# REDUCED MODELS FOR LINEARLY ELASTIC THIN FILMS ALLOWING FOR FRACTURE, DEBONDING OR DELAMINATION

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Abstract. This work is devoted so show the appearance of different cracking modes in linearly elastic thin film systems by means of an asymptotic analysis as the thickness tends to zero. By superposing two thin plates, and upon suitable scaling law assumptions on the elasticity and fracture parameters, it is proven that either debonding or transverse cracks can emerge in the limit. A model coupling debonding, transverse cracks and delamination is also discussed.

Contents



## 1. Introduction

It is experimentaly observed that thin films systems can essentially develop two different crack patterns: either transverse cracks channeling through the thickness of the film, or planar debonding at the interface of two layers. In classical fracture mechanics, a threshold criterion on the energy release rate drives the propagation of a crack along a prescribed path. Within this framework, [31] described different possibilities of failure modes. In [42], a reduced two-dimensional model of a thin film system on an elastic foundation is proposed, and the propagation of different crack modes is discussed. This model is later

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recast as an energy minimization problem, based on the variational approach to fracture of [13], first in [32] in a simplified one-dimensional setting where transverse cracks are represented by a finite number of discontinuity points for the displacement, then in [33] in the full two-dimensional case. The total energy is the sum of a bulk energy (including the elastic energy of the film outside the transverse cracks) and a surface energy of Griffith type (including the area of the transverse fractures and of the debonded regions). Concretely, it takes the form

$$E(\bar{\boldsymbol{u}},\Gamma,\Delta)=\int_{\omega\setminus\Gamma}\underbrace{\left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f}e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}})+\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\right]}_{Ae(\bar{\boldsymbol{u}}):e(\bar{\boldsymbol{u}})}\,\mathrm{d}x$$
$$+\frac{\mu_b}{2}\int_{\omega\setminus\Delta}|\bar{\boldsymbol{u}}-\boldsymbol{w}|^2\,\mathrm{d}x+\kappa_f\,\mathrm{length}(\Gamma)+\kappa_b\,\mathrm{area}(\Delta);$$

we proceed to explain each term separately. The region $\omega\subset\mathbb{R}^2$ denotes the basis of a thin film $\Omega^\varepsilon=\omega\times(0,h_f)$ bonded on a infinitely rigid substrate, where $h_f$ is the thickness of the film and $\varepsilon=\frac{h_f}{L}$, $L=\mathrm{diam}\,\omega$, is a non-dimensional small parameter. The transverse cracks are of the form $\Gamma\times(0,h_f)$ where $\Gamma$ is a one-dimensional object which can be thought of as the union of a finite number of closed curves, which are themselves part of the unknowns of the problem. The delamination zone $\Delta\subset\omega$ is also an unknown. The in-plane displacement of the film at the interface with the substrate is denoted by $\bar{\boldsymbol{u}}:\omega\setminus\Gamma\to\mathbb{R}^2$. The fracture toughness $\kappa_f$ is a material property of the film, while $\kappa_b$ measures the strength of the bonding between the film and the substrate. The reduced linearly elastic energy $Ae(\bar{\boldsymbol{u}}):e(\bar{\boldsymbol{u}})$ is well-known and rigorously derived in the Kirchhoff-Love theory of elastic plates [18].

