the annealed survival probability, where the so-called Pascal principle will be introduced. In Section 3, we give a preliminary upper bound on the quenched survival probability in dimensions 1 and 2, as well as an upper bound for a semi-annealed system. Lastly, in Section 4, we prove the existence of the quenched Lyapunov exponent in Theorems 1.2 and 1.3 via a shape theorem, and we show that the quenched Lyapunov exponent is always positive.

2 Annealed survival probability

In this section, we prove Theorem 1.1. We start with a proof in Section 2.1 of the existence of the annealed Lyapunov exponent $\lambda_{d, \gamma, \kappa, \rho, \nu}$. Our proof follows the same argument as for the catalytic model with $\gamma < 0$ in Gärtner and den Hollander [GH06], which is based on a special representation of $\mathbb{E}^\xi [Z_{t, \xi}^\gamma]$ after integrating out the Poisson random field ξ , which then allows us to apply the subadditivity lemma. In Section 2.2, we prove Theorem 1.1 for the special case $\kappa = 0$, i.e., $X \equiv 0$, relying on exact calculations. Sections 2.3 and 2.4 prove respectively the lower and upper bound on $\mathbb{E}^\xi [Z_{t, \xi}^\gamma]$ in Theorem 1.1, for $d = 1, 2$ and general $\kappa > 0$. The lower bound is obtained by creating a space-time box void of traps and forcing X to stay inside the box, while the upper bound is based on the so-called Pascal principle, first introduced in the physics literature by Moreau et al [MOBC03, MOBC04]. In Section 2.4, we will also prove the aforementioned Corollary 2.1 on the range of a symmetric random walk.

2.1 Existence of the annealed Lyapunov exponent

In this section, we prove the existence of the annealed Lyapunov exponent

$$\lambda = \lambda_{d, \gamma, \kappa, \rho, \nu} := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^\xi [Z_{t, \xi}^\gamma]. \quad (2.10)$$

Proof. The proof is similar to that for the catalytic model with $\gamma < 0$ in [GH06]. As in [GH06], we can integrate out the Poisson system ξ to obtain

$$\mathbb{E}^\xi [Z_{t, \xi}^\gamma] = \mathbb{E}^\xi [u(t, 0)] = \mathbb{E}_0^X \mathbb{E}^\xi \left[\exp \left\{ -\gamma \int_0^t \xi(t-s, X(s)) ds \right\} \right] = \mathbb{E}_0^X \left[\exp \left\{ \nu \sum_{y \in \mathbb{Z}^d} (v_X(t, y) - 1) \right\} \right], \quad (2.11)$$

where conditional on X ,

$$v_X(t, y) = \mathbb{E}_y^Y \left[\exp \left\{ -\gamma \int_0^t \delta_0(Y(s) - X(t-s)) ds \right\} \right] \quad (2.12)$$

with $\mathbb{E}_y^Y[\cdot]$ denoting expectation with respect to a simple symmetric random walk Y with jump rate ρ and $Y(0) = y$. By the Feynman-Kac formula, $(v_X(t, y))_{t \geq 0, y \in \mathbb{Z}^d}$ solves the equation

$$\begin{aligned} \frac{\partial}{\partial t} v_X(t, y) &= \rho \Delta v_X(t, y) - \gamma \delta_{X(t)}(y) v_X(t, y), & y \in \mathbb{Z}^d, t \geq 0, \\ v_X(0, \cdot) &\equiv 1, \end{aligned} \quad (2.13)$$

which implies that $\Sigma_X(t) := \sum_{y \in \mathbb{Z}^d} (v_X(t, y) - 1)$ is the solution of the equation

$$\begin{aligned} \frac{d}{dt} \Sigma_X(t) &= -\gamma v_X(t, X(t)), \\ \Sigma_X(0) &= 0. \end{aligned} \quad (2.14)$$

Hence, $\Sigma_X(t) = -\gamma \int_0^t v_X(s, X(s)) ds$, and the representation (2.11) becomes

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \mathbb{E}_0^X \left[\exp \left\{ -\nu \gamma \int_0^t v_X(s, X(s)) ds \right\} \right]. \quad (2.15)$$

We now observe that for $t_1, t_2 > 0$,

$$\begin{aligned} \mathbb{E}^\xi[Z_{t_1+t_2,\xi}^\gamma] &= \mathbb{E}_0^X \left[\exp \left\{ -\nu \gamma \int_0^{t_1} v_X(s, X(s)) ds \right\} \exp \left\{ -\nu \gamma \int_{t_1}^{t_1+t_2} v_X(s, X(s)) ds \right\} \right] \\ &\geq \mathbb{E}_0^X \left[\exp \left\{ -\nu \gamma \int_0^{t_1} v_X(s, X(s)) ds \right\} \exp \left\{ -\nu \gamma \int_0^{t_2} v_{\theta_{t_1} X}(s, (\theta_{t_1} X)(s)) ds \right\} \right] \\ &= \mathbb{E}^\xi[Z_{t_1,\xi}^\gamma] \mathbb{E}^\xi[Z_{t_2,\xi}^\gamma], \end{aligned} \quad (2.16)$$

where $\theta_{t_1} X := ((\theta_{t_1} X)(s))_{s \geq 0} = (X(t_1 + s) - X(t_1))_{s \geq 0}$, we used the independence of $(X(s))_{0 \leq s \leq t_1}$ and $((\theta_{t_1} X)(s))_{0 \leq s \leq t_2}$, and the fact that for $s > t_1$,

$$\begin{aligned} v_X(s, X(s)) &= \mathbb{E}_{X(s)}^Y \left[\exp \left\{ -\gamma \int_0^s \delta_0(Y(r) - X(s-r)) dr \right\} \right] \\ &\leq \mathbb{E}_{X(s)}^Y \left[\exp \left\{ -\gamma \int_0^{s-t_1} \delta_0(Y(r) - X(s-r)) dr \right\} \right] = v_{\theta_{t_1} X}(s-t_1, (\theta_{t_1} X)(s-t_1)). \end{aligned}$$

From (2.16), we deduce that $-\log \mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ is subadditive in t , and hence the limit in (2.10) exists and

$$\lambda_{d,\gamma,\kappa,\rho,\nu} = -\sup_{t>0} \frac{1}{t} \log \mathbb{E}^\xi[Z_{t,\xi}^\gamma]. \quad (2.17)$$

Clearly $\lambda \geq 0$, and Theorem 1.1 will imply that λ always equals 0 in dimensions $d = 1, 2$. For $d \geq 3$, the lower bound for the quenched survival probability in Theorem 1.2 will imply that $\lambda < \gamma\nu + \kappa < \infty$, while an exact calculation of λ for the case $\kappa = 0$ in Section 2.2 and the Pascal principle in Section 2.4 will imply that $\lambda > 0$ for all $\gamma, \nu, \rho > 0$ and $\kappa \geq 0$. \blacksquare

2.2 Special case $\kappa = 0$

In this section, we prove Theorem 1.1 for the case $\kappa = 0$, which will be useful for lower bounding $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ for general $\kappa > 0$, as well as for providing an upper bound on $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ by the Pascal principle.

Proof of Theorem 1.1 for $\kappa = 0$. We first treat the case $\gamma \in (0, \infty)$. When $\kappa = 0$, (2.15) becomes

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \exp \left\{ -\nu\gamma \int_0^t v_0(s, 0) ds \right\}, \quad (2.18)$$

where v_0 is the solution of (2.13) with $X \equiv 0$. It then suffices to analyze the asymptotics of $v_0(t, 0)$ as $t \rightarrow \infty$. Note that the representation (2.12) for $v_0(t, 0)$ becomes

$$v_0(t, 0) = \mathbb{E}_0^Y [e^{-\gamma \int_0^t \delta_0(Y(s)) ds}], \quad (2.19)$$

which is the Laplace transform of the local time of Y at the origin. For $d = 1, 2$, $v_0(t, 0) \downarrow 0$ as $t \uparrow \infty$ by the recurrence of simple random walks, while for $d \geq 3$, $v_0(t, 0) \downarrow C_d$ for some $C_d > 0$ by transience.

By Duhamel's principle, we have the following integral representation for the solution v_X of (2.13),

$$v_X(t, y) = 1 - \gamma \int_0^t p_{\rho s}(y - X(t-s)) v_X(t-s, X(t-s)) ds, \quad (2.20)$$

where $p_s(\cdot)$ is the transition probability kernel of a rate 1 simple symmetric random walk on \mathbb{Z}^d . When $X \equiv 0$, we obtain

$$v_0(t, 0) = 1 - \gamma \int_0^t p_{\rho s}(0) v_0(t-s, 0) ds. \quad (2.21)$$

Denote the Laplace transforms (in t) of $v_0(t, 0)$ and $p_t(0)$ by

$$\hat{v}_0(\lambda) = \int_0^\infty e^{-\lambda t} v_0(t, 0) dt, \quad \hat{p}(\lambda) = \int_0^\infty e^{-\lambda t} p_t(0) dt. \quad (2.22)$$

Taking Laplace transform in (2.21) and solving for $\hat{v}_0(\lambda)$ then gives

$$\hat{v}_0(\lambda) = \frac{1}{\lambda} \cdot \frac{\rho}{\rho + \gamma \hat{p}(\lambda/\rho)}. \quad (2.23)$$

We can apply the local central limit theorem for continuous time simple random walks in $d = 1$ and 2 (i.e., $p_t(0) = \left(\frac{d}{2\pi t}\right)^{d/2} (1 + o(1))$ as $t \rightarrow \infty$) to obtain the following asymptotics for $\hat{p}(\lambda)$ as $\lambda \downarrow 0$,

$$\hat{p}(\lambda) = \begin{cases} \frac{1}{\sqrt{2\lambda}} (1 + o(1)), & d = 1, \\ \frac{\ln(\frac{1}{\lambda})}{\pi} (1 + o(1)), & d = 2, \\ G_d(0) (1 + o(1)), & d \geq 3, \end{cases} \quad (2.24)$$

with $G_d(0) = \int_0^\infty p_t(0) dt$, which translates into the following asymptotics for $\hat{v}_0(\lambda)$ as $\lambda \downarrow 0$:

$$\hat{v}_0(\lambda) = \begin{cases} \frac{\sqrt{2\rho}}{\gamma} \cdot \frac{1}{\sqrt{\lambda}} (1 + o(1)), & d = 1, \\ \frac{\pi\rho}{\gamma} \cdot \frac{1}{\lambda \ln(\frac{1}{\lambda})} (1 + o(1)), & d = 2, \\ \frac{\rho}{\rho + \gamma G_d(0)} \cdot \frac{1}{\lambda} (1 + o(1)), & d \geq 3. \end{cases} \quad (2.25)$$

Since $v_0(t, 0)$ is monotonically decreasing in t by (2.19), by Karamata's Tauberian theorem (see e.g. [F66, Chap. XIII.5, Thm. 4]), we have the following asymptotics for $v_0(t, 0)$ as $t \rightarrow \infty$,

$$v_0(t, 0) = \begin{cases} \frac{1}{\gamma} \sqrt{\frac{2\rho}{\pi}} \cdot \frac{1}{\sqrt{t}} (1 + o(1)), & d = 1, \\ \frac{\pi\rho}{\gamma} \cdot \frac{1}{\ln t} (1 + o(1)), & d = 2, \\ \frac{\rho}{\rho + \gamma G_d(0)} (1 + o(1)), & d \geq 3, \end{cases} \quad (2.26)$$

which by (2.18) implies Theorem 1.1 for $\kappa = 0$ and $\gamma \in (0, \infty)$.

