Effective Polynomial Ballisticity Condition for Random Walk in Random Environment

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Abstract

The conditions $(T)_\gamma$, $\gamma \in (0,1)$, which have been introduced by Sznitman in 2002, have had a significant impact on research in random walk in random environment. Among others, these conditions entail a ballistic behaviour as well as an invariance principle. They require the stretched exponential decay of certain slab exit probabilities for the random walk under the averaged measure and are asymptotic in nature.

The main goal of this paper is to show that in all relevant dimensions (i.e., $d \geq 2$), in order to establish the conditions $(T)_\gamma$, it is actually enough to check a corresponding condition $(\mathcal{P})$ of polynomial type. In addition to only requiring an a priori weaker decay of the corresponding slab exit probabilities than $(T)_\gamma$, another advantage of the condition $(\mathcal{P})$ is that it is effective in the sense that it can be checked on finite boxes.

In particular, this extends the conjectured equivalence of the conditions $(T)_\gamma$, $\gamma \in (0,1)$, to all relevant dimensions.

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1 Introduction and statement of the main result Theorem 1.6 (Polynomial decay is enough)

1.1 Introduction

Random walk in random environment (RWRE) is a generalisation of simple random walk which serves as a model for describing transport processes in inhomogeneous media. Its study has originally been motivated by its role as a toy model in the replication of DNA chains as well as by the investigation of phase transitions in alloys (in particular the growth of crystals) in the late 60's and early 70's of the last century, see e.g. Chernov [Che67] and Temkin [Tem72]. In addition, the model is related to Anderson's tight-binding model for disordered electron systems as well as to a deterministic motion among random scatterers (such as the Lorentz gas, see Sinaĭ [Sin82b]). Furthermore, it

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serves as a theoretical model exhibiting $1/f$-noise — a phenomenon frequently occurring in physics but hard to be established in theoretical models (see Marinari et al. [MPRW83]).
The model has attracted significant mathematical attention and has undergone a major development during the last decades, establishing results on limiting velocities, diffusive behaviour and substantially differing limiting laws, for example.
In particular, the model exhibits appealing phenomena not present in simple random walk. For instance, the question of whether RWRE exhibits diffusive behaviour has attracted considerable attention, and in fact in [Sin82a], Sinaĭ showed that in a standard one-dimensional setting, RWRE (X_n) has fluctuations of scale $(\log n)^2$ only, in contrast to the diffusive scale $\sqrt{n}$; see Kesten, Kozlov and Spitzer [KKS75] for further results in this direction as well as Bricmont and Kupiainen [BK91] (and references therein) also for a discussion of the multi-dimensional situation, where understanding is still far from complete.
As another intriguing example, consider for an element $l \in \mathbb{S}^{d-1}$ of the $d-1$-dimensional unit sphere in $\mathbb{R}^d$, the event $A_l := \{X_n \cdot l = \infty\}$ of *transience in direction* l. Then Kalikow's zero-one law states that $P_0(A_l \cup A_{-l}) \in \{0, 1\}$ (cf. Kalikow [Kal81], Sznitman and Zerner [SZ99], as well as Zerner and Merkl [ZM01]), where P_0 is the averaged probability defined in (1.1) below; however, in dimensions larger than two it is not known whether $P_0(A_l) \notin \{0, 1\}$ can occur or a corresponding zero-one law holds for $P_0(A_l)$ also.
Two of the main difficulties in investigating RWRE are the fact that under the averaged measure, the walk is not Markovian anymore as well as its strongly non-self-adjoint character. As a consequence, the power of spectral theoretic tools is of limited scope only.
In particular, coming back to the above-named difficulties in understanding the higher-dimensional situation, there is still no handy criterion to *characterise* the situations in which the walk exhibits a non-vanishing limiting velocity (i.e. ballisticity). However, the conditions $(T)_\gamma$, $\gamma \in (0, 1]$, introduced by Sznitman in [Szn01] and [Szn02] have proven to be useful in deriving many interesting results concerning the ballistic and diffusive behaviour of RWRE.

1.2 Basic notation and known results

In order to be more precise, we now give a short introduction to the model, thereby fixing some of the notation we employ. We use $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for the 1-norm and the infinity-norm on $\mathbb{Z}^d$, respectively. By $\mathcal{M}_d$ we denote the space of probability measures on the measurable space $(\{e \in \mathbb{Z}^d : \|e\|_1 = 1\}, \mathcal{A})$ of canonical unit vectors, with $\mathcal{A}$ denoting the power set of $\{e \in \mathbb{Z}^d : \|e\|_1 = 1\}$, and we set $\Omega := (\mathcal{M}_d)^{\mathbb{Z}^d}$. Elements of Ω will be referred to as *environments*, and for any $\omega = (\omega(x, \cdot))_{x \in \mathbb{Z}^d} \in \Omega$ one can consider a Markov chain $(X_n)_{n \in \mathbb{N}}$ with transition probabilities from x to $x + e$ given by $\omega(x, e)$ if $\|e\|_1 = 1$, and 0 otherwise. We denote by $P_{x,\omega}$ the law of this Markov chain conditional on $\{X_0 = x\}$. By $\mathcal{F}$ we will denote the σ-algebra on $\mathcal{M}_d$ induced through the Borel-σ-algebra on $\mathbb{R}^{2d}$ (with elements of $\mathcal{M}_d$ identified with elements of $\mathbb{R}^{2d}$ with non-negative entries summing up to 1). Furthermore, to account for the randomness of the environments,

(IID) we assume $\mathbb{P}$ to be a probability measure on $(\Omega, \mathcal{F}^{\mathbb{Z}^d})$ such that the coordinates $(\omega(x, \cdot))_{x \in \mathbb{Z}^d}$ of the environment ω are independent identically distributed under $\mathbb{P}$.

In this context, $\mathbb{P}$ is called *elliptic*, if $\mathbb{P}(\min_{\|e\|_1 = 1} \omega(0, e) > 0) = 1$, and it is called *uniformly elliptic* if there is a constant $\kappa > 0$ such that $\mathbb{P}(\min_{\|e\|_1 = 1} \omega(0, e) \geq \kappa) = 1$. We refer to $P_{x,\omega}$ as the *quenched law* of the RWRE starting from x, and correspondingly we define the *averaged* (or *annealed*) law of the RWRE by

$$P_x := \int_\Omega P_{x,\omega} \, \mathbb{P}(\mathrm{d}\omega). \tag{1.1}$$

As mentioned above, by $\mathbb{S}^{d-1}$ we denote the $(d-1)$-dimensional unit-sphere in $\mathbb{R}^d$. Given a direction $l \in \mathbb{S}^{d-1}$, one refers to the RWRE as being *transient in the direction* l if

$$P_0\big(\lim_{n\to\infty} X_n \cdot l = \infty\big) = 1,$$

and as being *ballistic in the direction* l if P_0-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

In this context, the case $d = 1$ has been resolved by Solomon [Sol75] who has given concise and useful characterisations of the situations in which the walk exhibits transient and ballistic behaviour, respectively.

Theorem 1.1 ([Sol75]). *Let $d = 1$ and $\rho(0) := \omega(0,1)/\omega(0,-1)$. If $\mathbb{E}\ln\rho(0)$ is well-defined (possibly taking the values $\pm\infty$), then the events $\{\lim X_n = \infty\}$, $\{\liminf X_n = -\infty, \liminf X_n = -\infty\}$, and $\{\lim X_n = -\infty\}$, have full P_0-probability according to whether $\mathbb{E}\ln\rho(0) > 0$, $\mathbb{E}\ln\rho(0) = 0$, and $\mathbb{E}\ln\rho(0) < 0$, respectively. Similarly, writing $v^+ := (1 - \mathbb{E}\rho)/(1 + \mathbb{E}\rho)$ and $v^- := (\mathbb{E}(\rho^{-1}) - 1)/(1 + \mathbb{E}(\rho^{-1}))$, the events $\{\lim X_n/n = v^+\}$, $\{\lim X_n/n = 0\}$, and $\{\lim X_n/n = v^-\}$, have full P_0-probability according to whether $\mathbb{E}\rho(0) > 0$, $\mathbb{E}\rho(0) = 0$, and $\mathbb{E}\rho(0) < 0$, respectively.*

In particular, from this result one easily infers that in $d = 1$, there exists RWRE that is transient but not ballistic to the right. The picture is much more involved in dimensions larger than one, though. In fact, there it has also been established that there exist elliptic RWRE in independent identically distributed environments which are transient but not ballistic in a given direction, see for example Sabot and Tournier [ST11]. However, there are still no useful characterisations of the situations in which RWRE is transient or ballistic. To be more precise, let us introduce the following condition.

(UE) We will assume $d \geq 2$ and $\mathbb{P}$ to be uniformly elliptic with ellipticity constant $\kappa > 0$.

Then the following fundamental conjecture remains open.

