Level 1 quenched large deviation principle for random walk in dynamic random environment

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Abstract

Consider a random walk in a time-dependent random environment on the lattice $\mathbb{Z}^d$. Recently, Rassoul-Agha, Seppäläinen and Yilmaz [RSY11] proved a general large deviation principle under mild ergodicity assumptions on the random environment for such a random walk, establishing first level 2 and 3 large deviation principles. Here we present two alternative short proofs of the level 1 large deviations under mild ergodicity assumptions on the environment: one for the continuous time case and another one for the discrete time case. Both proofs provide the existence, continuity and convexity of the rate function. Our methods are based on the use of the sub-additive ergodic theorem as presented by Varadhan in [V03].

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1 Introduction

We consider uniformly elliptic random walks in space-time random environment both in continuous and discrete time. We prove a level 1 quenched large deviation principle under mild conditions on the environment. For the continuous time case, our method is an extension of the technique presented by Drewitz, Gärtner, Ramírez and Sun in [DGRS11] to prove the existence of the quenched Lyapunov exponent of the survival probability of a random walk in a system of moving traps. In the discrete time case we develop a different approach which relies directly on sub-additivity. Nevertheless, both methods are based on the use of the sub-additive ergodic theorem as presented by Varadhan in [V03].

Let $\kappa_2 > \kappa_1 > 0$. Denote by $G := \{e_1, e_{-1}, \dots, e_d, e_{-d}\}$ the set of unit vectors in $\mathbb{Z}^d$. Define $\mathcal{Q} := \{v = \{v(e) : e \in G\} : \kappa_1 \le \inf_{e\in G} v(e) \le \sup_{e\in G} v(e) \le \kappa_2\}$. Consider a continuous time Markov process $\omega := \{\omega_t : t \ge 0\}$ with state space $\Omega_c := \mathcal{Q}^{\mathbb{Z}^d}$, so that $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ with $\omega_t(x) := \{\omega_t(x, e) : e \in G\} \in \mathcal{Q}$. We call ω the *continuous time environmental process*. We assume that for each initial condition ω_0, the process ω defines a probability measure $Q^c_{\omega_0}$ on the Skorokhod space $D([0, \infty); \Omega_c)$. Let μ be an invariant measure for the environmental process ω so that for every bounded continuous function $f : \Omega_c \to \mathbb{R}$ and $t \ge 0$ we have that

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$$\int f(\omega_t)d\mu = \int f(\omega_0)d\mu.$$

Assume that μ is also invariant under the action of space-translations. Furthermore, we define $Q^c_\mu := \int Q^c_\omega d\mu$, where with a slight abuse of notation here $\omega \in \Omega_c$. For a given trajectory $\omega \in D([0,\infty);\Omega_c)$ consider the process $\{X_t : t \geq 0\}$ defined by the generator

$$L_s f(x) := \sum_{e\in G} \omega_s(x,e)(f(x+e) - f(x)),$$

where $s \geq 0$. We call this process a *continuous time random walk in a uniformly elliptic time-dependent random environment* and denote for each $x \in \mathbb{Z}^d$ by $P^c_{x,\omega}$ the law on $D([0,\infty);\mathbb{Z}^d)$ of this random walk with initial condition $X_0 = x$. We call $P^c_{x,\omega}$ the *quenched law* starting from x of the random walk.

We will also consider a discrete version of this model which we define as follows. Let $\kappa > 0$ and $M \in \mathbb{N}$. Define $U := [-M,M]^d$ and $R := [-M,M]^d \cap \mathbb{Z}^d$ and define $\mathcal{P} := \{v = \{v(e) : e \in R\} : \inf_{e\in R} v(e) \geq \kappa, \sum_{e\in R} v(e) = 1\}$. Consider a discrete time Markov process $\omega := \{\omega_n : n \geq 0\}$ with state space $\Omega_d := \mathcal{P}^{\mathbb{Z}^d}$, so that $\omega_n := \{\omega_n(x) : x \in \mathbb{Z}^d\}$ with $\omega_n(x) := \{\omega_n(x,e) : e \in R\} \in \mathcal{Q}$. We call ω the *discrete time environmental process*. Let us denote by Q^d_ω the corresponding law of the process defined on the space $\Omega_d^{\mathbb{N}}$. Let μ be an invariant measure for the environmental process ω so that for every bounded continuous function $f : \Omega_d \to \mathbb{R}$ and $n \geq 0$ we have that

$$\int f(\omega_n)d\mu = \int f(\omega_0)d\mu.$$

Assume that μ is also invariant under the action of space-translations. Furthermore, we define $Q^d_\mu := \int Q^d_\omega d\mu$. Given $\omega \in \Omega_d$ and $x \in \mathbb{Z}^d$, consider now the discrete time random walk $\{X_n : n \geq 0\}$ with a law $P^d_{x,\omega}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ defined through $P^d_{x,\omega}(X_0 = x) = 1$ and the transition probabilities

$$P^d_{x,\omega}(X_{n+1} = x + e | X_n = x) = \omega_n(x,e),$$

for $n \geq 0$ and $e \in R$. We call this process a *discrete time random walk in space-time random environment* and call $P^d_{x,\omega}$ the *quenched law* of the discrete time random walk starting from x.

For $x \in \mathbb{R}^d$, $|x|_2$, $|x|_1$ and $|x|_\infty$ denote respectively, their Euclidean, l_1 and l_∞-norm. Also, for $r > 0$, we define $B_r(x) := \{y \in \mathbb{Z}^d : |y - x|_2 \leq r\}$. Furthermore, given any topological space T, we will denote by $\mathcal{B}(T)$ the corresponding Borel sets. Throughout we will make the following ergodicity assumption. Note that we do not demand the environment to be necessarily ergodic under time shifts.

Assumption (EC). Consider the continuous time environmental process ω. For each $s > 0$ and $x \in \mathbb{Z}^d$ define the transformation $T_{s,x} : D([0,\infty);\Omega_c) \to D([0,\infty);\Omega_c)$ by $(T_{s,x}\omega)_t(y) := \omega_{t+s}(y + x)$. We say that the environmental process ω satisfies *assumption (EC)* if $\{T_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ is an ergodic family of transformations acting on the space $(D([0,\infty);\Omega_c), \mathcal{B}(D([0,\infty);\Omega_c)), Q^c_\mu)$. In other words, the latter means that whenever $A \in \mathcal{B}(D([0,\infty);\Omega_c))$ is such that $T^{-1}_{s,x}A = A$ for every $s > 0$ and $x \in \mathbb{Z}^d$, then $Q^c_\mu(A)$ is 0 or 1.

Assumption (ED). Consider the discrete time environmental process ω. For $x \in \mathbb{Z}^d$ define the transformation $T_{1,x} : D([0,\infty);\Omega_d) \to D([0,\infty);\Omega_d)$ by $(T_{1,x}\omega)_n(y) := \omega_{n+1}(y+x)$. We say that the environmental process ω satisfies *assumption (ED)* if $\{T_{1,x} : x \in R\}$ is an ergodic family of transformations acting on the space $(\Omega_d^{\mathbb{N}}, \mathcal{B}(\Omega_d^{\mathbb{N}}), Q^d_\mu)$. In other words, whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $T^{-1}_{1,x}A = A$ for every $x \in R$, then $Q^d_\mu(A)$ is 0 or 1.

