Random walk in the low disorder ballistic regime

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We consider a random walk in in $\mathbb{Z}^d$ which jumps from a site x to a nearest neighboring site $x+e$, with e an element of the sites V at distance 1 from the origin, with probability $p_0(e)+\epsilon\xi(x,e)$. Here $\sum_e p_0(e) = 1$ and $p_0(e) > 0$, ϵ is a small parameter while $\{\{\xi(x,e) : e \in V\} : x \in \mathbb{Z}^d\}$ are i.i.d. random variables with an absolute value bounded by 1. We review recent progress in the non-vanishing velocity case, giving an asymptotic expansion in ϵ of the invariant measure of the environmental process, and bounds for the velocity.

Keywords: Random walk in random environment; Ballisticity; Invariant probability measure; low disorder; asymptotic expansions.

1. Introduction

Random perturbations of random walks have attracted attention both from the mathematical and physical literature, as a natural model of movement in a disordered media (see for example the reviews Ref. 3 and 13). Fundamental questions about its behavior remain open[3,13]. In this article, we will review the actual state of understanding of the so called ballistic case (non-vanishing velocity) at low disorder and will specially describe recent results obtained providing asymptotic expansions for the velocity and the invariant measure[2,5].

For any $x \in \mathbb{Z}^d$, define as $|x|_1$ its l_1 norm. Let $V := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ and $\mathcal{P} := \{p = \{p(e) : e \in V\} : p(e) \geq 0, \sum_{e\in V} p(e) = 1\}$. We define $\Omega := \mathcal{P}^{\mathcal{Z}^d}$ and call it the *environmental space*. For each environment $\omega \in \Omega$ and $x \in \mathbb{Z}^d$, we define the *random walk in the environment* ω starting from x, as the Markov chain with transition probabilities

$$P_{x,\omega}(X_{n+1} = y + e | X_n = y) = \omega(y, e) \qquad \text{for} \quad y \in \mathbb{Z}^d, e \in V.$$

Let $\mathbb{P}$ be a probability measure defined on Ω, denote by $\mathbb{E}$ the corresponding expectation associated to $\mathbb{P}$ and call $P_{x,\omega}$ the *quenched law* of the random walk in random environment and $P_x := \int P_{x,\omega} d\mathbb{P}$ the *averaged* or *annealed* law of the random walk starting from x. Throughout this article, we will assume that under $\mathbb{P}$ the random variables $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d.

We say that a random walk in random environment is *ballistic* if $\mathbb{P}_0$-a.s. we have that

$$\liminf_{n\to\infty} \frac{X_n}{n} > 0. \tag{1}$$

It is well known (see for example[3]) that whenever a random walk in random environment is ballistic, the *velocity*

$$v := \lim_{n\to\infty} \frac{X_n}{n}$$

exists.

We will consider perturbations of the dynamics of a random walk in a deterministic environment with jump rates $p_0 \in \mathcal{P}$ at every site, where

$$\mathcal{P}_0 := \{p \in \mathcal{P} : \min_{e\in V} p(e) > 0\}. \tag{2}$$

Given $\epsilon > 0$, define

$$\Omega_{p_0,\epsilon} := \left\{ \omega \in \Omega : \sup_{x\in\mathbb{Z}^d, e\in V} |\omega(x,e) - p_0(e)| \leq \epsilon \right\}.$$

Furthermore, we will define the constant

$$\kappa := \min\{p_0(e) : e \in V\} - \epsilon,$$

which is a lower bound for the jump probabilities of environments $\omega \in \Omega_{p_0,\epsilon}$, so that $\omega(x,e) \geq \kappa$ for every $x \in \mathbb{Z}^d$ and $e \in V$.

