Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **42**, 2007

Transition Asymptotics for Reaction – Diffusion in Random Media

Gérard Ben Arous, Stanislav Molchanov, and Alejandro F. Ramírez

Abstract. We describe a universal transition mechanism between annealed and quenched regimes in the context of reaction – diffusion in random media. We study the total population size for random walks which branch and annihilate on $\mathbb{Z}^d$, with time-independent random rates. The random walks are independent, continuous time, rate $2d\kappa$, simple, symmetric, with $\kappa \geq 0$. A random walk at $x \in \mathbb{Z}^d$, binary branches at rate $v_+(x)$, and annihilates at rate $v_-(x)$. The random environment w has coordinates $w(x) = (v_-(x), v_+(x))$ which are i.i.d. We identify a natural way to describe the annealed-Gaussian transition mechanism under mild conditions on the rates. Indeed, we introduce the exponents $F_\theta(t) := \big(H_1\big((1+\theta)t\big) - (1+\theta)H_1(t)\big)/\theta$, and assume that $\big(F_{2\theta}(t) - F_\theta(t)\big)/\big(\theta\log(\kappa t + e)\big) \to \infty$ for $|\theta| > 0$ small enough, where $H_1(t) := \log\langle m(0,t)\rangle$ and $\langle m(0,t)\rangle$ denotes the average of the expected value of the number of particles $m(0,t,w)$ at time t and an environment of rates w, given that initially there was only one particle at 0. Then the empirical average of $m(x,t,w)$ over a box of side $L(t)$ has different behaviors: if $L(t) \geq \mathrm{e}^{F_\epsilon(t)/d}$ for some $\epsilon > 0$ and large enough t, a law of large numbers is satisfied; if $L(t) \geq \mathrm{e}^{F_\epsilon(2t)/d}$ for some $\epsilon > 0$ and large enough t, a CLT is satisfied. These statements are violated if the reversed inequalities are satisfied for some negative ϵ. As corollaries, we obtain more explicit statements under regularity conditions on the tails of the random rates, including examples in the four universality classes defined in [14]: potentials which are unbounded of Weibull type, of double exponential type, almost bounded, and bounded of Fréchet type. For them we also derive sharper results in the nonannealed regime.

2000 *Mathematics Subject Classification.* Primary 82B41, 82B44; Secondary 60J45, 60J65, 82C22.

Key words and phrases. parabolic Anderson model, random walk, branching processes, Feynman – Kac formula, principal eigenvalue.

This work was partially supported by Fondo Nacional de Desarrollo Científico y Tecnológico grant 7020686.

The work of the last author is partially supported by Fondo Nacional de Desarrollo Científico y Tecnológico grant 1020686.

This is the final form of the paper.

1. Introduction

When studying the long time behavior of Markovian dynamics in random media, one is faced with an important distinction: quenched vs. averaged estimates. Should one work in the so-called quenched regime, where the randomness of the medium is frozen, i.e., where the dynamics are studied in one fixed random realization of the medium/environment? Or should one work in the averaged regime where both the randomness of the dynamics and of the medium are considered, i.e., when one studies the dynamics in a given realization but then also averages over the randomness of the medium?[1]

The true significance of this distinction "quenched vs. averaged" is important when these two regimes give different answers, which is the case in many situations where the extreme values of the random environment might play an important role. A good generic class of examples where this distinction is significant is given by models of reaction – diffusion in random media ([12, 13, 17]).

There are two opposing views about this distinction. The first approach is to think that the relevant and important asymptotic long-time estimates are the quenched ones. But, recognizing the obvious fact that the quenched estimates are the most intractable ones mathematically, the averaged/annealed results are seen as a welcome first approximation. The second and opposing view is that, in many applications, the averaged asymptotic estimates are the only relevant ones. The quenched results, though mathematically more challenging, are not seen as useful or relevant. This second view is naturally based on the idea that some mechanism must be at work, which allows for averaging in the medium randomness.

Some years ago, the authors of this paper held the two opposing opinions expressed above, based on their former collaborations with different domains of applications (specifically physics of pollution by underground waste storage on the one hand and chemical kinetics on the other). In the recent years we have built a common answer, which we believe provides not only a natural resolution of the scientific debate, but also introduces the idea that there exists in fact a very rich universal transition between the averaged and quenched results, which goes far beyond the reaction – diffusion context. This transition also explains, in our view,the true relevance of these regimes in various applications.

The key idea is the following: one should work with a fixed realization of the medium but introduce a new parameter, say L, which will be the scale of the spatial extent of the initial distribution of the dynamics. Then, depending on the respective sizes of the time scale t and the space scale L (when both t and L diverge), one should see the transition we mentioned between the quenched and averaged asymptotic results. More precisely one should expect the following transition: for time scales $t(L)$ short enough the averaged asymptotic results should be valid, whereas for very long time scales $t(L)$ the quenched results should hold, and our new intermediate asymptotics should emerge in between. The mechanism of this transition is the following: for short time scales $t(L)$, or equivalently for large space scales $L(t)$ for the initial distribution, the spatial ergodic theorem should ensure the needed averaging mechanism in order to enforce the validity of averaged/annealed results. For very large time scales no averaging is possible and the extreme values

[1]This regime is often called the annealed regime in the mathematical literature. This is an entrenched but misleading vocabulary since it is not the usual convention in physics

of the environment play the prominent role. The transition regimes are then easily understood, they consist in regions of parameters t and L where one sees a gradual emergence of the extreme values versus the average.

This scheme has been first established in the simplest possible context, i.e., i.i.d. samples [4]. More precisely sums of exponential of i.i.d. random variables are shown to exhibit this transition. In the context of reaction diffusion this could simply be seen as reaction with no diffusion! In this simple context a full transition is given from the Gaussian asymptotics to extreme value theory: one sees in [4] the gradual emergence of the importance of extreme values of the i.i.d. sample which gradually destroys the validity of the Central Limit Theorem and then of the Law of Large Numbers by enforcing α-stable fluctuations where the exponent α decreases through the whole possible range, i.e., from 2 to 0. This mechanism is analogous to the phase transition description of mean-field spin-glass equilibrium models such as the Random Energy Model ([8, 9]).

We then proceeded ([5]) to one important case of reaction – diffusion, i.e., annihilation (or absorption) for random walks in a random environment, more precisely random walks killed on random obstacles, building on the work of [17]. There, we studied the same transition mechanism for the natural quantity, which is the probability of survival. Our picture was less precise than in the i.i.d. context, in the sense that even though we get the proper scales for the intermediate regimes we cannot establish the stable nature of the fluctuations, due to a lack of precise enough understanding of the edge of the spectrum (for the generator of the dynamics which is the discrete Dirichlet Laplacian on a random domain of $\mathbb{Z}^d$).

In this paper we address a rather general case of reaction – diffusion in random environments, i.e., of a system of noninteracting continuous-time Random Walks on the lattice $\mathbb{Z}^d$ branching and annihilating with stationary random rates ([12, 13]).

Let us first describe the random environment $\{w(x) : x \in \mathbb{Z}^d\}$, with $w(x) := (v_-(x), v_+(x))$ where $v_+ := \{v_+(x) : x \in \mathbb{Z}^d\}$ represents the branching rates with $v_+(x) \in [0, \infty)$, while $v_- := \{v_-(x) : x \in \mathbb{Z}^d\}$ represents the annihilation rates, with $v_-(x) \in [0, \infty]$ so that we admit the possibility of hard core obstacles. We assume that the random variables $\{w(x) : x \in \mathbb{Z}^d\}$ are i.i.d. and call their distribution μ.

We consider the following dynamics in a fixed random realization of this environment. We start with one particle at each site of the box Λ_L of side L in $\mathbb{Z}^d$. Each random walk moves independently of the others according to a continuous time simple symmetric rate $2d\kappa$ dynamics for some $\kappa \geq 0$ (we admit the possibility that $\kappa = 0$, i.e., no diffusion at all, as in [4]). A random walk at a site $x \in \mathbb{Z}^d$, branches at rate $v_+(x)$, disappearing and producing two new independent offsprings, and annihilates at rate $v_-(x)$ (note that $v_-(x) = \infty$ means the annihilation is instantaneous and certain as in [5]).

We study the asymptotic behavior, when both t and L go to infinity, of the following observable of our system of random walkers in random environment:

$$m_L(0, t, w) := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} m(x, t, w).$$

Note that $m_L(0, t, w)$ is simply the (normalized) size of the total population.

Our main result, Theorem 2.3, describes sharp conditions on the scales of t and L for the validity of annealed asymptotics of $m_L(0, t, w)$ and for these annealed asymptotics to cease to be true. It also describes the scales for t and L where the fluctuations about these annealed asymptotics are Gaussian and when they cease to be so. We intend to give a full and complete picture of the expected stable fluctuations, all the way to quenched asymptotics, for a wide class of branching rates in a forthcoming work.

The results we obtain here are general, in the sense that they are valid for a large class of product distributions for the random environment, under mild conditions on the branching and annihilation rates. They include examples in the four universality classes recently introduced by van den Hofstad et al. [14] describing all the cases of the random environment. For example, potentials which are unbounded of Weibull type, of double exponential type, almost bounded and bounded of Fréchet type. Furthermore, our results include and generalize both[4, Theorems 2.1 and 2.2] and [5, Theorem 1(ii), (iii) and Theorem 2(i)].

Let us now be more precise about our assumptions about the distribution of the environment. In the recent paper [14], it is shown that under regularity assumptions on the tail of the law of the *effective potential* $v(0) := v_+(0) - v_-(0)$, exactly four universality classes of environments can occur. Their assumptions are formulated in terms of the *cumulant generating function*,

$$H(t) = \log\langle e^{v(0)t} \rangle, \quad t \geq 0, \tag{1.1}$$

of the law of the effective potential $v(0)$, where for any function of the environment f, we define $\langle f \rangle := \int f \, d\mu$. Their basic assumption is that this function is defined and finite for every $t \geq 0$. Then, under two regularity assumptions on H they show that four universality classes can occur: (1) a first class where v is unbounded and has "heavy tails" at infinity, and which includes Weibull-type tails; (2) a second class of unbounded potentials with "lighter" tails which includes the double exponential law; (3) a class containing bounded and unbounded potentials; (4) a class of bounded potentials including those which have Fréchet-type tails near their essential supremum, and the degenerate case of random walks on hard core random obstacles with $\mu[v(0) = -\infty] = p < 1$. In this paper we generalize [5, Theorem 1(ii) and (iii)] describing the passage to an annealed and Gaussian regime, to the previously described system of random walks on the random environment w, under mild conditions which include cases in these four universality classes.

Throughout this article the following will be assumed.

Assumption E. The law of the effective potential is such that $\mu[v(0) = -\infty] < 1$ and

$$\langle e^{v(0)t} \rangle < \infty, \quad t \geq 0.$$

Assumption E ensures that μ-a.s. the stochastic process of random walks on the random environment w can be constructed on infinite volume ([12]), as a limit of processes defined on finite boxes corresponding to continuous time Markov branching processes, as defined in Athreya and Ney in [3]. Furthermore, if $\zeta(t)$ denotes the total number of random walks at time t on a random environment w, given that initially there was only a single one at site x, and E_x^w the expectation defined by its law, Condition E ensures the existence of the first moment $m(x,t,w) = E_x^w[\zeta(t)]$. This is the content of Proposition 2.1 of this paper. This first moment will be the central object of our study. We will see that Assumption E ensures the finiteness for $t \geq 0$ of the annealed first moments $\langle m(0,t) \rangle = \int m(0,t,w) \, d\mu$, with which we will state our main assumptions. We will need to define the *growth functions*

$\{H_1(t) : t \geq 0\}$ by

$$H_1(t) := \log\langle m(0,t)\rangle. \tag{1.2}$$

Our main assumption will be formulated with the help of a family of functions $\{F_\theta : \theta \in \mathbb{R}\}$ which we call the *intermittency exponents*, defined for every $\theta \neq 0$ and $t \geq 0$ as,

$$F_\theta(t) := \frac{H_1\big((1+\theta)t\big) - (1+\theta)H_1(t)}{\theta}. \tag{1.3}$$

Assumption MI. For all $|\theta| > 0$ small enough,

$$\lim_{t\to\infty} \frac{F_{2\theta}(t) - F_\theta(t)}{\theta \log t} = \infty.$$

As it will be shown in Corollary 4.6, this assumption implies the occurrence of the so called *intermittent* behavior of the random field w [12]. It encompasses examples falling in the four universality classes of [14].

We will show that it is possible to formulate an assumption directly in terms of the cumulant generating function H, which is sufficient for Assumption MI to be satisfied, and includes the first class of [14]. For this we need to define the *cumulant exponents*, parametrized by $\theta \neq 0$, for $t \geq 0$ as,

$$G_\theta(t) := \frac{H\big((1+\theta)t\big) - (1+\theta)H(t)}{\theta}. \tag{1.4}$$

By Jensen's inequality it can be seen that $G_\theta(t) \geq 0$ whenever $0 < |\theta| \leq 1$. We will see that Condition MI is satisfied whenever the following happens.

Assumption SI. For all $|\theta| > 0$ small enough,

$$\lim_{t\to\infty} \frac{G_{2\theta}(t) - G_\theta(t)}{\theta t} = \infty.$$

Condition SI includes the first universality class of [14] and can be viewed as a strong intermittency requirement. It implies $H(t)/t \gg 1$. Hence, using the bounds $\mathrm{e}^{H(t)-2d\kappa t} \leq \langle m(0,t)\rangle \leq \mathrm{e}^{H(t)}$ (see, e.g., Theorem 3.1 of Gärtner and Molchanov [12]), we see that if Assumption SI is satisfied, we have the asymptotics

$$\log\langle m(0,t)\rangle \sim H(t), \tag{1.5}$$

which is much faster than the a.s. one (see [13]).

As already mentioned the interest of the results of the present paper are their generality. Namely, their are valid only under the Assumptions E and MI. Part (i) of Theorem 2.3, states that if for some $\epsilon > 0$ we have $L(t) \geq \mathrm{e}^{F_\epsilon(t)/d}$ eventually in t, the law of large numbers $m^L/\langle m\rangle \sim 1$ is satisfied in probability: hence we have the annealed behavior $\log m^L(0,t) \sim \log\langle m(0,t)\rangle$. On the other hand, if for some $\epsilon > 0$ we have $L(t) \leq \mathrm{e}^{F_{-\epsilon}(t)/d}$ eventually in t, in probability $m^L/\langle m\rangle \ll 1$. Part (ii) says that if for some $\epsilon > 0$ we have $L(t) \geq \mathrm{e}^{F_\epsilon(2t)/d}$ eventually in t, then $(m^L - (2L+1)^d\langle m\rangle)/\operatorname{Var}_\mu m^L$ converges in distribution to a centered normal law of unit variance $\mathcal{N}(0,1)$, where Var_μ denotes the variance. Also, if for some $\epsilon > 0$ we have $L(t) \leq \mathrm{e}^{F_{-\epsilon}(2t)/d}$ eventually in t, in probability $(m^L - (2L+1)^d\langle m\rangle)/\operatorname{Var}_\mu m^L \ll 1$. This discussion is summarized in Table 1.

Under an additional regularity assumption on the intermittency exponents (Assumption RI of Section 2.4) it will be shown in Corollary 2.6 that there exist two constants γ_1 and γ_2, called *transition exponents* and a function $J(t)\colon [0,\infty) \to [0,\infty)$

Table 1. Large time asymptotic behavior of the averaged first moments

Annealed behavior	$d\log L(t)\ge F_\epsilon(t)$	$m^L/\langle m\rangle\sim 1$
Non-annealed behavior	$d\log L(t)\le F_{-\epsilon}(t)$	$m^L/\langle m\rangle\ll 1$
Gaussian behavior	$d\log L(t)\ge F_\epsilon(2t)$	$\dfrac{m^L-(2L+1)^d\langle m\rangle}{\mathrm{Var}_\mu m^L}\to\mathcal{N}(0,1)$
Non-Gaussian behavior	$d\log L(t)\le F_\epsilon(2t)$	$\dfrac{m^L-(2L+1)^d\langle m\rangle}{\mathrm{Var}_\mu m^L}\ll 1$

with $J(t)\gg 1$, called the *growth exponent*, describing more explicitly the transition of Theorem 2.3. Indeed, in this case, a law of large numbers is satisfied when $d\log L(t)\ge\gamma J(t)$ for some $\gamma>\gamma_1$; the CLT when $d\log L(t)\ge\gamma J(t)$ for some $\gamma>\gamma_2$. Furthermore, if $d\log L(t)\le\gamma J(t)$ for some $\gamma<\gamma_1$, the law of large numbers is not satisfied, while if $d\log L(t)\le\gamma J(t)$ for some $\gamma<\gamma_2$, the CLT is not satisfied. Propositions 2.7, 2.8, 2.9 and 2.10 give the explicit value of γ_1, γ_2 and J in the case of unbounded potentials with Weibull-type tails, unbounded potentials with double exponential type tails, potentials in the third universality class of [14] and bounded potentials with Fréchet-type tails including the pure hard core case. Table 2 below summarizes those results.

Also, in Theorem 2.11, we obtain sharper upper bounds for the order of magnitude of the averaged first moments in the nonannealed regime for the examples treated in Propositions 2.7, 2.8, 2.9 and 2.10. This theorem generalizes Case 3 of [5, Theorem 2].

The special case in which Assumption SI is satisfied expressed as Corollary 2.4. This includes the case $\kappa=0$, corresponding to sums of i.i.d. random exponentials where Condition MI reduces to,

$$\lim_{t\to\infty}\frac{G_{2\theta}(t)-G_\theta(t)}{\theta}=\infty, \tag{1.6}$$

for $|\theta|>0$ large enough, and $H_1(t)=H(t)$. Corollary 2.4 is a result complementary to theorem of [4], generalizing [4, Theorems 2.1 and 2.2] where Weibull and Fréchet-type tails are assumed on $v(0)$.