It is straightforward to check that assumption (ED) is equivalent to asking that whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $A = T^{-1}_{n,x}A$ for every $x \in R$ and $n \in \mathbb{N}$ then $Q^d_\mu(A)$ is 0 or 1.

In this paper we present a level 1 quenched large deviation principle for both the continuous and the discrete time random walk in space-time random environment. Similar results have been obtained in Rassoul-Agha, Seppäläinen and Yilmaz [RSY11] establishing level 2 and 3 large deviations, and then level 1 through contraction, for discrete time random walks on space-time random environments and potentials along with variational expressions for the rate functions. Our proof is obtained by directly establishing the level 1 large deviation principle and is based on the sub-additive ergodic theorem as used by Varadhan in [V03]. (Similar ideas were already presented in the context of the homogenization of the stochastic Hamilton-Jacobi equation for example by Rezakhanlou and Tarver in [RT00] and for the totally asymmetric simple K-exclusion processes and growth processes by Seppäläinen in [S99] and Rezakhanlou in [R02]). While our methods do not give any explicit information about the rate function, besides its convexity and continuity, the proofs are short and simple.

Theorem 1.1 *Consider a continuous time random walk $\{X_t : t \geq 0\}$ in a uniformly elliptic time-dependent environment ω satisfying assumption (EC). Then, there exists a convex continuous rate function $I_c(x) : \mathbb{R}^d \to [0, \infty)$ such that the following are satisfied.*

(i) For every open set $G \subset \mathbb{R}^d$ we have that Q^c_μ-a.s.

$$\liminf_{t\to\infty} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in G\right) \geq -\inf_{x\in G} I_c(x).$$

(ii) For every closed set $C \subset \mathbb{R}^d$ we have that Q^c_μ-a.s.

$$\limsup_{t\to\infty} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in C\right) \leq -\inf_{x\in C} I_c(x).$$

Theorem 1.2 *Consider a discrete time random walk $\{X_n : n \geq 0\}$ in a uniformly elliptic time-dependent environment ω satisfying assumption (ED). Then, there exists a convex rate function $I_d(x) : \mathbb{R}^d \to [0, \infty]$ such that $I_d(x) \leq |\log \kappa|$ for $x \in U$, $I_d(x) = \infty$ for $x \notin U$, I is continuous for every $x \in U^o$ and the following are satisfied.*

(i) For every open set $G \subset \mathbb{R}^d$ we have that Q^d_μ-a.s.

$$\liminf_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in G\right) \geq -\inf_{x\in G} I_d(x).$$

(ii) For every closed set $C \subset \mathbb{R}^d$ we have that Q^d_μ-a.s.

$$\limsup_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{x\in C} I_d(x).$$

A particular case of Theorem 1.1 is the case of a random walk which has a drift in a given direction on occupied sites and in another given direction on unoccupied sites, where the environment is generated by an attractive spin-flip particle system or a simple exclusion process (see Avena, den Hollander and Redig [ARDH10] for the case of a one-dimensional attractive spin-flip dynamics, and also [ARDH11, ADSV11, DHDSS11]). This case is also included in the results presented in [RSY11]. Another particular case of Theorem 1.1 is a continuous time random walk in a static random environment with a law which is ergodic under spatial translations: two of these cases are the Bouchaud trap random walk with bounded jump rates (see for example [BC06]) and the continuous time random conductances model (see for example [DFGW89]). Our proof would also apply to the polymer measure defined by a continuous time random walk in time-dependent random environment and bounded random potential (see [RSY11]). Theorem 1.2 does not include the classical nearest neighbor case, because of how R is defined, but it is not difficult with little additional work to extend the proof to include this case (an example would be the random walk on a space-time i.i.d. environment studied by Yilmaz [Y09]).

The main difficulty in the proofs of Theorems 1.1 and 1.2 is an equicontinuity estimate on the transition probabilities of the random walk. In the case of Theorem 1.1 we follow the method presented in [DGRS11]: we first express the transition probabilities of the walk in terms of those of a simple symmetric random walk through a Radon-Nykodym derivative, then through the use of Chapman-Kolmogorov equation we rely on standard large deviation estimates for the continuous time simple symmetric random walk.

In section 2 we present the proof of Theorem 1.1. In section 3 we continue with the proof of Theorem 1.2. Throughout the rest of the paper we will use the notations c, C, C', C'' to refer to different positive constants.

2 Proof of Theorem 1.1

For each $s \geq 0$, let $\theta_s : D([0,\infty);\Omega_c) \to D([0,\infty);\Omega_c)$ denote the canonical time shift. As in [DGRS11], we first define for each $0 \leq s < t$ and $x, y \in \mathbb{Z}^d$ the quantities

$$e(s,t,x,y) := P^c_{x,\theta_s\omega}\left(X_{t-s} = y\right),$$

and

$$a_c(s,t,x,y) := -\log e(s,t,x,y),$$

where the subscript c in a_c is introduced to distinguish this quantity from the corresponding discrete time one. Note that these functions still depend on the realization of ω. We call $a_c(s,t,x,y)$ the point to point passage function from x to y between times s and t. Due to the fact that we are considering a continuous time random walk, here we do not need to smooth out the point to point passage functions (see [V03]). Nevertheless, there is an equicontinuity issue that should be resolved. It is straightforward to check that Theorem 1.1 will follow directly from the following shape theorem. A version of this shape theorem for a random walk in random potential has been established as Theorem 4.1 in [DGRS11] (see also Theorem 2.5 of Chapter 5 of Sznitman [S98]).

Theorem 2.1 [Shape theorem] *There exists a deterministic convex function $I_c : \mathbb{R}^d \to [0,\infty)$ such that $Q^c_\mu - a.s.$, for any compact set $K \subset \mathbb{R}^d$*

$$\lim_{t\to\infty} \sup_{y \in tK\cap\mathbb{Z}^d} \left| t^{-1} a_c(0,t,0,y) - I_c\left(\frac{y}{t}\right)\right| = 0. \tag{2.1}$$

Furthermore, for any $M > 0$, we can find a compact $K \subset \mathbb{R}^d$ such that $Q^c_\mu - a.s.$

$$\limsup_{t\to\infty} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \notin K\right) \leq -M. \tag{2.2}$$

Display (2.2) of Theorem 2.1 follows from standard large deviation estimates for the process $\{N_t : t \geq 0\}$, where N_t is the total number of jumps up to time t of the random walk $\{X_t : t \geq 0\}$, which can be coupled with a Poisson process of parameter $2d\kappa_2$. To prove the first statement (2.1) of Theorem 2.1 we first observe that for every $0 \leq t_1 < t_2 < t_3$ and $x_1, x_2, x_3 \in \mathbb{Z}^d$ one has that Q^c_μ-a.s.

$$a_c(t_1,t_3,x_1,x_3) \leq a_c(t_1,t_2,x_1,x_2) + a_c(t_2,t_3,x_2,x_3). \tag{2.3}$$

We will also need to obtain bounds on the point to point passage functions which will be eventually used to prove some crucial equicontinuity estimates. To prove these bounds, we first state the following large deviation estimate for the simple symmetric random walk, whose proof can be found in [DGRS11].