When $\kappa = 0$ and $\gamma = \infty$, we have

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \mathbb{P}\left(\xi(s, 0) = 0 \forall s \in [0, t]\right) = \exp\left\{-\nu \sum_{y \in \mathbb{Z}^d} \psi(t, y)\right\},$$

where $\psi(t, y) = \mathbb{P}_y^Y(\exists s \in [0, t] : Y(s) = 0)$ for a jump rate ρ simple symmetric random walk Y starting from y . Note further that $\psi(t, y)$ solves the parabolic equation

$$\frac{\partial}{\partial t} \psi(t, y) = \rho \Delta \psi(t, y), \quad y \neq 0, t \geq 0, \quad (2.27)$$

with boundary conditions $\psi(\cdot, 0) \equiv 1$ and $\psi(0, \cdot) \equiv 0$. Therefore $\sum_{y \in \mathbb{Z}^d} \psi(t, y)$ solves the equation

$$\frac{d}{dt} \sum_{y \in \mathbb{Z}^d} \psi(t, y) = -\rho \Delta \psi(t, 0) = \rho(1 - \psi(t, e_1)) = \rho \phi(t, e_1), \quad (2.28)$$

where $e_1 = (1, 0, \dots, 0)$, $\phi(t, e_1) := 1 - \psi(t, e_1)$, and we have used the fact that $\sum_{x \in \mathbb{Z}^d} \Delta \psi(t, x) = 0$ and the symmetry of the simple symmetric random walk. Therefore

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \exp\left\{-\nu \rho \int_0^t \phi(s, e_1) ds\right\}. \quad (2.29)$$

By generating function calculations and Tauberian theorems (see e.g. [L96, Sec. 2.4] or [S76, Sec. 32, P3]), it is known that $\phi(t, e_1)$, which is the probability that a rate 1 simple random walk starting from e_1 does not hit 0 before time ρt , has the asymptotics $\phi(t, e_1) = \sqrt{\frac{2}{\pi \rho t}}(1 + o(1))$ for $d = 1$, $\phi(t, e_1) = \frac{\pi}{\ln t}(1 + o(1))$ for $d = 2$, and $\phi(t, e_1) = G_d(0)^{-1}(1 + o(1))$ for $d \geq 3$. Therefore as $t \rightarrow \infty$,

$$\log \mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \begin{cases} -\nu \sqrt{\frac{8\rho t}{\pi}}(1 + o(1)), & d = 1, \\ -\nu \pi \frac{\rho t}{\ln t}(1 + o(1)), & d = 2, \\ -\nu \frac{\rho t}{G_d(0)}(1 + o(1)), & d \geq 3, \end{cases} \quad (2.30)$$

which proves Theorem 1.1 for $\kappa = 0$ and $\gamma = \infty$. ■

Remark. When $\kappa = 0$ so that $X \equiv 0$, the representation (2.18) allows us to easily compute the Laplace transform of $D_t := \frac{1}{t} \int_0^t \xi(s, 0) ds$, since $\mathbb{E}^\xi[Z_{t,\xi}^\gamma] = \mathbb{E}^\xi[\exp\{-\gamma t D_t\}]$. By replacing γt with a suitable scale $\lambda t/a_t$, where $\lambda \in \mathbb{R}$, $a_t = \sqrt{t}$ for $d = 1$, $a_t = \log t$ for $d = 2$, and $a_t = 1$ for $d \geq 3$, we can identify

$$\Psi(-\lambda) := \lim_{t \rightarrow \infty} \frac{a_t}{t} \mathbb{E}^\xi\left[\exp\left\{-\frac{\lambda t}{a_t} D_t\right\}\right]$$

using the asymptotics in (2.26). As shown in Cox and Griffeath [CG84], applying Gärtner-Ellis theorem then leads to a large deviation principle for D_t with scale t/a_t , except that in [CG84], the derivation of $\Psi(-\lambda)$ was by Taylor expansion in λ , which can be greatly simplified if we use the representation (2.18) instead.

2.3 Lower bound on the annealed survival probability

In this section, we prove the lower bound on $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ in Theorem 1.1 for dimensions $d = 1$ and 2, i.e.,

Lemma 2.1 *For all $\gamma \in (0, \infty]$, $\kappa \geq 0$, $\rho > 0$ and $\nu > 0$, we have*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \mathbb{E}^\xi[Z_{t,\xi}^\gamma] &\geq -\nu \sqrt{\frac{8\rho}{\pi}}, & d = 1, \\ \liminf_{t \rightarrow \infty} \frac{\ln t}{t} \log \mathbb{E}^\xi[Z_{t,\xi}^\gamma] &\geq -\nu \pi \rho, & d = 2. \end{aligned} \quad (2.31)$$

Proof. The basic strategy is the same as for the case of immobile traps, namely, we force the environment ξ to create a ball B_{R_t} of radius R_t around the origin, which remains void of traps up to time t , and we force the random walk X to stay inside B_{R_t} up to time t . This leads to a lower bound on the survival probability that is independent of $\gamma \in (0, \infty]$ and $\kappa \geq 0$. Surprisingly, in dimensions $d = 1$ and 2 , this lower bound turns out to be sharp, which can be attributed to the larger fluctuation of the random field ξ in $d = 1$ and 2 , which makes it easier to create space-time regions void of traps. Note that it is clearly more costly to maintain the same space-time region void of traps than in the case when the traps are immobile.

Recall that ξ is the counting field of a family of independent random walks $\{Y_j^y\}_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y}$, where $\{N_y\}_{y \in \mathbb{Z}^d}$ are i.i.d. Poisson random variables with mean ν . Let B_r denote the ball of radius r , i.e., $B_r = \{x \in \mathbb{Z}^d : \|x\|_\infty \leq r\}$. For a scale function $1 \ll R_t \ll \sqrt{t}$ to be chosen later, let E_t denote the event that $N_y = 0$ for all $y \in B_{R_t}$. Let F_t denote the event that $Y_j^y(s) \notin B_{R_t}$ for all $y \notin B_{R_t}$, $1 \leq j \leq N_y$, and $s \in [0, t]$; furthermore, let G_t denote the event that X with $X(0) = 0$ does not leave B_{R_t} before time t . Then by (1.3),

$$\mathbb{E}^\xi[Z_{t,\xi}^\gamma] \geq \mathbb{P}(E_t \cap F_t \cap G_t) = \mathbb{P}(E_t)\mathbb{P}(F_t)\mathbb{P}(G_t). \quad (2.32)$$

Note that $\mathbb{P}(E_t) = e^{-\nu(2R_t+1)^d}$. To estimate $\mathbb{P}(G_t)$, note that by Donsker's invariance principle, if $1 \ll R_t \ll \sqrt{t}$ as $t \rightarrow \infty$, then there exists $\alpha > 0$ such that for all t sufficiently large,

$$\inf_{x \in B_{\sqrt{t}/2}} \mathbb{P}\left(X(s) \in B_{\sqrt{t}} \forall s \in [0, t], X(t) \in B_{\sqrt{t}/2} \mid X(0) = x\right) \geq \alpha. \quad (2.33)$$

By partitioning $[0, t]$ into intervals of length R_t^2 and applying the Markov property, we obtain

$$\begin{aligned} \mathbb{P}(G_t) &\geq \mathbb{P}\left(X(s) \in B_{R_t} \forall s \in [(i-1)R_t^2, iR_t^2], \text{ and } X(iR_t^2) \in B_{R_t/2}, i = 1, 2, \dots, \lceil t/R_t^2 \rceil\right) \\ &\geq \alpha^{t/R_t^2} = e^{t \ln \alpha / R_t^2}. \end{aligned} \quad (2.34)$$

To estimate $\mathbb{P}(F_t)$, let \tilde{F}_t denote the event that $Y_j^y(s) \neq 0$ for all $y \in \mathbb{Z}^d$, $1 \leq j \leq N_y$, and $s \in [0, t]$. Note that $\mathbb{P}(\tilde{F}_t)$ is precisely the annealed survival probability $\mathbb{E}^\xi[Z_{t,\xi}^\gamma]$ when $\kappa = 0$ and $\gamma = \infty$, which satisfies the asymptotics in Theorem 1.1 by our calculations in Section 2.2. We next compare $\mathbb{P}(F_t)$ with $\mathbb{P}(\tilde{F}_t)$.

For a jump rate ρ simple random walk Y starting from $y \in \mathbb{Z}^d$, let $\tau_{B_{R_t}}$ denote the stopping time when Y first enters B_{R_t} , and τ_0 the stopping time when Y first visits 0 . Then standard computations yield

$$\ln \mathbb{P}(F_t) = -\nu \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y(\tau_{B_{R_t}} \leq t), \quad (2.35)$$

and a similar identity holds for $\ln \mathbb{P}(\tilde{F}_t)$ with B_{R_t} replaced by B_0 . Note that

$$\sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y(\tau_{B_{R_t}} \leq t) \geq \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y(\tau_0 \leq t) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^Y(\tau_0 \leq t) - \sum_{y \in B_{R_t}} \mathbb{P}_y^Y(\tau_0 \leq t).$$

Hence

$$\ln \mathbb{P}(F_t) \leq \ln \mathbb{P}(\tilde{F}_t) + \nu \sum_{y \in B_{R_t}} \mathbb{P}_y^Y(\tau_0 \leq t) \leq \ln \mathbb{P}(\tilde{F}_t) + \nu(2R_t + 1)^d. \quad (2.36)$$

On the other hand, for $\epsilon > 0$, we have

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^Y(\tau_0 \leq t + \epsilon t) \geq \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y(\tau_{B_{R_t}} \leq t, \tau_0 \leq t + \epsilon t) \geq \inf_{z \in \partial B_{R_t}} \mathbb{P}_z^Y(\tau_0 \leq \epsilon t) \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y(\tau_{B_{R_t}} \leq t),$$

where we used the strong Markov property. Therefore

$$\sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y(\tau_{B_{R_t}} \leq t) \leq \frac{\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^Y(\tau_0 \leq t + \epsilon t)}{\inf_{z \in \partial B_{R_t}} \mathbb{P}_z^Y(\tau_0 \leq \epsilon t)},$$

and hence by (2.35),

$$\ln \mathbb{P}(F_t) \geq \frac{\ln \mathbb{P}(\tilde{F}_{t+\epsilon t})}{\inf_{z \in \partial B_{R_t}} \mathbb{P}_z^Y(\tau_0 \leq \epsilon t)}. \quad (2.37)$$

We now choose R_t for $d = 1$ and 2 . For $d = 1$, let $R_t = \sqrt{t/\ln t}$, which is by no means the unique scale appropriate. Clearly $\inf_{z \in \partial B_{\sqrt{t/\ln t}}} \mathbb{P}_z^Y(\tau_0 \leq \epsilon t) \rightarrow 1$ as $t \rightarrow \infty$. By (2.36)–(2.37), the fact that $\mathbb{P}(\tilde{F}_t)$ satisfies the asymptotics in Theorem 1.1 for $\kappa = 0$ and $\gamma = \infty$, and that $\epsilon > 0$ can be made arbitrarily small, we obtain

$$\ln \mathbb{P}(F_t) = -\nu \sqrt{\frac{8\rho t}{\pi}} (1 + o(1)) = \ln \mathbb{P}(\tilde{F}_t).$$

Furthermore, for $R_t = \sqrt{t/\ln t}$ we have

$$\ln \mathbb{P}(E_t) = -\nu(2\sqrt{t/\ln t} + 1) \quad \text{and} \quad \ln \mathbb{P}(G_t) \geq \ln \alpha \ln t,$$

whence substituting these asymptotics into (2.32) gives (2.31) for $d = 1$.

For $d = 2$, let $R_t = \ln t$. Then we have $\inf_{z \in \partial B_{\ln t}} \mathbb{P}_z^Y(\tau_0 \leq \epsilon t) \rightarrow 1$ as $t \rightarrow \infty$, which is an easy consequence of [L96, Exercise 1.6.8]. By the same argument as for $d = 1$, we have

$$\ln \mathbb{P}(F_t) = -\nu\pi\rho \frac{t}{\ln t} (1 + o(1)) = \ln \mathbb{P}(\tilde{F}_t).$$

Together with the asymptotics

$$\ln \mathbb{P}(E_t) = -\nu(2 \ln t + 1)^2 \quad \text{and} \quad \ln \mathbb{P}(G_t) \geq \frac{t \ln \alpha}{\ln^2 t},$$

we deduce from (2.32) the desired bound in (2.31) for $d = 2$. ■

2.4 Upper bound on the annealed survival probability: the Pascal principle

In this section, we prove an upper bound on the annealed survival probability, called the Pascal principle.

Proposition 2.1 [Pascal principle] *Let ξ be the random field generated by a collection of irreducible symmetric random walks $\{Y_j^y\}_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y}$ on \mathbb{Z}^d with jump rate $\rho > 0$. Then for all piecewise constant $X : [0, t] \rightarrow \mathbb{Z}^d$ with a finite number of discontinuities, we have*

$$\mathbb{E}^\xi \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} \right] \leq \mathbb{E}^\xi \left[\exp \left\{ -\gamma \int_0^t \xi(s, 0) ds \right\} \right]. \quad (2.38)$$

In words, conditional on the random walk X , the annealed survival probability is maximized when $X \equiv 0$. This result was first established by Moreau et al in [MOBC03, MOBC04], where they named it the *Pascal principle*, because Pascal once asserted that *all misfortune of men comes from the fact that he does not stay peacefully in his room*. The Pascal principle together with the proof of Theorem 1.1 for $\kappa = 0$ in Section 2.2 imply the desired upper bound on the annealed survival probability in Theorem 1.1 for dimensions $d = 1, 2$, and it also shows that for $d \geq 3$, the annealed Lyapunov exponent $\lambda_{d,\gamma,\kappa,\rho,\nu}$ is always bounded from below by $\lambda_{d,\gamma,0,\rho,\nu} = \nu\gamma/(1 + \frac{\gamma G_d(0)}{\rho})$.

We will extract a proof of the Pascal principle from [MOBC04], which being written as a physics paper, can be hard for the reader to separate the rigorous arguments from the non-rigorous ones. As a byproduct, we will show in Corollary 2.1 that the expected cardinality of the range of a continuous time symmetric random walk increases under perturbation by a deterministic path.