One of the main ingredients of the proof of Theorem 2.3 is a coarse graining technique, necessary to reduce the asymptotics of the averaged first moments, to a sum of independent random exponentials. This technique, was introduced in [5],

Table 2. Transition and growth exponents in the four universality classes. In the pure hard core case $\mu[v(0)=-\infty]=p$ and c_2 is a constant depending on p and d

POTENTIAL	$-\log\mu[v(0)>x]$	$J(t)$	γ_1	γ_2
Weibull	x^ρ, $\rho>1$	$H(t)$	$\dfrac{1}{1-\rho}$	$2^{1-\gamma_1}\gamma_1$
Double exponential	$\mathrm{e}^{x/\rho}$	t	ρ	2ρ
Third class example	e^{x^2}	$\dfrac{t}{2\sqrt{\log t}}$	1	2
Pure hard core	—	$c_2t^{d/(d+2)}$	$\dfrac{2}{d+2}$	$2^{1-\gamma_1}\gamma_1$

but here it faces the extra difficulty that the terms of the sum defining the averaged probabilities are not uniformly bounded with respect to the time variable t or the scale L (whereas in [5], such a bound existed having the value 1). This requires more careful estimates on these quantities, which are performed, via the Feynman – Kac formula and spectral estimates. Once the reduction to a sum of i.i.d. exponentials is achieved, an analysis based on von Bahr – Esseen inequality finishes the proof (see also [4, 5]).

Besides this introduction, this paper has four other sections. The main results are stated in Section 2. We first introduce in Section 2.1 the main definitions. In Section 2.2 we formulate Proposition 2.1, stating that a growth of the form $\limsup_{|x|\to\infty} v_+(x)/(|x|\log|x|) = 0$ is enough to ensure well-defined first moments for the total number of particles. When this proposition is combined with [12, Proposition 2], one concludes that under the condition E, the reaction – diffusion process on the lattice $\mathbb{Z}^d$ is such that the total number of particles at any given time has a finite first moment, for initial conditions with a finite total number of particles. In particular, there is no explosion, and the process is well defined. We then state Theorem 2.3 in Section 2.3. Corollary 2.4, under the Assumption SI is stated and proved next. The applications of Theorem 2.3 are given in Section 2.4. First, the regularity condition RI is introduced. This is applied to the case of unbounded effective potentials with Weibull-type tails, through Proposition 2.7, using the Kasahara exponential Tauberian theorem [6]. Next, Corollary 2.6 is applied to the case of unbounded potentials with double exponentially decaying type tails, through Proposition 2.8. Then, we treat the case of potentials falling in the third universality class (almost bounded) of [14] through Proposition 2.9. We end Section 2.4 considering the case of bounded potentials with tails near their upper-bound which are of the Fréchet type (Proposition 2.10). In Section 2.5, we state Theorem 2.11, which improves the upper bounds describing the order of magnitude of the empirical average for the examples considered in Section 2.4. The proof of Proposition 2.1 is the content of the third section. In Section 4, the truncated first moments are introduced. These are the first moments of a reaction – diffusion process defined on a finite box, with Dirichlet boundary conditions. They are then used to approximate some important quantities related to the averaged first moments. Then, several important estimates for the moments and correlations of the first moments are derived. The proof of Theorems 2.3 and 2.11 are given in Section 5. In Section 5.1, the partition analysis method of [5] is recalled. This and together with the estimates of Section 4, and the von Bahr – Esseen inequality, is subsequently applied to prove Theorem 2.3. The paper finishes with Section 5.7, where Theorem 2.11 is proved.

2. Notation and results

Here we will state the results of this paper. After introducing most of the notation and giving the main definitions in the first subsection, in Section 2.2 we state Proposition 2.2, which ensures that a.s. there is no explosion for the reaction – diffusion process under Assumption E. Then, the principal result of this paper, Theorem 2.3 is stated in Section 2.3, together with Corollary 2.4. In Section 2.4, we state Corollary 2.6, giving the form of Theorem 2.3, under certain regularity assumptions. Here we will consider applications of this results to several specific examples of distributions of the effective potential. We end the presentation of our results with Section 2.5, where we state Theorem 2.11.

2.1. Definition of the reaction – diffusion process. We begin defining a reaction – diffusion model corresponding to a set of random walks on the lattice $\mathbb{Z}^d$ branching and annihilating at rates depending on the their position. Consider the set of natural numbers $\mathbb{N}$ endowed with the discrete topology. Define the set $\Omega := \mathbb{N}^{\mathbb{Z}^d}$ representing the possible configuration of particles on the lattice. In this paper we will be interested only on the subset of configurations $\Omega_0 \subset \Omega$ characterized by the property that $\{x : \eta(x) > 0\}$ has finite cardinality whenever $\eta \in \Omega_0$. Let $v_+ := \{v_+(x) : x \in \mathbb{Z}^d\}$ and $v_- := \{v_-(x) : x \in \mathbb{Z}^d\}$, where $v_+(x) \in [0, \infty)$ and $v_-(x) \in [0, \infty]$. Here $v_+(x)$ and $v_-(x)$ represent the rate at which particles branch and annihilate at site x, respectively. Note that we admit the value ∞ for the annihilation rate: this represents a hard core obstacle, where particles are instantly annihilated. Call an ordered pair $w := (v_-, v_+) \in W$, with coordinates $w(x) = \big(v_-(x), v_+(x)\big)$, a *field configuration*, where $W := ([0, \infty) \times [0, \infty])^{\mathbb{Z}^d}$. We will denote the set of *hard core obstacle sites* by $\mathcal{G}(w) := \{x \in \mathbb{Z}^d : v_-(x) = \infty\}$. Given $r \in [0, \infty)$ and $x \in \mathbb{Z}^d$ we will call $\Lambda(x, r) := \{y \in \mathbb{Z} : \|x - y\| \le r\}$ the ball of radius r centered at x under the norm $\|x\| := \sup_{1 \le i \le d} |x_i|$, where x_i are the coordinates of x. We will furthermore use the notation Λ_r in place of $\Lambda(0, r)$. In this subsection we will construct a stochastic process as the limit as $L \to \infty$ of processes defined on the boxes Λ_L. Throughout the sequel, given a subset $U \subset \mathbb{Z}^d$, we will denote by U^{c} the complement of U and $\delta U := \{x \in U^{\mathrm{c}} : \inf_{y \in U} |x - y| = 1\}$, where $|\cdot|$ denotes the Euclidean distance. So, for each finite subset $U \subset \mathbb{Z}$, we want to consider a process with state space $\Omega_0^w := \{\eta \in \Omega_0 : \eta(x) = 0, x \in \mathcal{G}(w)\}$ defined formally by the infinitesimal generator,

$$
\begin{aligned}
L_U f(\eta) := {} & \sum_{x \in U \cap \mathcal{G}(w)^{\mathrm{c}}} \sum_{\substack{y \in \mathcal{G}(w)^{\mathrm{c}} \\ \|x - y\| = 1}} \kappa \eta(x) \big(f(\eta^{x,y}) - f(\eta)\big) \\
& + \sum_{x \in U \cap \mathcal{G}(w)^{\mathrm{c}}} \sum_{\substack{y \in \mathcal{G}(w) \\ \|x - y\| = 1}} \kappa \eta(x) \big(f(\eta^{x,-}) - f(\eta)\big) \\
& + \sum_{x \in U \cap \mathcal{G}(w)^{\mathrm{c}}} v_+(x) \eta(x) \big(f(\eta^{x,+}) - f(\eta)\big) \\
& + \sum_{x \in \mathcal{G}(w)^{\mathrm{c}}} v_-(x) \eta(x) \big(f(\eta^{x,-}) - f(\eta)\big),
\end{aligned}
\tag{2.1}
$$

acting on an appropriate dense subset $\mathcal{D}_U$ of the space of real bounded functions $B(\Omega_0^w)$, defined on Ω_0^w, endowed with the uniform norm. In the above expression, $\kappa \ge 0$, $\eta^{x,y}$ is the configuration where a particle from site x has jumped to site y so that $\eta^{x,y}(z) = \eta(z)$ if $z \ne x, y$, $\eta^{x,y}(x) = \eta(x) - 1$, and $\eta^{x,y}(y) = \eta(y) + 1$; $\eta^{x,+}$ is the configuration where there is an extra particle at site x and $\eta^{x,-}$ the configuration where one particle at site x has disappeared. It is a well known fact that it is possible to construct a strong Markov process, denoted by $\eta^U := \{\eta^U(t) : t \ge 0\}$, corresponding to an infinitesimal generator of the form (2.1), and taking values on the Skorokhod space $\mathcal{S} := D([0, \infty); \Omega_0^w)$. In fact, such a process falls in the category called $|U|$*-dimensional continuous time Markov branching process* by Athreya and Ney (see Chapter V, Sections 7.1 – 7.2 of Athreya – Ney [3]). Furthermore, it can be shown that a.s. the expected value of each coordinate of such a process $\{\eta^U(t) : t \geq 0\}$, is finite, ensuring that there cannot be infinitely many particles produced in a finite time (see [3, Section 7.1]). Let us now call $\mathcal{P}(\Omega_0^w)$ the set of probability measures defined on Ω_0^w endowed with the Borel σ-algebra associated to the subspace topology of Ω_0^w as a subset of Ω with the product topology. Then, for each field configuration $w \in W$ and probability measure $\nu \in \mathcal{P}(\Omega_0^w)$ denote by $P_\nu^{U,w}$ the law of the process $\{\eta^U(t) : t \geq 0\}$ defined on $\mathcal{S}$ endowed with its Borel σ-field $\mathcal{B}(\mathcal{S})$. We will call this process the *reaction – diffusion process on* U with field w and initial condition ν. In the particular case in which $U = \Lambda_n$ we will use the obvious notations L_n and $P_\nu^{n,w}$. Furthermore, we will call the process on Λ_n, the *reaction – diffusion process at scale* n. Now, note that using the natural coupling [15] and Kolmogorov's extension theorem, it is possible to define for each field configuration $w \in W$ and initial condition $\nu \in \mathcal{P}(\Omega_0^w)$ a probability measure Q_ν^w on the product space $\mathcal{S}^{\mathbb{N}}$, endowed with its Borel σ-algebra induced by the product topology, in such a way that if $\eta^n \in \mathcal{S}$ denotes the nth coordinate of an element $\eta \in \mathcal{S}^{\mathbb{N}}$, $\eta^n(t) \in \Omega_0^w$ its value at time $t \geq 0$ and $\eta^n(t,x) \in \mathbb{N}$ the value at time t of the x-coordinate of $\eta^n(t)$, then,

(i) for every $A \in \mathcal{B}(\mathcal{S})$ and $n \geq 1$,
$$Q_\nu^w[\eta^n \in A] = P_\nu^{n,w}[A].$$
In particular, for every $B \in \mathcal{B}(\Omega_0)$ we have that, $Q_\nu^w[\eta^n(0) \in B] = \nu[B]$.

(ii) for every $n \geq 1$,
$$Q_\nu^w[\eta^{n+1}(t) \geq \eta^n(t)] = 1.$$

(iii) for each $n \geq 1$ define the first exit time from the box Λ_n as
$$T_n := \inf\{t \geq 0 : \sup_{x \in \Lambda_n^c} \eta^n(x,t) > 0\}.$$
Then, for every $n \geq 1$,
$$Q_\nu^w[\eta^n(t) = \eta^{n+1}(t), T_n > t] = Q_\nu^w[T_n > t].$$

Let us now remark that due to property (ii), for every $t \geq 0$ and $x \in \mathbb{Z}^d$ the limit,
$$\eta(t,x) := \lim_{n \to \infty} \eta^n(t,x),$$
exists, possibly taking the value ∞. Define $\eta(t) := \{\eta(t,x) : x \in \mathbb{Z}^d\}$. We denote the stochastic process $\{\eta(t) : t \geq 0\}$, taking values on the space $\overline{\mathbb{N}}^{\mathbb{Z}^d}$, where $\overline{\mathbb{N}}$ is the Alexandrov compactification of the natural numbers, and distributed according to the measure Q_ν^w, the *reaction – diffusion process with field* w *and initial law* ν. We will denote by E_ν^w the corresponding expectation. In addition, for each $t \geq 0$, we define the *total number of particles at time* t by
$$\zeta(t) := \sum_{x \in \mathbb{Z}^d} \eta(t,x).$$
Also, whenever it is true that,
$$Q_\nu^w[\eta(x,t) < \infty, \forall\ x \in \mathbb{Z}^d, t \geq 0] = 1,$$
we will say that with probability one there is *no explosion*. In the sequel we define for each $x \in \mathbb{Z}^d$ the probability measure δ_x on $(\Omega_0^w, \mathcal{B})$ which assigns probability 1 to configurations with one particle at site x and none elsewhere. We will be interested in initial configurations where $\nu = \delta_x$. In such a case we will use the

notation P_x^w instead of $P_{\delta_x}^w$ and E_x^w for the corresponding expectation. In the case where $x \in \mathcal{G}(w)$, we adopt the convention that P_x^w is the probability measure which has a unique atom at the configuration $\eta \equiv 0$ ($\eta(x) = 0$ for every $x \in \mathbb{Z}^d$).

Let us denote by $\mathcal{P}(W)$ the set of probability measures defined on the space $W = ([0,\infty) \times [0,\infty])^{\mathbb{Z}^d}$ endowed with its natural σ-algebra. In the sequel we will take fields v_+ and v_- which are random, assigning a probability measure $\mu \in \mathcal{P}(W)$ in such a way that the field configuration $\{w(x) : x \in \mathbb{Z}^d\}$ has independent coordinates with respect to μ. Furthermore, we will use the notation $\langle \cdot \rangle$ to denote expectation with respect to this law and $Var_\mu(\cdot)$ variance. Now, let us define the *quenched first moment* on $\mathbb{Z}^d$ of the total number of particles at time t starting from site x as, $m(x,t,w) := \mathrm{E}_x^w[\zeta(t)]$, and the *annealed first moment* on $\mathbb{Z}^d$ of the total number of particles at time t starting from site x as, $\langle m(x,t)\rangle := \int m(x,t,w)\,\mathrm{d}\mu$. Furthermore, we call the sets $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ and $\{\langle m(x,t)\rangle : x \in \mathbb{Z}^d\}$, the *fields of quenched first moments* and the *field of annealed first moments*, respectively. Depending on the context, we might write m or $m(x,t)$ in place of $m(x,t,w)$, dropping the dependence on the field configuration w, and $\langle m\rangle$ instead of $\langle m(x,t)\rangle$.

The quantity which will give us a transition mechanism between of the quenched first moments is the *averaged first moment* at scale L and time t defined for a reaction – diffusion process starting from site x as,

$$m^L(x,t,w) := \frac{1}{|\Lambda(x,L)|} \sum_{y \in \Lambda(x,L)} m(y,t,w).$$

2.2. Results giving conditions for no explosion. Here we will give a criteria on the field configuration w, stated as Proposition 2.1, which ensures that there is no explosion in the reaction – diffusion process with field w.

Proposition 2.1. *Consider the reaction diffusion process with field w and initial law ν. Assume that,*

$$\limsup_{|x|\to\infty} \frac{v_+(x)}{|x|\log|x|} = 0. \tag{2.2}$$

Then for every $t \geq 0$ we have that

$$\mathrm{E}_\nu^w[\zeta(t)] < \infty.$$

Hence, there is no explosion.

We state below with the name of Proposition 2.2, a result of Gärtner – Molchanov [12, Lemma 2.5] giving a sufficient condition on the law μ in order that the first moment of the total number of particles $\zeta(t)$ at time t, exists μ-a.s., and hence that there is no explosion. Given a field configuration $w = (v_+, v_-)$, we now need to introduce the *effective potential* $\{v(x) : x \in \mathbb{Z}^d\}$, defined by $v(x) := v_+(x) - v_-(x)$. Furthermore, set $\log_+ x = \log x$ if $x > e$ and $\log_+ x = 1$ otherwise, while define the positive part $x^+ := \max(0, x)$.

Proposition 2.2. *Consider the reaction diffusion process with field w and initial law ν. Assume that the field configuration w is distributed according to a product probability measure $\mu \in \mathcal{P}(W)$. Suppose that*

$$\left\langle \left(\frac{v^+(0)}{\log_+ v(0)} \right)^d \right\rangle < \infty. \tag{2.3}$$

Then μ-a.s. it is true that,

$$\limsup_{|x|\to\infty} \frac{v_+(x)}{|x|\log|x|} = 0,$$

and therefore, μ-a.s. there is no explosion for the reaction – diffusion process with field w and arbitrary initial law in $\mathcal{P}(\Omega_0)$.

Note that the last statement of Proposition 2.2 follows from Proposition 2.1. Furthermore, Assumption E is enough for (2.3) to be satisfied, and hence to ensure no explosion.

2.3. The Gaussian-annealed transition results. Here we will state the main result of this paper, which shows how under different growth of scales, the averaged first moment has an asymptotic behavior where a law of large numbers is satisfied, and a central limit theorem can describe the fluctuations around this law of large numbers. We will assume Condition E, ensuring the existence of the annealed first moments. We will also need to consider the *growth functions* $\{H_1(t) = \log\langle m(0,t)\rangle : t \geq 0\}$ already defined in equation (1.2) of the introduction and the *intermittency exponents* $\{F_\theta : \theta \in \mathbb{R}\}$, defined in equation (1.3). Let us now state the main result of this paper.

Theorem 2.3. *Consider a reaction – diffusion process with initial law δ_0 and field $w = (v_+, v_-)$ distributed according to a product measure $\mu \in \mathcal{P}(W)$. Consider the intermittency exponents $\{F_\theta : \theta \in \mathbb{R}\}$ defined in equation* (1.3) *and the growth functions $\{H_1(t) : t \geq 0\}$ defined in equation* (1.2)*. Assume that Conditions* E *and* MI *are satisfied. Then the following statements are true,*

(i) **Law of large numbers.** *Assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \geq \exp\{F_\epsilon(t)/d\}$. Then in μ-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1, \tag{2.4}$$

as $t \to \infty$. Furthermore, assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \leq \exp\{F_{-\epsilon}(t)/d\}$. Then in μ-probability we have

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \ll 1, \tag{2.5}$$

as $t \to \infty$.