Lemma 2.1 *Let X be a simple symmetric random walk on $\mathbb{Z}^d$ with jump rate κ and starting point $X(0) = 0$. For each $x \in \mathbb{Z}^d$ and $t > 0$ let $p(t, 0, x)$ be the probability that this random walk is at position x at time t starting from 0. Then for every $t > 0$ and $x \in \mathbb{Z}^d$, we have*

$$p(t,0,x) = \frac{e^{-J(\frac{x}{t})\,t}}{(2\pi t)^{\frac{d}{2}} \Pi_{i=1}^d \left(\frac{x_i^2}{t^2} + \frac{\kappa^2}{d^2}\right)^{1/4}} \left(1 + o(1)\right), \tag{2.4}$$

where

$$J(x) := \sum_{i=1}^d \frac{\kappa}{d} j\left(\frac{dx_i}{\kappa}\right) \qquad \text{with} \quad j(y) := y \sinh^{-1} y - \sqrt{y^2+1} + 1,$$

and the error term $o(1)$ tends to zero as $t \to \infty$ uniformly in $x \in tK \cap \mathbb{Z}^d$, for any compact $K \subset \mathbb{R}^d$. Furthermore the function j is increasing with $|y|$ and $j \geq 0$.

We will need the following estimates for the transition probabilities.

Lemma 2.2 *Consider the transition probabilities of a random walk on a uniformly elliptic time-dependent environment. The following hold Q_μ^c-a.s.*

(i) Let $C_3 > 0$. There exists a $t_0 > 0$ and constants C_1, C_1' and C_2 such that for $\epsilon > 0$ small enough and every $t \geq t_0$, $y, z \in \mathbb{Z}^d$ such that $|y - z|_2 \leq \epsilon t + \frac{tC_3}{|\log \epsilon|}$ we have that

$$C_1 e^{-C_1' t \frac{1}{|\log \epsilon|^{1/2}}} p(\epsilon t, z, y) \leq e(t(1-\epsilon), t, z, y) \leq C_2 e^{C_2 t \frac{1}{|\log \epsilon|^{1/2}}} p(\epsilon t, z, y).$$

(ii) Let $r > 0$. There exists a $t_0 > 0$ and a constant $C > 0$ such that for each $t \geq t_0$ and $x \in B_{tr}(0)$ one has that

$$e(0, t, 0, x) \geq e^{-Ct} p(t, 0, x).$$

(iii) There is a function $\alpha : (0, \infty) \times [0, \infty) \to (0, \infty)$ such that for each $x, y \in \mathbb{Z}^d$ and $t > s \geq 0$ one has that

$$e(s, t, x, y) \geq \alpha(t - s, |x - y|_1) > 0. \tag{2.5}$$

Proof. Part (i). Note that

$$e(t(1-\epsilon), t, z, y) = E_{z,t(1-\epsilon)}\left[e^{\int_{t(1-\epsilon)}^t \log(2d\omega_s(Y_{s\_}, Y_s - Y_{s\_}))dN_s - \int_{t(1-\epsilon)}^t (\omega_s(Y_s, G) - 1)ds} 1_{Y_t}(y) \right], \tag{2.6}$$

where $E_{z,s}$ is the expectation with respect to the law of a continuous time simple symmetric random walk $\{Y_t : t \geq 0\}$ of jump rate 1 starting from z at time s, N_t is the number of jumps up to time t of the walk, while for each $x \in \mathbb{Z}^d$ and $s > 0$, $\omega_s(x, G) := \sum_e \omega_s(x, e)$ is the total jump rate at site x and time s (see for example Proposition 2.6 in Appendix 1 of Kipnis-Landim [KL99]). Using the fact that the jump rates are bounded from above and from below, it is clear that there is a constant $C > 0$ such that

$$e^{\int_{t(1-\epsilon)}^t \log(2d\omega_s(Y_{s\_}, Y_s - Y_{s\_}))dN_s - \int_{t(1-\epsilon)}^t (\omega_s(Y_s, G) - 1)ds} \leq e^{C(N_t - N_{t(1-\epsilon)}) + C\epsilon t}.$$

Substituting this bound in (2.6), we see that

$$e(t(1-\epsilon), t, z, y) \leq e^{C\epsilon t} E\left[e^{CN_{\epsilon t}} p_{N_{\epsilon t}}(z, y) \right], \tag{2.7}$$

where now E is the expectation with respect to a Poisson process $\{N_t : t \geq 0\}$ of rate 1 and p_n is the n-step transition probability of a discrete time simple symmetric random walk. Let now $R_\epsilon := \frac{1}{\epsilon |\log \epsilon|^{1/2}}$. Note that

$$
\begin{aligned}
E\left[e^{CN_{\epsilon t}}p_{N_{\epsilon t}}(z,y)\right] &\le e^{CR_\epsilon t\epsilon}p(\epsilon t,z,y)+E[e^{N_{\epsilon t}C},N_{\epsilon t}>R_\epsilon t\epsilon] \\
&\le e^{CR_\epsilon t\epsilon}p(\epsilon t,z,y)+E[e^{2N_{\epsilon t}C}]^{1/2}P(N_{\epsilon t}>R_\epsilon t\epsilon)^{1/2}.
\end{aligned}
$$

Now, using the exponential Chebychev inequality with parameter $\log R_\epsilon$, we get

$$
P(N_{\epsilon t}>R_\epsilon \epsilon t)\le e^{-\epsilon t(R_\epsilon \log R_\epsilon-(R_\epsilon-1))} \tag{2.8}
$$

and we compute $E[e^{2N_{\epsilon t}C}]=e^{\epsilon t(e^{2C}-1)}$. Hence,

$$
E\left[e^{CN_{\epsilon t}}p_{N_{\epsilon t}}(z,y)\right]\le e^{CR_\epsilon t\epsilon}p(\epsilon t,z,y)+e^{\epsilon\frac{t}{2}(e^{2C}-1)}e^{-\epsilon\frac{t}{2}(R_\epsilon\log R_\epsilon-(R_\epsilon-1))}. \tag{2.9}
$$