An important quantity which will somehow give us a control on how close we are to a possible regime of vanishing velocity is the *local drift* of the random walk, defined on an environment ω and at a point $x \in \mathbb{Z}^d$ as

$$d(x,\omega) := \sum_{e\in V} e\omega(x,e). \tag{3}$$

Assumption (LD). *Let $\alpha > 0, C > 0$, $\epsilon > 0$ and $p_0 \in \mathcal{P}_0$. We say that the local drift condition **(LD)** with bound $C\epsilon^\alpha$ is satisfied if $\mathbb{P}(\Omega_{p_0,\epsilon}) = 1$ and*

$$\mathbb{E}\left[d(0,\omega)\cdot e_1\right] > C\epsilon^\alpha. \tag{4}$$

Given an environment in $\omega \in \Omega_{p_0,\epsilon}$, we will define

$$\xi(x,e) := \frac{1}{\epsilon}\left(\omega(x,e) - p_0(e)\right), \quad \text{for} \quad x \in \mathbb{Z}^d, e \in V,$$

and use the more suggestive writings

$$\omega(x,e) = p_0(e) + \epsilon\xi(x,e) \quad \text{and} \quad \omega(x,e) = p_\epsilon(e) + \epsilon\bar{\xi}(x,e), \tag{5}$$

where $p_\epsilon(e) := p_0(e) + \mathbb{E}[\xi(0,e)]$ and $\bar{\xi}(x,e) := \xi(x,e) - p_1(e)$.

2. Invariant measures and a heuristic derivation of their expansion

A fundamental and useful concept in the understanding of the behavior of a random walk in random environment is the invariant measure of the environment seen from the position of the random walk[3]. Here we will show how to formally derive an expansion in the parameter ϵ of it.

For each $x \in \mathbb{Z}^d$, we define the shift $t_x : \Omega \to \Omega$ as $t_x\omega(y) := \omega(x+y)$ for each $y \in \mathbb{Z}^d$. Now, given an environment $\omega \in \Omega$, we define the *environment seen from the random walk* or the *environmental process* as

$$\bar{\omega}_n := t_{X_n}\omega \quad \text{for} \quad n \geq 0.$$

Note that $\{\bar{\omega}_n : n \geq 0\}$ is a Markov process with state space Ω and transition kernel defined for $f : \Omega \to \mathbb{R}$ by

$$Rf(\omega) := \sum_{e \in V} \omega(0,e) f(t_e\omega).$$

We say that a probability measure $\mathbb{Q}$ defined in Ω is an *invariant* measure for the environmental process if for every bounded measurable function $f : \Omega \to \mathbb{R}$ we have that $\int Rf d\mathbb{Q} = \int f d\mathbb{Q}$. Now, in the case of an environment $\omega \in \Omega_{p_0,\epsilon}$, by (5), we can write the transition kernel as

$$R = R_\epsilon = R_0 + \epsilon R_1, \tag{6}$$

where for $f : \Omega \to \Omega$ measurable we have

$$R_0 f(\omega) = \sum_{e \in V} p_\epsilon(e) f(t_e\omega) \quad \text{and} \quad R_1 f(\omega) = \sum_{e \in V} \bar{\xi}(0,\omega) f(t_e\omega).$$

Assume now that the measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ with a Radon-Nykodim derivative g. We then have that

$$\int Rfg d\mathbb{P} = \int fg d\mathbb{P}. \tag{7}$$

If we furthermore assume that g has the analytic expansion

$$g = \sum_{k=0}^{\infty} \epsilon^k g_k, \tag{8}$$

we see inserting (6) and (8) into (7) that necessarily $g_0 = 1$ and that for each $k \geq 0$ one should have that

$$g_{k+1} = (R_0^*)^{-1} R_1^* g_k, \tag{9}$$

where R_0^* and R_1^* denotes the adjoint of R_0 and R_1 respectively with respect to the measure $\mathbb{P}$. Now,

$$(R_0^*)^{-1} f(\omega) = \sum_{z \in \mathbb{Z}^d} G^{p_\epsilon^*}(0, z) f(t_z \omega) \quad \text{and} \quad R_1^* f(\omega) = \sum_{e \in V} \bar{\xi}(-e, e) f(t_e \omega).$$

It follows that $g_1 = \sum_{z \in \mathbb{Z}^d, e \in V} \bar{\xi}(z, e) G^{p_\epsilon^*}(0, z + e)$, and hence

$$g = 1 + \epsilon \sum_{z \in \mathbb{Z}^d, e \in V} \bar{\xi}(z, e) G^{p_\epsilon^*}(0, z + e) + O(\epsilon^2). \tag{10}$$

Of course, for the above formal expansion to make sense, we will need to require certain conditions on the random walk which will ensure the existence of a non-vanishing speed.