(ii) **Central limit theorem.** *Assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \geq \exp\{F_\epsilon(2t)/d\}$. Then,*

$$\lim_{t\to\infty} \frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{\mathrm{Var}_\mu\big(m(0,t)\big)}} = \mathcal{N}(0,1), \tag{2.6}$$

where $\mathcal{N}(0,1)$ is a centered normalized normal law and the convergence is in the sense of distributions. Furthermore, assume that there is an $\epsilon > 0$ such that eventually in t, $L(t) \leq \exp\{F_{-\epsilon}(2t)/d\}$. Then in μ-probability we have that,

$$\frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{\mathrm{Var}_\mu\big(m(0,t)\big)}} \ll 1. \tag{2.7}$$

To provide a better insight on the meaning of Theorem 2.3, we will see the form that it takes under the stronger Assumption SI, which includes the first universality class of [14]. This will be formulated as a corollary, which in the case $\kappa = 0$

generalizes [4, Theorems 2.1 and 2.2] to include distributions μ of the field, which not necessarily have regularly varying log-tails.

Corollary 2.4. *Consider a reaction – diffusion process with initial law* δ_0 *and field* $w = (v_+, v_-)$ *distributed according to a product measure* $\mu \in \mathcal{P}(W)$. *Consider the cumulant intermittency exponents* $\{G_\theta : \theta \in \mathbb{R}\}$ *defined in equation* (1.4) *and the cumulant generating function* $\{H(t) : t \geq 0\}$ *defined in equation* (1.1). *Assume that Conditions* E *and* SI *are satisfied. Then the following statements are true,*

(i) **Law of large numbers.** *Assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \geq \exp\{G_\epsilon(t)/d\}$. *Then in* μ*-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1,$$

as $t \to \infty$. *In particular,* $\log m^L(0,t,w)/H(t) \sim 1$. *Furthermore, assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \leq \exp\{G_{-\epsilon}(t)/d\}$. *Then in* μ*-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \ll 1,$$

as $t \to \infty$.

(ii) **Central limit theorem.** *Assume that there is an* $\epsilon > 0$ *such that eventually in* t, $L(t) \geq \exp\{G_\epsilon(2t)/d\}$. *Then,*

$$\lim_{t\to\infty} \frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{\mathrm{Var}_\mu\big(m(0,t)\big)}} = \mathcal{N}(0,1),$$

where the convergence is in the sense of distributions. Furthermore, assume that there is an $\epsilon > 0$ *such that eventually in* t, $L(t) \leq \exp\{G_{-\epsilon}(2t)/d\}$. *Then in* μ*-probability we have that,*

$$\frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{\mathrm{Var}_\mu\big(m(0,t)\big)}} \ll 1.$$

The proof of Corollary 2.4 in the case $\kappa = 0$ follows from the observation that $G_\theta(t) = F_\theta(t)$ in this case. The case $\kappa > 0$ is a direct consequence of the fact that $e^{H(t)-2d\kappa t} \leq \langle m(0,t)\rangle \leq e^{H(t)}$, stated in [12, Theorem 3.1], and the observation that $G_\theta(t) \geq 0$ for $\theta > 0$ and $G_\theta(t) \leq 0$ for $\theta < 0$, which follows from Jensen's inequality. Now, the following proposition, which will be proved in Section 4, shows that the condition,

$$\lim_{t\to\infty} tH''(t) = \infty, \tag{2.8}$$

is sufficient for Assumption SI to be true. This condition implies a kind of domination of the branching over the annihilation.

Proposition 2.5. *Consider the cumulant exponents* $\{G_\theta : \theta \in \mathbb{R}\}$. *Assume that condition* (2.8) *is satisfied. Then, Condition* SI *is satisfied. Furthermore*: (i) *for every* $\theta \neq 0$, *there is a* $t_0 \geq 0$, *such that the function* $G_\theta(t)$ *is monotone in* t, *for* $t \geq t_0$; (ii) *there is a* $t_1 \geq 0$, *such that the function* $G_\theta(t)$ *is monotone in* θ *for* $t \geq t_1$.

Condition (2.8) implies that the branching dominates the annihilation, in the sense that $H(t)/t \to \infty$ as $t \to \infty$, which implies that the essential supremum of the random variable $v(0)$ is infinite (see [12]).

2.4. Regularity assumptions on the intermittency exponents. For the purpose of applications, it will be important to identify cases where the assumptions in Theorem 2.3 can be formulated in a more explicit way. As it will be shown, the following assumption on the intermittency exponents, turns out to fall in one of these situations.

Assumption RI. The intermittency exponents $\{F_\theta(t) : \theta \neq \mathbb{R}\}$ satisfy the mild intermittency condition MI. In addition, there exist two increasing functions $f_1, f_2 \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and a function $J(t) \colon [0,\infty) \to [0,\infty)$, such that for $\theta \neq 0$ small enough,

(i) $F_\theta(t) \sim f_1(\theta) J(t)$, and $F_\theta(2t) \sim f_2(\theta) J(t)$.
(ii) There exists two constants γ_1 and γ_2, such that $\lim_{\theta\to 0} f_1(\theta) = \gamma_1$ and $\lim_{\theta \to 0} f_2(\theta) = \gamma_2$.

Throughout the sequel, the constants γ_1 and γ_2 will be called *transition exponents* and the function J *growth exponent*. It will be shown that there exist several important cases of random fields which fall in this category. Furthermore, the following corollary of Theorem 2.3 shows the convenience of Assumption RI.

Corollary 2.6. *Consider a reaction – diffusion process with initial law δ_0 and field $w = (v_+, v_-)$ distributed according to a product measure $\mu \in \mathcal{P}(W)$. Suppose that Assumptions* E *and* RI *are satisfied with transition exponents γ_1 and γ_2 and growth function J. Then the following statements are true,*

(i) **Law of large numbers.** *Assume that there is a $\gamma > \gamma_1$ such that eventually in t, $\log L(t) \geq \gamma J(t)/d$. Then in μ-probability we have*

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1,$$

as $t \to \infty$. Furthermore, assume that there is a $0 < \gamma < \gamma_1$ such that eventually in t, $\log L(t) \leq \gamma J(t)/d$. Then in μ-probability we have

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \ll 1,$$

as $t \to \infty$.

(ii) **Central limit theorem.** *Assume that there is a $\gamma > \gamma_2$ such that eventually in t, $\log L(t) \geq \gamma J(t)/d$. Then,*

$$\lim_{t\to\infty} \frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{\mathrm{Var}_\mu\big(m(0,t)\big)}} = \mathcal{N}(0,1),$$

where the convergence is in the sense of distributions. Furthermore, assume that there is a $0 < \gamma < \gamma_2$ such that eventually in t, $\log L(t) \leq \gamma J(t)/d$. Then in μ-probability we have that,

$$\frac{m^L(0,t,w) - \langle m(0,t)\rangle}{\sqrt{\mathrm{Var}_\mu\big(m(0,t)\big)}} \ll 1.$$

In the next subsection we will apply Corollary 2.6 to four situations each one falling in one of the universality classes described by van den Hofstad et al.in [14]. These classes encompass all possible situations under three conditions. The first is Condition E, ensuring the existence of the positive moments defining the cumulant generating functions (1.1). The second and third condition avoids different qualitative behaviors of the potential at different scales. Let us formulate next the second condition of [14].

Assumption H. The function $H(t)/t$ is in the de Haan class.

A function f is said to be in the de Haan class if for some regularly varying function $g\colon (0,\infty) \to \mathbb{R}$, we have that $\big(f(\lambda t) - f(t)\big)/g(t)$ converges to a limit different from 0 as $t \to \infty$, for $\lambda > 0$. Let us recall that a function h is regularly varying at infinity with index ρ, if for any $a > 0$ we have $\lim_{x\to\infty} h(ax)/h(x) = a^\rho$. This property will be stated as $h \in R_\rho$. Whenever H is satisfied, then $H(t)$ is regularly varying with index $\gamma \geq 0$. Furthermore, in [14, Proposition 1.1], it is proved that under Assumption H there exist a function $\widehat{H}\colon (0,\infty) \to \mathbb{R}$ and a continuous function $k(t)\colon (0,\infty) \to (0,\infty)$ such that,

$$\lim_{t\to\infty} \frac{H(ty) - yH(t)}{k(t)} = \widehat{H}(y) \neq 0, \tag{2.9}$$

for $y \in (0,1) \cup (1,\infty)$. It is also shown that $k(t)$ is regularly varying of index γ. We can now recall the third assumption of [14].

Assumption K. The limit $k^* = \lim_{t\to\infty} k(t)/t$ exists in $[0,\infty]$.

Under Assumptions E, H, and K, the four universality classes defined in [14] are:

(1) $\gamma > 1$, or $\gamma = 1$ and $k^* = \infty$.
(2) $\gamma = 1$ and $k^* \in (0,\infty)$.
(3) $\gamma = 1$ and $k^* = 0$.
(4) $\gamma < 1$.

In what follows we exhibit examples in each one of these classes satisfying Assumption RI so that Corollary 2.6 can be applied, and the transition exponents can be explicitly written. In the sequel, following [14], we will call the third class the class of *almost bounded potentials.*

2.4.1. *Unbounded potentials with Weibull-type tails.* Our first application of Corollary 2.4 will be to an example falling in the first universality class of [14]. We will assume that the essential supremum of $v(0)$ is ∞, and the tails at ∞ of $v(0)$ follow a Weibull-type law, $\mu[v(0) > x] = \exp\{-h(x)\}$ for $x > 0$, with $h \in R_\rho$ for some $1 < \rho < \infty$. In the terminology of [4] in the context of i.i.d. random exponentials ($\kappa = 0$ in our situation), this is called *Case* B. The following proposition shows that Assumption RI is satisfied in this situation, and hence Corollary 2.6, which generalizes [4, Theorems 2.1 and 2.2] in Case B form $\kappa = 0$ to $\kappa > 0$.

Proposition 2.7. *Suppose that the essential supremum of the effective potential* $v(0)$ *is* ∞ *with Weibull-type tails* $\mu[v(0) > x] = \exp\{-h(x)\}$ *for* $x > 0$*, and* $h \in R_\rho$ *for some* $1 < \rho < \infty$*. Then Assumption* RI *is satisfied with transition exponents*

$$\gamma_1 = \frac{1}{\rho - 1}, \quad \gamma_2 = 2^{\frac{\rho}{\rho-1}} \frac{1}{\rho - 1},$$

and growth exponent,

$$J(t) = H(t).$$

Let us now prove Proposition 2.7. Note that under the conditions on the tail of the law of $v(0)$, the cumulant generating function $H(t)$ of $v(0)$ is well defined, smooth, nondecreasing and tends to infinity as $t \to \infty$. Furthermore, by the Kasahara exponential Tauberian theorem (see Bingham et al. [6, Theorem 4.12.7]), we know that $H \in R_{\rho'}$, where the index ρ' is defined by the equation,

$$\frac{1}{\rho} + \frac{1}{\rho'} = 1.$$

From this observation it is easy to check that Assumption SI of Corollary 2.4 is satisfied, and that for every $\epsilon \neq 0$, the cumulant growth exponents $G_\epsilon(t)$ and $G_\epsilon(2t)$ satisfy

$$\frac{G_\epsilon(t)}{H(t)} \sim \frac{(1+\epsilon)^{\rho'} - (1+\epsilon)}{\epsilon}$$

and

$$\frac{G_\epsilon(2t)}{H(t)} \sim 2^{\rho'} \frac{(1+\epsilon)^{\rho'} - (1+\epsilon)}{\epsilon}.$$

Now note that $\big((1+\epsilon)^{\rho'} - (1+\epsilon)\big)/\epsilon$ is increasing in ϵ and converges to $\gamma_1 = 1/(\rho-1)$ as $\epsilon \to 0$, and similarly $2^{\rho'}\big((1+\epsilon)^{\rho'} - (1+\epsilon)\big)/\epsilon$ is increasing in ϵ and converges to $\gamma_2 = 2^{\rho/(\rho-1)}\gamma_1$ as $\epsilon \to 0$.

2.4.2. *Unbounded potentials with double exponential type tails.* Here we consider the second universality class so that H is regularly varying with index $\gamma = 1$ and $k^* \in (0,\infty)$. As shown in [14, Proposition 1.1], this is equivalent to the existence of a constant $\rho \in (0,\infty)$ such that

$$\lim_{t\to\infty} \frac{H(yt) - yH(t)}{t} = \rho y \log y, \tag{2.10}$$

for all $y \in (0,1) \cup (1,\infty)$ (this and $0 < \rho < \infty$ is called Assumption H in [13]). Furthermore, under assumption (2.10) it is true that,

$$\lim_{t\to\infty} \frac{H(t)}{t} = \infty.$$

This second universality class includes the case of unbounded potentials which are double exponentially distributed with parameter ρ, $0 < \rho < \infty$,

$$\mu[v(0) > x] = \exp\{-\mathrm{e}^{x/\rho}\},$$

for $x \in \mathbb{R}$. We then have the following interesting proposition.

Proposition 2.8. *Suppose that assumption* (2.10) *is satisfied for some* $\rho \in (0,\infty)$*. Then Assumption* RI *is satisfied with transition exponents*

$$\gamma_1 = \rho, \quad \gamma_2 = 2\rho,$$

and growth exponent,

$$J(t) = t.$$

To prove Proposition 2.8, we quote [13, Theorem 1.2], which shows that under assumption (2.10) we have that,

$$\langle m(0,t)\rangle = \exp\left\{H(t) - 2d\kappa\chi\left(\frac{\rho}{\kappa}\right)t + o(t)\right\},$$

for $\kappa > 0$, where $\chi(x) := \frac{1}{2}\inf_{p\in\mathcal{P}(\mathbb{Z})}[S(p) + \rho I(p)]$ for $x \geq 0$, $\mathcal{P}(\mathbb{Z})$ is the space of probability measure on $\mathbb{Z}$, $S\colon \mathcal{P}(\mathbb{Z}) \to [0,\infty)$ is the Donsker−Varadhan functional defined by $S(p) := \sum_{x\in\mathbb{Z}}\big(\sqrt{p(x+1)} - \sqrt{p(x)}\big)^2$, while $I\colon \mathcal{P}(\mathbb{Z}) \to [0,\infty)$ is the entropy functional defined by $I(p) := -\sum_{x\in\mathbb{Z}} p(x)\log p(x)$. On the other hand, from the previous discussion we can conclude that, for every $\epsilon \neq 0$,

$$\frac{F_\epsilon(t)}{t} \sim \rho(1+\epsilon)\frac{\log(1+\epsilon)}{\epsilon},$$

and

$$\frac{F_\epsilon(2t)}{t} \sim 2\rho(1+\epsilon)\frac{\log(1+\epsilon)}{\epsilon}.$$

From these limiting behaviors, we see that Assumption MI is satisfied. Furthermore, form the fact that $(1+\epsilon)\log(1+\epsilon)/\epsilon$ is increasing for ϵ small enough and converges to 1 as $\epsilon \to 0$, we obtain the transition exponents at ρ and 2ρ of Proposition 2.8.

2.4.3. *Almost bounded potentials.* We now focus on the third universality class, where $\gamma = 1$ and $k^* = 0$. As shown in [14, Theorem 1.4], in this case it is true that,

$$\log\langle m(0,t)\rangle \sim \frac{H(t\alpha_t^{-d})}{\alpha_t^{-d}}, \tag{2.11}$$

where $\alpha_t\colon [0,\infty) \to [0,\infty)$ is the so called *scaling function*, which is implicitly defined for all $t > 0$ sufficiently large by the equation,

$$\frac{k(t\alpha_t^{-d})}{t\alpha_t^{-d}} = \frac{1}{\alpha_t^2}. \tag{2.12}$$

As shown in [14, Proposition 1.2], this function is unique up to asymptotic equivalence. Furthermore, for the third universality class, it is a slowly varying function (regularly varying of index 0).

Proposition 2.9. *Suppose that H is regularly varying of index* 1 *and that* (2.9) *is satisfied for $k(t)$ such that $k(t) \ll t$. Then Assumption* RI *is satisfied with transition exponents*

$$\gamma_1 = \rho, \quad \gamma_2 = 2\rho,$$

and growth exponent,

$$J(t) = \frac{t}{\alpha_t^2}. \tag{2.13}$$

The proof of Proposition 2.9 follows now applying the definition of α_t through (2.12), the asymptotics (2.11) and (2.9).

A specific example of a distribution falling in the third universality class is given by the *squared double exponential* law. In other words, an effective potential $v(0)$ which is unbounded, with law,

$$\mu[v(0) > x] = \exp\{-e^{x^2}\},$$

for $x \geq 0$. As it can be deduced from the discussion of [14, Example 1.4.3], in this case the scale function is given by $\alpha_t \sim 2^{1/2}(\log t)^{1/4}$ and $k(t) \sim t/(2\sqrt{\log t})$. Furthermore, $\rho = 1$ and the growth exponent can be chosen as $J(t) \sim t/(2(\log t)^{1/2})$ (see Table 2 of the introduction).

2.4.4. *Bounded potentials with Fréchet-type tails.* We now continue with an example falling in the universality class (4) of [14]. We will consider the case in which the essential supremum of $v(0)$ is 0 and the tails are of Fréchet type: $\mu[v(0) > -x] = \exp\{-h(x^{-1}\}$ for $x > 0$, with $h \in R_\rho$ for some $0 < \rho < \infty$. Using the terminology of [4] in the context of i.i.d. random exponentials, this is *Case* A. To state appropriately this result, we will need to recall the work of Biskup and König [7], who studied the asymptotics of the annealed and quenched first moments in the case of product environments μ such that the cumulant generating function $H(t)$ is in the so called γ-class for some $\gamma \in [0, 1)$. We say that H *is in the* γ-*class* if the essential supremum of $v(0)$ is 0 and if there is a nondecreasing function $\alpha_t \in (0, \infty)$ and a function $\widetilde{H}\colon [0, \infty) \to (-\infty, 0]$, $\widetilde{H} \neq 0$, such that

$$\lim_{t\to\infty} \frac{\alpha_t^{d+2}}{t} H\left(\frac{t}{\alpha_t^d} y\right) = \widetilde{H}(t), \tag{2.14}$$

for $y \geq 0$, uniformly on compact sets in $(0, \infty)$. We will denote the function α_t the *scale function.* It is not difficult to show, using the de Bruijn exponential Tauberian theorem [6], that if $v(0)$ is in Case A for some $\rho > 0$, then H is in the ρ'-class for $\rho' = \rho/(\rho + 1)$, or $1/\rho' - 1/\rho = 1$. We can now sate the following proposition.