Now, by Lemma 2.1 we know that $j(y)$ is increasing with $|y|$, so that

$$
\sup_{y,z:|y-z|_2\le\epsilon t+\frac{C_3 t}{|\log\epsilon|}}\epsilon t j\left(\frac{|z-y|}{\epsilon t}\right)\le \epsilon t j\left(\frac{C_3}{\epsilon|\log\epsilon|}+1\right)\le t\left(\frac{C_3}{|\log\epsilon|}+\epsilon\right)\log\left(3+\frac{2C_3}{\epsilon|\log\epsilon|}\right)
$$

for $t\ge 1$. Hence, again by Lemma 2.1 with $\kappa=1$, we see that for any constant $c>0$ we can choose ϵ small enough such that

$$
\lim_{t\to\infty}\frac{e^{\epsilon\frac{t}{2}(e^{2C}-1)}e^{-\epsilon tc(R_\epsilon\log R_\epsilon-(R_\epsilon-1))}}{\inf_{y,z}p(\epsilon t,z,y)}=0, \tag{2.10}
$$

where the infimum is taken over y,z as in the previous display. Applying (2.10) with $c=1/2$, we see that the second term of the right-hand side of inequality (2.9), after taking the supremum over y,z such that $|y-z|_2\le\epsilon t+\frac{C_3 t}{|\log\epsilon|}$, is negligible with respect to the first one. Hence, for ϵ small enough, there is a constant C and a $t_0>0$ such that for y,z such that $|y-z|_2\le\epsilon t+\frac{C_3 t}{|\log\epsilon|}$ and $t\ge t_0$ one has

$$
e(t(1-\epsilon),t,z,y)\le Ce^{(R_\epsilon+1)Ct\epsilon}p(\epsilon t,z,y).
$$

Similarly, using the fact that the jump rates are bounded from above and from below it can be shown that for y,z such that $|y-z|_2\le\epsilon t+\frac{C_3 t}{|\log\epsilon|}$ and t large enough

$$
\begin{aligned}
e(t(1-\epsilon),t,z,y) &\ge e^{-C'\epsilon t}E[e^{-C'N_{\epsilon t}}p_{N_{\epsilon t}}(z,y)1_{N_{\epsilon t}\le R_\epsilon \epsilon t}] \\
&\ge e^{-(R_\epsilon+1)\epsilon tC'}E[p_{N_{\epsilon t}}(z,y)1_{N_{\epsilon t}\le R_\epsilon\epsilon t}]\ge e^{-(R_\epsilon+1)\epsilon tC'}\left(p(\epsilon t,z,y)-P(N_{\epsilon t}>R_\epsilon\epsilon t)\right) \\
&\ge C''e^{-(R_\epsilon+1)\epsilon tC'}p(\epsilon t,z,y),
\end{aligned}
$$

where we have used (2.8) and (2.10) with $c=1$.
Part (ii). The proof of part (ii) is analogous to the proof of the lower bound of part (i).
Part (iii). By the same argument as the last part of the proof of part (i), there is a constant $C'>0$ such that

$$
e(s,t,x,y)\ge e^{-C'(t-s)}E[e^{-C'N_{t-s}}p_{N_{t-s}}(x,y),N_{t-s}=|x-y|_1]
$$

But $P(N_{t-s}=|x-y|_1)>0$ (there is, with positive probability, a trajectory from 0 to x such that $N_{t-s}=|x-y|_1$). Thus,

$$
\begin{aligned}
e(s,t,x,y) &\ge e^{-C'(t-s)-C'|x-y|_1}p_{|x-y|_1}(x,y)P(N_{t-s}=|x-y|_1) \\
&= e^{-C'(t-s)-C'|x-y|_1}\frac{1}{(2d)^{|x-y|_1}}P(N_{t-s}=|x-y|_1)>0.
\end{aligned}
$$

∎

We can now apply Kingman's sub-additive ergodic theorem (see for example Liggett [L85]), to prove the following lemma.

Lemma 2.3 *There exists a deterministic function $I_c : \mathbb{Q}^d \to [0, \infty)$ such that for every $y \in \mathbb{Q}^d$, Q_μ^c-a.s. we have that*

$$\lim_{\substack{t\to\infty \\ ty\in\mathbb{Z}^d}} \frac{a_c(0,t,0,ty)}{t} = I_c(y). \tag{2.11}$$

Proof. Assume first that $y \in \mathbb{Z}^d$. Let $q \in \mathbb{N}$. We will consider for $m > n \geq 1$ the random variables

$$X_{n,m}(y) := a_c(nq, mq, ny, my).$$

By (2.3), we have

$$X_{0,m}(y) \leq X_{0,n}(y) + X_{n,m}(y).$$

By part *(iii)* of Lemma 2.2, we see that the random variables $\{X_{n,m}(y)\}$ are integrable. Hence, by Kingman's sub-additive ergodic theorem (see Liggett [L85]) we can then conclude that the limit

$$\hat{I}(q, y, \omega) := \lim_{m\to\infty} \frac{a_c(0, mq, 0, my)}{m} \tag{2.12}$$

exists for $y \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. We have to show that it is deterministic. For this reason, let $r > 0$, $z \in \mathbb{Z}^d$ be arbitrary. It suffices to prove that

$$\hat{I}(q, y, \omega) \leq \hat{I}(q, y, T_{r,z}\omega) = \lim_{m\to\infty} \frac{a_c(r, mq + r, z, my + z)}{m}.$$

First, we have that

$$\frac{a_c(0, mq, 0, my)}{m} \leq \frac{a_c(0, r, 0, z)}{m} + \frac{a_c(r, mq, z, my)}{m}.$$

By part *(iii)* of Lemma 2.2, the first term of the right-hand side of the last equation tends to 0 as $m \to \infty$. Therefore,

$$\hat{I}(q, y, \omega) = \lim_{m\to\infty} \frac{a_c(0, mq, 0, my)}{m} \leq \liminf_{m\to\infty} \frac{a_c(r, mq, z, my)}{m}. \tag{2.13}$$

On the other hand, for $u \in \mathbb{N}$ such that $m > u > r$ we have that

$$\begin{aligned} \frac{a_c(r, mq, z, my)}{m} \leq\ & \frac{a_c(r, (m-u)q + r, z, (m-u)y + z)}{m} \\ +\ & \frac{a_c((m-u)q + r, mq, (m-u)y + z, my)}{m}. \end{aligned}$$

Again, by part *(iii)* of Lemma 2.2, the last term tends to 0 as $m \to \infty$. Therefore

$$\liminf_{m\to\infty} \frac{a_c(r, mq, z, my)}{m} \leq \lim_{m\to\infty} \frac{a_c(r, (m-u)q + r, z, (m-u)y + z)}{m} = \hat{I}(q, y, T_{r,z}\omega). \tag{2.14}$$

Hence $\hat{I}(q, y, \omega) \leq \hat{I}(q, y, T_{r,z}\omega)$. Since $r > 0$ and $z \in \mathbb{Z}^d$ are arbitrary, $\hat{I}(q, y)$ is shift-invariant under each transformation $T_{r,z}$. By assumption (EC), $\hat{I}(q, y)$ is Q_μ^c-a.s equal to a constant for each y. Now, if $y \in \mathbb{Q}^d$, choose the smallest $q \in \mathbb{N}$ such that $qy \in \mathbb{Z}^d$. Then by (2.12), we conclude that

$$\lim_{m\to\infty} \frac{a_c(0, mq, 0, mqy)}{mq} = \frac{1}{q}\hat{I}(q, qy, \omega) =: I_c(y), \tag{2.15}$$

exists (and is well-defined) and is Q_μ^c-a.s. equal to a constant. ∎

We now need to extend the definition of the function $I_c(x)$ for all $x \in \mathbb{R}^d$ and prove the uniform convergence in (2.1). To do this, we will prove that for each compact K there is a $t_0 > 0$ such that the family of functions $\{t^{-1}a_c(0, t, 0, ty) : t \geq t_0\}$ defined on K is equicontinuous. We can now proceed to the main step of the proof of Theorem 2.1.