Conjecture 1.2. *Assume* **(IID)** *and* **(UE)** *to hold. Then RWRE which is transient in a given direction is necessarily ballistic in the same direction.*

Some partial progress has been made towards the resolution of this conjecture by studying RWRE satisfying the conditions $(T)_\gamma$ alluded to above. To rigorously formulate this condition, let $L \geq 0$ and $l \in \mathbb{S}^{d-1}$ an element of the unit sphere. Then we write

$$H_L^l := \inf\{n \in \mathbb{N}_0 : X_n \cdot l > L\} \tag{1.2}$$

for the first entrance time of (X_n) into the half-space $\{x \in \mathbb{Z}^d : x \cdot l > L\}$.

Definition 1.3 ([Szn02]). *Let $\gamma \in (0,1]$ and $l \in \mathbb{S}^{d-1}$. We say that* condition $(T)_\gamma$ *is satisfied with respect to l (written $(T)_\gamma|l$ or $(T)_\gamma$) if for each l' in a neighborhood of l and each $b > 0$ one has that*

$$\limsup_{L\to\infty} L^{-\gamma} \ln P_0(H_L^{l'} > H_{bL}^{-l'}) < 0.$$

We say that condition (T') *is satisfied with respect to l (written $(T')|l$ or (T')), if for each $\gamma \in (0,1)$, condition $(T)_\gamma|l$ is fulfilled.*

In the following we will shortly explain the importance of the conditions $(T)_\gamma$.

It is known that in dimensions $d \geq 2$, the validity of the condition (T') already implies the existence of a deterministic $v \in \mathbb{R}^d \backslash \{0\}$ such that P_0-a.s. $\lim_{n\to\infty} \frac{X_n}{n} = v$, as well as an invariance principle for the RWRE so that under the annealed law P_0,

$$B^n_\cdot := \frac{X_{\lfloor \cdot n \rfloor} - \lfloor \cdot n \rfloor v}{n}$$

converges in distribution to a Brownian motion in the Skorokhod space $D([0,\infty), \mathbb{R}^d)$ as $n \to \infty$; see for instance Theorem 4.1 in Sznitman [Szn04] for further details. Recently, this condition has also been used to obtain further knowledge about large deviations for RWRE, see e.g. Berger, Peterson and Zeitouni, as well as Yilmaz [Ber12, PZ09, Yil11].

While $(T)_\gamma$ a priori is a stronger condition the larger γ is, it has been shown in Sznitman [Szn02] by a detour along the so-called *effective criterion* that for $d \geq 2$, the conditions $(T)_\gamma$ are equivalent for all $\gamma \in (\frac{1}{2}, 1)$. This equivalence has been further improved in Drewitz and Ramírez [DR11] to all $\gamma \in (\gamma_d, 1)$ for some constant $\gamma_d \in (0.366, 0.388)$. For dimensions larger or equal to four, it has been established in Drewitz and Ramírez [DR12] by different methods that the conditions $(T)_\gamma$ are actually equivalent for all $\gamma \in (0,1)$. It has been conjectured by Sznitman in [Szn04] that for any $d \geq 2$ fixed, the conditions $(T)_\gamma$ are equivalent for all $\gamma \in (0,1]$.

1.3 Main result

The goal of this paper is to significantly weaken the condition that has to be checked in order to establish (T'). For this purpose we set

$$c_0 := 2^{3(d-1)} \vee \exp\Big\{2\Big(\ln 90 + \sum_{j=1}^{\infty} \frac{\ln j}{2^j}\Big)\Big\}, \tag{1.3}$$

and introduce the following definition.

Definition 1.4. *Let $M > 0$, $l \in \mathbb{S}^{d-1}$. We say that* condition $(\mathcal{P})_M|l$ *is satisfied with respect to l (written $(\mathcal{P})_M|l$ or $(\mathcal{P})_M$) if for some $v \in \{\pm e_1, \dots, \pm e_d\}$ the following holds: For some $N_0 \geq c_0$ and some choice of $b_1, b_2 > 0$ as well as vectors $l_1, \dots, l_d \in \mathbb{S}^{d-1}$ which satisfy*

(a)

$$\inf_{x \in \left\{x \in \mathbb{R}^d : x \cdot l_i > -b_1 N_0 \ \forall i \in \{1,\dots,d\}\right\}} x \cdot l > -5N_0/6,$$

(b)

$$\inf_{x \in \left\{x \in \mathbb{R}^d : \exists i \in \{1,\dots,d\} \text{ such that } x \cdot l_i > b_2 N_0 \text{ and } x \cdot l_j > -b_1 N_0 \ \forall j \neq i\right\}} x \cdot l > 5N_0/6,$$

and

(c)

$$\sup_{x \in \left\{x \in \mathbb{R}^d : x \cdot l_i \in [-b_1 N_0, b_2 N_0] \ \forall i \in \{1,\dots,d\}\right\}} \Big\| x - \frac{x \cdot v}{l \cdot v} l \Big\|_\infty < 20 N_0^3,$$

the inequalities

$$P_0\big(H^{-l_i}_{b_1 N_0} < H^{l_i}_{b_2 N_0}\big) < \frac{(b_2 N_0/b_1)^{-M}}{d},$$

hold for all $i \in \{1, \dots, d\}$.

Remark 1.5. *(a) In fact, in the whole paper (in particular, in the main result Theorem 1.6) the condition* $(\mathcal{P})_M$ *can be replaced by the weaker condition given in Remark 3.1 (a). The reason for giving Definition 1.4 of* $(\mathcal{P})_M$ *here instead of the one from Remark 3.1 (a) is that the latter requires quite some notation which will only be introduced later on.*

(b) Note that if $(\mathcal{P})_M|l$ *holds, then* $(\mathcal{P})_M|l'$ *holds for all directions* l' *out of a neighbourhood of* l *in* $\mathbb{S}^{d-1}$ *also.*

It is straightforward that condition $(T')|l$ implies $(\mathcal{P})_M|l$ for any $M \in (0, \infty)$. The main result of the paper states that the converse is true also, provided that M is large enough.

Theorem 1.6 (Polynomial decay is enough)**.** *Assume* **(IID)** *and* **(UE)** *to be fulfilled. Let* $l \in \mathbb{S}^{d-1}$ *and assume that* $(\mathcal{P})_M|l$ *holds for some* $M > 15d + 5$*. Then* $(T')|l$ *holds.*

The importance of this result also stems from the multitude of results that so far have been known to hold under the condition (T') only. Using Theorem 1.6, it is now sufficient to establish the polynomial decay of the exit probabilities corresponding to $(\mathcal{P})_M$ instead of the a priori stronger stretched exponential decay.
In addition, in contrast to the conditions $(T)_\gamma$, the condition $(\mathcal{P})_M$ can be checked on finite boxes (without a detour along an analogue to the *effective criterion* of [Szn02]), which emphasises its effective character.
Furthermore, combining Theorem 1.6 with the above remark that $(T')|l$ implies $(\mathcal{P})_M|l$, we directly obtain the following corollary.

Corollary 1.7. *Assume* **(IID)** *and* **(UE)** *to be fulfilled. Then for any* $l \in \mathbb{S}^{d-1}$*, the conditions* $(T)_\gamma|l$*,* $\gamma \in (0, 1)$*, are equivalent.*

1.4 Some further notation

For $k \in \mathbb{N}$, we define the canonical left shift

$$\theta_k : \mathbb{R}^{\mathbb{N}} \ni (x_n)_{n\in\mathbb{N}} \mapsto (x_{k+n})_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}. \tag{1.4}$$

Throughout the rest of the paper, C will denote differing strictly positive and finite constants. Their precise values may change from one side of an inequality to the other; however, in particular, they do not depend on the parameter L that will be employed frequently in the paper. If we want to refer to constants that depend on the dimension but otherwise are absolute, we put indices as in c_3 for example.
For a $\mathbb{Z}^d$-valued discrete time stochastic process (Y_n) and $B \subset \mathbb{Z}^d$ we define the entrance time into B as

$$H^Y_B := H_B(Y) := \inf\{n \in \mathbb{N}_0 : Y_n \in B\}, \tag{1.5}$$

and for singletons $B = \{z\}$ we denote $H_z(Y) := H_{\{z\}}(Y)$. Furthermore, for $l \in \mathbb{S}^{d-1}$ and $L \in \mathbb{R}$, in accordance with (1.2), we define the entrance time

$$H^l_L(Y) := \inf\{n \in \mathbb{N}_0 : Y_n \cdot l > L\}, \tag{1.6}$$

into the half-space $\{x \in \mathbb{Z}^d : x \cdot l > L\}$, with the usual convention that $\inf \emptyset := \infty$. Similarly, the exit time is defined as

$$T_B^Y := T_B(Y) := \inf\{n \in \mathbb{N}_0 : Y_n \notin B\},$$

When referring to the canonical RWRE (X_n) that we will be dealing with, then for the sake of simplicity we will often omit X as an argument of the entrance and exit times.
For $1 \leq j \leq d$ and $l \in \mathbb{S}^{d-1}$, we will use the notation

$$\pi_l : \mathbb{R}^d \ni x \mapsto (x \cdot l)\, l \in \mathbb{R}^d \tag{1.7}$$