Moreau et al proved in [MOBC04] the Pascal principle for a discrete time random walk among a Poisson field of moving traps, defined as follows. Let \bar{X} be a discrete time mean zero random

walk on \mathbb{Z}^d with $\bar{X}_0 = 0$. Let $\{N_y\}_{y \in \mathbb{Z}^d}$ be i.i.d. Poisson random variables with mean ν , and let $\{\bar{Y}_j^y\}_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y}$ be a family of independent *symmetric* random walks on \mathbb{Z}^d where \bar{Y}_j^y denotes the j -th random walk starting from y at time 0. Let

$$\bar{\xi}(n, x) := \sum_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y} \delta_x(\bar{Y}_j^y(n)). \quad (2.39)$$

Fix $0 \leq q \leq 1$, which will be the trapping probability. The dynamics of \bar{X} is such that \bar{X} moves independently of the traps $\{\bar{Y}_j^y\}_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y}$, and at each time $n \geq 0$, \bar{X} is killed with probability $1 - (1 - q)^{\bar{\xi}(n, \bar{X}(n))}$. Namely, each trap at the time-space lattice site $(n, \bar{X}(n))$ tries independently to capture \bar{X} with probability q . Given a realization of \bar{X} , let $\bar{\sigma}^{\bar{X}}(n)$ denote the probability that \bar{X} has survived till time n . Then analogous to (2.11), we have

$$\bar{\sigma}^{\bar{X}}(n) = \mathbb{E}^{\bar{\xi}} \left[(1 - q)^{\sum_{i=0}^n \bar{\xi}(i, \bar{X}(i))} \right] = \exp \left\{ -\nu \sum_{y \in \mathbb{Z}^d} \bar{w}^{q, \bar{X}}(n, y) \right\}, \quad (2.40)$$

where if we let \bar{Y} denote a random walk with the same jump kernel as \bar{Y}_j^y , then

$$\bar{w}^{q, \bar{X}}(n, y) := 1 - \mathbb{E}_y^{\bar{Y}} \left[(1 - q)^{\sum_{i=0}^n 1_{\{\bar{Y}(i) = \bar{X}(i)\}}} \right]. \quad (2.41)$$

The main result we need from Moreau et al [MOBC04] is the following discrete time Pascal principle.

Lemma 2.2 [Pascal principle in discrete time [MOBC04]]

Let \bar{Y} be an irreducible symmetric random walk on \mathbb{Z}^d with $\mathbb{P}_0^{\bar{Y}}(\bar{Y}(1) = 0) \geq 1/2$. Then for all $q \in [0, 1]$, $n \in \mathbb{N}_0$ and $\bar{X} : \mathbb{N}_0 \rightarrow \mathbb{Z}^d$, we have

$$\sum_{y \in \mathbb{Z}^d} \bar{w}^{q, \bar{X}}(n, y) \geq \sum_{y \in \mathbb{Z}^d} \bar{w}^{q, 0}(n, y), \quad (2.42)$$

and hence $\bar{\sigma}^{\bar{X}}(n) \leq \bar{\sigma}^0(n)$, where $\bar{w}^{q, \bar{X}}$ and $\bar{\sigma}^{\bar{X}}$ with $\bar{X} \equiv 0$.

Proof. The argument we present here is extracted from [MOBC04]. First note that the assumption \bar{Y} is symmetric implies that the Fourier transform $f(k) := \mathbb{E}_0^{\bar{Y}}[e^{i\langle k, \bar{Y}(1) \rangle}]$ is real for all $k \in [-\pi, \pi]^d$. The assumption $\mathbb{P}_0^{\bar{Y}}(\bar{Y}(1) = 0) \geq 1/2$ guarantees that $f(k) \in [0, 1]$. If we let $p_n^{\bar{Y}}(y)$ denote the n -step transition probability kernel of \bar{Y} , then by Fourier inversion, we have

$$\begin{aligned} p_n^{\bar{Y}}(0) &\geq p_n^{\bar{Y}}(y), \\ p_n^{\bar{Y}}(0) &\geq p_{n+1}^{\bar{Y}}(0) \end{aligned} \quad \text{for all } n \geq 0, y \in \mathbb{Z}^d. \quad (2.43)$$

If we now regard \bar{X} as a trap, then $\bar{w}^{q, \bar{X}}(n, y)$ can be interpreted as the probability that a random walk \bar{Y} starting from y gets trapped by \bar{X} by time n , where each time \bar{Y} and \bar{X} coincide, \bar{Y} is trapped by \bar{X} with probability q . More precisely, let $Z_i, i \in \mathbb{N}_0$, be i.i.d. Bernoulli random variables with mean q , where $Z_i = 1$ means that the trap at $(i, \bar{X}(i))$ is open. Then \bar{X} is killed at the stopping time

$$\tau_{\bar{X}}(\bar{Y}) := \min\{i \geq 0 : \bar{Y}(i) = \bar{X}(i), Z_i = 1\}, \quad (2.44)$$

and $\bar{w}^{q, \bar{X}}(n, y) = \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} \leq n)$.

We examine the following auxiliary quantity, where by decomposition with respect to $\tau_{\bar{X}}$, we have

$$\begin{aligned} q &= \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\bar{Y}(n) = \bar{X}(n), Z_n = 1) = \sum_{k=0}^{n-1} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} = k) p_{n-k}^{\bar{Y}}(\bar{X}(n) - \bar{X}(k)) q + \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} = n) \\ &\leq q \sum_{k=0}^{n-1} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} = k) p_{n-k}^{\bar{Y}}(0) + \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} = n), \end{aligned} \quad (2.45)$$

where in the inequality we used (2.43). Similarly, when \bar{X} is replaced by $\bar{X} \equiv 0$, we have

$$q = \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} \left(\bar{Y}(n) = 0, Z_n = 1 \right) = q \sum_{k=0}^{n-1} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} (\tau_0 = k) p_{n-k}^{\bar{Y}}(0) + \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} (\tau_0 = n). \quad (2.46)$$

Denote

$$\begin{aligned} S_n^{\bar{X}} &:= \sum_{y \in \mathbb{Z}^d} \bar{w}^{q, \bar{X}}(n, y) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} (\tau_{\bar{X}} \leq n), \\ S_n^0 &:= \sum_{y \in \mathbb{Z}^d} \bar{w}^{q, 0}(n, y) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} (\tau_0 \leq n). \end{aligned} \quad (2.47)$$

Note that $S_0^{\bar{X}} = S_0^0 = q$, and $\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} (\tau_{\bar{X}} = k) = S_k^{\bar{X}} - S_{k-1}^{\bar{X}}$, $\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}} (\tau_0 = k) = S_k^0 - S_{k-1}^0$, where we set $S_{-1}^{\bar{X}} = S_{-1}^0 = 0$. Together with (2.45) and (2.46), this gives

$$q \sum_{k=0}^{n-1} p_{n-k}^{\bar{Y}}(0) (S_k^0 - S_{k-1}^0) + S_n^0 - S_{n-1}^0 \leq q \sum_{k=0}^{n-1} p_{n-k}^{\bar{Y}}(0) (S_k^{\bar{X}} - S_{k-1}^{\bar{X}}) + S_n^{\bar{X}} - S_{n-1}^{\bar{X}}.$$

Rearranging terms, we obtain

$$S_n^{\bar{X}} - S_n^0 \geq (1 - qp_1^{\bar{Y}}(0)) (S_{n-1}^{\bar{X}} - S_{n-1}^0) + q \sum_{k=0}^{n-2} \left(p_{n-k-1}^{\bar{Y}}(0) - p_{n-k}^{\bar{Y}}(0) \right) (S_k^{\bar{X}} - S_k^0). \quad (2.48)$$

This sets up an induction bound for $S_n^{\bar{X}} - S_n^0$. Since $S_0^{\bar{X}} - S_0^0 = 0$, $1 - qp_1^{\bar{Y}}(0) \geq 0$, and $p_k^{\bar{Y}}(0)$ is decreasing in k by (2.43), it follows that $S_n^{\bar{X}} \geq S_n^0$ for all $n \in \mathbb{N}_0$, which is precisely (2.42). \blacksquare

Proof of Proposition 2.1. Integrating out ξ on both sides of (2.38) as in (2.11) shows that (2.38) is equivalent to

$$\sum_{y \in \mathbb{Z}^d} w^{\gamma, X}(t, y) \geq \sum_{y \in \mathbb{Z}^d} w^{\gamma, 0}(t, y), \quad (2.49)$$

where

$$w^{\gamma, X}(t, y) := 1 - \mathbb{E}_y^Y \left[\exp \left\{ -\gamma \int_0^t \delta_0(Y(s) - X(s)) ds \right\} \right]. \quad (2.50)$$

For $n \in \mathbb{N}$, let $Y^{(n)}(k) = Y(\frac{kt}{n})$ and $X^{(n)}(k) = X(\frac{t}{n})$ for $k \in \mathbb{N}_0$. Clearly $Y^{(n)}$ is symmetric, and for n sufficiently large, $\mathbb{P}_0^{Y^{(n)}}(Y^{(n)}(1) = 0) \geq 1/2$. Therefore we can apply Lemma 2.2 with $\bar{Y} = Y^{(n)}$, $\bar{X} = X^{(n)}$ and $q = q^{(n)} = \gamma t/n$ to obtain

$$\sum_{y \in \mathbb{Z}^d} \bar{w}^{\gamma t/n, X^{(n)}}(n, y) \geq \sum_{y \in \mathbb{Z}^d} \bar{w}^{\gamma t/n, 0}(n, y). \quad (2.51)$$

By (2.41) and the definition of $Y^{(n)}$ and $X^{(n)}$, we have

$$\bar{w}^{\gamma t/n, X^{(n)}}(n, y) = 1 - \mathbb{E}_y^{Y^{(n)}} \left[\left(1 - \frac{\gamma t}{n} \right)^{\sum_{k=0}^n 1_{\{Y^{(n)}(k) = X^{(n)}(k)\}}} \right] = 1 - \mathbb{E}_y^Y \left[\left(1 - \frac{\gamma t}{n} \right)^{\sum_{k=0}^n 1_{\{Y(kt/n) = X(kt/n)\}}} \right].$$

By the assumption that X is a random walk path which is necessarily piecewise constant with a finite number of discontinuities, for a.s. all realization of Y , we have

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\gamma t}{n} \right)^{\sum_{k=0}^n 1_{\{Y(kt/n) = X(kt/n)\}}} = \exp \left\{ -\gamma \int_0^t \delta_0(Y(s) - X(s)) ds \right\}.$$

Therefore by the bounded convergence theorem, $\lim_{n \rightarrow \infty} \bar{w}^{\gamma t/n, X^{(n)}}(n, y) = w^{\gamma, X}(t, y)$. By the same argument, $\lim_{n \rightarrow \infty} \bar{w}^{\gamma t/n, 0}(n, y) = w^{\gamma, 0}(t, y)$. Next we note that $w^{\gamma t/n, X^{(n)}}(n, y)$ is the probability that $Y^{(n)}$ is trapped by $X^{(n)}$ before time n . Since $Y^{(n)}$ and $X^{(n)}$ are embedded in Y and X , we

have $w^{\gamma t/n, X^{(n)}}(n, y) \leq \mathbb{P}_y^Y(\tau_X \leq t)$ uniformly in n , where $\tau_X = \inf\{s \geq 0 : Y(s) = X(s)\}$. Clearly $\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^Y(\tau_X \leq t) < \infty$. Similarly $w^{\gamma t/n, 0}(n, y) \leq \mathbb{P}_y^Y(\tau_0 \leq t)$ uniformly in n and $\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^Y(\tau_0 \leq t) < \infty$. Therefore we can send $n \rightarrow \infty$ and apply the dominated convergence theorem in (2.51), from which (2.49) then follows. \blacksquare

The Pascal principle in Lemma 2.2 and Proposition 2.1 have the following interesting consequence for the range of a symmetric random walk, which we denote by $R_t(X) = \{y \in \mathbb{Z}^d : X(s) = y \text{ for some } 0 \leq s \leq t\}$.

Corollary 2.1 [Increase of expected cardinality of range under perturbation]

Let \bar{Y} and \bar{X} be discrete time random walks as in Lemma 2.2. Let Y be a continuous time irreducible symmetric random walk on \mathbb{Z}^d with jump rate $\rho > 0$, and let $X : [0, t] \rightarrow \mathbb{Z}^d$ be piecewise constant with a finite number of discontinuities. Then for all $n \in \mathbb{N}_0$, respectively $t \geq 0$, we have

$$\begin{aligned} \mathbb{E}_0^{\bar{Y}} [|R_n(\bar{Y} - \bar{X})|] &\geq \mathbb{E}_0^{\bar{Y}} [|R_n(Y)|], \\ \mathbb{E}_0^Y [|R_t(Y - X)|] &\geq \mathbb{E}_0^Y [|R_t(Y)|], \end{aligned} \tag{2.52}$$

where $|\cdot|$ denotes the cardinality of the set.

Proof. The first inequality in (2.52) for discrete time random walks follows from the observation that

$$\begin{aligned} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} \leq n) &= \sum_{y \in \mathbb{Z}^d} \mathbb{P}_0^{\bar{Y}}(\bar{Y}(i) - \bar{X}(i) = y \text{ for some } 0 \leq i \leq n) = \mathbb{E}_0^{\bar{Y}} [|R_n(\bar{Y} - \bar{X})|], \\ \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_0 \leq n) &= \sum_{y \in \mathbb{Z}^d} \mathbb{P}_0^{\bar{Y}}(\bar{Y}(i) = y \text{ for some } 0 \leq i \leq n) = \mathbb{E}_0^{\bar{Y}} [|R_n(\bar{Y})|], \end{aligned} \tag{2.53}$$

where $\tau_{\bar{X}} = \min\{i \geq 0 : \bar{Y}_i = \bar{X}_i\}$ and $\tau_0 = \min\{i \geq 0 : \bar{Y}_i = 0\}$, which combined with Lemma 2.2 for $q = 1$ gives precisely

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_{\bar{X}} \leq n) \geq \sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^{\bar{Y}}(\tau_0 \leq n). \tag{2.54}$$

The continuous time case follows by similar considerations, where we apply Proposition 2.1 with $\gamma = \infty$, or rather $\gamma > 0$ with $\gamma \uparrow \infty$. \blacksquare

3 Quenched and semi-annealed upper bounds

In this section, we prove sub-exponential upper bounds on the quenched survival probability in dimensions 1 and 2 (the exponential upper bound in dimensions 3 and higher follows trivially from the annealed upper bound by Jensen's inequality and Borel-Cantelli). Although they will be superseded later by a proof of exponential decay using difficult results of Kesten and Sidoravicius [KS05], the proof we present here is relatively simple and self-contained. Along the way, we will also prove an upper bound (Proposition 3.2) on the annealed survival probability of a random walk in a random field of traps ξ with deterministic initial condition, which we call a semi-annealed bound.