Proposition 2.10. *Suppose that the essential supremum of the effective potential* $v(0)$ *is* 0 *with Fréchet-type tails* $\mu[v(0) > -x] = \exp\{-h(x^{-1})\}$ *for* $x > 0$, *and* $h \in R_\rho$ *for some* $0 < \rho < \infty$. *Then Assumption* RI *is satisfied with transition exponents*

$$\gamma_1 = \left(\frac{1}{d + 2 + 2\rho}\right)^2, \quad \gamma_2 = 2^{1-\gamma_1}\gamma_1,$$

and growth exponent,

$$J(t) = \chi \frac{t}{\alpha_t^2},$$

for some constant $\chi \in (0, \infty)$.

As a consequence of Proposition 2.10 we can now apply Corollary 2.6, generalizing Case A of [4, Theorems 2.1 and 2.2] from $\kappa = 0$ to $\kappa \geq 0$. In contrast to Proposition 2.7, where there is no change in the value of the transition exponents γ_1 and γ_2 from $\kappa = 0$ to $\kappa > 0$, here there is.

Let us now prove Proposition 2.10. In [7, Proposition 2.1], it is shown that whenever μ is in the ρ'-class, then the scaling function $\alpha_t \in R_\nu$, for,

$$\nu := \frac{1 - \rho'}{d + 2 - d\rho'}.$$

Furthermore, [7, Theorem 1.2] states that when μ is in the ρ'-class, there exists a $\chi \in (0, \infty)$ such that,

$$\log\langle m(0, t)^\beta\rangle \sim -\chi \frac{\beta t}{\alpha_{\beta t}^2},$$

for every $\beta \in (0, \infty)$. Then Condition MI of Theorem 2.3 is satisfied, and for every $\epsilon \neq 0$, the growth exponents $F_\epsilon(t)$ and $F_\epsilon(2t)$ satisfy

$$\frac{F_\epsilon(t)}{t/\alpha_t^2} \sim \frac{(1 + \epsilon) - (1 + \epsilon)^{1-\nu^2}}{\epsilon}$$

and

$$\frac{F_\epsilon(2t)}{t/\alpha_t^2} \sim 2^{1-\nu^2} \frac{(1+\epsilon)-(1+\epsilon)^{1-\nu^2}}{\epsilon}.$$

As in the proof of Proposition 2.7, we can show that $\big((1+\epsilon)-(1+\epsilon)^{1-\nu^2}\big)/\epsilon$ is increasing in ϵ and converges to $\gamma_1=\nu^2$ as $\epsilon\to 0$, and similarly $2^{1-\nu^2}\big((1+\epsilon)-(1+\epsilon)^{1-\nu^2}\big)/\epsilon$ is increasing in ϵ and converges to $\gamma_2 = 2^{1-\nu^2}\gamma_1$ as $\epsilon\to 0$.

2.5. The critical regime. For the examples discussed in the four previous sections, it is possible to obtain the following improvement of Theorem 2.3.

Theorem 2.11. *Consider a reaction – diffusion process with initial law δ_0 and field $w=(v_+,v_-)$ distributed according to a product measure $\mu\in\mathcal{P}(W)$. Then the following statements are satisfied.*

(i) **Weibull type.** *Assume that μ has Weibull-type tails so that $\mu[v(0)>x]=\exp\{-h(x)\}$ for $x>0$, $h\in R_\rho$ for $1<\rho<\infty$. Then if $d\log L(t)\le\gamma H(t)$ eventually in t, and $0<\gamma<\gamma_1=1/(\rho-1)$, we have that for every $\delta>0$ in μ-probability,*

$$\frac{m_L(0,t)}{\mathrm{e}^{(a_{\mathrm{W}}(\gamma)+\delta)H(t)}} \ll 1,$$

where

$$a_{\mathrm{W}}(\gamma) := \frac{\rho}{\rho-1}[(\rho-1)\gamma]^{1/\rho}-\gamma.$$

(ii) **Double exponential type.** *Assume that μ satisfies assumption* (2.10) *for some constant $\rho\in(0,\infty)$. Then if, $d\log L(t)\le\gamma t$ eventually in t, and $\gamma<\gamma_1=\rho$, we have that for every $\delta>0$ in μ-probability,*

$$\frac{m_L(0,t)}{\exp\big\{H\big((a_{\mathrm{D}}(\gamma)+\delta)t\big)/(a_{\mathrm{D}}(\gamma)+\delta)\big\}} \ll 1,$$

where

$$a_{\mathrm{D}}(\gamma) := \gamma\mathrm{e}^{(\gamma-\rho)/\rho}.$$

(iii) **Almost bounded potentials.** *Assume that μ is such that H is regularly varying of index* 1 *and that* (2.9) *is satisfied for $k(t)$ such that $k(t)\ll t$. Consider the growth exponent $J(t)$ as defined in equation* (2.13). *Then if, $d\log L(t)\le\gamma J(t)$ eventually in t, and $\gamma<\gamma_1=1$, we have that for every $\delta>0$ in μ-probability,*

$$\frac{m_L(0,t)}{\exp\big\{H\big((a_{\mathrm{A}}(\gamma)+\delta)t\big)/(a_{\mathrm{D}}(\gamma)+\delta)\big\}} \ll 1,$$

where

$$a_{\mathrm{A}}(\gamma) := \gamma\mathrm{e}^{(\gamma-\rho)/\rho}.$$

(iv) **Fréchet type.** *Assume that μ is such that essupv$(0)=0$ and is of Fréchet type so that $\mu[v(0)>-x]=\exp\{-h(x^{-1})\}$ for $x>0$, and $h\in R_\rho$ for some $\rho\in(0,\infty)$. Then if $d\log L(t)\le\gamma J(t)$ eventually in t, and $\gamma<\gamma_1=\nu^2$, we have that for every $\delta>0$ in μ-probability,*

$$\frac{m_L(0,t)}{\mathrm{e}^{-(a_{\mathrm{F}}(\gamma)-\delta)J(t)}} \ll 1,$$

where

$$a_{\mathrm{F}}(\gamma) := (1-\nu^2)\left(\frac{\gamma}{\nu^2}\right)^{-\nu^2/(1-\nu^2)}+\gamma.$$

Let us remark that Theorem 2.11(iv) includes as a particular case, Case 3 of [5, Theorem 2(i)]. We believe that the four functions a_{W}, a_{D}, a_{A}, a_{F} are sharp, in the sense that the quantities of the four parts of Theorem 2.11 diverge if the sign of δ is changed. Also, these four functions have as maximum value 1, which is reached at γ_1.

In the special case in which μ is a double exponential law so that $\log \mu[v(0) > x] = -\mathrm{e}^{x^2}$ in Theorem 2.11(ii), the function $\exp\{H\big((a_{\mathrm{D}}(\gamma)+\delta)t\big)/(a_{\mathrm{D}}(\gamma)+\delta)\}$ takes the form $\exp\{t\log\big((a_{\mathrm{D}}(\gamma)+\delta)t/e\big)\}$, showing how the transition mechanism takes place at a logarithmic order in the exponent in contrast to the polynomial one of parts (i) and (iv). The whole picture suggested by Theorem 2.11 seems to indicate the presence of a phase transition type behavior, as it is found in some mean field statistical mechanics models like the Random Energy Model [8, 9]. Indeed, when combined with Theorem 2.3(i), we conclude that $\log m_L(0,t,w) \sim \bar{a}(\gamma)J(t)$, where $\bar{a}(\gamma)$ equals a_{W}, a_{D}, a_{A} or a_{F} depending on the potential for $\gamma < \gamma_1$, while $\bar{a}(\gamma) = 1$ for $\gamma > 1$. Thus, there is nonanalyticity at $\gamma = 1$ of a quantity playing the role of a "free energy."

3. The conditions for no explosion

In this section we will prove Proposition 2.1. Since the initial conditions $\nu \in \mathcal{P}(V)$ are concentrated on configurations with a finite number of particles, and by translation invariance of the dynamics of the reaction – diffusion process on $\mathbb{Z}^d$, note that it is enough to consider the case where $\nu = \delta_0$.

3.1. Preliminary lemmas. Let us consider the reaction – diffusion process at scale n with field w satisfying condition (2.2), and with the initial condition δ_0. Define the quantities,

$$\zeta^n(t) := \sum_{x\in\Lambda_n} \eta^n(t,x),$$

representing the total number of particles produced at time t and,

$$\bar{\zeta}^n(t) := \sum_{x\in\Lambda_n^{\mathrm{c}}} (\eta^n(t,x) - \eta^n(0,x)),$$

representing the total number of particles which have touched Λ_n^{c} in the time interval $[0,t]$. From this definition, we can conclude that for $m > n \geq 1$ it is true that,

$$\zeta^m(t) = \zeta^n(t) + \sum_{k=1}^{\bar{\zeta}^n(t)} \zeta^m_{x_k}(t-\tau_k), \tag{3.1}$$

where for $1 \leq k \leq \bar{\zeta}^n(t)$, $x_k \in \delta\Lambda_n$ is the set of exit sites from Λ_n of the random walks which have touched the set Λ_n^{c} in the time interval $[0,t]$, $0 \leq \tau_k \leq t$ the exit times of each one of these random walks and $\{\zeta^m_{x_k}(s) : s \geq 0\}$ is a set of independent processes such that $\zeta^m_{x_k}(s) = 0$ for $s < 0$, $\zeta^m_{x_k}(s) = \delta_{x_k}$ for $s = 0$ while $\{\zeta^m_{x_k}(s) : s \geq 0\}$ has the law $P^{m,w}_{x_k}$. Let us now define for each $n \in \mathbb{N}$ the maximum value of the field v_+ on the box Λ_n by $v_n := \max_{x\in\Lambda_n} v_+(x)$.

Lemma 3.1. *Consider the reaction – diffusion process at scale n with field w and initial condition δ_0. Then,*

(i) *For every $t \geq 0$ and $n \geq 1$,*

$$\mathrm{E}_0^w[\zeta^n(t)] \leq \exp\{v_n t\}.$$

(ii) *For every $t \geq 0$ and $n \geq 2\kappa t$,*

$$\mathrm{E}_0^w[\bar{\zeta}^n(t)] \leq 4d \exp\left\{(v_n - 2\kappa)t - n \log\left(\frac{n}{2e\kappa t}\right)\right\}.$$

(iii) *For every $t \geq 0$ and $m > n \geq 1$,*

$$\mathrm{E}_0^w[\zeta^m(t)] \leq \mathrm{E}_0^w[\zeta^n(t)] + \mathrm{E}_0^w[\bar{\zeta}^n(t)]\,\mathrm{E}_0^w[\zeta^{2n+m}(t)].$$

The following elementary lemma will be used to prove Lemma 3.1. We will need to define,

$$I(y) := \sup_{\lambda \in \mathbb{R}}\{\lambda y - (\cosh \lambda - 1)\} = y \sinh^{-1} y - \sqrt{1 + y^2} + 1, \tag{3.2}$$

Note that $I\colon [0, \infty) \to [0, \infty)$ is one to one and that $I(x) > 0$ for $x > 0$.

Lemma 3.2. *Let $\{X_t : t \geq 0\}$ be a simple symmetric continuous time random walk on $\mathbb{Z}$ of total jump rate $2\kappa > 0$ starting from 0. For a nonnegative real x define $\tau_x := \inf\{t \geq 0 : |X_t| \geq x\}$ as the first exit time of this random walk from the interval Λ_x. Then, if P is its law, we have*

$$P[\tau_x < t] \leq 4 \exp\left\{-2\kappa t I\left(\frac{x}{2\kappa t}\right)\right\} \leq 4 \exp\left\{-2\kappa t - x \log\left(\frac{x}{2e\kappa t}\right)\right\}, \tag{3.3}$$

the second inequality being satisfied only for $x \geq 2\kappa t$.

The proof of Lemma 3.2 is a simple large deviation estimate and will be omitted.

Proof of Lemma 3.1. (i) Let us remark that for every bounded nondecreasing function $f\colon \mathbb{N} \to \mathbb{R}$ which is eventually constant we have,

$$L_n f(\zeta^n) \leq v_n \zeta^n \big(f(\zeta^n + 1) - f(\zeta^n)\big).$$

For a natural $N \geq 1$ fixed, choose $f(m) = m \wedge N$. Using the fact that $\zeta^n(0) = 1$, we then conclude that,

$$\zeta^n(t) \wedge N - 1 - v_n \int_0^t \zeta^n(s)\theta_{[0,N-1]}\big(\zeta^n(s)\big)\,\mathrm{d}s, \tag{3.4}$$

where for $A \subset \mathbb{R}$, θ_A is the indicator function of the set A, is a super-martingale. Hence, since the integrand of the integral in (3.4) is a positive function, by Fubini's theorem,

$$\mathrm{E}_0^w\big[\zeta^n(t)\theta_{[0,N-1]}\big(\zeta^n(t)\big)\big] \leq 1 + v_n \int_0^t \mathrm{E}_0^w\big[\zeta^n(s)\theta_{[0,N-1]}\big(\zeta^n(s)\big)\big]\,\mathrm{d}s.$$

Therefore, by Gronwall's lemma,

$$\mathrm{E}_0^w[\zeta^n(t)\theta_{[0,N-1]}(\zeta^n(t)] \leq \exp\{v_n t\}.$$

Taking the limit when $N \to \infty$ and using the monotone convergence theorem we conclude the proof of part (i) of the lemma.

(ii) Let us note the following identity,

$$\mathrm{E}_0^w[\bar\zeta^n(t)] \le d\mathrm{e}^{v_n t} P[\tau_n < t], \tag{3.5}$$

where τ_n is the first exit time of a simple symmetric continuous time random walk of total jump rate $2d\kappa$, starting from the origin 0, from the box Λ_n and P is its law. Now, from the second inequality of equation (3.3) applied to each of the d coordinates of such a random walk, we conclude that,

$$P[\tau_n < t] \le 4d \exp\left\{-2\kappa t - n\log\left(\frac{n}{2e\kappa t}\right)\right\},$$

substituting the corresponding expression back in (3.5) and using part (i) of the proposition we conclude the proof of the lemma.

(iii) From (3.1) we have that,

$$\mathrm{E}_0^w[\zeta^m(t)] = \mathrm{E}_0^w[\zeta^n(t)] + \mathrm{E}_0^w\left[\sum_{k=1}^{\bar\zeta^n(t)} \mathrm{E}_0^w[\zeta^m_{x_k}(t-\tau_k) \mid \bar\zeta^n(t)]\right].$$

But, $\mathrm{E}_0^w[\zeta^m_{x_k}(t-\tau_k) \mid \bar\zeta^n(t)] \le \mathrm{E}_0^w[\zeta^{2n+m}(t)]$, which concludes the proof. □

3.2. Proof of Proposition 2.1. We will now prove Proposition 2.1 with the help of Lemma 3.1. Let $\delta = \frac{1}{5}$ and choose N so that $v_n \le \delta\big(n\log\big(n/(2e\kappa t)\big) - 4n\log 4\big)/t$ whenever $n \ge N$. By part (i) of the lemma we have that,

$$\mathrm{E}_x^w[\zeta^n(t)] \le \exp\left\{\delta\left(n\log\left(\frac{n}{2e\kappa t}\right) - 4n\log 4\right)\right\},$$

while by part (ii) we have,

$$\mathrm{E}_x^w[\bar\zeta^n(t)] \le 4d\exp\left\{-(1-\delta)n\log\left(\frac{n}{2e\kappa t}\right) - 4\delta n\log 4\right\},$$

whenever $n \ge N$. Choosing $m = 2n > N$ in part (iii) of the same lemma, it follows that,

$$\mathrm{E}_x^w[\zeta^{2n}(t)] \le A_n + B_n\,\mathrm{E}_x^w[\zeta^{4n}(t)],$$

where $A_n := \mathrm{e}^{\delta n\log(n/(2e\kappa t))}$ and $B_n := 4d\mathrm{e}^{-(1-\delta)n\log(n(2e\kappa t)) - 4\delta n\log 4}$. Repeating the bound for $2n$ and $m = 4n$ and substituting back we get,

$$\mathrm{E}_x^w[\zeta^{2n}(t)] \le A_n + B_nA_{2n} + B_nB_{2n}\,\mathrm{E}_x^w[\zeta^{8n}(t)].$$

Now, by induction on m, we get that,

$$\mathrm{E}_x^w[\zeta^{2n}(t)] \le \sum_{k=0}^{m-1} c_k + c_m\frac{\mathrm{E}_x^w[\zeta^{n2^{m+1}}(t)]}{A_{n2^m}}, \tag{3.6}$$

where $c_0 := A_n$ and for $k \ge 0$, $c_{k+1} := c_kB_{n2^k}A_{n2^{k+1}}/A_{n2^k}$. Now,

$$\frac{B_{n2^k}A_{n2^{k+1}}}{A_{n2^k}} \le 2d\mathrm{e}^{-(1-2\delta)n2^k\log(n2^k/(2e\kappa t))}.$$

Hence, by d'Alembert test and the fact that $1-2\delta > 0$ we know that the series $\sum_{k=0}^\infty c_k$ is convergent. On the other hand we have,

$$c_m\frac{\mathrm{E}_x^w[\zeta^{n2^{m+1}}(t)]}{A_{n2^m}} \le c_{m-1}\mathrm{e}^{\delta n2^m\log(n2^m/(2e\kappa t)) - (1-2\delta)n2^{m-1}\log(n2^{m-1}/(2e\kappa t))},$$

which tends to 0 since $2\delta < 1-2\delta$ and $c_m < 1$ for m large enough. Taking the limit when $m \to \infty$, then when $n \to \infty$ and using the monotone convergence theorem in inequality (3.6), we deduce that

$$\mathrm{E}_x^w[\zeta(t)] \le \sum_{k=0}^{\infty} c_k < \infty.$$

4. Moment and correlation estimates

Here we will obtain some important bounds for the large time asymptotic behavior of the field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$, the annealed first moment field $\{\langle m(x,t)\rangle : x \in \mathbb{Z}^d\}$, and their correlations. In the first subsection, we will prove Proposition 2.5.

4.1. Proof of Proposition 2.5. Our first lemma states a useful super-additivity and convexity property of the cumulant generating function of the random variable $v(0)$.