Lemma 2.4 *Let K be any compact subset of $\mathbb{R}^d$. There exist deterministic $\phi_K : (0,\infty) \to (0,\infty)$ with $\lim_{r\downarrow 0} \phi_K(r) = 0$, and $t_0 > 0$ such that for any $\epsilon > 0$ and $t \geq t_0$, Q_μ^c-a.s., we have*

$$\sup_{\substack{x,y\in tK\cap\mathbb{Z}^d \\ |x-y|_2\leq \epsilon t}} t^{-1}|a_c(0,t,0,x) - a_c(0,t,0,y)| \leq \phi_K(\epsilon). \tag{2.16}$$

Proof. Let us note that for every $\epsilon > 0$, t and $x \in \mathbb{Z}^d$ one has that

$$e(0,t,0,x) = \sum_{z\in\mathbb{Z}^d} e(0,t(1-\epsilon),0,z)e(t(1-\epsilon),t,z,x).$$

Let $R_K := \sup\{|x|_2 : x \in K\}$ be the maximal distance to 0 for any point in K and $r_K = \frac{C_K}{|\log\epsilon|}$, where C_K is a constant that will be chosen large enough. From part (i) of Lemma 2.2 and Lemma 2.1, note that for $t \geq t_0$ (where t_0 is given by part (i) of Lemma 2.2)

$$e(0,t,0,x) \leq \sum_{z\in B_{r_K t}(x)} e(0,t(1-\epsilon),0,z)e(t(1-\epsilon),t,z,x) + Ce^{\frac{1}{|\log\epsilon|^{1/2}}tC - \epsilon t\frac{1}{d}j\left(d\frac{r_K}{\epsilon}\right)}. \tag{2.17}$$

On the other hand by part (ii) of Lemma 2.2 we have that for $t \geq t_0$

$$e(0,t,0,x) \geq e^{-C't - tJ\left(\frac{x}{t}\right)}.$$

Using the upper bound $J\left(\frac{x}{t}\right) \leq dR_K \log(1+2dR_K)$ we see that if

$$\epsilon\frac{1}{d}j\left(d\frac{r_K}{\epsilon}\right) > C + C' + dR_K\log\left(1+2dR_K\right), \tag{2.18}$$

the second term of (2.17) is negligible. But (2.18) is satisfied for $C_K > 2(C+C'+dR_K\log(1+2dR_K))$ and then $\epsilon > 0$ small enough. Hence, it is enough to prove that, Q_μ^c-a.s. we have that

$$\sup_{\substack{x,y\in tK\cap\mathbb{Z}^d \\ |x-y|_2\leq\epsilon t}} \sup_{z\in B_{r_K t}(x)} \frac{e(t(1-\epsilon),t,z,x)}{e(t(1-\epsilon),t,z,y)} \leq Ce^{t\phi_K(\epsilon)}. \tag{2.19}$$

To this end, by Lemmas 2.1 and 2.2

$$\frac{e(t(1-\epsilon),t,z,x)}{e(t(1-\epsilon),t,z,y)} \leq Ce^{2tC\frac{1}{|\log\epsilon|^{1/2}}} e^{-\epsilon t\left(J\left(\frac{x-z}{\epsilon t}\right) - J\left(\frac{y-z}{\epsilon t}\right)\right)}. \tag{2.20}$$

But,

$$J\left(\frac{z-x}{t\epsilon}\right) - J\left(\frac{z-y}{t\epsilon}\right) = \sum_{i=1}^d \frac{1}{d}\left[j\left(d\frac{z_i-x_i}{t\epsilon}\right) - j\left(d\frac{z_i-y_i}{t\epsilon}\right)\right]$$
$$\leq \sum_{i=1}^d \left|\frac{1}{d}\int_{d\frac{z_i-x_i}{t\epsilon}}^{d\frac{z_i-y_i}{t\epsilon}} \log\left(1+2|u|\right)du\right| \leq d\log\left(1+\frac{2dC_K}{\epsilon|\log\epsilon|}\right).$$

Substituting this estimate back into (2.20) we obtain (2.19) with $\phi_K(\epsilon) = C\frac{1}{|\log\epsilon|^{1/2}}$. ∎

Using this lemma, we can extend I_c to a continuous function on $\mathbb{R}^d$. It remains to show the convexity of I_c. For this purpose, let $\lambda \in (0,1)$, $x, y \in \mathbb{R}^d$ and let $(\lambda_n) \subset (0,1)\cap\mathbb{Q}$, $(x_n), (y_n) \subset \mathbb{Q}^d$ such that $\lambda_n \to \lambda$, $x_n \to x$, and $y_n \to y$. In addition let $r_n \in \mathbb{N}$ be such that $r_n(\lambda_n x_n + (1-\lambda_n)y_n)$, $\lambda_n m r_n$, and $\lambda_n m r_n x_n$, are contained in $\mathbb{Z}^d$. Then for any $n \in \mathbb{N}$ one has

$$\begin{aligned}
I_c(\lambda_n x_n + (1-\lambda_n)y_n) &= \lim_{m\to\infty} \frac{a_c(0, mr_n, 0, mr_n(\lambda_n x_n + (1-\lambda_n)y_n))}{mr_n} \\
&\leq \lim_{m\to\infty} \frac{a_c(0, \lambda_n mr_n, 0, \lambda_n mr_n x_n)}{mr_n} \\
&\quad + \lim_{m\to\infty} \frac{a_c(\lambda_n mr_n, mr_n, \lambda_n mr_n x_n, mr_n(\lambda_n x_n + (1-\lambda_n)y_n))}{mr_n}.
\end{aligned}$$

Now taking $n \to \infty$, the continuity of I_c yields that the left-hand side converges to $I_c(\lambda x+(1-\lambda)y)$. Taking advantage of the continuity of I_c and (2.15), the first summand on the right-hand side converges to $\lambda I_c(x)$ a.s., while in combination with the fact that the transformations $T_{\lambda_n mr_n, \lambda_n mr_n x_n}$ are measure preserving, the second summand converges in probability to $(1-\lambda)I_c(y)$; from the last fact we deduce a.s. convergence along an appropriate subsequence and hence the convexity of I_c.