to denote the orthogonal projection on the space $\{\lambda l : \lambda \in \mathbb{R}\}$ as well as

$$\pi_{l^\perp} : \mathbb{R}^d \ni x \mapsto x - \pi_l(x) \in \mathbb{R}^d \tag{1.8}$$

for the projection on the corresponding orthogonal subspace. Furthermore, if $e_j \cdot l > 0$ holds, then we define the projections

$$\widetilde{\pi}_l^j : x \mapsto \frac{x \cdot e_j}{l \cdot e_j} l$$

and

$$\widetilde{\pi}_{l^\perp}^j : x \mapsto x - \widetilde{\pi}_l^j(x) \tag{1.9}$$

on the spaces $\{\lambda l : \lambda \in \mathbb{R}\}$ and $\{\lambda e_j : \lambda \in \mathbb{R}\}^\perp$, respectively. For $j = 1$ we will abbreviate these notations by $\widetilde{\pi}_l$ and $\widetilde{\pi}_{l^\perp}$.
For $L > 0$, define

$$\mathcal{D}_L^l := \big\{x \in \mathbb{Z}^d : -L \leq \pi_l(x) \cdot l \leq 10L, \|\pi_{l^\perp}(x)\|_\infty \leq L^3(\ln L)^4\big\}$$

as well as its right boundary part

$$\partial_+ \mathcal{D}_L^l := \big\{x \in \partial \mathcal{D}_L^l : \pi_l(x) \cdot l > 10L\big\},$$

where for any subset $B \subset \mathbb{Z}^d$ its outer boundary ∂B is defined to be

$$\partial B := \big\{x \in \mathbb{Z}^d \backslash B : \exists y \in B \text{ such that } \|x - y\|_1 = 1\big\}. \tag{1.10}$$

We introduce the following condition for further reference. Its validity under $(\mathcal{P})_M|l'$ will be the content of Proposition 2.1.

$$\text{For } l' \in \mathbb{S}^{d-1} \text{ one has } P_0\big(H_{\partial \mathcal{D}_L^{l'}} \neq H_{\partial_+ \mathcal{D}_L^{l'}}\big) \leq \exp\big\{ - L^{\frac{(1+o(1))\ln 2}{\ln \ln L}}\big\}, \tag{1.11}$$

as $L \to \infty$.

Remark 1.8. *If* (1.11) *holds, in correspondence to condition* $(T)_\gamma$ *of Definition 1.3, we write*

$$\gamma_L := \frac{\ln 2}{\ln \ln L}$$

to denote the effective γ.

Definition 1.9. *If* (1.11) *holds for all* l' *out of a neighbourhood of* $l \in \mathbb{S}^{d-1}$, *then we say that condition* $(T)_{\gamma_L}|l$ *is fulfilled.*

2 Proof of Theorem 1.6 (Polynomial decay is enough)

2.1 Auxiliary results (Propositions 2.1 and 2.3)

In this subsection we state two results that play a key role in proving Theorem 1.6. Their proofs will be the subject of Sections 3 and 4.

Proposition 2.1 (Sharpened annealed exit estimates)**.** *Assume* **(IID)** *and* **(UE)** *to be fulfilled. Let* $M > 15d + 5$, $l \in \mathbb{S}^{d-1}$ *and assume that condition* $(\mathcal{P})_M|l$ *is satisfied. Then* $(T)_{\gamma_L}|l$ *holds.*

The previous proposition will be proven in Section 3.
To be able to formulate the second essential ingredient we have to recall the effective criterion which has been introduced in [Szn02] and can be seen as an analogue to the conditions of Solomon (cf. Theorem 1.1) in higher dimensions.
For positive numbers L, L' and $\widetilde{L}$ as well as a space rotation R around the origin we define the

box specification $\mathcal{B}(R, L, L', \widetilde{L})$ as the box $B := \{x \in \mathbb{Z}^d : x \in R((-L, L') \times (-\widetilde{L}, \widetilde{L})^{d-1})\}$.

Furthermore, let

$$\rho_{\mathcal{B}}(\omega) := \frac{P_{0,\omega}(H_{\partial B} \neq H_{\partial_+ B})}{P_{0,\omega}(H_{\partial B} = H_{\partial_+ B})}.$$

Here, $\partial_+ B := \{x \in \partial B : R(e_1) \cdot x \geq L', |R(e_j) \cdot x| < \widetilde{L} \; \forall j \in \{2, \dots, d\}\}$. We will sometimes write ρ instead of $\rho_{\mathcal{B}}$ if the box we refer to is clear from the context and use $\hat{R}$ to label any rotation mapping e_1 to $\hat{v}$. Given $l \in \mathbb{S}^{d-1}$, the *effective criterion with respect to* l is satisfied if for some $L > c_2$ and $\widetilde{L} \in [3\sqrt{d}, L^3)$, we have that

$$\inf_{\mathcal{B},a} \Big\{ c_3 \Big(\ln \frac{1}{\kappa} \Big)^{3(d-1)} \widetilde{L}^{d-1} L^{3(d-1)+1} \mathbb{E} \rho_{\mathcal{B}}^a \Big\} < 1. \tag{2.1}$$

Here, when taking the infimum, a runs over $[0, 1]$ while $\mathcal{B}$ runs over the

$$\text{box specifications } \mathcal{B}(R, L-2, L+2, \widetilde{L}) \text{ with } R \text{ a rotation around the origin such that } R(e_1) = l. \tag{2.2}$$

Furthermore, c_2 and c_3 are dimension dependent constants.
The effective criterion is of significant importance due to its equivalence to (T') (cf. Theorem 2.2) and the fact that it can be checked on finite boxes (in comparison to (T') which is asymptotic in nature).

Theorem 2.2 ([Szn02])**.** *Assume* **(IID)** *and* **(UE)** *to be fulfilled. For each* $l \in \mathbb{S}^{d-1}$ *the following conditions are equivalent.*

(a) The effective criterion with respect to l *is satisfied.*

(b) $(T')|l$ *is satisfied.*

We can now formulate the second key-ingredient for our proof of Theorem 1.6.

Proposition 2.3 (Atypical quenched exit estimates)**.** *Assume* **(IID)** *and* **(UE)** *to be fulfilled. Furthermore, let* $(T)_{\gamma_L}|l$ *be fulfilled. Then, for* $\epsilon(L) := \frac{1}{(\ln \ln L)^2}$, *and each function* $\beta : (0, \infty) \to (0, \infty)$, *one has that*

$$\mathbb{P}\Big(P_{0,\omega}(H_{\partial B} = H_{\partial_+ B}) \leq \frac{1}{2} \exp\big\{ -c_1 L^{\beta(L)} \big\} \Big) \leq 2d \frac{e}{\lceil L^{\beta(L)-\epsilon(L)}/2d \rceil !}, \tag{2.3}$$

where B *is a box specification as in* (2.2) *with* $\widetilde{L} = L^2$, *and*

$$c_1 := -2d \ln \kappa > 1. \tag{2.4}$$

The proof of this result is the subject of Section 4.

2.2 Proof of Theorem 1.6 (assuming Propositions 2.1 and 2.3)

Our proof of Theorem 1.6 goes along establishing the effective criterion and in the following we will give some lemmas that will prove useful in this.
For that purpose, we define the quantities

$$\beta_1(L) := \frac{\gamma_L}{2} = \frac{\ln 2}{2 \ln \ln L}, \tag{2.5}$$

$$\alpha(L) := \frac{\gamma_L}{3} = \frac{\ln 2}{3 \ln \ln L}, \tag{2.6}$$

$$a := L^{-\alpha(L)} \tag{2.7}$$

and write ρ for $\rho_{\mathcal{B}}$ with some arbitrary box specification of (2.2) with $\widetilde{L} = L^2$. We split $\mathbb{E}\rho^a$ according to

$$\mathbb{E}\rho^a = \mathcal{E}_0 + \sum_{j=1}^{n-1} \mathcal{E}_j + \mathcal{E}_n, \tag{2.8}$$

where

$$n := n(L) := \Big\lceil \frac{4(1 - \gamma_L/2)}{\gamma_L} \Big\rceil + 1,$$

$$\mathcal{E}_0 := \mathbb{E}\Big(\rho^a, P_{0,\omega}(H_{\partial B} = H_{\partial_+ B}) > \frac{1}{2} \exp\big\{ - c_1 L^{\beta_1} \big\}\Big),$$

$$\mathcal{E}_j := \mathbb{E}\Big(\rho^a, \frac{1}{2} \exp\big\{ - c_1 L^{\beta_{j+1}} \big\} < P_{0,\omega}(H_{\partial B} = H_{\partial_+ B}) \le \frac{1}{2} \exp\big\{ - c_1 L^{\beta_j} \big\}\Big)$$

for $j \in \{1, \dots, n-1\}$, and

$$\mathcal{E}_n := \mathbb{E}\Big(\rho^a, P_{0,\omega}(H_{\partial B} = H_{\partial_+ B}) \le \frac{1}{2} \exp\big\{ - c_1 L^{\beta_n} \big\}\Big),$$

with parameters

$$\beta_j(L) := \beta_1(L) + (j-1)\frac{\gamma_L}{4}, \tag{2.9}$$

for $2 \le j \le n(L)$; for the sake of brevity we may sometimes omit the dependence on L of the parameters if that does not cause any confusion. Furthermore, in order to verify that equality (2.8) is indeed true, note that due to the uniform ellipticity assumption **(UE)** and the choice of c_1 (cf. (2.4)), one has for $\mathbb{P}$-a.a. ω that

$$P_{0,\omega}(H_{\partial B} = H_{\partial_+ B}) > e^{-c_1 L},$$

as well as that

$$\beta_n > 1.$$

To bound $\mathcal{E}_0$ we employ the following lemma.