Lemma 4.1. *Consider the cumulant generating function* $H(t)\colon [0,\infty) \to \mathbb{R}$ *of the random variable* $v(0)$*, defined in equation* (1.1)*. Then, the following statements are true.*

(i) *H is super-additive. In other words, if $t_1, \dots, t_n$ are nonnegative reals then,*

$$H(t_1 + \cdots + t_n) \ge H(t_1) + \cdots + H(t_n).$$

(ii) *Assume that* $\lim_{t\to\infty} tH''(t) = \infty$*. Then, for every* $\alpha > 1$,

$$\lim_{t\to\infty} \frac{H(\alpha t) - \alpha H(t)}{t} = \infty. \tag{4.1}$$

Proof. (i) Let $t_1, t_2 \ge 0$. From Hölder's inequality, we have that $\phi(t_1)\phi(t_2) \le \phi(t_1 + t_2)$. Hence, $H(t_1 + t_2) \ge H(t_1) + H(t_2)$. By induction on n we conclude the proof.

(ii) From the assumption, note that we can write $H''(t) = f(t)/t$, where $\lim_{t\to\infty} f(t) = \infty$. Integrating the function H'' from t to αt, it follows that,

$$H'(\alpha t) - H'(t) \ge \inf_{s\ge t} f(s) \cdot \log \alpha. \tag{4.2}$$

Integrating again we obtain for $u > t$ that,

$$H(\alpha t) - \alpha H(t) \ge (t-u) \inf_{s\ge u} f(s) \cdot \alpha \log \alpha + c(u),$$

where $c(u) = H(\alpha u) - \alpha H(u)$. Dividing by t, taking the limit when $t \to \infty$ and then the limit when $u \to \infty$, we obtain (4.1). □

Let us now prove Proposition 2.5(i) and (ii). Note that,

$$\frac{\partial G}{\partial t} = \frac{(1+\theta)H'\big((1+\theta)t\big) - (1+\theta)H'(t)}{\theta}.$$

Then, inequality (4.2) implies part (i) of the lemma. On the other hand we have that,

$$\frac{\partial G}{\partial \theta} = \frac{\theta t H'((1+\theta)t) - H\big((1+\theta)t\big) + H(t)}{\theta^2},$$

By the mean value theorem there exists a $\bar\theta$ such that $0 < \bar\theta < \theta$ and $H\big((1+\theta)t\big) - H(t) = \theta t H'\big((1+\bar\theta)t\big)$ and hence

$$\theta t H'\big((1+\theta)t\big) - H\big((1+\theta)t\big) + H(t) = \theta t\big[H'\big((1+\theta)t\big) - H'\big((1+\bar\theta)t\big)\big],$$

which by inequality (4.2) is positive if $t \geq t_1$, where t_1 is independent of θ and $\bar\theta$. This proves that $G_\theta(t)$ is monotone in θ for $t \geq t_1$.

We now show that condition (2.8) ensures SI. We will without loss of generality assume that $\kappa > 0$. Let us define real valued functions f, g by $f(x) := H(\theta t + xt) - H(xt)$ and $g(x) := 2x$ for real x. By the generalized mean value theorem applied in the interval $[1, 1+\theta]$, there exists a $\theta_1 \in (0, \theta)$, such that $\big(f(1+\theta) - f(1)\big)/\big(g(1+\theta) - g(1)\big) = f'(1+\theta_1)/g'(1+\theta_1)$. In other words, the expression $G_{2\theta}(t) - G_\theta(t) = \big(H\big((1+2\theta)t\big) - 2H\big((1+\theta)t\big) + H(t)\big)/(2\theta)$, equals,

$$\frac{t}{2}\left(H'\big((1+\theta+\theta_1)t\big) - H'\big((1+\theta_1)t\big)\right) = \frac{\theta t^2}{2} H''\big((1+\theta_1+\theta_2)t\big),$$

where in the last equality we have applied the mean value theorem and $\theta_2 \in (0, \theta)$. It therefore follows that there is a function $\bar\theta\colon [0,\infty) \to (0, 2\theta)$ such that,

$$\frac{G_{2\theta}(t) - G_\theta(t)}{t} = \frac{\theta t}{2} H''((1+\bar\theta(t))t).$$

Our hypothesis $\lim_{t\to\infty} tH''(t) = \infty$, shows that the expression above tends to ∞ as $t \to \infty$. The proof of the case in which $\theta < 0$ is similar and the details will be omitted.

We now continue in Section 4.2, defining the truncated quenched first moments, and then describing the parabolic Anderson equation satisfied by the quenched first moments and the corresponding Feynman – Kac representations.

4.2. Truncated quenched first moments. In the sequel, given a real function $f(x)$ defined on $\mathbb{Z}^d$, we will define the discrete Laplacian by,

$$\Delta f(x) := \sum_{e \in \mathbb{Z}^d : |e| = 1} \big(f(x+e) - f(x)\big). \tag{4.3}$$

Let us now for each finite set $U \subset \mathbb{Z}^d$ and environment $w \in W$, define the field $w^U := (v_-^U, v_+)$ with $v_-^U(x) = v_-(x)$ for $x \in U$, while $v_-^U(x) = \infty$ for $x \notin U$. We now define for $x \in \mathbb{Z}^d$ and $t \geq 0$,

$$\widetilde{m}_U(x,t,w) := m(x,t,w^U).$$

As it will be seen later, this expression satisfies the parabolic Anderson equation with Dirichlet boundary conditions. We will denote this quantity the *truncated quenched first moment* on U at time t for a reaction – diffusion process starting from x. Also, we will call the set $\{\widetilde{m}_U(x,t,w) : x \in \mathbb{Z}^d\}$, the *field of truncated first moments* on U at time t. Now, in the particular case in which $U = \Lambda(x, r)$ for some $r > 0$, we will use the notation $\widetilde{m}_r(x,t,w)$ instead of $\widetilde{m}_U(x,t,w)$. We will refer to this quantity as the *truncated quenched first moment* at scale r at time t for a reaction – diffusion process starting from site x. Furthermore, we will call the sets $\{\widetilde{m}_r(x,t,w) : x \in \mathbb{Z}^d\}$, the *field of truncated quenched first moments* at scale r at time t.

4.3. The parabolic Anderson equations. Here we will recall the moment equations satisfied by the field of quenched first moments $\{m(x,t,w)\}$ and by the corresponding truncated fields. Following [12], we have the proposition.

Proposition 4.2. *Let $U \subset \mathbb{Z}^d$ be a finite set and $w \in W$ an environment. Consider the field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ on $\mathbb{Z}^d$ at time t and the field of truncated quenched first moments $\{\widetilde{m}_U(x,t,w) : x \in \mathbb{Z}^d\}$ on U at time t. Then the following statements are true.*

(i) *The field of truncated quenched first moments $\{\widetilde{m}_U(x,t,w) : x \in \mathbb{Z}^d\}$ of the total number of particles at time t on U, satisfies the equation,*

$$\begin{aligned} &\frac{\partial \widetilde{m}_U}{\partial t} = \kappa\Delta\widetilde{m}_U + v(x)\widetilde{m}_U, && \text{for } x \in U \cap \mathcal{G}(w)^{\mathrm{c}} \\ &\widetilde{m}_U(x,0,w) = 1, && \text{for } x \in \mathbb{Z}^d \\ &\widetilde{m}_U(x,t,w) = 0, && \text{for } x \notin U \cap \mathcal{G}(w)^{\mathrm{c}},\ t > 0. \end{aligned}$$

(ii) *The field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ of the total number of particles at time t on $\mathbb{Z}^d$, satisfies the equation,*

$$\begin{aligned} &\frac{\partial m}{\partial t} = \kappa\Delta m + v(x)m, && \text{for } x \in \mathcal{G}(w)^{\mathrm{c}} \\ &m(x,0,w) = 1, && \text{for } x \in \mathbb{Z}^d \\ &m(x,t,w) = 0, && \text{for } x \notin \mathcal{G}(w)^{\mathrm{c}},\ t > 0. \end{aligned}$$

Proof. Consider the family of functions $\{u_z(x,t) := \mathrm{E}_x^w[z^{\zeta(t)}]\}$, parametrized by complex z such that $0 < |z| \le 1$. It is easy to see that,

$$\begin{aligned} \frac{\partial u_z}{\partial t} &= \kappa\Delta u_z + v_+(x)u_z^2 - (v_+(x) + v_-(x))u_z + v_-(x), \\ u_z(x,0) &= z, \end{aligned}$$

for $x \in \mathcal{G}(w)^{\mathrm{c}}$, while $u_z(x,t) = 1$ for $t \ge 0$ and $x \in \mathcal{G}(w)$. Differentiating the above equation with respect to z we obtain part (i). A similar proof can be carried out for part (ii). □

4.4. Bounds on the quenched first moments. We will now obtain upper and lower bounds for the annealed moments of the quenched first moments. Let us first recall two elementary inequalities. For n natural, let $a_1, \dots, a_n$ be arbitrary real numbers. Then, for $r \ge 1$, we have Jensen's inequality,

$$\left|\sum_{i=1}^n a_i\right|^r \le n^{r-1}\sum_{i=1}^n |a_i|^r, \tag{4.4}$$

while for $0 \le r \le 1$ we have,

$$\left|\sum_{i=1}^n a_i\right|^r \le \sum_{i=1}^n |a_i|^r. \tag{4.5}$$

We will also need to introduce for $L \ge 0$ the notation,

$$M_L := \max_{x \in \Lambda_L} |v(x)|. \tag{4.6}$$

Let us recall the following lemma, contained in the statement of [12, Theorem 2.1].

Lemma 4.3. *Consider a finite subset $U \subset \mathbb{Z}^d$ and $\mu \in \mathcal{P}(W)$. Assume that μ satisfies Condition* E. *Then, μ-a.s. for every $x \in \mathbb{Z}^d$ and $t \geq 0$, the quenched first moment $m(x,t,w)$ on $\mathbb{Z}^d$ at time t starting from site x admits the Feynman – Kac representation*

$$m(x,t,w) = \mathrm{E}_x\Big[\mathrm{e}^{\int_0^t v(X_s)\,\mathrm{d}s} 1\big(\tau_{\mathcal{G}(w)} > t\big)\Big], \tag{4.7}$$

and the truncated quenched first moment $\widetilde{m}_U(x,t,w)$ on U at time t starting from x also,

$$\widetilde{m}_U(x,t,w) = \mathrm{E}_x\Big[\mathrm{e}^{\int_0^t v(X_s)\,\mathrm{d}s} 1\big(\tau_{U^c \cup \mathcal{G}(w)} > t\big)\Big], \tag{4.8}$$

where in both (4.7) *and* (4.8), *$\{X_t : t \geq 0\}$ is a simple symmetric random walk of total jump rate $2d\kappa$ starting from x, of law P_x, E_x is the expectation related to this law, and for $A \subset \mathbb{Z}^d$ we define $\tau_A := \inf\{t \geq 0 : X_t \notin A\}$.*

We can now apply Lemma 4.3 to obtain the first estimates on the quenched first moments.

Proposition 4.4. *Consider a finite subset $U \subset \mathbb{Z}^d$ and $\mu \in \mathcal{P}(W)$. Assume that μ satisfies Condition* E. *Then,*

(i) *For each $x \in \mathbb{Z}^d$, $t \geq 0$ and $\beta > 0$ there exists a constant C such that,*

$$\mathrm{e}^{H(\beta t) - 2d\kappa t} \leq \langle m(x,t)^\beta \rangle \leq C(\kappa + t)^d \mathrm{e}^{H(\beta t)}, \tag{4.9}$$

(ii) *For each $x \in U$, $t \geq 0$ and $\beta > 0$,*

$$\mathrm{e}^{H(\beta t) - 2d\kappa t} \leq \langle \widetilde{m}_U(x,t)^\beta \rangle \leq C(\kappa + t)^d \mathrm{e}^{H(\beta t)}, \tag{4.10}$$

where C is the constant of part (i).

(iii) *For each $\beta > 0$, $\gamma > 0$ and $a > 0$ we have that,*

$$\langle |m(x,t) - \widetilde{m}_{\gamma(\kappa t)^a}(x,t)|^\beta \rangle \leq C(\gamma(\kappa t)^a + 1)^d \mathrm{e}^{-2\beta\kappa t I(\gamma(\kappa t)^{a-1}/2)} \mathrm{e}^{H(\beta t)}, \tag{4.11}$$

for some constant $C > 0$, where $I\colon [0,\infty) \to [0,\infty)$ is defined in equation (3.2).

Proof. (i) The first inequality of equation (4.9) can be obtained from the Feynman – Kac representation (4.7), taking only into account the contribution of the path X_s which stays during the whole time interval $[0,t]$ at x ([12, p. 637]). To prove the second inequality of (4.9), let us note that by translation invariance it is enough to prove the estimate for $\langle m(0,t) \rangle$. On the other hand,

$$1(\tau_{\mathcal{G}(w)} > t)\mathrm{e}^{\int_0^t v(X_s)\,\mathrm{d}s} \leq \sum_{n=0}^{\infty} \mathrm{e}^{M_{R_n} t} 1(T_{n-1} \leq t < T_n),$$

where $T_{-1} = 0$, while for n natural T_n is the first exit time of the random walk $\{X_t : t \geq 0\}$ from the box Λ_{R_n}, with $R_n := R_0 2^n$ and $R_0 := \max\{\kappa t, 1\}$, while $M_{R_n} := \max_{x : \|x\| \leq R_n} |v(x)|$ as defined in equation (4.6). It follows that,

$$m(0,t,w) \leq \sum_{n=0}^{\infty} \mathrm{e}^{M_{R_n} t} P[T_{n-1} \leq t]. \tag{4.12}$$

Let us also remark that since $\beta > 0$, for each natural n we have the following inequality which will be used soon,

$$\langle \mathrm{e}^{\beta t M_{R_n}} \rangle \leq (2(R_n + 1))^d \exp\{H(\beta t)\}. \tag{4.13}$$

In fact, $\langle e^{\beta t M_{R_n}}\rangle \le \sum_{x\in[-R_n,R_n]^d}\langle e^{\beta t M_{R_n}} 1(v(x)=M_{R_n})\rangle$. Let us now consider the case $0<\beta\le 1$. Then, by an application of inequality (4.5) to estimate (4.12), we conclude that,

$$m(0,t,w)^\beta \le e^{\beta t M_{R_0}} + \sum_{n=1}^\infty e^{\beta t M_{R_n}} P[T_{n-1}\le t]^\beta.$$

Taking expectations on both sides of this inequality and applying the estimate (4.13) and the second inequality of equation (3.3) for each of the d coordinates of the underlying random walk, we obtain,

$$\langle m(0,t)^\beta\rangle \le 2^d(R_0+1)^d e^{H(\beta t)}\left(1+4d\sum_{n=1}^\infty 2^{nd} e^{-\beta R_0 2^{n-1}\log(R_0 2^{n-1}/(2e\kappa t))}\right),$$

Now, since $e^{-\beta R_0 2^{n-1}\log(R_0 2^{n-1}/(2e\kappa t))} \le e^{-\beta 2^{n-1}\log(2^{n-1}/(2e))}$, and $R_0+1\le 2R_0$, we see that there is a constant C such that inequality (4.9) is satisfied for $0<\beta\le 1$.

Let us now consider the case $\beta>1$. Let $\beta'>1$ be defined by $1/\beta+1/\beta'=1$. Then, if we represent the left-hand side of (4.12) as

$$\sum_{k=0}^\infty e^{t M_{R_n}} P[T_{n-1}<t\le T_n]^{1/\beta'} P[T_{n-1}<t\le T_n]^{1/\beta},$$

by Hölder's inequality we get that,

$$m(0,t,w) \le \left(\sum_{n=0}^\infty e^{t\beta M_{R_n}} P[T_{n-1}<t\le T_n]\right)^{1/\beta}.$$

A computation similar to the case $0<\beta\le 1$ finishes the proof of part (i).

(ii) The first inequality of equation (4.10) can be deduced by an argument analogous to the one leading to the first inequality of equation (4.9). Now, note from the representations (4.7) and (4.8) that $\widetilde{m}_U(x,t,w)\le m(x,t,w)$. Hence, the second inequality of equation (4.10) is a corollary of the second inequality of equation (4.9).