Proposition 3.1 [Sub-exponential upper bound on $Z_{t,\xi}^\gamma$] *There exist constants $C_1, C_2 > 0$ depending on $\gamma, \kappa, \rho, \nu > 0$ such that a.s. with respect to ξ , we have*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log t}{t} \log Z_{t,\xi}^\gamma &\leq -C_1, & d = 1, \\ \limsup_{t \rightarrow \infty} \frac{\log \log t}{t} \log Z_{t,\xi}^\gamma &\leq -C_2, & d = 2. \end{aligned} \tag{3.55}$$

The same bounds hold if we replace $Z_{t,\xi}^\gamma$ by $u(t, 0)$ as in Theorem 1.3.

Proof. The proof is based on coarse graining combined with the annealed bound in Theorem 1.1. Let us focus on dimension $d = 1$ first. Let X be a random walk as in (1.2), and let $M(t) := \sup_{0 \leq s \leq t} |X(s)|_\infty$. The first step is to note that by basic large deviation estimates for X ,

$$\mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{M_t \geq t\}} \right] \leq \mathbb{P}_0^X(M_t \geq t) \leq e^{-Ct}$$

for some $C > 0$ depending only on κ . Therefore to show (3.55), it suffices to prove that

$$\mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{M_t < t\}} \right] \leq e^{-\frac{Ct}{\log t}} \quad (3.56)$$

for some $C > 0$ for all t sufficiently large. Since the integrand in the definition of $Z_{t,\xi}^\gamma$ is monotone in t , we may even restrict our attention to $t \in \mathbb{N}$.

The second step is to introduce a coarse graining scale $L_t := A \log t$ for some $A > 0$, and partition the space-time region $[-2t, 2t] \times [0, t]$ into blocks of the form $\Lambda_{i,k} := [(i-1)L_t, iL_t] \times [(k-1)L_t^2, kL_t^2]$ for $i, k \in \mathbb{Z}$ with $-\frac{2t}{L_t} + 1 \leq i \leq \frac{2t}{L_t}$ and $1 \leq k \leq \frac{t}{L_t^2}$. We say a block

$$\Lambda_{i,k} \text{ is good if } \sum_{(i-1)L_t \leq x < iL_t} \xi((k-1)L_t^2, x) \geq \frac{\nu L_t}{2}.$$

Since for each $s \geq 0$, $(\xi(s, x))_{x \in \mathbb{Z}}$ are i.i.d. Poisson distributed with mean ν , by basic large deviation estimates for Poisson random variables, there exists $C > 0$ such that for all $t > 1$,

$$\mathbb{P}(\Lambda_{i,k} \text{ is bad}) \leq e^{-C\nu L_t}.$$

Let $G_t(\xi)$ be the event that all the blocks $\Lambda_{i,k}$ in $[-2t, 2t] \times [0, t]$ are good. Then

$$\mathbb{P}(G_t^c(\xi)) \leq \frac{4t^2}{L_t^3} e^{-C\nu L_t} = \frac{4}{A^3 (\log t)^{3tC\nu A - 2}},$$

which is summable in $t \geq 2$, $t \in \mathbb{N}$, if A is chosen sufficiently large. Therefore by Borel-Cantelli, a.s. with respect to ξ , for all $t \in \mathbb{N}$ sufficiently large, the event $G_t(\xi)$ occurs. To prove (3.55), it then suffices to prove

$$1_{G_t(\xi)} \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{M_t < t\}} \right] \leq e^{-\frac{Ct}{\log t}} \quad (3.57)$$

almost surely for all $t \in \mathbb{N}$ sufficiently large.

The third step is applying an annealing bound. More precisely, to show (3.57), it suffices to average over ξ and show that

$$\mathbb{E}^\xi \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{M_t < t\}} 1_{G_t(\xi)} \right] \leq e^{-\frac{2Ct}{\log t}} \quad (3.58)$$

for some $C > 0$ for all $t \in \mathbb{N}$ sufficiently large. Indeed, (3.58) implies that

$$\mathbb{P}^\xi \left(1_{G_t(\xi)} \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{M_t < t\}} \right] > e^{-\frac{Ct}{\log t}} \right) \leq e^{-\frac{Ct}{\log t}},$$

from which (3.57) then follows by Borel-Cantelli.

To prove (3.58), let us denote $Z_k := \exp \left\{ -\gamma \int_{(k-1)L_t^2}^{kL_t^2} \xi(s, X(s)) ds \right\}$, and let \mathcal{F}_k be the σ -field generated by $(X_s, \xi(s, \cdot))_{0 \leq s \leq kL_t^2}$. Replacing L_t^2 by $t/\lceil t/L_t^2 \rceil$ if necessary, we may assume without loss of generality that $t/L_t^2 = t/(A \log t)^2 \in \mathbb{N}$. Then

$$\begin{aligned} & \mathbb{E}^\xi \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{M_t < t\}} 1_{G_t(\xi)} \right] = \mathbb{E}^\xi \mathbb{E}_0^X \left[1_{\{M_t < t\}} 1_{G_t(\xi)} \prod_{k=1}^{t/L_t^2} Z_k \right] \\ &= \mathbb{E}^\xi \mathbb{E}_0^X \left[1_{\{M_t < t\}} 1_{G_t(\xi)} \prod_{k=1}^{t/L_t^2} \frac{Z_k}{\mathbb{E}^\xi \mathbb{E}_0^X [Z_k | \mathcal{F}_{k-1}]} \prod_{k=1}^{t/L_t^2} \mathbb{E}^\xi \mathbb{E}_0^X [Z_k | \mathcal{F}_{k-1}] \right]. \end{aligned} \quad (3.59)$$

By Proposition 3.2 below, on the event $|X((k-1)L_t^2)|_\infty < t$ and $\Lambda_{i,k}$ is good for all $-\frac{2t}{L_t} + 1 \leq i \leq \frac{2t}{L_t}$, which is an event in \mathcal{F}_{k-1} , we have

$$\mathbb{E}^\xi \mathbb{E}_0^X [Z_k | \mathcal{F}_{k-1}] = \mathbb{E}^\xi \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_{(k-1)L_t^2}^{kL_t^2} \xi(s, X(s)) ds \right\} \middle| \mathcal{F}_{k-1} \right] \leq e^{-CL_t} \quad (3.60)$$

for some $C > 0$ depending on $\gamma, \kappa, \rho, \nu$. substituting this bound into (3.59) for $1 \leq k \leq t/L_t^2$ and using the fact that $\prod_{k=1}^{t/L_t^2} \frac{Z_k}{\mathbb{E}^\xi \mathbb{E}_0^X [Z_k | \mathcal{F}_{k-1}]}$ is a martingale then gives the desired bound $e^{-Ct/L_t} = e^{-Ct/(A \log t)}$ for (3.58).

For dimension $d = 2$, the proof is similar. We choose $L_t = A \log t$ with A sufficiently large. We partition the space-time region $[-2t, 2t]^2 \times [0, t]$ into blocks of the form $\Lambda_{i,j,k} := [(i-1)L_t, iL_t) \times [(j-1)L_t, jL_t) \times [(k-1)L_t^2, kL_t^2)$, and we define good blocks and bad blocks as before. Applying Proposition 3.2 below then gives an upper bound of $\exp \left\{ -C \frac{t}{L_t^2} \frac{L_t^2}{\log L_t} \right\} = \exp \left\{ -\frac{Ct}{\log A + \log \log t} \right\}$, analogous to (3.58).

Lastly we note that the arguments also apply to the solution of the parabolic Anderson model

$$u(t, 0) = \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(t-s, X(s)) ds \right\} \right].$$

The only difference lies in passing the result (3.55) from $t \in \mathbb{N}$ to $t \in \mathbb{R}$, due to the lack of monotonicity of $u(t, 0)$ in t . This can be easily overcome by the observation that for $n-1 < t < n$ with $n \in \mathbb{N}$,

$$u(n, 0) \geq e^{-\kappa(n-t)} e^{-\gamma \int_t^n \xi(r, 0) dr} u(t, 0),$$

and the fact that almost surely $\int_i^{i+1} \xi(r, 0) dr \leq \sqrt{i}$ for all i large by Borel-Cantelli because $\int_0^1 \xi(r, 0) dr$ has finite exponential moments. \blacksquare

The following is a partial analogue of Theorem 1.1 for ξ with deterministic initial conditions.

Proposition 3.2 [Semi-annealed upper bound] *Let ξ be defined as in (1.1) with deterministic initial condition $(\xi(0, x))_{x \in \mathbb{Z}^d}$. For $L > 0$ and $\vec{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$, let $B_{L, \vec{i}} := [(i_1-1)L, i_1L) \times \dots \times [(i_d-1)L, i_dL)$. Assume that there exist $a > 2$ and $\nu > 0$ such that for all $\vec{i} \in [-3L^a, 3L^a]^d$, $\sum_{x \in B_{L, \vec{i}}} \xi(0, x) \geq \nu L^d$. Then there exist constants $C_d > 0$, $d \geq 1$, such that for all L sufficiently large and for all $x \in \mathbb{Z}^d$ with $|x|_\infty \leq L$, we have*

$$\mathbb{E}^\xi \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2} \xi(s, X(s)) ds \right\} \right] \leq \begin{cases} e^{-C_1 L}, & d = 1, \\ e^{-C_2 \frac{L^2}{\log L}}, & d = 2, \\ e^{-C_d L^2}, & d \geq 3. \end{cases} \quad (3.61)$$

The same is true if we replace $\int_0^{L^2} \xi(s, X(s)) ds$ by $\int_0^{L^2} \xi(s, X(L^2 - s)) ds$.

Proof. The basic strategy is to dominate $(\xi(L^2/2, x))_{|x|_\infty < 2L^a}$ from below by i.i.d. Poisson random variables, which then allows us to apply Theorem 1.1. We proceed as follows.

Let ξ be generated by independent random walks $(Y_j^y)_{y \in \mathbb{Z}^d, 1 \leq j \leq \xi(0, y)}$ as in (1.1), and let $\bar{\xi}$ be generated by a separate system of independent random walks $(\bar{Y}_j^y)_{y \in \mathbb{Z}^d, 1 \leq j \leq \bar{\xi}(0, y)}$, where $(\bar{\xi}(0, y))_{y \in \mathbb{Z}^d}$ are i.i.d. Poisson distributed with mean $\bar{\nu}$. Choose any $\bar{\nu} \in (0, \nu)$. Then by large deviation estimates for Poisson random variables,

$$\begin{aligned} \mathbb{P}^{\bar{\xi}}(G_L^c) &:= \mathbb{P}^{\bar{\xi}} \left(\sum_{x \in B_{L, \vec{i}}} \bar{\xi}(0, x) \geq \sum_{x \in B_{L, \vec{i}}} \xi(0, x) \text{ for some } \vec{i} \in [-3L^a, 3L^a]^d \right) \\ &\leq \sum_{\vec{i} \in [-3L^a, 3L^a]^d} \mathbb{P}^{\bar{\xi}} \left(\sum_{x \in B_{L, \vec{i}}} \bar{\xi}(0, x) \geq \nu L^d \right) \leq 6^d L^{ad} e^{-C_{\nu, \bar{\nu}} L^d}. \end{aligned} \quad (3.62)$$

On the event G_L , we will construct a coupling between $(Y_j^y)_{y \in \mathbb{Z}^d, 1 \leq j \leq \xi(0, y)}$ and $(\bar{Y}_j^y)_{y \in \mathbb{Z}^d, 1 \leq j \leq \bar{\xi}(0, y)}$ as follows. For each walk \bar{Y}_j^y with $1 \leq j \leq \bar{\xi}(0, y)$ and $y \in B_{L, \vec{i}}$ for some $\vec{i} \in [-3L^a, 3L^a]^d$, we can match \bar{Y}_j^y with a distinct walk Y_k^z for some $z \in B_{L, \vec{i}}$ and $1 \leq k \leq \xi(0, z)$, which is possible on the event G_L .