Lemma 2.4. *Let $(T)_{\gamma_L}$ be fulfilled. Then*

$$\mathcal{E}_0 \le \exp\big\{c_1 L^{\gamma_L/6} - L^{(1+o(1))\gamma_L/2}\big\},$$

as $L \to \infty$.

Proof. Jensen's inequality yields

$$\mathcal{E}_0 \le 2\exp\left\{c_1 L^{\beta_1-\alpha}\right\} P_0(H_{\partial B} \neq H_{\partial_+ B})^a.$$

Using (2.6) and (2.5), in combination with $(T)_{\gamma_L}$ we obtain the desired result. ☐

To deal with the middle summand in the right-hand side of (2.8), we use the following lemma.

Lemma 2.5. *Assume* **(IID)** *and* **(UE)** *to be fulfilled. Let* $(T)_{\gamma_L}|l$ *be fulfilled. Then for all* L *large enough we have uniformly in* $j \in \{1, \dots, n-1\}$ *that*

$$\mathcal{E}_j \le 4d\exp\left\{c_1 L^{\beta_j-\alpha}\right\} \frac{e}{\lceil L^{\beta_j-\epsilon(L)}/2d\rceil!}.$$

Proof. Using Markov's inequality, for $j \in \{1, \dots, n-1\}$ we obtain the estimate

$$\mathcal{E}_j \le 2\exp\left\{c_1 L^{\beta_{j+1}-\alpha}\right\} \mathbb{P}\Big(P_{0,\omega}(H_{\partial B} = H_{\partial_+ B}) \le \frac{1}{2}\exp\left\{-c_1 L^{\beta_j}\right\}\Big). \tag{2.10}$$

Thus, due to Proposition 2.3, the probability on the right-hand side of (2.10) can be estimated from above by

$$2d\frac{e}{\lceil L^{\beta_j-\epsilon(L)}/2d\rceil!},$$

☐

With respect to the term $\mathcal{E}_n$ in (2.8) we note that it vanishes due to the choice of c_1.

Proof of Theorem 1.6. It follows from Lemmas 2.4, 2.5, the choice of parameters in (2.5) to (2.7) and (2.9), and the fact that $\mathcal{E}_n$ vanishes, that for L large enough, (2.8) can be bounded from above by

$$\exp\left\{c_1 L^{\gamma_L/6} - L^{(1+o(1))\gamma_L/2}\right\} + 4dn(L)\exp\left\{c_1 L^{\beta_n-\alpha}\right\} \frac{e}{\lceil L^{\beta_1-\epsilon(L)}/2d\rceil!}.$$

Thus, we see that for our choice of parameters, (2.8) tends to zero faster than any polynomial in L. Hence, due to (2.1), the effective criterion holds and Theorem 2.2 then yields the desired result. ☐

3 Proof of Proposition 2.1 (Sharpened averaged exit estimates)

3.1 Renormalisation step

In this subsection we introduce a renormalisation scheme that will finally lead to the proof of Proposition 2.1. For the sake of notational simplicity we will assume that v from the Definition 1.4 of $(\mathcal{P})_M|l$ equals e_1, and we start with giving some auxiliary results. Let N_0 be an even integer larger than c_0, where we recall that the latter has been defined in (1.3). For $k \in \mathbb{N}_0$, define recursively the scales

$$N_{k+1} := 3(N_0 + k)^2 N_k. \tag{3.1}$$

We define for $k \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$ the boxes

$$B(x,k) := \left\{y \in \mathbb{Z}^d : -\frac{N_k}{2} < (y-x)\cdot e_1 < N_k, \|\widetilde{\pi}_{l^\perp}(y-x)\|_\infty < 25N_k^3\right\}, \tag{3.2}$$

as well as their frontal parts

$$\widetilde{B}(x,k) := \left\{y \in \mathbb{Z}^d : N_k - N_{k-1} \le (y-x)\cdot e_1 < N_k, \|\widetilde{\pi}_{l^\perp}(y-x)\|_\infty < N_k^3\right\}, \tag{3.3}$$

with the convention that $N_{-1} := 2N_0/3$. Furthermore, we define

$$\partial_+ B(x,k) := \{y \in \partial B(x,k) : (y-x)\cdot e_1 \geq N_k\}. \tag{3.4}$$

We will call $\widetilde{B}(x,k)$ its *middle frontal part* of $B(x,k)$. Furthermore, for $n_1, n_2 \in \mathbb{N}$ we define the sublattice

$$\mathcal{L}_{n_1,n_2} := (n_1, n_2, n_2, \dots, n_2)\mathbb{Z}^d$$

and refer to the elements of

$$\mathcal{B}_k := \left\{ B(x,k) : x \in \mathcal{L}_{N_{k-1}, N_k^3} \right\}$$

as *boxes of scale* k.

To simplify notation, throughout we will denote a typical box of scale k by B_k, and its middle frontal part by $\widetilde{B}_k$. The reader should clearly distinguish such boxes from the box configurations introduced around (2.1).

Remark 3.1. *(a) Condition $(\mathcal{P})_M|l$ can actually be replaced in the whole paper by the weaker condition that*

$$\sup_{x\in\widetilde{B}_0} P_x(H_{\partial B_0} \neq H_{\partial_+ B_0}) < N_0^{-M} \tag{3.5}$$

holds for some $N_0 \geq c_0$. See (1.5), (1.10), (3.2), (3.3), *and* (3.4) *for the notation in this definition, and in particular note that l comes into play in the definitions* (3.2) *to* (3.4).

Also, note that similarly to $(\mathcal{P})_M|l$, if (3.5) *is fulfilled for l, then it is fulfilled for all directions in a neighbourhood of l also.*

(b) For later reference note that $\cup_{B(x,k)\in\mathcal{B}_k} \widetilde{B}(x,k) = \mathbb{Z}^d$.

Definition 3.2. (Good boxes). *We say that a box $B_0 \in \mathcal{B}_0$ is* good (with respect to $\omega \in \Omega$) *if*

$$\inf_{x\in\widetilde{B}} P_{x,\omega}(H_{\partial B_0} = H_{\partial_+ B_0}) \geq 1 - N_0^{-5}. \tag{3.6}$$

Otherwise, we say that the box is bad. *For $k \geq 1$ we say that a box $B_k \in \mathcal{B}_k$ is* good (with respect to $\omega \in \Omega$)*, if there is a box $Q_{k-1} \in \mathcal{B}_{k-1}$ of scale $k-1$ such that every box $B_{k-1} \in \mathcal{B}_{k-1}$ of scale $k-1$ satisfying $B_{k-1} \cap Q_{k-1} = \emptyset$ and $B_{k-1} \cap B_k \neq \emptyset$, is good (with respect to $\omega \in \Omega$). Otherwise, we say that the box B_k is* bad.

We show that for M large enough, condition $(\mathcal{P})_M|l$ implies that boxes of scale k are bad with $\mathbb{P}$-probability decaying doubly-exponentially in k at most, and start with the case $k = 0$.