3. Asymptotic expansion of the invariant measure

Here we will show in what sense is the formal expansion derived in the previous section valid. Although it does not seem possible in general to obtain an analytic expansion of the Radon-Nykodim derivative of the invariant measure of the environmental process as given by (8) and (9), under the local drift condition **(LD)** it is possible to derive an asymptotic expansion at least up to first order. Given a $B \subset \mathbb{Z}^d$ and a probability measure $\mathbb{S}$ in Ω, we define $\mathbb{S}_B$ as the restriction of $\mathbb{S}$ to $\mathcal{P}^B$, and with a slight abuse of language we will call it just the *restriction of* $\mathbb{S}$ *to* B. Furthermore, we define for each $p \in \mathcal{P}$, $x \in \mathbb{Z}^d$ and $e \in V$,

$$J_p(x) := \lim_{n \to \infty} \sum_{k=0}^{n} (p_k(0, -x) - p(0, 0)),$$

where for each $n \geq 0$ and $x, y \in \mathbb{Z}^d$, we define $p_n(x, y)$ as the probability that a random walk with transition kernel p jumps from x to y after n steps. The following theorem has been proved in Ref. 2.

Theorem 3.1 (Campos-Ramírez). *Let $\eta > 0$ and B finite subset of $\mathbb{Z}^d$. Then, there is an $\epsilon_0 > 0$ such that whenever $\epsilon \leq \epsilon_0$, $p_0 \in \mathcal{P}_0$, and $\mathbb{P}$ satisfies the local drift condition **(LD)** with bound ϵ [c.f. (4)], the limiting invariant measure $\mathbb{Q}$ has a restriction $\mathbb{Q}_B$ to B which is absolutely continuous with respect to the restriction $\mathbb{P}_B$ to B of $\mathbb{P}$, with a Radon-Nikodym derivative admitting $\mathbb{P}$-a.s. the expansion*

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \epsilon \sum_{z \in B} \sum_{e \in V} \bar{\xi}(z,e) J_{p_\epsilon^*}(e+z) + O\left(\epsilon^{2-\eta}\right), \tag{11}$$

where $\left|O\left(\epsilon^{2-\eta}\right)\right| \leq c_1 \epsilon^{2-\eta}$, for some constant $c_1 = c_1(\eta, \kappa, d, B)$ depending only on η, κ, d and B.

As a corollary of Theorem 3.1, it is possible to obtain the asymptotic expansion of the velocity already obtained by Sabot[7] under the local drift condition **(LD)** with bound ϵ. Indeed, this condition implies that the random walk is ballistic (see Ref. 7) and hence by an argument presented by Sznitman and Zerner in Ref. 12 that the marginal law of the environmental process at time n converges to the invariant measure as $n \to \infty$ of Theorem 3.1. On the other hand since $X_n - \sum_{i=0}^{n-1} d(0, \bar{\omega}_i)$, $\quad i \geq 0$, is a martingale, we can derive Sabot's asymptotic expansion for the velocity

$$v = \int d(0,\omega) d\mathbb{Q} = d_0 + \epsilon d_1 + \epsilon^2 d_2^\epsilon + O\left(\epsilon^{3-\eta}\right), \tag{12}$$

where

$$d_0 := \sum_{e \in V} e p_0(e), \quad d_1 := \sum_{e \in V} e \mathbb{E}[\xi(0,e)], \tag{13}$$

$$d_2^\epsilon := \sum_{e \in V} e p_{2,\epsilon} \quad \text{and} \quad p_{2,\epsilon} := \sum_{e,e' \in V} C_{e,e'} J_{p_\epsilon^*}(e) \tag{14}$$

with $C_{e,e'} := Cov(\xi(0,e), \xi(0,e'))$, while $|O(\epsilon^{3-\eta})|_1 \leq c_1' \epsilon^{3-\eta}$ for some constant c_1'.