3 Proof of Theorem 1.2

Let us call $\pi_{n,m}(x, y)$, the probability that the discrete time random walk in space-time random environment jumps from time n to time m from site x to site y. Define

$$a_d(n, m, x, y) := -\log \pi_{n,m}(x, y).$$

As in the continuous time case, we have the following sub-additivity property for $n \leq p \leq m$ and $x, y, z \in \mathbb{Z}^d$,

$$a_d(n, m, x, y) \leq a_d(n, p, x, z) + a_d(p, m, z, y). \tag{3.21}$$

We first need to define some concepts that will be used throughout this section. An element (n, z) of the set $\mathbb{N} \times \mathbb{Z}^d$ will be called a *space-time point*. The space-time points of the form $(1, z)$ will be called *steps*. Furthermore, given two space-time points $(n_1, x^{(1)})$ and $(n_2, x^{(2)})$ a sequence of steps $(1, z^{(1)}), \ldots, (1, z^{(k)})$, with $k = n_2 - n_1$ will be called an *admissible path from* $(n_1, x^{(1)})$ *to* $(n_2, x^{(2)})$, if $x^{(2)} = x^{(1)} + z^{(1)} + \ldots + z^{(k)}$ and

$$\begin{aligned}
&\pi_{n_1,n_1+1}(x^{(1)}, x^{(1)} + z^{(1)})\pi_{n_1+1,n_1+2}(x^{(1)} + z^{(1)}, x^{(1)} + z^{(1)} + z^{(2)}) \times \cdots \\
&\cdots \times \pi_{n_2-1,n_2}(x^{(1)} + z^{(1)} + \cdots + z^{(k-1)}, x^{(1)} + z^{(1)} + \cdots + z^{(k)}) > 0.
\end{aligned} \tag{3.22}$$

In other words, there is a positive probability for the space-time random walk (n, X_n) to jump through the sequence of space-time points $(n_1, x^{(1)}), (n_1+1, x^{(1)}+z^{(1)}), \ldots, (n_2, x_2) = (n_2, x^{(1)} + z^{(1)} + \cdots + z^{(k)})$. Let us note that by uniform ellipticity asking that the left-hand side of (3.22) be positive is equivalent to asking that it be larger than or equal to $\kappa^{n_2-n_1}$. With a slight abuse of notation, we will adopt the convention that for $u \in \mathbb{R}$, $[u]$ is the integer closest to u that is between u and 0. Furthermore, we introduce for $x \in \mathbb{R}^d$, the notation $[x] := ([x_1], \ldots, [x_d]) \in \mathbb{R}^d$.

Lemma 3.1 *Fix $z \in U$ and let $K = \{i : |z_i| = M\}$ and $L = \{i : |z_i| < M\}$. Suppose $x \in U$ satisfies $x_i = z_i$ for $i \in K$ and $|x_i| < M$ for $i \in L$. Then for every $n \in \mathbb{N}$ there exists $n_2 \in \mathbb{N}$ such that*

$$n \leq n_2 \leq n + 1 + \bigvee_{i \in L} \left(\frac{1 + n(x_i - z_i)}{M - x_i} \vee \frac{1 + n(z_i - x_i)}{M + x_i} \right) \tag{3.23}$$

and

$$a_d(0, n_2, 0, [n_2 x]) \leq a_d(0, n, 0, [nz]) - \log \kappa^{n_2-n} \tag{3.24}$$

Similarly, for each $n \in \mathbb{N}$ there exists $n_1 \in \mathbb{N}$ such that

$$n - 1 + \bigwedge_{i \in L} \left(\frac{n(z_i - x_i) - 1}{M + x_i} \wedge \frac{n(x_i - z_i) - 1}{M - x_i} \right) \le n_1 \le n \tag{3.25}$$

and

$$a_d(0, n, 0, [nz]) \le a_d(0, n_1, 0, [n_1 x]) - \log \kappa^{n - n_1} \tag{3.26}$$

Proof. Let $n \in \mathbb{N}$ be fixed. Suppose that there exists an n_2 such that $n_2 > n$ and such that for each coordinate direction $i \in L$ one has that

$$-(n_2 - n)M + 1 < nz_i - n_2 x_i < (n_2 - n)M - 1. \tag{3.27}$$

Then, for each $1 \le i \le d$, we can choose increments $w_i^{(\ell)} \in [-M, M] \cap \mathbb{Z}$ for $1 \le \ell \le n_2 - n$ so that $[n_2 x_i] = [nz_i] + w_i^{(1)} + \ldots + w_i^{(n_2 - n)}$. For $i \in K$ we can choose $w_i^{(\ell)} = 0$. The steps $w^{(\ell)} = (1, w_1^{(\ell)}, \ldots, w_d^{(\ell)})$, $1 \le l \le n_2 - n$, form an admissible path from $(n, [nz])$ to $(n_2, [n_2 x])$. Then, (3.24) follows from the Markov property. Hence, to complete the proof of the inequality (3.24) it is enough to show that there exists some $n_2 > n$ such that (3.27) is satisfied. But the first inequality of (3.27) is equivalent to

$$n_2(M - x_i) > 1 + n(M - z_i) \Longleftrightarrow n_2 > n + \frac{1 + n(x_i - z_i)}{M - x_i}.$$

Similarly, the second inequality of (3.27) is equivalent to

$$n_2 > n + \frac{1 + n(z_i - x_i)}{M + x_i}.$$

Given n, we can find an integer n_2 that satisfies these inequalities for all $i \in L$ and is not more than 1 above the maximum of the right-hand sides. This is the content of (3.23). Finally, (3.25)-(3.26) are reasoned similarly.

∎

We are now ready to prove the following proposition.

Proposition 3.1 *For each $x \in \mathbb{R}^d$ we have that Q_μ^d-a.s. the limit*

$$I(x) := -\lim_{n \to \infty} \frac{1}{n} \log \pi_{0,n}(0, [nx]),$$

exists, is convex and deterministic. Furthermore, $I(x) < \infty$ if and only if $x \in U$.

Proof. Note that for $x \in \mathbb{Z}^d$ and $n \ge 1$ the following three statements are equivalent: (i) $\pi_{0,n}(0, x) > 0$; (ii) $\pi_{0,n}(0, x) \ge \kappa^n$; (iii) $x \in nU$. It follows that for $x \notin U$, for there exists an $1 \le i \le d$ such that $[nx_i] \notin [-nM, nM]$ for n large enough. Thus, $I(x) = \infty$. We divide the rest of the proof in four steps. In step 1 for each $x \in \mathbb{Q}^d \cap U$ we define a function $\tilde{I}(x)$. In step 2 we will show that $\tilde{I}$ is deterministic for $x \in \mathbb{Q}^d \cap U^o$. In step 3 we will show that $I(x)$ is well defined for $x \in \mathbb{Q}^d \cap U$ and hence that $I(x) = \tilde{I}(x)$ and in step 4, we extend the definition of $I(x)$ for $x \in \mathbb{R}^d$.