Lemma 3.3. *Let $l \in \mathbb{S}^{d-1}$ and assume that $(\mathcal{P})_M|l$ holds. Then for all $B_0 \in \mathcal{B}_0$ and $N_0 \geq c_0$ (where c_0 has been defined in* (1.3)*),*

$$\mathbb{P}(B_0 \text{ is good}) \geq 1 - 2^{d-1} N_0^{3d+4-M}.$$

Proof. Note that

$$\mathbb{P}(B_0 \text{ is bad}) \leq \sum_{x\in\widetilde{B}_0} \mathbb{P}\big(P_{x,\omega}(H_{\partial B_0} \neq H_{\partial_+ B_0}) \geq N_0^{-5}\big). \tag{3.7}$$

Now by Markov's inequality we have for $x \in \widetilde{B}_0$ that

$$\mathbb{P}\big(P_{x,\omega}(H_{\partial B_0} \neq H_{\partial_+ B_0}) \geq N_0^{-5}\big) \leq N_0^5 \sup_{x\in\widetilde{B}_0} P_x(H_{\partial B_0} \neq H_{\partial_+ B_0}). \tag{3.8}$$

In combination with (3.7) and (3.8), assumption $(\mathcal{P})_M$ implies that

$$\mathbb{P}(B_0 \text{ is bad}) \leq 2^{d-1}|\widetilde{B}_0| N_0^5 N_0^{-M} \leq 2^{d-1} N_0^{3d+4-M},$$

where we used that $|\widetilde{B}_0| \leq 2^{d-1} N_0^{3d-1}$ due to (3.3), which thus finishes the proof. □

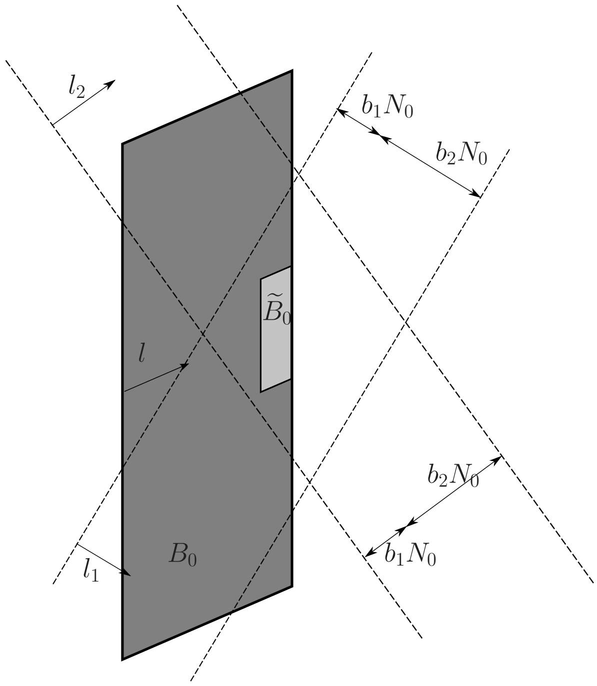

Figure 1: *A box B_0 and the middle frontal part $\widetilde{B}_0$. These boxes are much wider than they are long. The dashed lines illustrate the slabs from the definition of $(\mathcal{P})_M|l$, shifted by some $x \in \widetilde{B}_0$; see Remark 3.1 (a) also.*

Next we treat the case of a general $k \in \mathbb{N}_0$.

Proposition 3.4. *Let $l \in \mathbb{S}^{d-1}$, $M > 15d+5$, and assume that $(\mathcal{P})_M|l$ is satisfied. Then for $N_0 \geq c_0$ one has that for all $k \in \mathbb{N}_0$ and all $B_k \in \mathcal{B}_k$,*

$$\mathbb{P}(B_k \textit{ is good}) \geq 1 - \exp\{-2^k\}. \tag{3.9}$$

Proof. For the sake of simplicity, denote $p_k := \mathbb{P}(B_k \,\mathrm{is\,good})$ for B_k as in the assumptions as well as $q_k := 1 - p_k$. We will prove by induction that

$$\mathbb{P}(B_k \text{ is good}) \geq 1 - \exp\{-c'_k 2^k\} \tag{3.10}$$

for all $k \in \mathbb{N}_0$, where

$$c'_k := \Big(12d + \frac{2}{3}\Big) \ln N_0 - \sum_{j=1}^{k} \frac{\ln((90(j+N_0))^{12d})}{2^j}.$$

We will then show that for N_0 as in the assumptions, $\inf_{k\geq 0} c'_k \geq 1$, which will finish the proof.
Induction start:
Lemma 3.3 yields that

$$\mathbb{P}(B_0 \text{ is good}) \geq 1 - \exp\big\{ - (12d + 2/3)\ln(\kappa)\big\},$$

which in particular implies (3.9) for $k = 0$.
Induction step:
Assume $k \geq 1$ and that the statement holds for $k-1$. Let $q_{k-1} = \exp\{-c'_{k-1}2^{k-1}\}$ and let $B_{k-1,1}, B_{k-1,2}, \ldots, B_{k-1,m_k}$ be all the boxes of scale $k-1$ that intersect B_k. By Definition 3.2,

if each two bad boxes among $B_{k-1,1}, B_{k-1,2}, \dots, B_{k-1,m_k}$ have a non-empty intersection, then the box B_k is good.
Therefore, all that we need to do is to upper bound the probability that there exist two non-intersecting boxes among $B_{k-1,1}, B_{k-1,2}, \dots, B_{k-1,m_k}$ which are bad. By the union bound and **(IID)** we get that

$$q_k \le \binom{m_k}{2} q_{k-1}^2.$$

Noting that for all in $k \ge 1$ we have $m_k \le (30 \cdot 3(k+N_0))^{6d}$, the induction hypothesis yields

$$q_k \le (90(k+N_0))^{12d} \big(\exp\{-c'_{k-1}2^{k-1}\}\big)^2 = \exp\big\{\ln((90(k+N_0))^{12d}) - c'_{k-1}2^k\big\},$$

so $c'_k = c'_{k-1} - \frac{\ln((90(k+N_0))^{12d})}{2^k}$ and hence inductively for every k,

$$c'_k \ge c'_0 - \sum_{j=1}^{\infty} \frac{\ln((90(j+N_0))^{12d})}{2^j}. \tag{3.11}$$

The sum obviously converges, but we need to compare it with the value of c'_0. By Lemma 3.3 and since $M \ge 15d+5$, we deduce that for N_0 as in the assumptions,

$$c'_0 \ge (12d + 2/3)\ln N_0. \tag{3.12}$$

To estimate the sum, we note that due to $\ln(1+x+y) \le \ln(1+x) + \ln(1+y)$, for $x, y \ge 0$, we have for N_0 as in the assumptions that

$$\sum_{j=1}^{\infty} \frac{\ln((90(j+N_0))^{12d})}{2^j} \le 12d\Big(\ln 90 + \sum_{j=1}^{\infty} \frac{\ln(j+N_0)}{2^j}\Big) \le (12d+1/2)\ln N_0. \tag{3.13}$$

Therefore, in combination with (3.11) to (3.13) it follows that

$$c'_k \ge c := c'_0 - \sum_{j=1}^{\infty} \frac{\ln((90(j+N_0))^{12d})}{2^j} \ge \frac{1}{6}\ln N_0 > 1,$$

for every k, and where the last inequality holds since $N_0 \ge c_0$, where c_0 as in (1.3). Hence, $q_k \le \exp\{-2^k\}$ as desired. □

Next we show that with high probability, a walker starting in the middle frontal part of a good box leaves it through the right boundary part. For this purpose, we define the left boundary part

$$\partial_- B(x,k) := \{y \in \partial B(x,k) : (y-x)\cdot l < -N_k/2\}$$

as well as the side boundary part

$$\partial_s B(x,k) := \{y \in \partial B(x,k) : -N_k/2 \le (y-x)\cdot l \le N_k\}.$$

Proposition 3.5. *Let $N_0 \ge c_0$, with c_0 as in* (1.3)*. Then there is a constant $c_4 > 0$ such that for each $k \in \mathbb{N}_0$ and $B_k \in \mathcal{B}_k$ which is good with respect to ω, one has*

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} \ne H_{\partial_+ B_k}) \le \exp\{-c_4 N_k\}.$$

Proof. For the sake of simplicity, we assume without loss of generality that $B_k = B(0,k)$. Using that

$$P_{x,\omega}(H_{\partial B_k} \neq H_{\partial_+ B_k}) \leq P_{x,\omega}(H_{\partial B_k} = H_{\partial_s B_k}) + P_{x,\omega}(H_{\partial B_k} = H_{\partial_- B_k}),$$

we split the proof into two parts. We will first prove that

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} = H_{\partial_s B_k}) \leq \exp\{-cN_k\} \tag{3.14}$$

and then that

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} = H_{\partial_- B_k}) \leq \exp\{-cN_k\}, \tag{3.15}$$

for some constant $c > 0$, which will finish the proof. To prove (3.14) and (3.15) we proceed as follows: Define the sequences $(c'_k)_{k \in \mathbb{N}_0}$ and $(c''_k)_{k \in \mathbb{N}_0}$ via

$$c'_k := \frac{5 \ln N_0}{N_0} - \sum_{j=1}^{k} \frac{\ln 27 (N_0 + j)^4}{N_{j-1}}$$

and

$$c''_k := \frac{5 \ln N_0}{N_0} - \sum_{j=1}^{k} \frac{5N_{j-1} + \ln 24 - 3c_1 N_{j-1} \ln \kappa}{N_j} - \sum_{j=1}^{k} \frac{\ln 27 (N_0 + j)^4}{N_{j-1}}.$$

We will show that

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} = H_{\partial_s B_k}) \leq \exp\{-c'_k N_k\} \tag{3.16}$$

and

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} = H_{\partial_- B_k}) \leq \exp\{-c''_k N_k\} \tag{3.17}$$

hold true for all $k \in \mathbb{N}_0$. Displays (3.14) and (3.15) will then follow since

$$c := \inf_{k \in \mathbb{N}_0} (c'_k) \wedge \inf_{k \in \mathbb{N}_0} (c''_k) > 0$$

for $N_0 \geq c_0$.

Induction start:

For $k = 0$, displays (3.16) and (3.17) follow from the definition of a good box at scale 0.