(iii) Let us remark from the Feynman–Kac representations (4.7) for $m(0,t)$ and (4.8) for $\widetilde{m}_{\gamma(\kappa t)^a}(0,t)$ that,

$$m(0,t)-\widetilde{m}_{\gamma(\kappa t)^a}(0,t) = E_x\left[e^{\int_0^t v(X_s)\,ds} 1\big(\tau_{\Lambda^c_{\gamma(\kappa t)^a}}<t\big) 1\big(\tau_{\mathcal{G}(w)}>t\big)\right].$$

Hence, as in part (i) of the proof of this proposition,

$$|m(0,t)-\widetilde{m}_{\gamma(\kappa t)^a}(0,t)| \le \sum_{n=1}^\infty e^{M_{R'_n} t} P[T'_{n-1}\le t<T'_n], \tag{4.14}$$

where T'_n is the first exit time of the random walk $\{X_t: t\ge 0\}$ from the box $\Lambda_{R'_n}$, with $R'_n := R'_0 2^n$, $R'_0 := \max\{\gamma(\kappa t)^a, 1\}$ and $M_{R'_n} := \max_{x:\|x\|\le R'_n}|v(x)|$. Let us consider the case $0<\beta\le 1$. By inequality (4.5) we obtain that,

$$|m(0,t)-\widetilde{m}_{\gamma(\kappa t)^a}(0,t)|^\beta \le \sum_{n=1}^\infty e^{t\beta M_{R'_n}} P[T'_{n-1}\le t]^\beta.$$

But, as in (4.13) we can conclude that $\langle e^{\beta t M_{R'_n}}\rangle \le 2^d(R'_n+1)^d \exp\{H(\beta t)\}$. Hence, from the first inequality of equation (3.3) we see that,

$$\langle |m(0,t)-\widetilde{m}_{\gamma(\kappa t)^a}(0,t)|^\beta\rangle \le d4^\beta 2^d(R'_1+1)^d e^{H(\beta t)} e^{-2\beta\kappa t I(R'_0/(2\kappa t))} \times \sum_{n=1}^{\infty} \frac{(2(R'_0 2^n+1))^d}{(2(R'_0 2+1))^d} e^{2\kappa t\beta(I(R'_0/(2\kappa t))-I(R'_0 2^n/(2\kappa t)))}. \tag{4.15}$$

Now, using the fact that I is convex, nondecreasing and that $I(0)=0$, we see that,

$$I\left(\frac{R'_0}{2\kappa t}\right) - I\left(\frac{R'_0 2^n}{2\kappa t}\right) \le -(2^n-1) I\left(\frac{\gamma(\kappa t)^{a-1}}{2}\right).$$

Substituting this back in (4.15), we obtain inequality (4.11) in the case $0<\beta\le 1$. The case $\beta>1$ can be proved using similarly, using Hölder's inequality as in part (i). □

We end this section with important estimates involving the growth functions. Given a subset $U\subset\mathbb{Z}^d$, we define the discrete Laplacian operator with effective potential $v(0)=v_+(0)-v_-(0)\in[-\infty,\infty)$ by its action on functions $f:\mathbb{Z}^d\to\mathbb{R}$, which vanish outside $U_w := U\cap\mathcal{G}^c(w)$ ($f(x)=0$ for $x\notin U\cap\mathcal{G}^c(w)$) as,

$$(\Delta+v)f(x) = \sum_{e\in\mathcal{B}}(f(x+e)-f(x)) + v(x)f(x),$$

where $\mathcal{B}$ is the set formed by the elements of the basis of the free Abelian group $\mathbb{Z}^d$ and its inverses. Defining $L^2(U_w) := \{f : \sum_{x\in\mathbb{Z}^d} f(x)^2<\infty, f(x)=0 \text{ if } x\notin U_w\}$, we can check that $\Delta+v$ is a bounded self-adjoint operator on the Hilbert space $L^2(U_w)$ endowed with the inner product $(f,g) := \sum_{x\in\mathbb{Z}^d} f(x)g(x)$. We then define $\{\lambda_n(U,w) : 0\le n\le \mathcal{U}-1\}$ as the set of eigenvalues of $\Delta+v$ in $L^2(U_w)$ in decreasing order, where $\mathcal{U}$ is the total number of eigenvalues. We will denote by $\psi_n^{U,w}$ the corresponding normalized eigenfunctions. Let $r\ge 0$. In the case in which $U=\Lambda(x,r)$, we will employ the notation $\{\lambda_n(x,r,w)\}$ instead of $\{\lambda_n(U,w)\}$ and $\psi_n^{x,r,w}$ instead of $\psi_n^{U,w}$.

We can now state the following important lemma.

Lemma 4.5. *Consider the quenched first moment $m(0,t,w)$. Then, for every $\beta>0$, $a>1$ and $t\ge 1$, there is a constant $k_1(d,a,\beta)$ such that,*

$$\frac{k_1^{-1}}{(\kappa t)^{da(\beta+1)}+1}\langle m(0,\beta t)\rangle \le \langle m(0,t)^\beta\rangle \le k_1\big((\kappa t)^{da(\beta+1)}+1\big)\langle m(0,\beta t)\rangle, \tag{4.16}$$

and

$$\frac{k_1^{-1}}{(\kappa t)^{da(\beta+1)}+1}\langle \widetilde{m}_{(\kappa t)^a}(0,\beta t)\rangle \le \langle \widetilde{m}_{(\kappa t)^a}(0,t)^\beta\rangle \le k_1\big((\kappa t)^{da(\beta+1)}+1\big)\langle \widetilde{m}_{(\kappa t)^a}(0,\beta t)\rangle. \tag{4.17}$$

Proof. We will only prove (4.16), being the proof of equation (4.17) analogous. Let us first show that for every real $a>0$, $\beta>0$ and $t\ge 0$ there is a constant $c(d,\beta,a)$ such that,

$$m(x,t,w) \le ([2(\beta\kappa t)^a]+1)^{d/2} e^{t\lambda_0(x,(\beta\kappa t)^a,w)} + 4d e^{-2\kappa t I((\beta\kappa t)^{a-1}/2)} e^{tM_{(\beta\kappa t)^a}}. \tag{4.18}$$

Note that by inequality (4.14) with $\gamma=\beta^a$, we have that

$$m(x,t,w) \le 4d e^{-2\kappa t I((\beta\kappa t)^{a-1}/2)} e^{tM_{(\beta\kappa t)^a}} + \widetilde{m}_U(x,t,w),$$

with $U=\Lambda(x,(\beta\kappa t)^a,w)$. We then need to estimate the truncated quenched first moment at scale $(\beta\kappa t)^a$, $\widetilde{m}_{(\beta\kappa t)^a}(x,t,w)$. First remark the following expansion in terms of the eigenvalues $\{\lambda_n(x,(\beta\kappa t)^a,w)\}$ and the corresponding eigenfunctions $\{\psi_n^{x,(\beta\kappa t)^a,w}\}$,

$$\widetilde{m}_{(\beta\kappa t)^a}(x,t,w)=\sum_{n=0}^{\mathcal{U}-1}\mathrm{e}^{t\lambda_n(x,(\beta\kappa t)^a,w)}\psi_n^{x,(\beta\kappa t)^a,w}(x)\big(\psi_n^{x,(\beta\kappa t)^a,w},1(A)\big),\tag{4.19}$$

where $A:=\Lambda(x,(\beta\kappa t)^a,w)$. Now, by the Cauchy – Schwartz inequality we see that the right-hand side of equality (4.19) is upper-bounded by $\mathrm{e}^{t\lambda_0(x,(\beta\kappa t)^a,w)}\times\big(\sum_{n=0}^{\mathcal{U}-1}\big(\psi_n^{x,(\beta\kappa t)^a,w},1(\{x\})\big)^2\sum_{n=0}^{\mathcal{U}-1}\big(\psi_n^{x,(\beta\kappa t)^a,w},1(A)\big)^2\big)^{1/2}$ which in turn is upper-bounded by $\mathrm{e}^{t\lambda_0(x,(\beta\kappa t)^a,w)}\sqrt{|A|}$, where $1(\{x\})(y)$ equals 1 if $y=x$ and 0 otherwise. Using the fact that $|A|=|\Lambda(x,(\beta\kappa t)^a,w)|\le([2(\beta\kappa t)^a]+1)^d$, finishes the proof of (4.18).

Let us now show that for every finite subset $U\subset\mathbb{Z}^d$, it is true that,

$$\frac{1}{|U|}\sum_{z\in U}m(z,t)\ge\frac{1}{|U|}\mathrm{e}^{t\lambda_0(U,w)}.\tag{4.20}$$

First note the trivial inequality $m(z,t,w)\ge\widetilde{m}_U(z,t,w)$. We also have the expansion, $\widetilde{m}_U(z,t,w)=\sum_{n=0}^{\mathcal{U}-1}\mathrm{e}^{\lambda_n(U,w)t}\psi_n^{U,w}(z)\big(\psi_n^{U,w},1(U)\big)$. Therefore we can see that,

$$\begin{aligned}\frac{1}{|U|}\sum_{z\in U}m(z,t)&\ge\frac{1}{|U|}\mathrm{e}^{\lambda_0(U,w)t}\big(\psi_0^{U,w},1(U)\big)^2\\&\ge\frac{1}{|U|}\mathrm{e}^{\lambda_0(U,w)t}\sum_{z\in U}(\psi_0^{U,w})^2(z)=\frac{1}{|U|}\mathrm{e}^{\lambda_0(U,w)t},\end{aligned}$$

where we have used in the second inequality the fact that $\psi_0^{U,w}(x)\ge0$ and in the last inequality the normalization condition $\sum_{z\in U}(\psi_0^{U,w})^2(z)=1$.

Let us now prove the second inequality of (4.16). By Jensen's inequality (4.4), in the case $\beta\ge1$, or inequality (4.5), in the case $0<\beta<1$, applied to (4.18), note that for some constant $c(a,d,\beta)$,

$$\langle m(x,t)^\beta\rangle\le c\big((\kappa t)^{da\beta/2}+1\big)\big\langle\mathrm{e}^{\beta t\lambda_0(x,(\beta\kappa t)^a,w)}\big\rangle+4d\mathrm{e}^{-2\beta\kappa tI((\beta\kappa t)^{a-1}/2)}\big\langle\mathrm{e}^{\beta tM_{(\beta\kappa t)^a}}\big\rangle.$$

Now, by the first inequality of equation (4.9) and a computation similar to the one leading to (4.13), the second term of the right hand side of the above inequality, is upper bounded by,

$$4d2^d\big((\beta\kappa t)^a+1\big)^d\exp\bigg\{-2\beta\kappa tI\bigg(\frac{(\beta\kappa t)^{a-1}}{2}\bigg)+2d\kappa t\bigg\}\langle m(x,t)^\beta\rangle.\tag{4.21}$$

On the other hand, by inequality (4.20) with $U=\Lambda(0,(\kappa t)^a)$, we see that,

$$\big\langle\mathrm{e}^{\beta t\lambda_0(x,(\beta\kappa t)^a,w)}\big\rangle\le(2[(\beta\kappa t)^a]+1)^d\langle m(x,\beta t)\rangle.$$

It follows that,

$$\langle m(x,t,w)^\beta\rangle\le c\big((\kappa t)^{da(\beta+1)}+1\big)\langle m(x,\beta t)\rangle,$$

where we have used the fact that for $a>1$, the term (4.21) is negligible with respect to $\langle m^\beta\rangle$. The second inequality of equation (4.16) now follows noting that for $a>1$ the second term in the right-hand factor of the above inequality is negligible with respect to the first term.

Let us now prove the first inequality of equation (4.16). In the case $0 < \beta \leq 1$, by inequality (4.5) and the bound (4.20), we have that,

$$\sum_{x \in U} m^\beta(x,t,w) \geq \mathrm{e}^{\beta t \lambda_0(U,w)}. \tag{4.22}$$

But when $\beta > 1$, by Jensen's inequality we have that

$$\sum_{x \in U} m^\beta(x,t,w) \geq |U|^{-(\beta-1)} \left(\sum_{x \in U} m(x,t,w) \right)^\beta,$$

and (4.22) is still satisfied. Choosing $U = \Lambda(0, (\beta\kappa t)^a)$, and using translation invariance we get,

$$\langle m(0,t)^\beta \rangle \geq (2[(\beta\kappa t)^a] + 1)^{-d} \big\langle \mathrm{e}^{\beta t \lambda_0(0,(\beta\kappa t)^a, w)} \big\rangle.$$

Using again the bound (4.18), neglecting the second term, we finish the proof in the case $0 < \beta \leq 1$. □

An important consequence of Lemma 4.5, is that it shows that Assumption MI implies the so called *intermittency effect* [12]. Let us define for $\theta \neq 0$, the functions,

$$\bar{F}_\theta(t) := \frac{\log\langle m(0,t)^{1+\theta}\rangle - (1+\theta)\log\langle m(0,t)\rangle}{\theta}.$$

Note that by Jensen's inequality, $\bar{F}_\theta(t) \geq 0$.

Corollary 4.6. *Suppose that Assumptions* (E) *and* MI *are satisfied. Then, for every $\theta \neq 0$,*

$$\lim_{t \to \infty} \frac{\bar{F}_{2\theta}(t) - \bar{F}_\theta(t)}{\theta \log(\kappa t + e)} = \infty. \tag{4.23}$$

4.5. Correlation and variance estimates on the field of quenched first moments. In order to prove Theorem 2.3(ii), it will be important to have a control on the variance of the the quenched first moments. In the sequel of this paper, to avoid heavy notation, we will use m_a instead of $m_{(\kappa t)^a}$. Given $x, y \in \mathbb{Z}^d$ and $t \geq 0$, let us define,

$$c(x,y,t) := \langle m(x,t)m(y,t)\rangle - \langle m(x,t)\rangle\langle m(y,t)\rangle,$$

which we will call the *correlation between sites x and y at time t* of the field of quenched first moments. Similarly let us define for $a > 0$,

$$c_a(x,y,t) := \langle \widetilde{m}_a(x,t)\widetilde{m}_a(y,t)\rangle - \langle \widetilde{m}_a(x,t)\rangle\langle \widetilde{m}_a(y,t)\rangle,$$

the *correlation between sites x and y at time t* of the truncated field of quenched first moments at scale $(\kappa t)^a$. Let us begin with the following lemma.

Lemma 4.7. *Let $t \geq 0$. Consider the fields of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ and truncated quenched first moments at scale $(\kappa t)^a$, $\{\widetilde{m}_a(x,t,w) : x \in \mathbb{Z}^d\}$. Then the following statements are true.*

(i) *The sum of the correlations between site 0 and the other sites of the field of quenched first moments, behaves asymptotically as $t \to \infty$ like the sum of the correlations between site 0 and the other sites of the truncated field of quenched first moments at scale $(\kappa t)^a$. In other words,*

$$\sum_{y \in \mathbb{Z}^d} c(0,y,t) \sim \sum_{y \in \mathbb{Z}^d} c_a(0,y,t).$$

(ii) *Let $\{U_t : t > 0\}$ be a collection of subsets of the lattice $\mathbb{Z}^d$ indexed by $t > 0$. Assume that $|U_t| \sim |U_{t,(\kappa t)^a}|$ as $t \to \infty$, where $U_{t,r} := \{x \in U_t : \mathrm{dist}(x, U_t^{\mathrm{c}}) \geq 2r\}$, for $r > 0$. Then,*

$$\mathrm{Var}_\mu \sum_{x \in \Lambda_{U_t}} m(x,t) \sim |U_t| \sum_{y \in \mathbb{Z}^d} c(0,y,t), \tag{4.24}$$

$$\mathrm{Var}_\mu \sum_{x \in \Lambda_{U_t}} \widetilde{m}_a(x,t) \sim |U_t| \sum_{y \in \mathbb{Z}^d} c_a(0,y,t), \tag{4.25}$$

and

$$\mathrm{Var}_\mu \sum_{x \in \Lambda_{U_t}} m(x,t) \sim \mathrm{Var}_\mu \sum_{x \in \Lambda_{U_t}} \widetilde{m}_a(x,t). \tag{4.26}$$

PROOF. From the Feynman – Kac representation (4.7) of Lemma 4.3, note that it is possible to write,

$$m(x,t,w) = \mathrm{E}_x\Big[\mathrm{e}^{\sum_{z \in \mathbb{Z}^d} v(z)\mathcal{L}(z,t)} 1(A)\Big],$$

where $A := \{\tau_{\mathcal{G}^{\mathrm{c}}(w)} > t\}$, $\mathcal{L}(z,t) := \int_0^t \delta_z(X_s)\, \mathrm{d}s$ is the local time at the point z of the random walk $\{X_t : t \geq 0\}$ starting from x, and $\delta_z : \mathbb{Z}^d \to \{0,1\}$ is the indicator function of the set $\{z\}$. From here, using Fubini's theorem it follows that we have the following representation for the correlations between site x and y of the quenched field of first moments.

$$\begin{aligned} &c(x,y,t) \\ &\quad = \mathrm{E}_{x,y}\Big[\Big\langle \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} v(\mathcal{L}+\tilde{\mathcal{L}})} 1(A \cap \tilde{A})\Big\rangle - \Big\langle \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} v\mathcal{L}} 1(A)\Big\rangle \Big\langle \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} v\tilde{\mathcal{L}}} 1(\tilde{A})\Big\rangle\Big] \end{aligned} \tag{4.27}$$

where $\tilde{\mathcal{L}}(z,t) := \int_0^t \delta_z(\widetilde{X}_s)\, \mathrm{d}s$ is the local time at the point z of the random walk $\{\widetilde{X}_t : t \geq 0\}$, independent of $\{X_t : t \geq 0\}$, starting from y, with law P_y, and $\tilde{A}$ is an identical copy of A, but defined in terms of the random walk $\{\widetilde{X}_t : t \geq 0\}$. Furthermore, $\mathrm{E}_{x,y} := \mathrm{E}_x \otimes \mathrm{E}_y$, denotes the expectation with respect to the law of the independent random walks $\{X_t\}$ and $\{\widetilde{X}_t\}$. Now, the expression (4.27) for the correlations can be written in terms of the cumulant generating function defined in equation (1.1),

$$c(x,y,t) = \mathrm{E}_{x,y}\Big[\mathrm{e}^{\sum_{z \in \mathbb{Z}^d} H(\mathcal{L}+\tilde{\mathcal{L}})} - \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} H(\mathcal{L})} \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} H(\tilde{\mathcal{L}})}\Big], \tag{4.28}$$

where we have used the independence of the coordinates of the effective potential $\{v(x) : x \in \mathbb{Z}^d\}$ under μ. Note that the super-additivity of H (Lemma 4.1(i)), implies that this expression is nonnegative. On the other hand, a reasoning similar to the one leading to the representation (4.28), this time based on the Feynman – Kac representation (4.8) of Lemma 4.3, enables us to deduce that,

$$\begin{aligned} &c_a(x,y,t) \\ &\quad = \mathrm{E}_{x,y}\Big[1(\tau_a > t) 1(\tilde{\tau}_a > t)\Big(\mathrm{e}^{\sum_{z \in \mathbb{Z}^d} H(\mathcal{L}+\tilde{\mathcal{L}})} - \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} H(\mathcal{L})} \mathrm{e}^{\sum_{z \in \mathbb{Z}^d} H(\tilde{\mathcal{L}})}\Big)\Big], \end{aligned} \tag{4.29}$$

where $\tau_a := \tau_{\Lambda(x,(\kappa t)^a)} = \inf\{t \geq 0 : X_t \notin \Lambda(x,(\kappa t)^a)\}$ and $\tilde{\tau}_a := \tilde{\tau}_{\Lambda(x,(\kappa t)^a)} := \inf\{t \geq 0 : \widetilde{X}_t \notin \Lambda(y,(\kappa t)^a)\}$. From (4.28) and (4.29) it follows that,

$$\sum_{y \in \mathbb{Z}^d} c(0,y,t) \geq \sum_{y \in \mathbb{Z}^d} c_a(0,y,t). \tag{4.30}$$

But note that in reality, due to the independence between any pair of truncated quenched first moments at time t at two points at a distance larger than $2(\kappa t)^a$, we have $\sum_{y\in\mathbb{Z}^d} c_a(0,y,t) = \sum_{y\in\Lambda_{2(\kappa t)^a}} c_a(0,y,t)$. Furthermore,

$$\sum_{y\in\mathbb{Z}^d} c(0,y,t) = \sum_{y\in\Lambda_{2(\kappa t)^a}} c(0,y,t) + \sum_{y\notin\Lambda_{2(\kappa t)^a}} c(0,y,t). \tag{4.31}$$

Now, an application of the first inequality of equation (4.9) and Proposition 4.4(ii), shows that since $a > 0$, it is true that $\sum_{y\in\Lambda_{2(\kappa t)^a}} c(0,y,t) \sim \sum_{y\in\Lambda_{2(\kappa t)^a}} c_a(0,y,t)$. And a second application of equation (4.9) and the first inequality in equation (3.3), shows that $\sum_{y\notin\Lambda_{2(\kappa t)^a}} c(0,y,t) \ll \sum_{y\in\Lambda_{2(\kappa t)^a}} c(0,y,t)$. This, together with inequality (4.30), ends up the proof of Lemma 4.7(i).