Step 1. Here we will define for each $x \in \mathbb{Q}^d \cap U$ a function $\tilde{I}(x)$. Given $x \in \mathbb{Q}^d \cap U$, there exist a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap kU$ such that $x = k^{-1} y$. Then, using the sub-additive ergodic theorem and (3.21) we can define Q_μ^d-a.s.

$$\tilde{I}(k^{-1} y) := -\lim_{m \to \infty} \frac{1}{mk} \log \pi_{0,mk}(0, my).$$

This definition is independent of the representation of x. Indeed, assume that $x = k^{-1} y_1 = l^{-1} y_2$ for some $k, l \in \mathbb{N}$, $y_1 \in \mathbb{Z}^d \cap kU$ and $y_2 \in \mathbb{Z}^d \cap lU$. Then, passing to subsequences,

$$\begin{aligned}\tilde{I}(k^{-1}y_1) &= -\lim_{n\to\infty}\frac{1}{nlk}\log\pi_{0,nlk}(0,nly_1)\\ &= -\lim_{n\to\infty}\frac{1}{nlk}\log\pi_{0,nlk}(0,nky_2)=\tilde{I}(l^{-1}y_2).\end{aligned}$$

Step 2. Here we will show that $\tilde{I}$ is deterministic in $\mathbb{Q}^d\cap U^o$. Let $x\in\mathbb{Q}^d\cap U^o$. We know that there exists a $k\in\mathbb{N}$ and a $y\in\mathbb{Z}^d\cap kU^o$ such that $x=k^{-1}y$. Let us now fix $z\in R$. It suffices to prove that

$$\tilde{I}(x,\omega)\le\tilde{I}(x,T_{1,z}\omega)=\lim_{m\to\infty}\frac{a_d(1,mk+1,z,my+z)}{mk}.$$

First, for each $n\in\mathbb{N}$, we have that

$$\frac{a_d(0,mnk,0,mny)}{mnk}\le\frac{a_d(0,1,0,z)}{mnk}+\frac{a_d(1,mnk,z,mny)}{mnk}.$$

By uniform ellipticity, the first term of the right-hand side of the last inequality tends to 0 as $m\to\infty$. Therefore,

$$\tilde{I}(x,\omega)=\lim_{m\to\infty}\frac{a_d(0,mnk,0,mny)}{mnk}\le\liminf_{m\to\infty}\frac{a_d(1,mnk,z,mny)}{mnk}. \tag{3.28}$$

On the other hand,

$$\begin{aligned}\frac{a_d(1,mnk,z,mny)}{mnk} &\le \frac{a_d(1,(m-1)nk+1,z,(m-1)ny+z)}{mnk}\\ &+\frac{a_d((m-1)nk+1,mnk,(m-1)ny+z,mny)}{mnk}.\end{aligned} \tag{3.29}$$

Let us now assume that there is an admissible path from $(0,z+(m-1)ny)$ to $(nk-1,mny)$. This is equivalent to asking that z satisfies the following condition:

$$\pi_{0,nk-1}(z+(m-1)ny,mny)>0 \qquad \text{for some } n\in\mathbb{N}. \tag{3.30}$$

Then, by uniform ellipticity, the last term of (3.29) tends to 0 as $m\to\infty$. Therefore, if $z\in R$ satisfies condition (3.30), by (3.28) and (3.29) we have that

$$\tilde{I}(x,\omega)\le\tilde{I}(x,T_{1,z}\omega). \tag{3.31}$$

Hence, to finish the proof it is enough to show that every $z\in R$ satisfies (3.30). Now, z satisfies (3.30) if and only if there exists an $n\in\mathbb{N}$ such that

$$|z_i-ny_i|\le M(nk-1),\quad \forall i=1,\dots,d. \tag{3.32}$$

We will show by contradiction that every $z\in R$ satisfies (3.32). Indeed, assume that for each n there exists $i=i(n)\in\{1,\dots,d\}$ such that

$$|z_i-ny_i|>M(nk-1).$$

Since $|z_i-ny_i|\le M(nk-1)$ implies $|z_i-ly_i|\le M(lk-1)$, $\forall l\ge n$, we conclude that there exists an $i\in\{1,\dots,d\}$ such that

$$|z_i-ny_i|>M(nk-1),\quad \forall n\in\mathbb{N}.$$

Therefore, taking the limit $n\to\infty$, we see that $|x_i|=\left|\frac{y_i}{k}\right|\ge M$, which is a contradiction. This proves that for every $z\in R$ condition (3.30) is satisfied and hence (3.31) is also valid. It follows now by the ergodicity assumption (ED), that for each $x\in\mathbb{Q}^d\cap(-M,M)^d$, $\tilde{I}(x)$ is Q^d_μ-a.s equal to a constant.

Step 3. Here we will show that I is well defined in $\mathbb{Q}^d \cap U$ and hence equals $\tilde{I}$ there. Let $x \in \mathbb{Q}^d \cap U$. Let k be such that $kx \in \mathbb{Z}^d$. Given n, choose m so that $mk \leq n < (m+1)k$. By ellipticity, for each $1 \leq i \leq d$ there exists a sequence of increments $z_i^{(j)} \in \mathbb{Z} \cap [-M, M]$, $1 \leq j \leq n - mk$, such that

$$[nx_i] = mkx_i + z_i^{(1)} + \cdots + z_i^{(n-mk)}.$$

For each $1 \leq j \leq n - mk$ we define $z^{(j)} = (z_1^{(j)}, \ldots, z_d^{(j)})$. Hence, by sub-additivity and considering the admissible path $(1, z^{(1)}), \ldots, (1, z^{(n-mk)})$ from $[nx]$ to mkx, we conclude that

$$\frac{a_d(0, n, 0, [nx])}{n} \leq \frac{a_d(0, mk, 0, mkx)}{n} - \frac{\log \kappa^{n-mk}}{n}.$$

It follows that

$$\limsup_{n\to\infty} \frac{a_d(0, n, 0, [nx])}{n} \leq \tilde{I}(x).$$

For the upper bound, first note that similarly there exists an admissible path of $(m+1)k - n$ steps from $[nx]$ to $(m+1)kx$. Hence,

$$\frac{a_d(0, (m+1)k, 0, (m+1)kx)}{n} \leq \frac{a_d(0, n, 0, [nx])}{n} - \frac{\log \kappa^{(m+1)k-n}}{n}.$$

Taking the limit when $n \to \infty$ we obtain

$$\liminf_{n\to\infty} \frac{a_d(0, n, 0, [nx])}{n} \geq \tilde{I}(x).$$

Step 4. Here we will show that I is well defined in the set $(\mathbb{R}^d \backslash \mathbb{Q}^d) \cap U$. Let $z \in (\mathbb{R}^d \backslash \mathbb{Q}^d) \cap U$, $K = \{i : |z_i| = M\}$ and $L = \{i : |z_i| < M\}$. Pick a rational point x such that $x_i = z_i$ if $i \in K$, $|x_i| < M$ if $i \in L$ and

$$\bigvee_{i\in L} \left(\frac{1}{M - x_i} \vee \frac{1}{M + x_i} \right) \leq 2 \bigvee_{i\in L} \left(\frac{1}{M - z_i} \vee \frac{1}{M + z_i} \right). \tag{3.33}$$