Induction step:

Now assume that (3.16) and (3.17) hold for scale $k-1$ where $k \geq 1$.

Proof of (3.16) *for* k :

Let κ_1 be the first time that the random walk leaves one of the boxes of scale $k-1$ whose middle frontal parts contain the starting point $x \in \widetilde{B}_k$. Define recursively for $n \geq 1$ the stopping time κ_{n+1} as the first time that the random walk leaves the box of scale $k-1$ whose middle frontal part contains the point X_{κ_n} (this is where we take advantage of Remark 3.1 (b)). If there is more than one such box, then we choose one arbitrarily. We now consider the sequence defined by

$$Y_0 := x \quad \text{and} \quad Y_n := X_{\kappa_n}, \text{ for } n \in \mathbb{N}, \tag{3.18}$$

and call (Y_n) the *rescaled random walk.*

Since the box B_k is good, we know that there exists a box $Q_{k-1} \in \mathcal{B}_{k-1}$ such that every box of scale $k-1$, intersecting B_k but not Q_{k-1}, is good. With this notation we define

$$\mathcal{B}_{Q_{k-1}} := \big\{ B_{k-1} \in \mathcal{B}_{k-1} : B_{k-1} \cap B_k \neq \emptyset \text{ and}$$
$$\text{there exists } x \in B_{k-1} \text{ such that } \widetilde{\pi}_{l^\perp} x = \widetilde{\pi}_{l^\perp} y \text{ for some } y \in Q_{k-1} \big\},$$

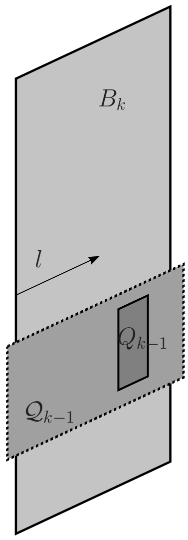

Figure 2: *The bad box Q_{k-1} and its superset $\mathcal{Q}_{k-1}$ inside the box B_k.*

i.e. $\mathcal{B}_{Q_{k-1}}$ is the collection of boxes of scale $k-1$ having at least one site whose coordinates orthogonal to l coincide with the coordinates orthogonal to l of some site in Q_{k-1}. Next, we define $\mathcal{Q}_{k-1} := \cup_{B_{k-1}\in\mathcal{B}_{Q_{k-1}}} B_{k-1}$, see Figure 2.
Let now m_1 be the first time such that the random walk (Y_n) is at a distance larger than $7N_k^3$ from $\mathcal{Q}_{k-1}$ and from the sides $\partial_s B_k$ of the box B_k, so that

$$m_1 := \min\left\{n \in \mathbb{N}_0 : \text{dist}(Y_n, \mathcal{Q}_{k-1}) \geq 7N_k^3 \quad \text{and} \quad \text{dist}(Y_n, \partial_s B_k) \geq 7N_k^3\right\}.$$

Define m_2 as the first time that (Y_n) exits the box B_k so that

$$m_2 := \min\{n \in \mathbb{N}_0, Y_n \notin B_k\}.$$

Furthermore, we define

$$m_3 := \inf\{n > m_1 : Y_n \in \mathcal{Q}_{k-1}\} \leq \infty$$

and note that on the event $\{H_{\partial B_k} = H_{\partial_s B_k}\}$, we have that $P_{x,\omega}$-a.s., $m_1 < m_2 < \infty$. Therefore, $m' := (m_2 \wedge m_3) \circ \theta_{m_1}$ is well-defined on that event and writing

$$J_k := \frac{3N_k/2}{N_{k-2}} + 1,$$

one has $P_{x,\omega}(\cdot \mid H_{\partial B_k} = H_{\partial_s B_k})$-a.s. that

$$m' - 1 \geq \frac{7N_k^3}{2N_{k-1}^3} - 1 \geq 2J_k \frac{N_k}{N_{k-1}}. \tag{3.19}$$

Next observe that if (Y_n) starts from some $y \in B_k$ such that $\text{dist}(y, \mathcal{Q}_{k-1}) \geq 2J_k N_{k-1}^3$ and $\text{dist}(y, \partial_s B_k) \geq 2J_k N_{k-1}^3$, and if it consecutively leaves J_k boxes fo scale $k-1$ through the frontal parts of their boundaries, then it will have left B_k through $\partial_+ B_k$. Thus, we have that

$$P_{y,\omega}\big(Y_j \in B_k \ \forall j \in \{1, \ldots, J_k\}\big) \leq J_k \exp\{-c'_{k-1} N_{k-1}\}.$$

This in combination with (3.19) and the Markov property at multiple times of J_k supplies us with

$$\begin{aligned} P_{x,\omega}(H_{\partial B_k} = H_{\partial_s B_k}) &= P_{x,\omega}(m' \geq 2J_k N_k / N_{k-1}, H_{\partial B_k} = H_{\partial_s B_k}) \\ &\leq \big(\exp\{-c'_{k-1} N_{k-1} + \ln J_k\}\big)^{\frac{N_k}{N_{k-1}}} = \exp\{-c'_k N_k\}. \end{aligned}$$

This completes the proof of (3.16) for k.

Proof of (3.17) *for* k :
The proof is based on a comparison of the exit probabilities of the rescaled random walk (Y_n) with those of a one-dimensional walk with drift.
Induction start:
By Definition 3.2, the statements hold true for $k = 0$.
Induction step:
Assume the statement holds for $k-1$ with $k \geq 1$. Let $B_k \in \mathcal{B}_k$ be good (which again for the sake of simplicity is supposed to be of the form $B_k = B(0,k)$ w.l.o.g.) and $Q_{k-1} \in \mathcal{B}_{k-1}$ be a bad box such that every box of scale $k-1$ that intersects B_k but not Q_{k-1} is good. Let

$$\begin{aligned} L_{Q_{k-1}} &:= \min\{l \cdot z - N_{k-2} : z \in Q_{k-1}\}, \\ R_{Q_{k-1}} &:= \max\{l \cdot z + N_{k-1}/2 : z \in Q_{k-1}\} \leq L_{Q_{k-1}} + 3N_{k-1}, \\ T^{(Q_{k-1})} &:= \inf\{n \in \mathbb{N}_0 : X_n \cdot l \in [L_{Q_{k-1}}, R_{Q_{k-1}}]\} \leq \infty. \end{aligned} \tag{3.20}$$

As alluded to, we will make use of a one-dimensional random walk with drift which, at every unit of time, being at $x \in \mathbb{Z}\backslash[L_{Q_{k-1}}, R_{Q_{k-1}}]$, moves N_{k-2} steps to the right with probability $1 - e^{-c''_{k-1}N_{k-1}}$ and N_{k-1} steps to the left with probability $e^{-c''_{k-1}N_{k-1}}$. From any $x \in [L_{Q_{k-1}}, R_{Q_{k-1}}]$, it jumps N_{k-2} steps to the right with probability $\kappa^{c_1 N_{k-2}}$ and N_{k-1} steps to the left with probability $1 - \kappa^{c_1 N_{k-2}}$. Denote such a walk by (Z_n) and by P_y the corresponding probability measure conditional on $\{Z_0 = y\}$.
We start with proving the estimates

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}\big(T^{(Q_{k-1})} < H_{\partial_+ B_k} \wedge H_{\partial_s B_k}\big) \leq 4 \exp\big\{ - c''_{k-1}\big(N_k - 2N_{k-1} - R_{Q_{k-1}}\big)\big\} \tag{3.21}$$

and

$$\sup_{y \in [L_{Q_{k-1}}, R_{Q_{k-1}}]} P_y\big((H_{-N_k/2}(Z) < H_{N_k}(Z)\big) \leq 6\kappa^{-3c_1 N_{k-1}}\big(\exp\{-c''_{k-1}N_{k-1}\}\big)^{\frac{L_{Q_{k-1}} + N_k/2}{N_{k-1}}}, \tag{3.22}$$

from the combination of which we will be able to deduce (3.17).
To see (3.21), observe that the left-hand side of (3.21) can be estimated from above by

$$\sup_{x \in \widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} = H_{\partial_s B_k}) + \sup_{x \in \widetilde{B}_k} P_{x,\omega}\big(T^{(Q_{k-1})} < H_{\partial_+ B_k}, H_{\partial B_k} \neq H_{\partial_s B_k}\big). \tag{3.23}$$