In dimensions $d \geq 2$, it is possible to expand $J_{p_\epsilon^*}$ in ϵ to rewrite the expansion (11) as

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \epsilon \sum_{z \in B} \sum_{e \in V} \bar{\xi}(z,e) J_{p_0^*}(e+z) + O\left(\epsilon^{2-\eta}\right), \tag{15}$$

where $\left|O\left(\epsilon^{2-\eta}\right)\right| \leq c_2 \epsilon^{2-\eta}$, for some constant $c_2 = c_2(\eta, \kappa, d, B)$ depending only on η, κ, d and B. Note that in the particular case when p_0 are the jump probabilities of a simple symmetric random walk and the dimension $d = 2$, as shown in Ref. 2, we can use explicit expressions for the potential kernel of the random walk (see Ref. 6,9), to obtain for example the following explicit expansion from (15),

$$\frac{d\mathbb{Q}_{z_0,z_1}}{d\mathbb{P}_{z_0,z_1}} = 1 - \frac{4}{\pi}\left(\bar{\xi}(z_1,e_1) + \bar{\xi}(z_1,-e_1)\right)\epsilon + \left(\frac{8}{\pi} - 4\right)\bar{\xi}(z_1,e_2)\epsilon + O\left(\epsilon^{2-\eta}\right),$$

where $z_0 := (0,0)$ and $z_1 := (1,0)$.

To prove Theorem 3.1 in Ref. 2, we first express the limiting invariant measure as a Cesàro average up to a random time distributed as a geometric random variable. Indeed, let $\delta \in (0,1)$, and τ_δ be a random time with a geometric distribution of parameter $1-\delta$, independent of the random walk and of the environment. Define for $x, y \in \mathbb{Z}^d$ the Green function

$$g_\delta^\omega(x,y) := E'_{x,\omega}\left[\sum_{n=0}^{\tau_\delta} 1_y(X_n)\right],$$

where the expectation is taken both over τ_δ and the random walk. We now define the probability measure μ_δ by the equality valid for every continuous function $f : \Omega \to \mathbb{R}$ as

$$\int f d\mu_\delta = \frac{\sum_{x\in\mathbb{Z}^d} \mathbb{E}[g_\delta^\omega(0,x) f(t_x\omega)]}{\sum_{x\in\mathbb{Z}^d} \mathbb{E}[g_\delta^\omega(0,x)]}. \tag{16}$$

It turns out that whenever the small local drift condition **(LD)** is satisfied with bound ϵ, the limit

$$\mu := \lim_{\delta\to 1^-} \mu_\delta \tag{17}$$

exists weakly and is the limiting invariant measure for the environmental process. In fact, this is true only under the assumption that the polynomial ballisticity condition $(P)_M$ for $M \geq 15d + 5$ is satisfied[2]. In section 4 we will give a precise definition of the polynomial ballisticity condition. The representation of the limiting invariant measure given by (17) and (16) is then used in Ref. 2 to derive the asymptotic expansion of Theorem 3.1, through the use of Green function expansions. On the other hand, this method produces badly converging series when the perturbation is around a $p_0 \in \mathcal{P}_0$ [c.f. (2)] which has a vanishing local drift.

4. The very small drift ballistic regime

It is natural to wonder, up to which point can the local drift condition **(LD)** with bound ϵ be relaxed, and still obtain the asymptotic expansion of the invariant measure given by Theorem 3.1 and of the velocity of Sabot[7]. It is still an open problem to give a complete answer to this question. We can nevertheless easily prove that under **(LD)** with bounds of order ϵ^2, one still has a ballistic behavior whenever the unperturbed random walk has a vanishing velocity.