Independently for each pair of walks (\bar{Y}_j^y, Y_k^z) , we will couple their coordinates as follows: For $1 \leq i \leq d$, the i -th coordinates of the two walks evolve independently until the first time that their difference is of even parity. Note that this is the case either at time 0 already or at the first time when one of the coordinates changes. From then on the i -th coordinates are coupled in such a way that they always jump at the same time and their jumps are always opposites until the first time when the two coordinates coincide. From that time onward the two coordinates always perform the same jumps at the same time. For walks in the ξ and $\bar{\xi}$ system which have not been paired up, we let them evolve independently. Note that such a coupling preserves the law of ξ (resp. $\bar{\xi}$), and each coupled pair (\bar{Y}_j^y, Y_k^z) is successfully coupled in the sense that $\bar{Y}_j^y(L^2/2) = Y_k^z(L^2/2)$ if the trajectory of \bar{Y}_j^y is in the event

$$E_j^y := \left\{ \sup_{0 \leq t \leq L^2/2} (\bar{Y}_j^y(t) - \bar{Y}_j^y(0))_i \in [L/2, L] \text{ and } \inf_{0 \leq t \leq L^2/2} (\bar{Y}_j^y(t) - \bar{Y}_j^y(0))_i \in [-L, -L/2] \quad \forall 1 \leq i \leq d \right\},$$

because $|y - z|_\infty \leq L$ by our choice of pairing of \bar{Y}_j^y and Y_k^z . Then by our coupling of $\bar{\xi}$ and ξ , on the event G_L , we have

$$\xi(L^2/2, x) \geq \zeta(x) := \sum_{\substack{y \in \mathbb{Z}^d, \\ 1 \leq j \leq \xi(0, y)}} 1_{E_j^y} 1_{\{\bar{Y}_j^y(L^2/2) = x\}} \quad \text{for all } |x|_\infty \leq 2L^a. \quad (3.63)$$

Now observe that, because $(\bar{\xi}(0, x))_{x \in \mathbb{Z}^d}$ are i.i.d. Poisson with mean $\bar{\nu}$, and $(\bar{Y}_j^y)_{y \in \mathbb{Z}^d, 1 \leq j \leq \bar{\xi}(0, y)}$ are independent, $(\zeta(x))_{x \in \mathbb{Z}^d}$ are also i.i.d. Poisson distributed with mean $\alpha := \bar{\nu} \mathbb{P}^{\bar{\xi}}(E_1^y) = \bar{\nu} \mathbb{P}^{\bar{\xi}}(E_1^0)$, which is bounded away from 0 uniformly in L by the properties of simple symmetric random walks. This achieves the desired stochastic domination of ξ at time $L^2/2$. Let $\zeta_L(t, \cdot)$ denote the counting field of independent random walks as in (1.1) with initial condition $\zeta_L(0, y) = \zeta(y) 1_{\{|y|_\infty \leq 2L^a\}}$. Then using (3.63), uniformly in $x \in \mathbb{Z}^d$ with $|x|_\infty \leq L$, we have

$$\begin{aligned} & \mathbb{E}^\xi \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2} \xi(s, X(s)) ds \right\} \right] = \mathbb{E}^{\xi, \bar{\xi}} \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2} \xi(s, X(s)) ds \right\} \right] \\ & \leq \mathbb{P}^{\bar{\xi}}(G_L^c) + \mathbb{P}_x^X(|X(L^2/2)|_\infty > L^2) + \sup_{|x|_\infty \leq L^2} \mathbb{E}^{\zeta_L} \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2/2} \zeta_L(s, X(s)) ds \right\} \right] \\ & \leq 6^d L^{ad} e^{-C\nu, \bar{\nu} L^d} + e^{-CL^2} + \sup_{|x|_\infty \leq L^2} \mathbb{E}^{\zeta_L} \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2/2} \zeta_L(s, X(s)) ds \right\} \right]. \end{aligned} \quad (3.64)$$

By the same argument as for (2.11), we have

$$\mathbb{E}^{\zeta_L} \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2/2} \zeta_L(s, X(s)) ds \right\} \right] = \mathbb{E}_x^X \left[\exp \left\{ -\alpha \sum_{|y|_\infty \leq 2L^a} (1 - v_X(L^2/2, y)) \right\} \right], \quad (3.65)$$

where

$$v_X(L^2/2, y) = \mathbb{E}_y^Y \left[\exp \left\{ -\gamma \int_0^{L^2/2} \delta_0(Y(s) - X(s)) ds \right\} \right].$$

To bound (3.65), note that by large deviation estimates,

$$\sup_{|x|_\infty \leq L^2} \mathbb{P}_x^X \left(\sup_{0 \leq s \leq L^2/2} |X(s)|_\infty > 2L^2 \right) \leq e^{-CL^2}. \quad (3.66)$$

On the complementary event $\{\sup_{0 \leq s \leq L^2/2} |X(s)|_\infty \leq 2L^2\}$, we have

$$1 - v_X(L^2/2, y) \leq \mathbb{P}_y^Y(\tau_{2L^2} \leq L^2/2) \leq \mathbb{P}(\mathcal{P}_{L^2/2} \geq |y|_\infty - 2L^2),$$

where $\tau_{2L^2} := \inf\{s \geq 0 : |Y(s)|_\infty \leq 2L^2\}$, and $\mathcal{P}_{L^2/2}$ is a Poisson random variable with mean $\rho L^2/2$, which counts the number of jumps of Y before time $L^2/2$. Therefore for L sufficiently large,

$$\begin{aligned} & \sum_{|y|_\infty > 2L^a} (1 - v_X(L^2/2, y)) \leq \sum_{|y|_\infty > 2L^a} \mathbb{P}(\mathcal{P}_{L^2/2} \geq |y|_\infty - 2L^2) \\ & \leq C \sum_{r=2L^a}^{\infty} \mathbb{P}(\mathcal{P}_{L^2/2} \geq r/2) r^{d-1} \leq C \mathbb{E}[\mathcal{P}_{L^2/2}^k] \sum_{r=2L^a}^{\infty} r^{d-k-1} \\ & \leq C(\rho L^2/2)^k (2L^a)^{d-k} \leq CL^{-(a-2)k+ad} \leq 1, \end{aligned} \quad (3.67)$$

where we have changed the values of the constant C (independent of L) from line to line, and the last inequality holds for all L large if we choose k large enough. Substituting the bounds (3.66)–(3.67) into (3.65) then gives the following bound uniformly for $x \in \mathbb{Z}^d$ with $|x|_\infty \leq L^2$:

$$\begin{aligned} & \mathbb{E}^{\zeta_L} \mathbb{E}_x^X \left[\exp \left\{ -\gamma \int_0^{L^2/2} \zeta_L(s, X(s)) ds \right\} \right] \\ & \leq \mathbb{P}_x^X \left(\sup_{0 \leq s \leq L^2/2} |X(s)|_\infty > 2L^2 \right) + \mathbb{E}_x^X \left[\exp \left\{ -\alpha \sum_{y \in \mathbb{Z}^d} (1 - v_X(L^2/2, y)) + 1 \right\} \right] \\ & \leq e^{-CL^2} + e \mathbb{E}_x^X \left[\exp \left\{ -\alpha \sum_{y \in \mathbb{Z}^d} (1 - v_X(L^2/2, y)) \right\} \right], \end{aligned}$$

where by the representation (2.11), the expectation is precisely the annealed survival probability of a random walk among a Poisson field of traps with density α , for which the bounds in (1.4) apply with ν replaced by α and t by $L^2/2$. Substituting this bound back into (3.64) then gives (3.61). The same proof applies when we reverse the time direction of X in (3.61). \blacksquare

4 Existence and positivity of the quenched Lyapunov exponent

In this section, we prove Theorems 1.2 and 1.3. In Section 4.1, we state a shape theorem which implies the existence of the quenched Lyapunov exponent for the quenched survival probability $Z_{t,\xi}^\gamma$. In Section 4.2, we prove the stated shape theorem for bounded ergodic random fields. In Section 4.3, we show how to deduce the existence of the quenched Lyapunov exponent for the solution of the parabolic Anderson model from what we already know for the quenched survival probability. Lastly in Section 4.4, we prove the positivity of the quenched Lyapunov exponent, which concludes the proof of Theorems 1.2 and 1.3.

4.1 Shape theorem and the quenched Lyapunov exponent

In this section, we focus exclusively on the quenched survival probability $Z_{t,\xi}^\gamma$. The approach we adopt to proving the existence of the quenched Lyapunov exponent for $Z_{t,\xi}^\gamma$ uses the subadditive ergodic theorem and follows ideas used by Varadhan in [V03] to prove the quenched large deviation principle for random walks in random environments.

For $s \geq 0$ and $x \in \mathbb{Z}^d$, let $\mathbb{P}_{x,s}^X$ and $\mathbb{E}_{x,s}^X$ denote respectively probability and expectation for a jump rate κ simple symmetric random walk X , starting from x at time s . For each $0 \leq s < t$ and $x, y \in \mathbb{Z}^d$, define

$$\begin{aligned} e(s, t, x, y, \xi) &:= \mathbb{E}_{x,s}^X \left[\exp \left\{ -\gamma \int_s^t \xi(u, X(u)) du \right\} 1_{\{X(t)=y\}} \right], \\ a(s, t, x, y, \xi) &:= -\log e(s, t, x, y, \xi). \end{aligned} \quad (4.68)$$

We call $a(s, t, x, y, \xi)$ the *point to point* passage function from x to y between times s and t . We will prove the following shape theorem for $a(0, t, 0, y, \xi)$.

Theorem 4.1 [Shape theorem] *There exists a deterministic convex function $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$, which we call the shape function, such that \mathbb{P}^ξ -a.s., for any compact $K \subset \mathbb{R}^d$,*

$$\lim_{t \rightarrow \infty} \sup_{y \in tK \cap \mathbb{Z}^d} |t^{-1}a(0, t, 0, y, \xi) - \alpha(y/t)| = 0. \quad (4.69)$$

Furthermore, for any $M > 0$, we can find a compact $K \subset \mathbb{R}^d$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{X(t) \notin tK\}} \right] \leq -M. \quad (4.70)$$

Remark. Note that (1.5) in Theorem 1.2 follows easily from Theorem 4.1, which we leave to the reader as an exercise. In particular, the quenched Lyapunov exponent satisfies

$$\tilde{\lambda}_{d, \gamma, \kappa, \rho, \nu} = \inf_{y \in \mathbb{R}^d} \alpha(y) = \alpha(0) = \lim_{t \rightarrow \infty} t^{-1}a(0, t, 0, 0, \xi), \quad (4.71)$$

where $\inf_{y \in \mathbb{R}^d} \alpha(y) = \alpha(0)$ follows from the convexity and symmetry of α , since ξ is symmetric.

The unboundedness of the random field ξ creates complications for the proof of Theorem 4.1. Therefore we first replace ξ by $\xi_N := (\max\{\xi(s, x), N\})_{s \geq 0, x \in \mathbb{Z}^d}$ for some large $N > 0$ and prove a corresponding shape theorem, then use almost sure properties of ξ established by Kesten and Sidoravicius in [KS03] to control the error caused by the truncation.

Theorem 4.2 [Shape theorem for bounded ergodic potentials] *Let $\zeta := (\zeta(s, x))_{s \geq 0, x \in \mathbb{Z}^d}$ be a real-valued random field which is ergodic with respect to the shift map $\theta_{r, z} \zeta := (\zeta(s+r, x+z))_{s \geq 0, x \in \mathbb{Z}^d}$, for all $r \geq 0$ and $z \in \mathbb{Z}^d$. Assume further that $|\zeta(0, 0)| \leq A$ a.s. for some $A > 0$. Then the conclusions of Theorem 4.1 hold with ξ replaced by ζ .*

Remark. Note that Theorem 4.2 can be applied to the occupation field of the exclusion process or the voter model in an ergodic equilibrium, which in particular implies the existence of the corresponding quenched Lyapunov exponents.

Before we prove Theorem 4.2 in the next section, let us first show how to deduce Theorem 4.1 from Theorem 4.2, using almost sure bounds on ξ from [KS03].

Proof of Theorem 4.1. Note that, since ξ is non-negative, (4.70) follows from elementary large deviation estimates for the random walk X , if we take K to be a large enough closed ball centered at the origin, which we fix for the rest of the proof.

By Theorem 4.2 applied to the truncated random field ξ_N , we have that for each $N > 0$, there exists a convex shape function $\alpha_N : \mathbb{R}^d \rightarrow \mathbb{R}$ such that (4.69) holds with ξ replaced by ξ_N and α replaced by α_N . Note that α_N is monotonically increasing in N , and its limit α is necessarily convex. To prove (4.69), it then suffices to show that, for any $\epsilon > 0$, we can choose N sufficiently large such that \mathbb{P}^ξ -a.s.,

$$\frac{1}{t} \sup_{y \in tK \cap \mathbb{Z}^d} |a(0, t, 0, y, \xi) - a(0, t, 0, y, \xi_N)| \leq \epsilon \quad \text{for all } t \text{ sufficiently large.} \quad (4.72)$$

To prove (4.72), we will need Lemma 15 of [KS03], which by Borel-Cantelli implies that there exist positive constants C_0, C_1, C_2, C_3, C_4 with $C_0 > 1$, such that if Ξ_l denotes the space of all possible random walk trajectories $\pi : [0, t] \rightarrow \mathbb{Z}^d$, which contain exactly l jumps and are contained in the rectangle $[-C_1 t \log t, C_1 t \log t]^d$, then \mathbb{P}^ξ -a.s., for all $t \in \mathbb{N}$ sufficiently large, we have

$$\sup_{\pi \in \Xi_l} \int_0^t \xi(s, \pi(s)) 1_{\{\xi(s, \pi(s)) \geq C_2 \nu C_0^{dm}\}} ds \leq (t+l) \sum_{r=m}^{\infty} C_3 C_0^{r(d+6)+d} e^{-C_4 C_0^{r/4}} \quad \forall m \in \mathbb{N}, l \geq 0, \quad (4.73)$$

where $A_m := \sum_{r=m}^{\infty} C_3 C_0^{r(d+6)+d} e^{-C_4 C_0^{r/4}} \rightarrow 0$ as $m \rightarrow \infty$.