We will first prove (4.25). Remark that,

$$\mathrm{Var}_\mu \sum_{x\in U_t} \widetilde{m}_a(x,t) = \sum_{x,y\in U_t} c_a(x,y,t) = \sum_{\substack{x,y\in U_t\\ \|x-y\|\le 2(\kappa t)^a}} c_a(x,y,t),$$

where we used as in the proof of part (i), the fact that $\widetilde{m}_a(x,t,v)$ and $\widetilde{m}_a(y,t,v)$ are independent for $\|x-y\| \ge 2(\kappa t)^a$. Now, the right-most member of the above inequality is bounded by, $|U_t|\sum_{y:\|y\|\le 2t^a} c_a(0,y,t)$, which gives us the inequality,

$$\mathrm{Var}_\mu \sum_{x\in U_t} \widetilde{m}_a(x,t) \le |U_t| \sum_{y\in\mathbb{Z}^d} c_a(0,y,t).$$

Similarly we can conclude that,

$$\mathrm{Var}_\mu \sum_{x\in U_t} \widetilde{m}_a(x,t) \ge |U_{t,(\kappa t)^a}| \sum_{y\in\mathbb{Z}^d} c_a(0,y,t).$$

This finishes the proof of the statement of equation (4.25). The statement of equation (4.24) now follows from equation (4.25), Lemma 4.7(i) proved above, and the inequalities,

$$\mathrm{Var}_\mu \sum_{x\in U_t} \widetilde{m}_a(x,t) \le \mathrm{Var}_\mu \sum_{x\in U_t} m_a(x,t) \le |U_t| \sum_{y\in\mathbb{Z}^d} c(0,y,t).$$

Finally, note that (4.26) is a trivial consequence of (4.24) and (4.25). □

5. The annealed and Gaussian regimes

Here we will prove Theorem 2.3(i) and (ii), making use of the estimates obtained in the previous section, and Theorem 2.11. The main argument which will be used to prove Theorem 2.3 is a renormalization method, which we call *partition analysis*, as developed in [5]. In order to present a self-contained proof, in the first subsection we recall this technique. Then, in the second subsection, we derive some important estimates via the partition analysis technique, which will enable us to reduce the proofs to sums of independent random variables. In the third subsection we prove the law of large numbers, stated in equation (2.4). In the fourth subsection we will prove the negative part (absence of a law of large numbers) stated in equation (2.5). In Section 5.5, we will prove the central limit theorem stated in equation (2.6). Then, in Section 5.6 we prove the absence of a central limit behavior stated in equation (2.7). Finally, in Section 5.7, we prove Theorem 2.11.

5.1. Partition analysis. Here we shall follow closely Section 5.1 of Ben Arous et al. [5]. For a fixed natural L we consider the box $\Lambda_L = \{x \in \mathbb{Z}^d : \|x\| \le L\}$. We will define two related but different kinds of partitions of Λ_L. The first one shows that Λ_L can be decomposed into disjoint *partition boxes* $\{\Lambda'_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$, indexed by some set $\mathcal{I}$, so that $\Lambda_L = \bigcup_i \Lambda'_i$. The second one defines a partition of Λ_L in a *strip set* and *main boxes* $\{\Lambda''_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$. In the first case, the index set $\mathcal{I}$ will be partitioned in disjoint subsets $\{\mathcal{I}_K : K \in \mathcal{K}\}$, where the cardinality of $\mathcal{K}$ is 2^d, in such a way that for each $K \in \mathcal{K}$ any pair of elements of the collection of partition boxes $\{\Lambda'_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}_K\}$ is at a large Euclidean distance. In the second partition case, it turns out that the survival probabilities corresponding to sites in the strip set have a total sum which is negligible, while the main boxes happen to be essentially independent. To proceed we will need to introduce some notation defining the corresponding scales and subsets.

Our first parameter is a natural number L' smaller than or equal to L, which will be called the *mesoscopic scale*. By the division algorithm, we know that there exist natural numbers q and $\bar{q}$ such that $2L + 1 = qL' + \bar{q}$, with $0 \le \bar{q} < q$. Note that this last equation can be written in the form

$$2L + 1 = \sum_{i=1}^{q} L'_i, \tag{5.1}$$

with $L'_i = L' + \theta_{\bar{q}}(i)$ and $\theta_{\bar{q}}(i) = 1$ for $i \le \bar{q}$ and $\theta_{\bar{q}}(i) = 0$ for $i > \bar{q}$. For our purposes, the relevant fact is that $L' \le L'_i \le L' + 1$. In the sequel, for any given pair of real numbers a, b we will use the notation $[a, b]_l$ for $[a, b] \cap \mathbb{Z}$. We now will subdivide the box $[-L, L]_l$ in intervals according to equation (5.1). Thus, we define $I_1 := [-L, -L + L'_1 - 1]_l$ and for $1 < i \le q$ we let $I_i := [-L + \sum_{j=1}^{i-1} L'_i, -L + \sum_{j=1}^{i} L'_i - 1]_l$. Note that $I_q = [L - L'_q + 1, L]_l$ and $|I_i| = L'_i$. Next, we introduce a second parameter r which is a natural number smaller than or equal to L'. We will call r the *fine scale*. Then, for each I_i we define an interval J_i such that $J_i \subset I_i$, $|J_i| = L' - 2r$ and the endpoints of J_i are at a distance larger than r to the endpoints of I_i. To do so, first let $r_i := r + \theta_{\bar{q}}(i)$. Then define $J_1 := [-L + r, -L + L'_1 - 1 - r_1]_l$ and for $1 < i \le q$ we let $J_i := [-L + \sum_{j=1}^{i-1} L'_i + r, -L + \sum_{j=1}^{i} L'_i - 1 - r_i]_l$.

We now proceed to define the partition in Λ_L in partition boxes and define the corresponding decomposition of the index set. First we define the set $\mathcal{I} := \{1, 2, \ldots, q\}^d$, which will correspond to the indexes parametrizing the sub-boxes. For a given element $i \in \mathcal{I}$, of the form $\mathbf{i} = (i_1, \ldots, i_d)$ with $1 \le i_k \le q$, $1 \le k \le d$, we define

$$\Lambda'_{\mathbf{i}} := I_{i_1} \times I_{i_2} \times \cdots \times I_{i_d}.$$

We will call such a set a *partition box*. By definition the cardinality $|\Lambda'_{\mathbf{i}}|$ of a partition box satisfies,

$$(L')^d \le |\Lambda'_{\mathbf{i}}| \le (L' + 1)^d. \tag{5.2}$$

Note also that the partition boxes defines a partition of Λ_L so that $\Lambda_L = \bigcup_{\mathbf{i} \in \mathcal{I}} \Lambda'_{\mathbf{i}}$ where the union is disjoint.

Next we define a partition of the index set $\mathcal{I}$. Consider the collection $\mathcal{K}$ of subsets of $\{1, 2, \ldots, d\}$. Note that $|\mathcal{K}| = 2^d$. Now given $K \in \mathcal{K}$ we define $\mathcal{I}_K$ as the subset of $\mathcal{I}$ having coordinates which are even for $k \in K$ and odd for $k \notin K$. In other words, if we define $\mathbb{E}$ as the set of even natural numbers and $\mathbb{O}$ as the set of odd natural numbers then,

$$\mathcal{I}_K := \{\mathbf{i} = (i_1, \ldots, i_d) \in \mathcal{I} : i_k \in \mathbb{E} \text{ if } k \in K, i_k \in \mathbb{O} \text{ if } k \notin K,\ 1 \leq k \leq d\}.$$

Note that $\{\mathcal{I}_K : K \in \mathcal{K}\}$ defines a partition of $\mathcal{I}$ so that $\mathcal{I} = \bigcup_{K\in\mathcal{K}} \mathcal{I}_K$ is a disjoint union. Hence,

$$\sum_{K\in\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{I}_K} \sum_{x\in\Lambda'_{\mathbf{i}}} f(x) = \sum_{x\in\Lambda_L} f(x), \tag{5.3}$$

for every function $f\colon \Lambda_L \to \mathbb{R}$, which is a consequence of the fact that $\Lambda_L = \bigcup_{K\in\mathcal{K}} \bigcup_{\mathbf{i}\in\mathcal{I}_K} \Lambda'_{\mathbf{i}}$ is a disjoint union. We will refer to such a decomposition as the *parity partition at scale* L' of Λ_L. Furthermore, given $K \in \mathcal{K}$ and any pair of boxes $\Lambda'_{\mathbf{i}}$ and $\Lambda'_{\mathbf{j}}$ with $\mathbf{i}, \mathbf{j} \in \mathcal{I}_K$ and $\mathbf{i} \neq \mathbf{j}$ we have that,

$$\operatorname{dist}(\Lambda'_{\mathbf{i}}, \Lambda'_{\mathbf{j}}) \geq L'. \tag{5.4}$$

Here for any pair of subsets $A, B \subset \mathbb{Z}^d$ we define $\operatorname{dist}(A, B) := \inf_{x\in A, y\in B} |x-y|$. In other words (5.4) expresses the fact that the distance between any pair of partition boxes with different sub-indexes in $\mathcal{I}_K$ is larger than or equal to L'. This completes the description of the partition of Λ_L in partition boxes.

Next, we describe the partition of Λ_L into the strip set and main boxes. Given an $\mathbf{i} \in \mathcal{I}$ we let,

$$\Lambda''_{\mathbf{i}} := J_{i_1} \times J_{i_2} \times \cdots \times J_{i_d}.$$

Such a box will be called a *main box*. Its cardinality is $|\Lambda''_{\mathbf{i}}| = (L' - 2r)^d$. Now let,

$$S_L := \Lambda_L - \bigcup_{\mathbf{i}\in\mathcal{I}} \Lambda''_{\mathbf{i}}.$$

Such a set will be called the *strip set*. Note that S_L and $\{\Lambda''_L : \mathbf{i} \in \mathcal{I}\}$ define a partition of Λ_L. We will refer to such a partition as the *strip-box partition at scale* L' of Λ_L. We furthermore remark the following cardinality estimate for the strip set which will be useful later,

$$\frac{|S_L|}{(2L+1)^d} \leq \frac{\big((L'+1)^d - (L'-2r)^d\big)}{(L')^d}, \tag{5.5}$$

where we have used the fact that $|\mathcal{I}| = q^d$.

5.2. Moment and decoupling inequalities. Following [5], we recall here some standard inequalities and derive inequalities involving sums of the quenched first moments that will be necessary to perform the partition analysis.

Let us first recall the well-known inequality of von Bahr and Esseen (p. 82, exercise 2.6.20, of Petrov [16]).

Lemma 5.1 (von Bahr − Esseen). *Let $X_1, \ldots, X_n$ be mean zero independent random variables and $S_n := \sum_{k=1}^n X_i$. Then if* E *denotes the expectation with respect to the joint law of these random variables, and $1 \leq r \leq 2$, it is true that*

$$\mathrm{E}\,|S_n| \leq 2 \sum_{k=1}^{n} \mathrm{E}\,|X_k|^r. \tag{5.6}$$

We continue with the following lemma (analogous to [5, Lemma 6]), which is a consequence of (5.6) and (4.5).

Lemma 5.2. *Let $a > 1$. Consider the field of truncated quenched first moments $\{\widetilde{m}_a(x,t,w) : x \in \mathbb{Z}^d\}$ at scale $(\kappa t)^a$. Let $L(t), L'(t)\colon [0,\infty) \to \mathbb{N}$ be functions such that $(\kappa t)^a \le L'(t) \le L(t)$. Then if $1 \le r \le 2$, it is true that,*

$$\left\langle \left| \sum_{x\in\Lambda_L} (\widetilde{m}_a - \langle \widetilde{m}_a\rangle) \right|^r \right\rangle \le 2(2L'+2)^{d(r-1)}(2L+1)^d \langle |\widetilde{m}_a - \langle \widetilde{m}_a\rangle|^r\rangle. \tag{5.7}$$

(i) Proof. The proof of this result is analogous to that of [5, Lemma 6], requiring the use of the decomposition (5.3) with $f(x) = \widetilde{m}_1(x,t,w) - \langle \widetilde{m}_1\rangle(x,t)$, von Bahr – Esseen inequality (5.6) and Jensen's inequality (4.4). □

Next, we have the following estimate analogous to [5, Lemma 7].

Lemma 5.3. *Let $a > 1$. Consider the field of quenched first moments $\{m(x,t,w) : x \in \mathbb{Z}^d\}$ and of truncated quenched first moments $\{m_a(x,t,w) : x \in \mathbb{Z}^d\}$ at scale $(\kappa t)^a$. Let $L(t)\colon [0,\infty) \to \mathbb{N}$ be such that $(\kappa t)^a \ll L$. Then the following statements are true.*

(i) *Assume that,*

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} - 1 \right| \right\rangle \ll 1.$$

Then,

$$\left\langle \left| \frac{m^L}{\langle m\rangle} - 1 \right| \right\rangle \ll 1.$$

(ii) *Asymptotically as $t \to \infty$ we have,*

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L}(m - \langle m\rangle)}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} m}} - \frac{\sum_{x\in\Lambda_L}(\widetilde{m}_a - \langle \widetilde{m}_a\rangle)}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} \widetilde{m}_a}} \right| \right\rangle \ll 1, \tag{5.8}$$

and

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L}(m - \langle m\rangle)}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} m}} - \frac{\sum_{x\in\Lambda_L}(\widetilde{m}_a - \langle \widetilde{m}_a\rangle)}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} m}} \right| \right\rangle \ll 1. \tag{5.9}$$

Proof. The case $\kappa = 0$ is trivial, so we will assume that $\kappa > 0$.

(i) A direct calculation shows us that,

$$\left| \frac{m^L}{\langle m\rangle} - \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} \right| \le \frac{\sum_{x\in\Lambda_L} |m - \widetilde{m}_a|}{(2L+1)^d \langle m\rangle} + \frac{\langle |m - \widetilde{m}_a|\rangle}{\langle m\rangle} \left| \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} \right|.$$

Now, an application of inequality (4.11) and of the first inequality in equation (4.9) shows us that, $\sum \langle |m - \widetilde{m}_a|\rangle / \big((2L+1)^d \langle m\rangle\big) \le \varepsilon_1(t)$, where $\varepsilon_1(t) := d42^d\big((\kappa t)^a + 1\big)^d \times \mathrm{e}^{-2\kappa t(I((\kappa t)^{a-1}/2) - d)}$. Similarly we have that $\langle |m - \widetilde{m}_a|\rangle / \langle m\rangle \le \varepsilon_1(t)$. By the triangle inequality it follows that,

$$\left\langle \left| \frac{m^L}{\langle m\rangle} - \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} \right| \right\rangle \le 2\varepsilon_1(t) + \left\langle \left| \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} - 1 \right| \right\rangle. \tag{5.10}$$

Now, for $a > 1$, we have $I\big((\kappa t)^{a-1}/2\big) \to \infty$. Hence, $\varepsilon_1(t) \to 0$ as $t \to \infty$. This clearly implies the statement of part (i) of the lemma.