The first probability can be estimated from above by $\exp\{-c' N_k\}$ using (3.16).
Note that on the event in the second probability, up to time $T^{(Q_{k-1})}$ the random walk only visits good boxes of scale $k-1$. Therefore, using the induction hypothesis (3.17) in combination with a comparison of the exit probabilities for $Y_\cdot \cdot l$ with those for $Z_\cdot$, we get that with (3.23), the left-hand side of (3.21) can be estimated from above by

$$\begin{aligned} &\sup_{x \in \widetilde{B}_k} P_{x,\omega}\big(T^{(Q_{k-1})} < H_{\partial_+ B_k} \wedge H_{\partial_s B_k}\big) \\ &\quad \leq \exp\{-c'_k N_k\} + \sup_{x \in \widetilde{B}_k} P_{x \cdot e_1}\big(H^{-e_1}_{R_{Q_{k-1}}}(Z) < H^{e_1}_{N_k}(Z)\big) \\ &\quad \leq \exp\{-c'_k N_k\} + 3 \sup_{x \in \widetilde{B}_k} \big(\exp\{-c''_{k-1}N_{k-1}\}\big)^{\frac{x \cdot e_1 - (R_{Q_{k-1}} + N_{k-1})}{N_{k-1}}} \\ &\quad \leq \exp\{-c'_k N_k\} + 3\exp\{-c''_{k-1}(N_k - 2N_{k-1} - R_{Q_{k-1}})\}, \end{aligned}$$

for N_0 as in the assumptions, and (3.21) follows.
To see (3.22), let $y \in [L_{Q_{k-1}}, R_{Q_{k-1}}]$ and define the events

$$D^+ := \{H_{N_k}(Z) < H_y \circ \theta_1(Z)\} \quad \text{and} \quad D^- := \big\{H_{-N_k/2}(Z) < H_y \circ \theta_1(Z)\big\},$$

with θ as defined in (1.4). We now estimate the probabilities of the events D^+ and D^-. Assumption **(UE)** in combination with the choices in (3.20) yields

$$P_y(D^+) \geq (1 - \exp\{-c''_{k-1}N_{k-1}\})^{3N_k/(2N_{k-2})}\kappa^{3c_1N_{k-1}} \geq \frac{1}{2}\kappa^{3c_1N_{k-1}} \tag{3.24}$$

for N_0 large enough, while the strong Markov property supplies us with

$$P_y(D^-) \leq 3\big(\exp\{-c''_{k-1}N_{k-1}\}\big)^{\frac{L_{Q_{k-1}}+N_k/2}{N_{k-1}}}. \tag{3.25}$$

Combining (3.24) and (3.25) with the fact that

$$\sup_{y\in[L_{Q_{k-1}},R_{Q_{k-1}}]} P_y(H_{-N_k/2}(Z) < H_{N_k}(Z)) \leq \sup_{y\in[L_{Q_{k-1}},R_{Q_{k-1}}]} \frac{P_y(D^-)}{P_y(D^+)},$$

(3.22) follows.
Noting that for $x \in \widetilde{B}_k$, on the event $\{H_{\partial B_k} = H_{\partial_- B_k}\}$ we have $P_{x,\omega}$-a.s. that $T^{(Q_{k-1})} < H_{\partial_+ B_k} \wedge H_{\partial_s B_k}$, we can now apply the strong Markov property and (3.22) as well as (3.21) to obtain

$$\begin{aligned}
&\sup_{x\in\widetilde{B}_k} P_{x,\omega}(H_{\partial B_k} = H_{\partial_- B_k})\\
&\quad\leq \sup_{x\in\widetilde{B}_k} P_{x,\omega}\big(T^{(Q_{k-1})} < H_{\partial_+ B_k} \wedge H_{\partial_s B_k}\big) \times \sup_{y\in[L_{Q_{k-1}},R_{Q_{k-1}}]} P_y\big(H_{-N_k/2}(Z) < H_{N_k}(Z)\big)\\
&\quad\leq 4\exp\{-c''_{k-1}(N_k - 2N_{k-1} - R_{Q_{k-1}})\}6\kappa^{-3c_1N_{k-1}}\big(3\exp\{-c''_{k-1}N_{k-1}\}\big)^{\frac{L_{Q_{k-1}}+N_k/2}{N_{k-1}}}\\
&\quad\leq \exp\Big\{-N_k\Big(3c''_{k-1}/2 - (5N_{k-1} + \ln 24 - 3c_1N_{k-1}\ln\kappa)/N_k\Big)\Big\},
\end{aligned}$$

and (3.17) follows for k. □

3.2 Proof of Proposition 2.1

Proof of Proposition 2.1. In order to apply our previous results, for L given we implicitly define k_L via $N_{k_L+1} \geq L > N_{k_L}$, which provides us with

$$k_L \sim \frac{\ln L}{\ln\ln L}. \tag{3.26}$$

Recall the definition of π_l and $\pi_{l^\perp}$ from (1.7) and (1.8), and set

$$\mathcal{E}_L^l := \Big\{x \in \mathbb{Z}^d : -N_{k_L} \leq \pi_l(x)\cdot l \leq 10N_{k_L+1} \text{ and } \|\pi_{l^\perp}(x)\|_\infty \leq 2250N_{k_L}^3(N_0+k_L)^2(N_0+k_L+1)^2\Big\}$$

as well as $\partial_+\mathcal{E}_L^l := \{x \in \partial\mathcal{E}_L^l : \pi_l(x)\cdot l > 10N_{k_L+1}\}$. For L large enough, one has

$$\begin{aligned}
P_0\big(H_{\partial\mathcal{D}_L^l} \neq H_{\partial_+\mathcal{D}_L^l}\big) &\leq P_0\big(H_{\partial\mathcal{E}_L^l} \neq H_{\partial_+\mathcal{E}_L^l}\big)\\
&\leq P_0\Big(\text{All } B_{k_L} \in \mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ are good}, H_{\partial\mathcal{E}_L^l} \neq H_{\partial_+\mathcal{E}_L^l}\Big)\\
&\quad + P_0\Big(\text{There exists } B_{k_L} \in \mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ that is bad}\Big).
\end{aligned} \tag{3.27}$$

For large L and using Proposition 3.4, the second summand of the above we estimate by

$$\begin{aligned}
P_0\Big(&\text{There exists } B_{k_L} \in \mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ that is bad}\Big)\\
&\le 2\big|\mathcal{L}_{N_{k_L-1},N_{k_L}^3} \cap \mathcal{E}_L^l\big| \exp\{-2^{k_L}\}\\
&\le 2(2250)^d (N_0+k_L)^{2d}(N_0+k_L+1)^{2d}\exp\{-2^{k_L}\}\\
&\le \exp\big\{-L^{\frac{(1+o(1))\ln 2}{\ln\ln L}}\big\},
\end{aligned}$$

using (3.26) in the last line.

We now bound the first summand of (3.27). For that purpose, note that if the walk leaves $(N_0+k_L)^2(N_0+k_L+1)^2$ blocks of $\mathcal{B}_{k_L}$ consecutively through their ∂_+-part, then $\{H_{\partial\mathcal{E}_L^l} = H_{\partial_+\mathcal{E}_L^l}\}$ occurs. Therefore, we can dominate the event $\{H_{\partial\mathcal{E}_L^l} \neq H_{\partial_+\mathcal{E}_L^l}\}$ from above by the event that one of the blocks B_{k_L} of scale k_L the walk encounters is left not through $\partial_+ B_{k_L}$. To make this formal, for each $k\in\mathbb{N}$ associate to $x\in\mathbb{Z}^d$ an element $\pi_k(x)\in\mathcal{L}_{N_{k-1},N_{k^3}}$ such that $x\in\widetilde{B}(\pi_k(x),k)$. Define the sequence of stopping times for (X_n) given by

$$D_0^k := 0,$$

$$D_j^k := \begin{cases} \inf\Big\{m\in\mathbb{N} : X_{m+D_{j-1}^k} \notin B\big(\pi_k(X_{D_{j-1}^k}),k\big)\Big\} + D_{j-1}^k, & \text{for } j\ge 1 \text{ if } D_{j-1}^k<\infty,\\ \infty, & \text{otherwise.}\end{cases}$$

Using this terminology and the strong Markov property at times $D_j^{k_L}$, $j\in\mathbb{N}$, we can upper bound the first summand of (3.27) by

$$\begin{aligned}
P_0\Big(&\text{All } B_{k_L}\in\mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ are good}, H_{\partial\mathcal{E}_L^l}\neq H_{\partial_+\mathcal{E}_L^l}\Big)\\
&\le \mathbb{E}\Big(P_{x,\omega}(H_{\partial\mathcal{E}_L^l}\neq H_{\partial_+\mathcal{E}_L^l}), \text{all } B_{k_L}\in\mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ are good}\Big)\\
&\le \mathbb{E}\Bigg(P_{0,\omega}\Big(\exists\, 1\le j\le 10(N_0+k_L)^2(N_0+k_L+1)^2 : X_{D_j^{k_L}} \notin \partial_+ B\big(\pi_{k_L}(X_{D_{j-1}^{k_L}}),k_L\big),\ \Big),\\
&\qquad \text{all } B_{k_L}\in\mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ are good}\Bigg)\\
&\le 10(N_0+k_L)^2(N_0+k_L+1)^2\exp\{-cN_{k_L}\}\mathbb{P}\big(\text{All } B_{k_L}\in\mathcal{B}_{k_L} \text{ intersecting } \mathcal{E}_L^l \text{ are good}\big)\\
&\le 10(N_0+k_L)^2(N_0+k_L+1)^2\exp\{-cN_{k_L}\}\\
&\le \exp\big\{-L^{\frac{(1+o(1))\ln 2}{\ln\ln L}}\big\},
\end{aligned}$$

where to obtain the second inequality we took advantage of Proposition 3.5. This finishes the proof. □