One important consequence of (4.73) is that

$$0 < \sup_{y \in K} \alpha(y) < \infty. \quad (4.74)$$

Indeed, if $l_t(X)$ denotes the number of jumps of X on the time interval $[0, t]$, then

$$\begin{aligned} \sup_{y \in K} \alpha(y) &\leq \lim_{t \rightarrow \infty} -t^{-1} \log \inf_{y \in tK \cap \mathbb{Z}^d} \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{X(t)=y\}} \right] \\ &\leq \lim_{t \rightarrow \infty} -t^{-1} \log \inf_{y \in tK \cap \mathbb{Z}^d} \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\} 1_{\{X(t)=y, l_t(X) \leq 2D(K)t, X \in \Xi_{l_t(X)}\}} \right], \end{aligned}$$

where $D(K) := \sup_{y \in K} |y|_1$. We can then apply (4.73) and large deviation estimates for random walks to the above bound to deduce $\sup_{y \in K} \alpha(y) < \infty$. The fact that $\sup_{y \in K} \alpha(y) > 0$ for a large ball K again follows from basic large deviation estimates.

By large deviation estimates, we can find B large enough such that

$$\mathbb{P}_0^X(l_t(X) \geq Bt \text{ or } X \notin \Xi_{l_t(X)}) \leq e^{-2 \sup_{y \in K} \alpha(y) t} \quad \text{for all } t \text{ sufficiently large.} \quad (4.75)$$

Let $N = C_2 \nu C_0^{dm}$. Then by (4.73), \mathbb{P}^ξ -a.s., uniformly in $y \in \mathbb{Z}^d$ and for all t large, we have

$$\begin{aligned} e(0, t, 0, y, \xi) &\geq e^{-(1+B)A_m \gamma t} \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi_N(s, X(s)) ds \right\} 1_{\{X(t)=y, l_t(X) \leq Bt, X \in \Xi_{l_t(X)}\}} \right] \\ &\geq e^{-(1+B)A_m \gamma t} (e(0, t, 0, y, \xi_N) - e^{-2 \sup_{y \in K} \alpha(y) t}), \end{aligned} \quad (4.76)$$

where in the last inequality we applied (4.75). Since $-t^{-1} \log e(t, 0, y, \xi_N) \rightarrow \alpha_N(y/t)$ uniformly for $y \in tK \cap \mathbb{Z}^d$ by Theorem 4.2, and $\sup_{y \in K} \alpha_N(y) \leq \sup_{y \in K} \alpha(y)$, (4.76) implies that \mathbb{P}^ξ -a.s., uniformly in $y \in tK \cap \mathbb{Z}^d$ and for all t large, we have

$$t^{-1} a(0, t, 0, y, \xi) \leq t^{-1} a(0, t, 0, y, \xi_N) + (1+B)A_m \gamma + o(1).$$

Since $a(0, t, 0, y, \xi) \geq a(0, t, 0, y, \xi_N)$, and A_m can be made arbitrarily small by choosing m sufficiently large, (4.72) then follows. \blacksquare

Remark. Theorem 4.1 in fact holds for the catalytic case as well, where we take $\gamma < 0$ in (4.68) and (4.70). This implies the existence of the quenched Lyapunov exponent in Theorem 1.2 for the catalytic case, where we set $\gamma < 0$ in the definition of $Z_{t,\xi}^\gamma$. Indeed, Theorem 4.2 still applies to the truncated field ξ_N . To control the error caused by the truncation, the following modifications are needed in the proof of Theorem 4.1. To prove (4.70), we need to apply (4.73). More precisely, we need to first consider trajectories $(X_x)_{0 \leq s \leq t}$ which are not contained in $[-C_1 t \log t, C_1 t \log t]^d$. The contribution from these trajectories can be shown to decay super-exponentially in t by large deviation estimates and a bound on ξ given in (2.37) of [KS03, Lemma 4]. For X which lies inside $[-C_1 t \log t, C_1 t \log t]^d$, we can then use (4.73) and large deviations to deduce (4.70). In contrast to (4.76), we need to upper bound $e(0, t, 0, y, \xi)$ in terms of $e(0, t, 0, y, \xi_N)$. The proof is essentially the same, except that in place of (4.75), we need to show that we can choose B large enough, such that \mathbb{P}^ξ -a.s.,

$$\sup_{y \in tK \cap \mathbb{Z}^d} \mathbb{E}_0^X \left[\exp \left\{ |\gamma| \int_0^t \xi(s, X(s)) ds \right\} 1_{\{X(t)=y\}} 1_{\{l_t(X) \geq Bt \text{ or } X \notin \Xi_{l_t(X)}\}} \right] \leq \inf_{y \in tK \cap \mathbb{Z}^d} \mathbb{P}_0^X(X_t = y). \quad (4.77)$$

This can be proved by appealing to (4.70), and applying (4.73) and large deviation estimates.

4.2 Proof of shape theorem for bounded ergodic potentials

In this section, we prove Theorem 4.2. From now on, let \mathbb{Q}_+ denote the set of positive rationals, and let \mathbb{Q}^d denote the set of points in \mathbb{R}^d with rational coordinates. We start with the following auxiliary result.

Lemma 4.1 *There exists a deterministic function $\alpha : \mathbb{Q}^d \rightarrow [-\gamma A, \infty)$ such that for every $y \in \mathbb{Q}^d$,*

$$\lim_{\substack{t \rightarrow \infty \\ ty \in \mathbb{Z}^d}} t^{-1} a(0, t, 0, ty, \zeta) = \alpha(y) \quad \mathbb{P}^\zeta - a.s. \quad (4.78)$$

Furthermore, for each $r \in \mathbb{Q}_+$ and $y \in \mathbb{Q}^d$, we have

$$\alpha(ry) = r\alpha(y). \quad (4.79)$$

Proof. Since we assume $y \in \mathbb{Q}^d$ and $ty \in \mathbb{Z}^d$ in (4.78), without loss of generality, it suffices to consider $y \in \mathbb{Z}^d$ and $t \in \mathbb{N}$. Note that by the definition of the passage function a in (4.68), \mathbb{P}^ζ -a.s.,

$$a(t_1, t_3, x_1, x_3, \zeta) \leq a(t_1, t_2, x_1, x_2, \zeta) + a(t_2, t_3, x_2, x_3, \zeta) \quad \forall t_1 < t_2 < t_3, x_1, x_2, x_3 \in \mathbb{Z}^d. \quad (4.80)$$

Together with our assumption on the ergodicity of ζ , this implies that the two-parameter family $a(s, t, sy, ty, \zeta)$, $0 \leq s \leq t$ with $s, t \in \mathbb{Z}$, satisfies the conditions of Kingman's subadditive ergodic theorem (see e.g. [L85]). Therefore, there exists a deterministic constant $\alpha(y)$ such that (4.78) holds. The fact that $\alpha(y) \geq -\gamma A$ follows by bounding ζ from above by the uniform bound A , and $\alpha(y) < \infty$ follows from large deviation bounds for the random walk X .

The scaling relation (4.79) follows from the definition of a , and the fact that for any $y \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$, we can find a sequence $t_n \rightarrow \infty$ with $t_n y, t_n r y \in \mathbb{Z}^d$. ■

To extend the definition of $\alpha(y)$ in Lemma 4.1 to $y \notin \mathbb{Q}^d$ and to prove the uniform convergence in (4.69), we need to establish equicontinuity of $t^{-1} a(0, t, 0, ty, \zeta)$ in y , as $t \rightarrow \infty$. For that, we first need a large deviation estimate for the random walk X .

Lemma 4.2 *Let X be a jump rate κ simple symmetric random walk on \mathbb{Z}^d with $X(0) = 0$. Then for every $t > 0$ and $x \in \mathbb{Z}^d$, we have*

$$\mathbb{P}_0^X(X(t) = x) = \frac{e^{-J(\frac{x}{t})t}}{(2\pi t)^{\frac{d}{2}} \prod_{i=1}^d \left(\frac{x_i^2}{t^2} + \frac{\kappa^2}{d^2}\right)^{1/4}} (1 + o(1)), \quad (4.81)$$

where

$$J(x) := \sum_{i=1}^d \frac{\kappa}{d} j\left(\frac{dx_i}{\kappa}\right) \quad \text{with} \quad j(y) := y \sinh^{-1} y - \sqrt{y^2 + 1} + 1,$$

and the error term $o(1)$ tends to zero as $t \rightarrow \infty$ uniformly in $x \in tK \cap \mathbb{Z}^d$, for any compact $K \subset \mathbb{R}^d$.

Proof. Since the coordinates of X are independent, it suffices to consider the case X is a rate κ/d simple symmetric random walk on \mathbb{Z} . Let $\sigma := t/\lceil t \rceil$. Let $Z_1^\lambda, \dots, Z_{\lceil t \rceil}^\lambda$ be i.i.d. with

$$\mathbb{P}(Z_1^\lambda = y) = \mathbb{P}(X(\sigma) = y) e^{\lambda y - \Phi(\lambda)}, \quad y \in \mathbb{Z},$$

where

$$\Phi(\lambda) = \log \mathbb{E}[e^{\lambda X(\sigma)}] = \frac{\sigma \kappa}{d} (\cosh \lambda - 1).$$

Note that

$$\mathbb{E}[Z_1^\lambda] = \frac{d\Phi}{d\lambda}(\lambda) = \frac{\sigma \kappa}{d} \sinh \lambda \quad \text{and} \quad \text{Var}(Z_1^\lambda) = \frac{d^2\Phi}{d^2\lambda}(\lambda) = \frac{\sigma \kappa}{d} \cosh \lambda.$$

We shall set $\lambda = \sinh^{-1}(\frac{dx}{\kappa t})$ so that $\mathbb{E}[Z_1^\lambda] = x/\lceil t \rceil$. If we let $S_{\lceil t \rceil} := \sum_{i=1}^{\lceil t \rceil} Z_i^\lambda$, then observe that

$$\mathbb{P}_0^X(X(t) = x) = \mathbb{P}(S_{\lceil t \rceil} = x)e^{-\lambda x + \lceil t \rceil \Phi(\lambda)} = \mathbb{P}(S_{\lceil t \rceil} = x)e^{-\frac{\kappa}{d}j(\frac{dx}{\kappa t})t}.$$

Note that $S_{\lceil t \rceil} - x$ has mean 0, variance $t\sqrt{\frac{x^2}{t^2} + \frac{\kappa^2}{d^2}}$, and characteristic function

$$e^{\lceil t \rceil (\Phi(ik+\lambda) - \Phi(\lambda)) - ikx} = e^{ix(\sin k - k) - t\sqrt{\frac{x^2}{t^2} + \frac{\kappa^2}{d^2}}(1 - \cos k)}.$$

Applying Fourier inversion then gives (4.81). ■

With the help of Lemma 4.2, we can control the modulus of continuity of $t^{-1}a(0, t, 0, ty, \zeta)$.

Lemma 4.3 *Let K be any compact subset of \mathbb{R}^d . There exists $\phi_K : (0, \infty) \rightarrow (0, \infty)$ with $\lim_{r \downarrow 0} \phi_K(r) = 0$, such that for any $\epsilon > 0$, \mathbb{P}^ζ -a.s., we have*

$$\limsup_{t \rightarrow \infty} \sup_{\substack{x, y \in tK \cap \mathbb{Z}^d \\ \|x - y\| \leq \epsilon t}} t^{-1} |a(0, t, 0, x, \zeta) - a(0, t, 0, y, \zeta)| \leq \phi_K(\epsilon). \quad (4.82)$$

Proof. Let $K \subset \mathbb{R}^d$ be compact. It suffices to consider $\epsilon \in (0, 1/2)$, which we also fix from now on. First note that, by Lemma 4.2, \mathbb{P}^ζ -a.s.,

$$\inf_{z \in tK \cap \mathbb{Z}^d} e(0, t, 0, z, \zeta) \geq e^{-At} \inf_{z \in tK \cap \mathbb{Z}^d} \mathbb{P}_0^X(X_t = z) \geq e^{-(A+1)t - \sup_{u \in K} J(u)t} \quad (4.83)$$

for all t sufficiently large.