Lemma 4.1. *Let $d \geq 2$ and $\eta \in (0,1)$. Let $\epsilon \in (0,1)$ and consider a random walk whose law satisfies the small drift condition **(LD)** with bound $C\epsilon^2$, where $C := 2/\min_{e\in V} p_0(e)^2$. Then the random walk is ballistic.*

Proof. We use the notation $d = d(0,\omega)$ for the local drift [c.f. (3)]. Let G be the set of vectors $g = \{g(e) : e \in V\}$ such that each component is in $[0,1]$. To prove that a random walk is ballistic, it is enough to show that the so called *Kalikow's criterion* is satisfied (see Ref. 4,10):

$$\inf_{g\in G} \mathbb{E}\left[\frac{d(0,\omega)\cdot e_1}{\sum_{e\in V}\omega(0,e)g(e)}\right] > 0.$$

Now, for each $g \in G$ one has that if ϵ is small enough

$$\begin{aligned}
&E\left[\frac{d\cdot e_1}{\sum_{e\in V}\omega(0,e)g(e)}\right] = E\left[\frac{(d\cdot e_1)_+}{\sum_{e\in V}\omega(0,e)g(e)}\right] - E\left[\frac{(d\cdot e_1)_-}{\sum_{e\in V}\omega(0,e)g(e)}\right]\\
&\geq E\left[\frac{(d\cdot e_1)_+}{\sum_{e\in V}p_0(e)g(e)+\epsilon\sum_{e\in V}g(e)}\right] - E\left[\frac{(d\cdot e_1)_-}{\sum_{e\in V}p_0(e)g(e)-\epsilon\sum_{e\in V}g(e)}\right]\\
&\geq \frac{\lambda}{\sum_{e\in V}p_0(e)g(e)} - \frac{\epsilon\sum_{e\in V}g(e)}{\left(\sum_{e\in V}p_0(e)g(e)\right)^2}\mathbb{E}[(d\cdot e_1)_+ + (d\cdot e_1)_-]\\
&\geq \frac{1}{\sum_{e\in V}g(e)}\left(\lambda - \epsilon\frac{\mathbb{E}[(d\cdot e_1)_+ + (d\cdot e_1)_-]}{\min_{e\in V}p_0(e)^2}\right) > \frac{1}{2d}\left(\lambda - \epsilon^2\frac{2}{\min_{e\in V}p_0(e)^2}\right) > 0,
\end{aligned}$$

where the last two inequalities are clearly satisfied if $\lambda > C\epsilon^2$. □

Let us now try to explore if in the case of perturbations of random walks with vanishing velocity, it would be possible to improve the bound of Lemma 4.1 to local drifts smaller than ϵ^α with $\alpha < 2$. We will look at the velocity expansion (12) for the case in which the perturbation is performed on a simple symmetric random walk, so that $p_0(e) = 1/2d$ for all $e \in V$. To simplify notation we define $J_0 := J_{p_0}$. Note that the factor $d_2^\epsilon = \sum_{e\in V} ep_{2,\epsilon}$ [c.f. (14)] in the term of second order of (12) can be expanded as

$$p_{2,\epsilon} = \sum_{e,e'\in V} C_{e,e'}J_0(e) + O(\epsilon) = O(\epsilon),$$

where we have used the isotropy of the Green function which implies that $J_0(e)$ is independent of $e \in V$. Furthermore, since $d_0 = 0$ [c.f. (13)] for a simple symmetric random walk, it follows by the expansion (12), that in this case we have that

$$v = \epsilon d_1 + O(\epsilon^{3-\eta}) = \mathbb{E}[d(0,\omega)] + O(\epsilon^{3-\eta}). \tag{18}$$

From (18) we see that at least formally, the velocity does not vanish as long as we have

$$\mathbb{E}[d(0,\omega)] \cdot e_1 > \epsilon^{3-\eta}, \tag{19}$$

for some $\eta > 0$. Let

$$\alpha(d) := \begin{cases} 2 & \text{if } d = 2 \\ 2.5 & \text{if } d = 3 \\ 3 & \text{if } d \geq 4. \end{cases}$$

The following theorem shows that at least in dimensions $d \geq 4$ condition (19) does imply that the random walk is ballistic [c.f. (1)]. In its statement we incorporate the case $d = 2$ of Lemma 4.1.