4 Proof of Proposition 2.3

Proof of Proposition 2.3. Let l be as in the assumptions of Proposition 2.3 and without loss of generality assume $e_1\cdot l>0$. For each $n,r\in\mathbb{N}$ consider the coarse-grained sublattice

$$\mathbb{L}_n^r := \big\{x\in\mathbb{Z}^d : x=(2ny_1, 2nry_2,\ldots,2nry_d) \text{ for some } y\in\mathbb{Z}^d\big\}.$$

of $\mathbb{Z}^d$, and, recalling $\widetilde{\pi}_{l^\perp}$ from (1.9), for each $x\in\mathbb{L}_n^r$ define the parallelograms

$$\mathcal{R}_n^r(x) := \big\{y\in\mathbb{Z}^d : -2n<(x-y)\cdot e_1<2n, \|\widetilde{\pi}_{l^\perp}(y-x)\|_\infty \le 2rn\big\},$$

and their corresponding central parts

$$\widetilde{\mathcal{R}}_n^r(x) := \big\{ y \in \mathbb{Z}^d : -n \le (x-y)\cdot e_1 \le n, \|\widetilde{\pi}_{l^\perp}(y-x)\|_\infty \le rn \big\}.$$

We chose the term 'parallelogram' and the corresponding notation $\mathcal{R}_n^r$ in order for the reader to be able to distinguish this setting more easily from Section 3. We will denote by $\mathcal{B}_n^r$ the set of parallelograms $\{\mathcal{R}_n^r(x) : x \in \mathbb{L}_n^r\}$. Denote by $J_{L,n}^r$ the number of parallelograms in $\mathcal{B}_n^r$ that intersect B, i.e.,

$$J_{L,n}^r := \big|\{\mathcal{R}_n^r(x) \in \mathcal{B}_n : \mathcal{R}_n^r(x) \cap B \neq \emptyset\}\big|.$$

Due to Proposition 2.1 applied to finitely many suitable vectors neighbouring l, if r is chosen large enough and fixed, we obtain

$$\sup_{y \in \widetilde{\mathcal{R}}_n^r(0)} P_y\big(H_{\partial \mathcal{R}_n^r(0)} \neq H_{\partial_+ \mathcal{R}_n^r(0)}\big) \le \exp\Big\{ - n^{\frac{(1+o(1))\ln 2}{\ln\ln n}} \Big\}, \tag{4.1}$$

as $n \to \infty$. Fix such r for the rest of this proof. We will now perform a one-step normalisation involving parallelograms of side-length $n := \lfloor L^{\epsilon(L)} \rfloor$. A parallelogram $\mathcal{R}_n^r(x) \in \mathcal{B}_n^r$ is defined to be *good (with respect to ω)* if

$$\inf_{y \in \widetilde{\mathcal{R}}_n^r(x)} P_{y,\omega}\big(H_{\partial \mathcal{P}_n^r(x)} = H_{\partial_+ \mathcal{R}_n^r(x)}\big) \ge 1 - L^{-\epsilon(L)^{-1}}.$$

Otherwise, $\mathcal{R}_n^r(x)$ is defined to be *bad (with respect to ω)*. Note now that by Markov's inequality and the invariance of $\mathbb{P}$ under translations of $\mathbb{Z}^d$,

$$\begin{aligned}
\mathbb{P}(\mathcal{R}_n^r(x) \text{ is bad}) &= \mathbb{P}\Big(\sup_{y \in \partial \widetilde{\mathcal{P}}_n^r(x)} P_{y,\omega}(H_{\mathcal{R}_n^r(x)} \neq H_{\partial_+ \mathcal{R}_n^r(x)}) > L^{-\epsilon(L)^{-1}} \Big) \\
&\le L^{d\epsilon(L) + \epsilon(L)^{-1}} \sup_{y \in \widetilde{\mathcal{R}}_n^r(0)} P_y\big(H_{\mathcal{R}_n^r(0)} \neq H_{\partial_+ \mathcal{R}_n^r(0)}\big) \\
&\le L^{d\epsilon(L) + \epsilon(L)^{-1}} \exp\Big\{ - L^{\frac{(1+o(1))\epsilon(L)\ln 2}{\ln\ln L^{\epsilon(L)}}} \Big\} \\
&\le \exp\Big\{ (d\epsilon(L) + \epsilon(L)^{-1}) \ln L - L^{\frac{(1+o(1))\ln 2}{(\ln\ln L)^3}} \Big\},
\end{aligned} \tag{4.2}$$

where the second inequality follows from (4.1). Next, we consider the event $G_{\beta,L} \subset \Omega$ defined via

$$G_{\beta,L} := \Big\{ \text{the number of bad parallelograms in } \mathcal{B}_n^r \text{ that intersect } B \text{ is less than } L^{\beta(L)} \Big\}.$$

A crude strategy for X starting in B to exit B through $\partial_+ B$ is to exit all $\mathcal{R}_n^r(x)$'s encountered through $\partial_+ \mathcal{R}_n(x)$. To make this formal, for each $n \in \mathbb{N}$ associate to $x \in \mathbb{Z}^d$ one of the elements $y \in \mathbb{L}_n^r$ minimising $\|x-y\|_\infty$ and denote this element by $\pi_n(x)$. In a fashion reminiscent of the end of Subsection 3.2, we define the sequence of stopping times for (X_n) given by

$$D_0^n := 0,$$
$$D_j^n := \begin{cases} \inf\big\{ k \in \mathbb{N} : X_{k+D_{j-1}^n} \notin \mathcal{R}_n^r(\pi_k^r(X_{D_{j-1}})) \big\} + D_{j-1}^n, & \text{for } j \ge 1 \text{ if } D_{j-1}^n < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Note that following the above crude strategy, the number of bad parallelograms of the type $\mathcal{R}_n^r(x)$ encountered by the random walk is at most $L^{\beta(L)}$. Thus, using the strong Markov property and **(UE)** we observe that for $\omega \in G_{\beta,L}$,

$$P_{0,\omega}\big(H_{\partial B} = H_{\partial_+ B}\big) \geq P_{0,\omega}\left(X_{D_j} \in \partial_+ \mathcal{R}_n^r(\pi_n(X_{D_{j-1}})),\ \forall\, 1 \leq j \leq \left\lceil \frac{L}{\lfloor L^{\epsilon(L)} \rfloor} \right\rceil\right)$$
$$\geq \left(\exp\{-c_1 L^{\epsilon(L)}\}\right)^{L^{\beta(L)}} \big(1 - L^{-\epsilon(L)^{-1}}\big)^L \tag{4.3}$$
$$> \frac{1}{2} \exp\big\{ - c_1 L^{\epsilon(L)+\beta(L)}\big\},$$

for all $L \geq c_4$, some absolute constant c_4, and where c_1 has been defined in (2.4). On the other hand, one has that

$$\mathbb{P}(G^c_{\beta,L}) \leq 2d \frac{e}{\lceil L^{\beta(L)}/2d \rceil !}. \tag{4.4}$$

Indeed, writing N_L for the number of bad parallelograms in $\mathcal{B}_n^r$ that intersect B, we get $G^c_{\beta,L} = \{N_L \geq L^{\beta(L)}\}$. Since non-overlapping boxes depend on disjoint parts of the environment, the assumption **(IID)** yields that N_L can be stochastically dominated by $\sum_{j=1}^{2d} N_L^j$, where the N_L^j, $1 \leq j \leq 2d$, are independent identically distributed binomial random variables (defined on some probability space with probability measure P) with parameters $J^n_{L,n}$ and $\mathbb{P}(\mathcal{R}_n^r(0) \text{ is bad})$. In particular,

$$J^n_{L,n} \leq C L^d \tag{4.5}$$

for some constant C and all L.
Next, note that for any binomially distributed random variable with expectation 1, i.e. of the type $Y_n \sim \text{Bin}(n, n^{-1})$, we have for all $n \in \mathbb{N}$ and $0 \leq k \leq n$ that

$$P(Y_n \geq k) \leq \frac{e}{k!}.$$

Indeed, we compute

$$P(Y_n \geq k) \leq \sum_{j=k}^{n} \binom{n}{j} n^{-j} \leq \sum_{j=k}^{n} \frac{1}{j!} \leq \frac{e}{k!}.$$

Now due to (4.2) and (4.5), for L large enough, the N_L^j are stochastically dominated by binomial random variables of the type Y_n. Thus, we obtain that

$$\mathbb{P}\big(N_L \geq L^{\beta(L)}\big) \leq P\Big(\sum_{j=1}^{2d} N_L^j \geq L^{\beta(L)}\Big) \leq 2d \frac{e}{\lceil L^{\beta(L)}/2d \rceil !} \tag{4.6}$$

Hence, inequality (4.4) follows and combining (4.3) with (4.4), we finish the proof of Proposition 2.3. □

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