Also note that for all $z \in \mathbb{Z}^d$ and $t > 0$,

$$e(0, t, 0, z, \zeta) = \sum_{w \in \mathbb{Z}^d} e(0, (1 - \epsilon)t, 0, w, \zeta) e((1 - \epsilon)t, t, w, z, \zeta). \quad (4.84)$$

By large deviation estimates, we can choose a ball B_R centered at the origin with radius R large enough and independent of ϵ , such that $K \subset B_R$ and \mathbb{P}^ζ -a.s.,

$$\begin{aligned} & \sup_{z \in \mathbb{Z}^d} \sum_{\substack{w \in \mathbb{Z}^d \\ w \notin tB_R}} e(0, (1 - \epsilon)t, 0, w, \zeta) e((1 - \epsilon)t, t, w, z, \zeta) \\ & \leq \mathbb{P}_0^X(X((1 - \epsilon)t) \notin tB_R) e^{At} \leq e^{-(A+2)t - \sup_{u \in K} J(u)t} \end{aligned}$$

for all t sufficiently large. In view of (4.83), the dominant contribution in (4.84) comes from $w \in tB_R \cap \mathbb{Z}^d$. Therefore to prove (4.82), it suffices to verify

$$\limsup_{t \rightarrow \infty} \sup_{\substack{x, y \in tK \cap \mathbb{Z}^d \\ \|x - y\| \leq \epsilon t}} t^{-1} \left| \log \frac{\sum_{w \in tB_R \cap \mathbb{Z}^d} e(0, (1 - \epsilon)t, 0, w, \zeta) e((1 - \epsilon)t, t, w, y, \zeta)}{\sum_{w \in tB_R \cap \mathbb{Z}^d} e(0, (1 - \epsilon)t, 0, w, \zeta) e((1 - \epsilon)t, t, w, x, \zeta)} \right| \leq \phi_K(\epsilon). \quad (4.85)$$

Note that \mathbb{P}^ζ -a.s., and uniformly in $x, y \in tK \cap \mathbb{Z}^d$ with $\|x - y\| \leq \epsilon t$,

$$\begin{aligned} & \frac{\sum_{w \in tB_R \cap \mathbb{Z}^d} e(0, (1 - \epsilon)t, 0, w, \zeta) e((1 - \epsilon)t, t, w, y, \zeta)}{\sum_{w \in tB_R \cap \mathbb{Z}^d} e(0, (1 - \epsilon)t, 0, w, \zeta) e((1 - \epsilon)t, t, w, x, \zeta)} \\ & \leq \sup_{w \in tB_R \cap \mathbb{Z}^d} \frac{e((1 - \epsilon)t, t, w, y, \zeta)}{e((1 - \epsilon)t, t, w, x, \zeta)} \leq \sup_{w \in tB_R \cap \mathbb{Z}^d} \frac{\mathbb{P}_0^X(X(\epsilon t) = y - w)}{\mathbb{P}_0^X(X(\epsilon t) = x - w)} e^{2A\epsilon t} \\ & \leq \exp \left\{ \epsilon t \sup_{w \in tB_R \cap \mathbb{Z}^d} \left(J\left(\frac{x - w}{\epsilon t}\right) - J\left(\frac{y - w}{\epsilon t}\right) \right) + 3A\epsilon t \right\} \\ & \leq \exp \left\{ \epsilon t \sup_{u, v \in B_{2R/\epsilon}, \|u - v\| \leq 1} |J(u) - J(v)| + 3A\epsilon t \right\} \end{aligned}$$

for all t sufficiently large, where we applied Lemma 4.2, and $B_{2R/\epsilon}$ denotes the ball of radius $2R/\epsilon$, centered at the origin. Therefore (4.85) holds with

$$\phi_K(\epsilon) = 3A\epsilon + \epsilon \sup_{u,v \in B_{2R/\epsilon}, \|u-v\| \leq 1} |J(u) - J(v)|.$$

It only remains to verify that $\phi_K(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$, which is easy to check from the definition of J . \blacksquare

Proof of Theorem 4.2. Because ζ is uniformly bounded, (4.70) follows by large deviation estimates for the number of jumps of X up to time t . Lemma 4.3 implies that for each compact $K \subset \mathbb{R}^d$, the function α in Lemma 4.1 satisfies

$$\sup_{\substack{u,v \in K \cap \mathbb{Q}^d \\ \|u-v\| \leq \epsilon}} |\alpha(u) - \alpha(v)| \leq \phi_K(\epsilon) \quad \text{for all } \epsilon > 0. \quad (4.86)$$

This allows us to extend α to a continuous function on \mathbb{R}^d .

To prove (4.69), it suffices to show that for each $\delta > 0$,

$$\limsup_{t \rightarrow \infty} \sup_{y \in tK \cap \mathbb{Z}^d} |t^{-1}a(0, t, 0, y, \xi) - \alpha(y/t)| \leq \delta. \quad (4.87)$$

We can choose an ϵ such that $\phi_K(\epsilon) < \delta/3$. We can then find a finite number of points $x_1, \dots, x_m \in \mathbb{Q}^d$ which form an ϵ -net in K , and along a subsequence of times of the form $t_n = n\sigma$ with $\sigma x_i \in \mathbb{Z}^d$ for all x_i , we have $t_n^{-1}a(0, t_n, 0, t_n x_i) \rightarrow \alpha(x_i)$ a.s. The uniform control of modulus of continuity provided by Lemma 4.3 and (4.86) then implies (4.87) along t_n . This can be transferred to $t \rightarrow \infty$ along \mathbb{R} using

$$e(0, t, 0, y, \zeta) \geq e(0, s, 0, y, \zeta) e(s, t, y, y, \zeta) \geq e(0, s, 0, y, \zeta) e^{-(\kappa+\gamma A)(t-s)} \quad \text{for } s < t.$$

Lastly, to prove the convexity of α , let $x, y \in \mathbb{R}^d$ and $\beta \in (0, 1)$. Then \mathbb{P}^ζ -a.s., we have

$$a(0, t_n, 0, \beta y_n + (1 - \beta)x_n, \zeta) \leq a(0, \beta t_n, 0, \beta y_n, \zeta) + a(\beta t_n, t_n, \beta y_n, \beta y_n + (1 - \beta)x_n, \zeta),$$

where we take sequences t_n, x_n, y_n with $t_n \rightarrow \infty$, $x_n/t_n \rightarrow x$, $y_n/t_n \rightarrow y$, and $\beta y_n, (1 - \beta)x_n \in \mathbb{Z}^d$. We have shown that the first two terms above converge a.s. to $\alpha(\beta y + (1 - \beta)x)$ and $\alpha(\beta y) = \beta\alpha(y)$ respectively, while the third term converges in probability to $\alpha((1 - \beta)x) = (1 - \beta)\alpha(x)$ by translation invariance. The inequality also holds for the limiting constants, which implies the convexity of α . \blacksquare

4.3 Existence of the quenched Lyapunov exponent for the PAM

Proof of (1.9) in Theorem 1.3. Since $Z_{t,\xi}^\gamma$ is equally distributed with $u(t, 0)$ for each $t \geq 0$, $-t^{-1} \log u(t, 0)$ converges in probability to the quenched Lyapunov exponent $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$. It only remains to verify the almost sure convergence. We will bound the variance of $\log u(t, 0)$, which is the same as that of $\log Z_{t,\xi}^\gamma$, and then apply Borel-Cantelli.

Assume that $t \in \mathbb{N}$. Note that we can write ξ as a sum of i.i.d. random fields $(\xi_i(s, x))_{s \geq 0, x \in \mathbb{Z}^d}$, each of which is defined from a Poisson system of independent random walks with density ν/t , in the same way as ξ . Then we can perform a martingale decomposition and write

$$\log Z_{t,\xi}^\gamma - \mathbb{E}^\xi[\log Z_{t,\xi}^\gamma] = \sum_{i=1}^t V_i := \sum_{i=1}^t \left(\mathbb{E}^\xi[\log Z_{t,\xi}^\gamma | \xi_1, \dots, \xi_i] - \mathbb{E}^\xi[\log Z_{t,\xi}^\gamma | \xi_1, \dots, \xi_{i-1}] \right),$$

and hence $\text{Var}(\log Z_{t,\xi}^\gamma) = \sum_{i=1}^t \mathbb{E}^\xi[V_i^2]$.

For each $1 \leq i \leq t$, we have

$$\begin{aligned}
V_i &= \mathbb{E}^{\xi_{i+1}, \dots, \xi_t} [\log Z_{t,\xi}^\gamma - \mathbb{E}^{\xi_i} [\log Z_{t,\xi}^\gamma]] \\
&= \mathbb{E}^{\xi_{i+1}, \dots, \xi_t} \mathbb{E}^{\xi_i'} \left[\log \frac{\mathbb{E}_0^X [e^{-\gamma \int_0^t (\sum_{1 \leq j \leq t, j \neq i} \xi_j(s, X(s)) + \xi_i(s, X(s))) ds}]}{\mathbb{E}_0^X [e^{-\gamma \int_0^t (\sum_{1 \leq j \leq t, j \neq i} \xi_j(s, X(s)) + \xi_i'(s, X(s))) ds}]} \right] \\
&= \mathbb{E}^{\xi_{i+1}, \dots, \xi_t} \mathbb{E}^{\xi_i'} \left[\log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i(s, X(s)) ds}] - \log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i'(s, X(s)) ds}] \right],
\end{aligned}$$

where ξ_i' denotes an independent copy of ξ_i , and $\mathbb{E}^{X,i}$ denotes expectation with respect to the Gibbs transform of the random walk path measure \mathbb{P}_0^X , with Gibbs weight $e^{-\gamma \int_0^t \sum_{1 \leq j \leq t, j \neq i} \xi_j(s, X(s)) ds}$. Then by Jensen's inequality,

$$\begin{aligned}
\mathbb{E}^\xi [V_i^2] &\leq \mathbb{E}^{\xi, \xi_i'} \left[\left(\log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i(s, X(s)) ds}] - \log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i'(s, X(s)) ds}] \right)^2 \right] \\
&\leq 2\mathbb{E}^{\xi, \xi_i'} \left[\left(\log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i(s, X(s)) ds}] \right)^2 \right] + 2\mathbb{E}^{\xi, \xi_i'} \left[\left(\log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i'(s, X(s)) ds}] \right)^2 \right] \\
&= 4\mathbb{E}^\xi \left[\left(\log \mathbb{E}^{X,i} [e^{-\gamma \int_0^t \xi_i(s, X(s)) ds}] \right)^2 \right] \\
&\leq 4\mathbb{E}^\xi \left[\left(\mathbb{E}^{X,i} \left[\gamma \int_0^t \xi_i(s, X(s)) ds \right] \right)^2 \right] \\
&\leq 4\gamma^2 \mathbb{E}^\xi \mathbb{E}^{X,i} \left[\left(\int_0^t \xi_i(s, X(s)) ds \right)^2 \right] = 4\gamma^2 \mathbb{E}^\xi \mathbb{E}^{X,i} \mathbb{E}^{\xi_i} \left[\left(\int_0^t \xi_i(s, X(s)) ds \right)^2 \right], \quad (4.88)
\end{aligned}$$

where in the third line we used the exchangeability of $\{\xi_i, \xi_i'\}$, and in the fourth line we applied Jensen's inequality¹ to the non-negative convex function $-\log x$ on the interval $(0, 1]$.

Note that for any realization of $((X(s))_{0 \leq s \leq t})$, we have

$$\begin{aligned}
\mathbb{E}^{\xi_i} \left[\left(\int_0^t \xi_i(s, X(s)) ds \right)^2 \right] &= 2 \iint_{0 < u < v < t} \mathbb{E}^{\xi_i} [\xi_i(u, X(u)) \xi_i(v, X(v))] du dv \\
&= 2 \iint_{0 < u < v < t} \left(\frac{\nu^2}{t^2} \sum_{\substack{y \in \mathbb{Z}^d \\ y \neq X(u)}} \mathbb{P}_{y,u}^Y(Y(v) = X(v)) + \left(\frac{\nu^2}{t^2} + \frac{\nu}{t} \right) \mathbb{P}_{X(u),u}^Y(Y(v) = X(v)) \right) du dv \\
&\leq 2\nu^2 + 2\nu \int_0^t \mathbb{P}_{0,0}^Y(Y(s) = 0) ds,
\end{aligned}$$

where $\mathbb{P}_{y,s}^Y$ denotes probability for a simple symmetric random walk on \mathbb{Z}^d with jump rate ρ , starting from y at time s , and in the last line we used that $\mathbb{P}_{0,0}^Y(Y(s) = y)$ is maximized at $y = 0$ for all $s \geq 0$. Combined with the previous bounds, we obtain

$$\text{Var}(\log u(t, 0)) = \text{Var}(\log Z_{t,\xi}^\gamma) = \sum_{i=1}^t \mathbb{E}^\xi [V_i^2] \leq 8\gamma^2 \nu^2 t + 8\gamma^2 \nu t \int_0^t \mathbb{P}_{0,0}^Y(Y(s) = 0) ds \leq Ct^{\frac{3}{2}}$$

for some $C > 0$, since $\int_0^t \mathbb{P}_{0,0}^Y(Y(s) = 0) ds$ is of order \sqrt{t} in dimension $d = 1$, of order $\log t$ in $d = 2$, and converges in $d \geq 3$. Therefore for any $\epsilon > 0$,

$$\mathbb{P}^\xi (|\log u(t, 0) - \mathbb{E}^\xi [\log u(t, 0)]| \geq \epsilon t) \leq \frac{C}{\epsilon^2 \sqrt{t}},$$

which by Borel-Cantelli implies that along the sequence $t_n = n^3$, $n \in \mathbb{N}$, we have almost sure convergence of $-t^{-1} \log u(t, 0)$ to the quenched Lyapunov exponent $\tilde{\lambda}_{d,\gamma,\kappa,\rho,\nu}$.

¹Note that this is where the proof fails for the $\gamma < 0$ case.