(ii) Let us first note that the left-hand side of equation (5.8) is upper-bounded by,

$$2\frac{\sum_{x\in\Lambda_L} \langle |m - \widetilde{m}_a|\rangle}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} m}} + \left\langle \frac{\sum_{x\in\Lambda_L} |\widetilde{m}_a - \langle \widetilde{m}_a\rangle|}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} \widetilde{m}_a}} \right\rangle \left(\frac{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} \widetilde{m}_a}}{\sqrt{\operatorname{Var}_\mu \sum_{x\in\Lambda_L} m}} - 1 \right). \tag{5.11}$$

Now, by Cauchy – Schwartz inequality we have that

$$\left\langle \frac{\sum_{x\in\Lambda_L} |\widetilde{m}_a - \langle \widetilde{m}_a\rangle|}{\sqrt{\mathrm{Var}_\mu \sum_{x\in\Lambda_L} \widetilde{m}_a}} \right\rangle \le 1.$$

On the other hand, by the assumption $t^a \ll L$ we know that the hypothesis of Lemma 4.7(ii) is satisfied for $U_t = \Lambda_{L(t)}$ so that the asymptotics (4.26) of this lemma holds, and hence the second term of (5.11) tends to 0 as $t \to \infty$. Furthermore, Cauchy – Schwartz inequality and Proposition 4.4(ii), imply that $\sum_{x\in\Lambda}\langle |m - \widetilde{m}_a|\rangle \le (2L+1)^d \varepsilon_2(t)\sqrt{\langle m(0,t)^2\rangle}$, where

$$\varepsilon_2(t) := d42^d\big((\kappa t)^a + 1\big)^{d/2} \exp\big\{-2\kappa t I\big((\kappa t)^{a-1}/2\big) + 2d\kappa t\big\}.$$

In addition, by Lemma 4.7(ii) we have $\mathrm{Var}_\mu \sum_{x\in\Lambda} m \sim (2L+1)^d \sum_{x\in\mathbb{Z}^d} c(0,y,t)$. Now, $\sum_{x\in\mathbb{Z}^d} c(0,y,t) \ge \sqrt{\langle m^2\rangle - \langle m\rangle^2}$. Thus, the first term of (5.11) is upper-bounded by a quantity which asymptotically behaves as $t \to \infty$ like,

$$\frac{\varepsilon_2(t)}{\sqrt{1 - \langle m\rangle^2/\langle m^2\rangle}}.$$

But by Corollary 4.6, and the fact that $\overline{F}_1(t) \ge 0$, we conclude that $\langle m\rangle^2/\langle m^2\rangle \ll 1$. In brief, the first term of (5.11) is upper-bounded by a quantity which asymptotically as $t \to \infty$ behaves like, $\varepsilon_2(t)$. Obviously, when $a > 1$ we have $\varepsilon_2(t) \ll 1$. □

5.3. The annealed asymptotics. Let us now prove the law of large numbers stated in equation (2.4). To simplify the writing of the expressions in the calculations, we will redefine ϵ as 2ϵ, assuming that

$$L(t) \ge \exp\left\{\frac{1}{d} F_{2\epsilon}(t)\right\}, \tag{5.12}$$

for some $\epsilon > 0$, and prove that then in μ-probability it is true that,

$$\frac{m^L(0,t,w)}{\langle m(0,t)\rangle} \sim 1. \tag{5.13}$$

To do so we first remark that by inequality (5.10) of Lemma 5.3, it is enough to show that for $a = \frac{3}{2}$,

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} - 1 \right|^{1+\epsilon} \right\rangle \ll 1. \tag{5.14}$$

Remark that the right-hand side of equation (5.14) can be rewritten as,

$$\left\langle \left| \frac{\sum_{x\in\Lambda_L} \widetilde{m}_a}{(2L+1)^d \langle \widetilde{m}_a\rangle} - 1 \right|^{1+\epsilon} \right\rangle = \frac{\langle |\sum_{x\in\Lambda_L} (\widetilde{m}_a - \langle \widetilde{m}_a\rangle)|^{1+\epsilon}\rangle}{(2L+1)^{d(1+\epsilon)}\langle \widetilde{m}_a\rangle^{1+\epsilon}}. \tag{5.15}$$

At this point we make use of the parity partition decomposition for Λ_L previously defined to deal with the numerator of the right-hand side of equation (5.15) via inequality (5.7) with $r = 1+\epsilon$. We will chose a time dependent mesoscopic scale $L'(t) = (\kappa t)^b$, where $0 < a < b$. Therefore, by Lemma 5.2, the right-hand side of equality (5.15) is upper-bounded by

$$\frac{2(2L'+2)^{d\epsilon}\langle |\widetilde{m}_a - \langle \widetilde{m}_a\rangle|^{1+\epsilon}\rangle}{(2L+1)^{d\epsilon}\langle \widetilde{m}_a\rangle^{1+\epsilon}}. \tag{5.16}$$

Now, since for any nonnegative reals x, y we have $|x-y|^{1+\epsilon} \le |x|^{1+\epsilon} + |y|^{1+\epsilon}$, it follows that $\langle |\widetilde{m}_a - \langle \widetilde{m}_a\rangle|^{1+\epsilon}\rangle \le \langle \widetilde{m}_a^{1+\epsilon}\rangle + \langle \widetilde{m}_a\rangle^{1+\epsilon} \le 2langle\widetilde{m}_a^{1+\epsilon}\rangle$, where in the last inequality we have used Jensen's inequality. Hence, since $\widetilde{m}_a \le m$, the expression of equation (5.16) is upper bounded by,

$$\frac{2(2L'+2)^{d\epsilon}\langle m^{1+\epsilon}\rangle}{(2L+1)^{d\epsilon}\langle \widetilde{m}_a\rangle^{1+\epsilon}}.$$

But by Proposition 4.4(ii) and (iii), we can replace in the denominator of the above expression the term $\langle \widetilde{m}_a\rangle$ by $\langle m\rangle$. Thus, (5.16) is upper-bounded by an expression which is asymptotically equivalent to,

$$\begin{aligned}(5.17)\quad &\frac{2(2L'+2)^{d\epsilon+1}k_1((\kappa t)^{da(2+\epsilon)}+1)\mathrm{e}^{H_1((1+\epsilon)t)-(1+\epsilon)H_1(t)}}{(2L+1)^{d\epsilon}}\\ &\qquad\le c((\kappa t)^{da(2+\epsilon)})+1)(L')^{d\epsilon}\mathrm{e}^{-\epsilon(F_{2\epsilon}(t)-F_\epsilon(t))},\end{aligned}$$

where we have used the second inequality of equation (4.16) in the first inequality, and assumption (5.12) in the last inequality. Note that when $\kappa = 0$ the last expression reduces to $\mathrm{e}^{-\epsilon(G_{2\epsilon}(t)-G_\epsilon(t))}$. Then, Assumption MI shows that the first and the second terms of the right-hand side of (5.17) tend to 0 as $t\to\infty$.

5.4. The nonannealed asymptotics. In this subsection we will prove the asymptotic behavior of equation (2.5). Again, we will redefine ϵ by 2ϵ, assuming that,

$$(5.18)\qquad L(t) \le \exp\left\{\frac{1}{d}F_{-2\epsilon}(t)\right\},$$

for some $\epsilon > 0$. Note that it will be enough to show that,

$$(5.19)\qquad \left\langle\left|\frac{m_L(0,t)}{\langle m(0,t)\rangle}\right|^{1-\epsilon}\right\rangle \ll 1.$$

To do so, note from inequality (4.5)) that the left-hand side of equation (5.19)) is upper-bounded by,

$$(5.20)\qquad (2L+1)^{d\epsilon}\frac{\langle m^{1-\epsilon}(0,t)\rangle}{\langle m(0,t)\rangle^{1-\epsilon}}.$$

Now, by the second inequality of equation (4.16) applied with $\beta = 1-\epsilon$ to the numerator of (5.20), we see that the left-hand side of equation (5.19) is upper bounded by,

$$(2L+1)^{d\epsilon}k_1((\kappa t)^{da(2-\epsilon)}+1)\mathrm{e}^{H_1((1-\epsilon)t)-(1-\epsilon)H_1(t)}.$$

Finally, by assumption (5.18), this expression is upper bounded by,

$$c((\kappa t)^{da(2-\epsilon)}+1)\mathrm{e}^{-\epsilon(F_{-\epsilon}(t)-F_{-2\epsilon}(t))}.$$

Now, the Assumption MI, implies that this expression converges to 0 as $t\to\infty$.

5.5. The Gaussian asymptotics. Here we prove the central limit theorem stated in equation (2.6). We will perform this time a strip-box partition of the box Λ_L into the strip set S_L and the main boxes. Let us fix $a > 1$. Let us fix b and c such that $c > b > a$, and choose the mesoscopic scale $L'(t) = (\kappa t)^c$ and the fine scale $r = (\kappa t)^b$. Note that by Lemma 5.3(ii), it is enough to prove that,

$$\frac{\sum_{x\in\Lambda_L}(\widetilde{m}_a(x,t,w)-\langle\widetilde{m}_a(x,t)\rangle)}{\sqrt{\mathrm{Var}_\mu\sum_{x\in\Lambda_L}\widetilde{m}_a(x,t)}},$$

converges in distribution to the normal law $\mathcal{N}(0,1)$. To do so, we write,

$$\frac{\sum_{x\in\Lambda_L}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)}{\sqrt{\mathrm{Var}_\mu\sum_{x\in\Lambda_L}\widetilde{m}_a}}=\frac{\sum_{x\in S_L}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)}{\sqrt{\mathrm{Var}_\mu\sum_{x\in\Lambda_L}\widetilde{m}_a}}+\frac{\sum_{\mathbf{i}\in\mathcal{I}}\sum_{x\in\Lambda''_{\mathbf{i}}}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)}{\sqrt{\mathrm{Var}_\mu\sum_{x\in\Lambda_L}\widetilde{m}_a}}. \tag{5.21}$$

We will first show that the strip component of the decomposition (5.21) converges to 0 in probability. In fact, note that for t large enough, we have by statement (4.25) applied with $U_t=S_{L(t)}$ and $U_t=\Lambda_{L(t)}$, that

$$\frac{\mathrm{Var}_\mu\sum_{x\in S_L}\widetilde{m}_a}{\mathrm{Var}_\mu\sum_{x\in\Lambda_L}\widetilde{m}_a}\sim\frac{|S_L|}{(2L+1)^d}\le\min\{\kappa,(\kappa t)^{-(c-b)}\},$$

where for the last inequality we have used estimate (5.5). Since $c>b$, this tends to 0 as $t\to\infty$.

Therefore, it is enough to prove that the second term of the right-hand side of equality (5.21), tends in law to $\mathcal{N}(0,1)$. For this purpose, since the random variables $\{\sum_{x\in\Lambda''_{\mathbf{i}}}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle):\mathbf{i}\in\mathcal{I}-\}$ are independent, it is enough to verify a version of the Lyapunov condition. Namely, we will show that,

$$\frac{\sum_{\mathbf{i}\in\mathcal{I}}\langle|\sum_{x\in\Lambda''_{\mathbf{i}}}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)|^{2+\epsilon}\rangle}{(\sum_{\mathbf{i}\in\mathcal{I}}\mathrm{Var}_\mu\sum_{x\in\Lambda''_{\mathbf{i}}}\widetilde{m}_a)^{1+\epsilon/2}}\ll 1, \tag{5.22}$$

and then apply again statement (4.25) to conclude that the variance of the denominator of the second term of equation can be substituted by $\sum_{\mathbf{i}\in\mathcal{I}}\mathrm{Var}_\mu\sum_{x\in\Lambda''_{\mathbf{i}}}\widetilde{m}_a$. Now, by the same token, we see that the denominator of the left-hand side of equation (5.22), behaves asymptotically as $t\to\infty$ like

$$(2L+1)^{d(1+\epsilon/2)}\Big(\sum_{x\in\mathbb{Z}^d}c_a(0,x,t)\Big)^{1+\epsilon/2}. \tag{5.23}$$

Thus, it is enough to prove the asymptotic behavior (5.22), with the denominator replaced by (5.23). Now, note that $\sum_{x\in\mathbb{Z}^d}c_a(0,x,t)$ is lower bounded by $\langle\widetilde{m}_a^2(0,t)\rangle-\langle\widetilde{m}_a(0,t)\rangle^2$. And by Proposition 4.4(i) and (iii) and Lemma 4.5, this variance is lower-bounded by an expression which is asymptotically equivalent to,

$$c\big((\kappa t)^{3da}+1\big)^{-1}\langle m(0,2t)\rangle-\langle m(0,t)\rangle^2\big). \tag{5.24}$$

Now, Assumption MI with $\theta=2$ and the inequality $\langle\widetilde{m}_a(0,t)\rangle\le\langle m(0,t)\rangle$, imply that, $(\log\langle m(0,2t)\rangle-2\log\langle m(0,t)\rangle\gg\log t$. This shows that the second term $c\langle m(0,t)\rangle^2$ of the expression (5.24), is negligible with respect to the first one $c\big((\kappa t)^{3da}+1\big)^{-1}\langle m(0,2t)\rangle$. Hence, we conclude that it is enough to prove that,

$$\big((\kappa t)^{3da}+1\big)\frac{\sum_{\mathbf{i}\in\mathcal{I}}\langle|\sum_{x\in\Lambda''_{\mathbf{i}}}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)|^{2+\epsilon}\rangle}{(2L+1)^{d(1+\epsilon/2)}\langle m(0,2t)\rangle^{(1+\epsilon/2)}}\ll 1. \tag{5.25}$$

By Jensen's inequality (4.4) and the upper-bound in equation (5.2), we see that, $\sum_{\mathbf{i}\in\mathcal{I}}\langle|\sum_{x\in\Lambda''_{\mathbf{i}}}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)|^{2+\epsilon}\rangle\le(L'+1)^{d(1+\epsilon)}(2L+1)^d\langle|\widetilde{m}_a-\langle\widetilde{m}_a\rangle|^{2+\epsilon}\rangle$. Using again the fact that for nonnegative reals x,y we have $|x-y|^{1+\epsilon}\le|x|^{1+\epsilon}+|y|^{1+\epsilon}$, it follows that $\langle|\widetilde{m}_a-\langle\widetilde{m}_a\rangle|^{2+\epsilon}\rangle\le\langle m^{2+\epsilon}\rangle+\langle m\rangle^{2+\epsilon}\le 2\langle m^{2+\epsilon}\rangle$, where we have used Jensen's inequality in the last inequality. We see that the left-hand side of (5.22) is upper-bounded by,

$$\big((\kappa t)^{3da}+1\big)(L'+1)^{d(1+\epsilon)}\frac{\langle m(0,t)^{2+\epsilon}\rangle}{(2L+1)^{d\epsilon/2}\langle m(0,2t)\rangle^{(1+\epsilon/2)}}.$$

Finally, by the hypothesis on the growth of L this can be bounded by,

$$\big((\kappa t)^{3da}+1\big)^2(L'+1)^{d(1+\epsilon)}\mathrm{e}^{\epsilon/2F_{\epsilon/2}(2t)-\epsilon/2F_\epsilon(2t)},$$

which by Condition MI, tends to 0 as $t\to\infty$.

5.6. The non-Gaussian asymptotics. Here we will prove the asymptotics of equation (2.7). It will be necessary to perform a parity partition of Λ_L with a mesoscopic scale $L'=(\kappa t)^b$ for some $b>1$. First note that by equation (5.9) it will be enough to show that for some $b>a>1$,

$$\left\langle\left|\frac{\sum_{x\in\Lambda_L}(\widetilde{m}_a-\langle\widetilde{m}_a\rangle)}{\sqrt{\mathrm{Var}_\mu\sum_{x\in\Lambda_L}m}}\right|^{2-\epsilon}\right\rangle\ll 1. \tag{5.26}$$

Now, by equation (4.24), the denominator $(\mathrm{Var}_\mu\sum_{x\in\Lambda_L}m)^{1-\epsilon/2}$, of the left-hand side of equation (5.26), can be lower-bounded by

$$\begin{aligned}(2L+1)^{d(1-\epsilon/2)}&(\langle m^2\rangle-\langle m\rangle^2)^{1-\epsilon/2}\\ &\geq c(2L+1)^{d(1-\epsilon/2)}\langle m^2\rangle^{1-\epsilon/2}\\ &\geq c\big((\kappa t)^{3da}+1\big)^{-1}(2L+1)^{d(1-\epsilon/2)}\langle m(0,2t)\rangle^{1-\epsilon/2},\end{aligned}$$

where in the second to last inequality we used the Assumption MI and Lemma 4.5 and in the last inequality assumption Lemma 4.5. On the other hand, inequality (5.7) of Lemma 5.2, applied with $r=2-\epsilon$, Jensen's inequality and $\widetilde{m}_a\leq m$, shows us that the numerator of the left-hand side of equation (5.26) is upper-bounded by $4(2L'+1)^{d(1-\epsilon)}(2L+1)^{d\epsilon/2}\langle m^{2-\epsilon}\rangle$. Using the upper-bound $\langle m^{2-\epsilon}(0,t)\rangle\leq k_1((\kappa t)^{3da}+1)\langle m(0,(2-\epsilon)t)\rangle$ of Lemma 4.5, the definition of the intermittency exponents $\{F_\theta\}$ in (1.3), and the assumption $L(t)\leq\mathrm{e}^{F_{-\epsilon}(2t)/2}$, we hence see that it is enough to show that,

$$(2L'+1)^{d(1-\epsilon)}((\kappa t)^{3da}+1)^2\mathrm{e}^{\epsilon/2F_{-\epsilon}(2t)}\mathrm{e}^{-\epsilon/2F_{-\epsilon/2}(2t)}\ll 1.$$

But this is a consequence of Assumption MI.

5.7. Proof of Theorem 2.11. Let us first prove part (i) for Weibull-type tails. Note that it is enough to find an $x>0$ such that,

$$\left\langle\left(\frac{m_L(0,t)}{\mathrm{e}^{(a(\gamma)+\delta)H(t)}}\right)^x\right\rangle\ll 1.$$

Now, this expression is upper bounded by,

$$\mathrm{e}^{-(x(a+\delta)H(t)-H(xt)-(1-x)\gamma H(t))+o(H(t))},$$

where we have used the asymptotics (1.5). Since $H\in R_{\rho'}$ for $\rho'=\frac{\rho}{\rho-1}$, we see that the above expression is upper bounded by,

$$\mathrm{e}^{-f_{\mathrm{W}}(x,a+\delta)H(t)+o(H(t))},$$

where $f_{\mathrm{W}}(x,b):=xb-x^{\rho'}-(1-x)\gamma$ for $x>0,b>0$. This function has a unique root at $x_0=\big((\rho-1)(b+\gamma)/\rho\big)^{\rho-1}$ when $b=a(\gamma)$. Choosing $x=x_0$, we get the upper bound,

$$\mathrm{e}^{-\delta x_0H(t)+o(H(t))}\ll 1.$$

This proves Theorem 2.11(i). The proof of part (iv) for Fréchet-type tails is completely analogous to the previous argument, so it will be omitted. To prove part (ii), note that in analogy to the proof of part (i), it is enough to show that,

$$\mathrm{e}^{-(xH((a+\delta)t)/(a+\delta)-H(xt)-(1-x)\gamma t)+o(t)} \ll 1. \tag{5.27}$$

Now, by supposition (2.10) we have

$$\frac{H(xt)-xH\big((a+\delta)t\big)/(a+\delta)}{t} \sim -\rho x \log \frac{x}{a+\delta}.$$

Thus, the expression of equation (5.27), is upper-bounded by

$$\mathrm{e}^{-f_{\mathrm{D}}(x,a+\delta)t+o(t)},$$

where $f_{\mathrm{D}}(x,b) := \rho x \log(x/b) - (1-x)\gamma$ for $x>0$, $b>0$. But this function has a single root at $x_0 = a\mathrm{e}^{(\gamma-1)/\rho}$ when $b=a$. Hence, we obtain the upper bound,

$$\mathrm{e}^{-x_0 \log(1+\delta/a)+o(t)} \ll 1.$$

This proves Theorem 2.11(ii). The proof of part (iii) for almost bounded potentials is analogous to the proof of part (ii) so it will be omitted. □

Acknowledgments. Alejandro Ramírez thanks the Courant Institute of Mathematical Science, New York City, for its kind hospitality, where part of this work was done.

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École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland & Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012-1185, USA
E-mail address: benarous@cims.nyu.edu

Department of Mathematics and Statistics, University of North Carolina — Charlotte, 376 Fretwell Bldg., 9201 University City Blvd., Charlotte, NC 28223-0001, USA
E-mail address: smolchan@uncc.edu

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Macul, Santiago 6904411, Chile
E-mail address: aramirez@mat.puc.cl
URL: http://www.mat.puc.cl/~aramirez