For each n, from Lemma 3.1, we can find n_1, n_2 such that $n_1 \leq n \leq n_2$,

$$\frac{n_2}{n} \cdot \frac{1}{n_2} a_d(0, n_2, 0, [n_2 x]) \leq \frac{1}{n} a_d(0, n, 0, [nz]) + b\left(\frac{n_2}{n} - 1\right)$$

and

$$\frac{1}{n} a_d(0, n, 0, [nz]) \leq \frac{n_1}{n} \cdot \frac{1}{n_1} a_d(0, n_1, 0, [n_1 x]) + b\left(1 - \frac{n_1}{n}\right),$$

where $b = -\log \kappa \in (0, \infty)$. Take $n \to \infty$. From (3.23) and (3.25) and taking $C(z) = 2\bigvee_{i\in L}\left(\frac{1}{M-z_i} \vee \frac{1}{M+z_i}\right)$, the limit points of $\frac{n_2}{n} - 1$ and $1 - \frac{n_1}{n}$ lie in the interval $[0, C(z)|x - z|_\infty]$ because x satisfies (3.33). Consequently from the last two inequalities we see that

$$I(x) \leq \liminf_{n\to\infty} \frac{1}{n} a_d(0, n, 0, [nz]) + C(z)b|x - z|_\infty \tag{3.34}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} a_d(0, n, 0, [nz]) \leq I(x) + C(z)b|x - z|_\infty. \tag{3.35}$$

Letting $x \to z$, we conclude that I is well defined in the set $(\mathbb{R}^d \backslash \mathbb{Q}^d) \cap U$.

∎

We are now in a position to introduce the rate function of Theorem 1.2. We define, for each $x \in U$,

$$I_d(x) := \begin{cases} I(x) & \text{for } x \in U^o \\ \liminf_{U^o \ni y \to x} I(y) & \text{for } x \in \partial U \\ \infty & \text{for } x \notin U. \end{cases} \tag{3.36}$$

We will now prove that I_d satisfies the requirements of Theorem 1.2. By uniform ellipticity, it is clear that $I(x) \le |\log \kappa|$ when $x \in U$. From (3.34) and (3.35), we see that I is continuous in the interior of R (in fact, Lipschitz continuous in any compact contained in U^o). These observations imply that I_d defined in (3.36) is bounded by $|\log \kappa|$ in U, is continuous in U^o, and is lower semi-continuous in U. The convexity of I_d is derived in a manner similar to the continuous time case. We now prove parts (i) and (ii) of Theorem 1.2.

Part (i) of Theorem 1.2 follows immediately from the definition of I_d and the fact that for open sets G, $\inf_{x\in G} I(x) = \inf_{x \in G} I_d(x)$. To prove part (ii) we first consider a compact set C contained in U^o. In this case, we have

$$\begin{gathered} \limsup_{n\to\infty} \tfrac{1}{n} \log P^d_{0,\omega}\left(\tfrac{X_n}{n} \in C\right) \le \limsup_{n\to\infty} \sup_{x\in C} \tfrac{1}{n} \log \pi_{0,n}(0,[nx]) \\ = \inf_n \sup_{m \ge n} \sup_{x \in C} \tfrac{1}{m} \log \pi_{0,m}(0,[mx]) = \inf_n \sup_{x\in C} \sup_{m\ge n} \tfrac{1}{m} \log \pi_{0,m}(0,[mx]) \\ = \inf_n \sup_{x \in C} a_n(x), \end{gathered}$$

where we have defined for $x \in U^o$,

$$a_n(x) := \sup_{m \ge n} \frac{1}{m} \log \pi_{0,m}(0,[mx]).$$

Hence, the upper bound follows if we can show that, for any given $\epsilon > 0$,

$$\sup_{x\in C} a_n(x) \le -\inf_{x\in C} I(x) + \epsilon$$

for large enough n. If we assume the opposite, we can find points $z_m \in C$ which have a subsequence converging to $z \in C$ and and such that along this subsequence one also has that

$$\frac{1}{m} \log \pi_{0,m}(0,[mz_m]) > -I(z) + \epsilon.$$

Applying the first part of Lemma 3.1 gives an index $m_2 > m$ such that

$$\frac{1}{m_2} \log \pi_{0,m_2}(0,[m_2 z]) \ge \frac{m}{m_2}(-I(z)+\epsilon) - b\left(1 - \frac{m}{m_2}\right).$$

Now, since $\lim_{m\to\infty} \frac{m}{m_2} = 1$ and since by Proposition 3.1 $\lim_{m_2 \to\infty} \frac{1}{m_2} \log \pi_{0,m_2}(0,[m_2 z]) = -I(z)$, we obtain that $-I(z) \ge -I(z) + \epsilon$, which is a contradiction.

In the general case, let $C \subset U$ be a compact set. Fix $\delta > 0$ and let $C_1 = \frac{1}{1+\delta} C$. Now C_1 is a compact set contained in U^o. Pick $\epsilon > 0$ small enough so that the closed $\epsilon-$fattening $C_2 = \overline{C_1^{(\epsilon)}}$ is still a compact set contained in U^o. Let $n_2 = \lfloor (1+\delta) n \rfloor$. Then for large enough n, $\frac{x}{n} \in C$ implies $\frac{x}{n_2} \in C_2$. By uniform ellipticity, we have that

$$\begin{gathered} P^d_{0,\omega}\left(\tfrac{X_n}{n} \in C\right) \kappa^{n_2 - n} = \textstyle\sum_{x \in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_n = x)\kappa^{n_2-n} \\ \le \textstyle\sum_{x\in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_n = x) \pi_{n,n_2}(x,x) = \sum_{x\in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_n = x, X_{n_2} = x) \\ \le \textstyle\sum_{x \in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_{n_2} = x) \le P^d_{0,\omega}\left(\tfrac{X_{n_2}}{n_2} \in C_2\right), \end{gathered}$$

where the last inequality is satisfied for n large enough. Then, from the first step of the proof of part (ii) of Theorem 1.2

$$\limsup_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \le -\inf_{x \in C_2} I(x) + \delta b.$$

By taking $\epsilon \searrow 0$ and using compactness and the continuity of I

$$\limsup_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \le -\inf_{x\in C_1} I(x) + \delta b.$$

Take $\delta \searrow 0$ along a subsequence δ_j. This takes C_1 to C. For each δ_j, let $z_j \in C_1 = C_1(\delta_j)$ satisfy $I(z_j) = \inf_{C_1(\delta_j)} I$. Pass to a further subsequence such that $\lim_{j\to\infty} z_j = z \in C$. Then regardless of whether z lies in the interior of U or not, by (3.36) $\liminf_{j\to\infty} I(z_j) \ge I_d(z) \ge \inf_C I_d$, and we get the final upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \le -\inf_{x\in C} I_d(x).$$

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