Theorem 4.1 (Sznitman). *Let $d \geq 2$ and $\eta \in (0,1)$. There exists an $\epsilon_0 \in (0,1)$ such that for every $\epsilon \in (0,\epsilon_0)$, whenever a random walk which is a random perturbation of a simple symmetric random walk has a law which satisfies the small drift condition **(LD)** with bound $\epsilon^{\alpha(d)-\eta}$, it is ballistic.*

The case $d = 2$ of Theorem 4.1 is contained in Lemma 4.1 above. In dimensions $d \geq 3$, Theorem 4.1 was proved by Sznitman in Ref. 11 and includes cases of random walks in random environments which do not satisfy Kalikow's condition[4,12]. Heuristically, its proof is based on showing that at scales of order ϵ^{-1} the behavior of the random walk is not far in terms of exit times from slabs from the behavior of a simple symmetric random walk. Then, through a renormalization type argument it is possible to derive a Solomon type criterion (originally for $d = 1$[8]), called the *effective criterion*[10] for slabs of order ϵ^{-4}, which then implies ballisticity.

5. Velocity estimates on the very small drift ballistic regime

In view of Theorem 4.1, it is natural to ask wether or not the velocity expansion (12) is still valid under the small drift condition **(LD)** with bound $\epsilon^{\alpha(d)}$ for $d \geq 3$. The following result proved by Laurent, Ramírez and Sabot in Ref. 5, shows that as an upper bound, the expansion (12) is still valid for perturbations around the simple symmetric random walk.

Theorem 5.1 (Laurent-Ramírez-Sabot). *Let $d \geq 3$. Consider a random walk in random environment satisfying the small drift condition **(LD)** with bound $\epsilon^{\alpha(d)}$. Then, for every $\eta > 0$ there is a constant $C_\eta > 0$ such that*

$$0 < v \cdot e_1 \leq \mathbb{E}[d(0,\omega)] \cdot e_1 + C_\eta \epsilon^{\alpha(d)-\eta}.$$

The proof of Theorem 5.1 combines elements of the proof of Theorem 4.1, with renormalization type arguments similar in spirit to the one used to establish ballisticity of random walks in random environments satisfying the polynomial ballisticity condition introduced in Ref. 1 and which we define below.

Let $l \in \mathbb{S}^{d-1}$, $L > 0$ and consider the box

$$B_{l,L} := R\left((-L, L) \times \left(-70L^3, 70L^3\right)^{d-1}\right) \cap \mathbb{Z}^d$$

where R is a rotation in $\mathbb{R}^d$ fixing the origin and defined by $R(e_1) = l$. Given $M \geq 1$ and $L \geq 2$, we say that the *polynomial condition* $(P)_M$ *in direction* l *is satisfied on a box of size* L (also written $(P)_M|l)$ if

$$P_0(X_{T_{B_{l,L}}} \cdot l < L) \leq \frac{1}{L^M}. \tag{20}$$

This condition was introduced Ref. 1, showing that whenever the environment is i.i.d. and uniformly elliptic and the polynomial condition is satisfied with $M \geq 15d + 5$ on a box of size $L \geq c_0$ with

$$c_0 := \frac{2}{3} 2^{3(d-1)} \wedge \exp\left\{2\left(\ln 90 + \sum_{j=1}^{\infty} \frac{\ln j}{2^j}\right)\right\},$$

the random walk is necesarilly ballistic and the decay in (20) is actually at least e^{-L^γ} for all $\gamma \in (0, 1)$. The proof of this statement involves a renormalization argument, where boxes are classified as *bad* or *good* according to the size of the quenched probability that the random walk exits the box through sides whose normal direction is not in the hyperspace defined by l. A modification of this argument which keeps track of the time spent on each box by the random walk, gives the velocity estimate of Theorem 5.1.

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