To extend the almost sure convergence to $t \rightarrow \infty$ along \mathbb{R} , consider $t \in [t_n, t_{n+1})$ for some $n \in \mathbb{N}$. As at the end of the proof of Proposition 3.1, we have

$$\begin{aligned} u(t, 0) &\geq e^{-\kappa(t-t_n)} e^{-\gamma \int_{t_n}^t \xi(s, 0) ds} u(t_n, 0), \\ u(t, 0) &\leq e^{\kappa(t_{n+1}-t)} e^{\gamma \int_t^{t_{n+1}} \xi(s, 0) ds} u(t_{n+1}, 0). \end{aligned}$$

Note that $(t_{n+1} - t_n)/t_n \rightarrow 0$ as $n \rightarrow \infty$, and we claim that also $t_n^{-1} \int_{t_n}^{t_{n+1}} \xi(s, 0) ds \rightarrow 0$ a.s. as $n \rightarrow \infty$, which then implies the desired almost sure convergence of $t^{-1} \log u(t, 0)$ as $t \rightarrow \infty$ along \mathbb{R} . Indeed, since $\int_0^1 \xi(s, 0) ds$ has finite exponential moments, as can be seen from (2.15) applied to the case $\gamma < 0$ and $X \equiv 0$, we have exponential tail bounds on $\int_0^1 \xi(s, 0) ds$, which by Borel-Cantelli implies that a.s. $\sup_{0 \leq i < m} \int_i^{i+1} \xi(s, 0) ds \leq \log m$ for all $m \in \mathbb{N}$ sufficiently large. The above claim then follows. \blacksquare

4.4 Positivity of the quenched Lyapunov exponent

In this section, we conclude the proof of Theorems 1.2 and 1.3 by showing that the quenched Lyapunov exponent $\tilde{\lambda}_{d, \gamma, \kappa, \rho, \nu}$ is positive in all dimensions. The strategy is as follows: Employing a result of Kesten and Sidoravicius [KS05, Prop. 8], we deduce that \mathbb{P}^ξ -a.s. for eventually all integer time points t , sufficiently many X paths encounter a ξ -particle close-by for of order t many integer time points. Using the Markov property, we then show that with positive \mathbb{P}_0^X probability, X moves to a close-by ξ -particle (which itself stays at its site for some time) within a very short time interval and collects some local time with this ξ particle. This then implies the desired exponential decay.

Proof of Theorems 1.2 and 1.3. Since we have shown the quenched Lyapunov exponent $\tilde{\lambda}_{d, \gamma, \kappa, \rho, \nu}$ in Theorems 1.2 and 1.3 to be the same, it suffices to consider only Theorem 1.2. Note that the upper bound on $\tilde{\lambda}_{d, \gamma, \kappa, \rho, \nu}$ in Theorem 1.2 follows trivially by requiring the walk X to stay at the origin. To show $\tilde{\lambda}_{d, \gamma, \kappa, \rho, \nu} > 0$, we will make the strategy outlined above precise. In compliance with [KS05] we let C_0 and $r > 0$ be large integers and for $\vec{i} \in \mathbb{Z}^d$ define the cubes

$$\mathcal{Q}_r(\vec{i}) := \prod_{j=1}^d [i_j, i_j + C_0^r).$$

In a slight abuse of common notation, let $D([0, \infty), \mathbb{Z}^d)$ denote the Skorohod space restricted to those functions that start in 0 at time 0 and have nearest neighbour jumps only. Then set

$$J_k := \{ \Phi \in D([0, \infty), \mathbb{Z}^d) : \Phi \text{ jumps at most } dC_0^r(\kappa \vee 1)k \text{ times up to time } k \}.$$

For integer times $t > 0$ define

$$\Xi(t) := \bigcap_{k=\lfloor t/4 \rfloor}^t J_k.$$

Then standard large deviation bounds yield

$$\mathbb{P}_0^X(X \in \Xi(t)^c) \leq e^{-c(t+o(t))}, \quad (4.89)$$

for some $c > 0$. In addition, define the cube

$$\mathcal{C}_t := [-dC_0^r(\kappa \vee 1)t, dC_0^r(\kappa \vee 1)t]^d \cap \mathbb{Z}^d,$$

as well as for arbitrary $t \in \mathbb{N}$, $k \in \{0, \dots, t\}$, $\Phi \in \Xi(t)$ and $\epsilon \geq 0$ the events

$$A(t, \Phi, k, \epsilon) := \{ \exists \vec{i} \in \mathcal{C}_t : \Phi(k) \in \mathcal{Q}_r(\vec{i}) \text{ and } \exists y \in \mathcal{Q}_r(\vec{i}) : \xi(s, y) \geq 1 \forall s \in [k, k + \epsilon/\rho] \}$$

and

$$G(t) := \bigcap_{\Phi \in \Xi(t)} \left\{ \sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, \epsilon)} \geq \epsilon t \right\},$$

which both depend on ξ .

For ϵ small enough, using Borel-Cantelli, it is a consequence of [KS05, Prop. 8] that \mathbb{P}^ξ -a.s., $G(t)$ occurs for eventually all $t \in \mathbb{N}$. Indeed, denoting by $\Xi(t)|_{\{[t/4], \dots, t\}}$ the subset of $(\mathbb{Z}^d)^{\{[t/4], \dots, t\}}$ obtained by restricting each element of $\Xi(t)$ to the domain $\{[t/4], \dots, t\}$, we estimate

$$\begin{aligned} \mathbb{P}^\xi(G(t)^c) &\leq \mathbb{P}^\xi \left(\bigcup_{\Phi \in \Xi(t)} \left\{ \sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, \epsilon)} \leq \epsilon t \right\} \right) \\ &\leq \mathbb{P}^\xi \left(\bigcup_{\Phi \in \Xi(t)} \left\{ \sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, 0)} \leq \frac{t}{2} \right\} \right) \\ &\quad + |\Xi(t)|_{\{[t/4], \dots, t\}} \times \max_{\Phi \in \Xi(t)} \mathbb{P}^\xi \left(\sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, 0)} \geq \frac{t}{2}, \sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, \epsilon)} \leq \epsilon t \right) \\ &\leq \mathbb{P}^\xi \left(\bigcup_{\Phi \in \Xi(t)} \left\{ \sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, 0)} \leq \frac{t}{2} \right\} \right) + |\Xi(t)|_{\{[t/4], \dots, t\}} \times \mathbb{P} \left(\sum_{i=1}^{t/2} p_{i, \epsilon} \leq \epsilon t \right), \quad (4.90) \end{aligned}$$

where in the last step we observed that, given $\Phi \in \Xi(t)$, by the strong Markov property of ξ applied successively to the stopping times $\tau_i := \inf\{j \geq [t/4] : \sum_{k=[t/4]}^j 1_{A(t, \Phi, k, 0)} = i\}$, we can couple ξ with a sequence of i.i.d. Bernoulli random variables $(p_{i, \epsilon})_{i \in \mathbb{N}}$ with

$$\mathbb{P}(p_{1, \epsilon} = 1) = \mathbb{P}^\xi(Y_1^0(s) = 0 \forall s \in [0, \epsilon/\rho] \mid \xi(0, 0) \geq 1),$$

such that $1_{A(t, \Phi, \tau_i, \epsilon)} \geq p_{i, \epsilon}$ a.s. for all $i \in \mathbb{N}$, and hence $\sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, \epsilon)} \geq \sum_{i=1}^{t/2} p_{i, \epsilon}$ on the event $\sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, \Phi, k, 0)} \geq \frac{t}{2}$. Here $p_{i, \epsilon}$ corresponds to the event that given $A(t, \Phi, \tau_i, 0)$, a chosen Y -particle, which is close to Φ at time τ_i , does not jump on the time interval $[\tau_i, \tau_i + \epsilon/\rho]$.

By [KS05, Prop. 8], the first term in (4.90) is bounded from above by $1/t^2$ for t large enough. For the second term we have $|\Xi(t)|_{\{[t/4], \dots, t\}} \leq e^{Ct}$ for some $C > 0$ and all t , while large deviations yield that we can find $\epsilon > 0$ such that

$$\mathbb{P} \left(\sum_{k=1}^{t/2} p_{k, \epsilon} \leq \epsilon t \right) \leq e^{-2Ct}$$

for t large enough. From now on we fix such an ϵ . Borel-Cantelli then yields that \mathbb{P}^ξ -a.s., $G(t)$ holds for all $t \in \mathbb{N}$ large enough.

Next observe that by the strong Markov property of X , we can construct a coupling such that on the event $\sum_{k \in \{[t/4], \dots, t-1\}} 1_{A(t, X, k, \epsilon)} \geq \epsilon t$, the random variable $\int_0^t \xi(s, X(s)) ds$ almost surely dominates the sum of i.i.d. random variables $(q_{i, \epsilon})_{1 \leq i \leq \epsilon t}$ with

$$\begin{aligned} \mathbb{P}(q_{1, \epsilon} = \epsilon/(2\rho)) &= \alpha := \inf_{y, z \in \mathcal{Q}_r(0)} \mathbb{P}_y^X(X(s) = z \forall s \in [\epsilon/(2\rho), \epsilon/\rho]) > 0, \\ \mathbb{P}(q_{1, \epsilon} = 0) &= 1 - \alpha; \end{aligned}$$

$q_{i, \epsilon}$ corresponds to the event that given $\tau_i := \inf\{j \geq [t/4] : \sum_{k=[t/4]}^j 1_{A(t, X, k, \epsilon)} = i\}$, X finds a Y -particle in the ξ field which guarantees the event $A(t, X, \tau_i, \epsilon)$, and then occupies the same position as that Y -particle on the time interval $[\tau_i + \epsilon/(2\rho), \tau_i + \epsilon/\rho]$. Since \mathbb{P}^ξ -a.s., $G(t)$ holds for all $t \in \mathbb{N}$ large enough, for such t , we have

$$\mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) ds \right\}, \Xi(t) \right] \leq \mathbb{E} \left[e^{-\gamma \sum_{i=1}^{\epsilon t} q_{i, \epsilon}} \right] = (\alpha e^{-\gamma \epsilon/(2\rho)} + 1 - \alpha)^{\epsilon t}. \quad (4.91)$$

Thus, with (4.89) and (4.91) we obtain that \mathbb{P}^ξ -a.s., for all $t \in \mathbb{N}$ large,

$$\begin{aligned} \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) \, ds \right\} \right] &\leq \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(s, X(s)) \, ds \right\}, \Xi(t) \right] + \mathbb{P}_0^X (X \in \Xi(t)^c) \\ &\leq e^{-\delta(t+o(t))} \end{aligned}$$

for some $\delta > 0$. This establishes the desired result along integer t . Since $Z_{t,\xi}^\gamma$ is monotone in t , we deduce that the result holds as stated. \blacksquare

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References

- [A94] P. Antal. *Trapping problem for the simple random walk*, Dissertation ETH, No 10759, 1994.
- [A95] P. Antal. Enlargement of obstacles for the simple random walk. *Ann. Probab.* 23, 1061–1101, 1995.
- [BK01] M. Biskup and W. König. Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* 29, 636–682, 2001.
- [B94] E. Bolthausen. Localization of a two-dimensional random walk with an attractive path interaction. *Ann. Probab.* 22, 875–918, 1994.
- [BL91] M. Bramson and J. Lebowitz. Asymptotic behavior of densities for two-particle annihilating random walks. *J. Statist. Phys.* 62, 297–372, 1991.
- [CG84] T. Cox and D. Griffeath. Large deviations for Poisson systems of independent random walks. *Z. Wahrsch. Verw. Gebiete* 66, 543–558, 1984.
- [DV75] M. Donsker and S.R.S. Varadhan. Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* 28, 525–565, 1975.
- [DV79] M. Donsker and S.R.S. Varadhan. On the number of distinct sites visited by a random walk. *Comm. Pure Appl. Math.* 32, 721–747, 1979.
- [F66] W. Feller. *An introduction to probability theory and its applications*. Vol. II. John Wiley & Sons, Inc., New York-London-Sydney, 1966.
- [GK05] J. Gärtner and W. König. The parabolic Anderson model. *Interacting stochastic systems*, 153–179, Springer, Berlin, 2005.
- [GH06] J. Gärtner and F. den Hollander. Intermittency in a catalytic random medium. *Ann. Probab.* 34, 2219–2287, 2006.
- [GHM09] J. Gärtner, F. den Hollander and G. Maillard. Intermittency on catalysts. *Trends in stochastic analysis*, 235–248, London Math. Soc. Lecture Note Ser., 353, Cambridge Univ. Press, Cambridge, 2009.
- [KS03] H. Kesten and V. Sidoravicius. Branching random walks with catalysts. *Electron. J. Probab.* 8, 1–51, 2003.

- [KS05] H. Kesten and V. Sidoravicius. The spread of a rumor or infection in a moving population. *Ann. Probab.* 33, 2402–2462, 2005.
- [L96] G. F. Lawler. *Intersections of random walks*, Birkhäuser Boston, 1996.
- [L85] T. Liggett. An improved subadditive ergodic theorem, *Ann. Probab.* 13, 1279–1285, 1985.
- [MOBC03] M. Moreau, G. Oshanin, O. Bénichou and M. Coppey. Pascal principle for diffusion-controlled trapping reactions. *Phys. Rev. E* 67, 045104(R), 2003.
- [MOBC04] M. Moreau, G. Oshanin, O. Bénichou and M. Coppey. Lattice theory of trapping reactions with mobile species. *Phys. Rev. E* 69, 046101, 2004.
- [R94] F. Redig. An exponential upper bound for the survival probability in a dynamic random trap model. *J. Stat. Phys.* 74, 815–827, 1994.
- [S76] F. Spitzer. *Principles of Random Walk*, 2nd edition, Springer-Verlag, 1976.
- [S98] A.S. Sznitman. *Brownian motion, obstacles and random media*, Springer-Verlag, 1998.
- [V03] S.R.S. Varadhan. Large deviations for random walks in a random environment. *Comm. Pure Appl. Math.* 56, 1222-1245, 2003.