It remains to explain the term $E_c:=\frac{\mu_b}{2}\int_{\omega\setminus\Delta}|\bar{\boldsymbol{u}}-\boldsymbol{w}|^2\,\mathrm{d}x$. The map $\boldsymbol{w}:\omega\to\mathbb{R}^2$ is given and represents the displacement of the substrate. Since the substrate is assumed to be infinitely rigid, $\boldsymbol{w}$ is the same displacement it would undergo if the film were not present. Note also that we are only considering planar displacements of the substrate. The energy $E_c$ represents the price to pay in order for the film to deform differently from the substrate. It only has to be paid in $\omega\setminus\Delta$ because in $\Delta$ the film is no longer attached to the substrate. By regarding the film and the substrate as a single elastic body, it is seen that $E_c$ has the form $\int_{\omega\setminus\Delta}g([\boldsymbol{u}])\,\mathrm{d}x$ of a Barenblatt's cohesive-zone surface energy [9], where $[\boldsymbol{u}]$ represents the jump of the displacement across the debonding zone, in this case with the integrand $g(s)=\frac{\mu_b}{2}s^2$. These cohesive energies are considered, in particular, in the existing analytical studies of delamination problems, e.g. [11, 37, 27, 28]. Apart from fracture mechanics, this type of integrals also appear in the study of Winkler foundations [41], with applications as varied as the understanding of the seismic response of piers, chromosome function, or the mechanical response of carbon nanotubes embedded in elastic media (see [34] and the references therein). In this setting, $E_c$ is interpreted as the effective energy of an elastic foundation, understood as a continuous bed of mutually independent, linear, elastic springs, hence the appearance of a reaction force of the form $\mu_b(\bar{\boldsymbol{u}}(x)-\boldsymbol{w}(x))$ (corresponding to the quadratic energy $\frac{\mu_b}{2}|\bar{\boldsymbol{u}}-\boldsymbol{w}|^2$) in response to the relative displacement of the body supported on the foundation.

The question addressed by this paper is the rigorous derivation of $E(\bar{\boldsymbol{u}},\Gamma,\Delta)$ from three-dimensional linearized elasticity in the limit as $\varepsilon\to 0$. This involves

(1) the derivation of a reduced Griffith model for the initiation and propagation of cracks in a thin film, and,
(2) the justification of the cohesive energy for the debonding at the interface.

Previous studies of the first problem include [14, 12, 6, 33], which consider scalar-valued problems or generalizations which are incompatible with 3D linear elasticity (coercivity assumptions of the form $W(F) \geq C(|F|^p - 1)$ are imposed on the stored-energy densities $W(\nabla \boldsymbol{u})$), and [26], where the thin film is linearly elastic but the path and the geometry of the crack are specified a priori (it has to be of the form $\Gamma \times (0, h_f)$ and it can only be a single crack). In Theorem 5.1 below, we present the first complete result for the reduction of dimension of a brittle linearly elastic thin film, without any prior assumption on the geometry or the topology of the cracks. As is well-known [13], this falls in the framework of free discontinuity problems, where a satisfactory mathematical treatment can be done in the space of special functions of bounded deformation. We adopt usual scalings for the elastic and fracture parameters, and show the convergence to a reduced model where admissible cracks are vertical, and admissible displacements have a Kirchhoff-Love type structure (the out-of-plane displacement is planar, while the in-plane displacement is affine with respect to the out-of-plane variable). The main difficulty is to establish a compactness result on minimizing sequences (Propositions 5.1 and 5.2) showing the structure of limit displacements and cracks with finite energy. It uses tools of geometric measure theory and fine properties of bounded deformation functions.

The justification of the cohesive energy $E_c$ is also very delicate. In [29, 21, 24, 19] a free discontinuity model with a cohesive fracture energy is obtained as the $\Gamma$-limit of an Ambrosio-Tortorelli functional in which the constraint $z \geq \sqrt{\varepsilon}$ is imposed on the internal damage variable, $\varepsilon$ being the width of the damage zones. This is in the spirit of considering the possibility that what macroscopically would be regarded as fracture is actually a strain mismatch that is continuously accommodated through a very thin layer of a very compliant material. However, it is unclear whether their approach is suitable for the study of thin films, in particular if it could explain that only transverse fracture and planar debonding at the interface can be observed in the limit. Cohesive-type energies have also been obtained by homogenization in [3, 4] as the limit of a Neumann sieve, debonding being regarded as the effect of the interaction of two films through a suitably periodically distributed contact zone. A different derivation of a cohesive fracture energy (albeit with a positive activation threshold) can also be found in [15], in this case as a result of the homogenization of brittle composites with soft inclusions.

Here we consider the problem of deriving $E_c$ based on conceiving the interface between a bimaterial system as a very thin layer of a third phase, occupying the region $\Omega_b = \omega \times [-h_b, 0]$, where $h_b$ represents its thickness. The expectation is to recover the cohesive delamination energy from the elastic energy of the bonding layer when it is made of a material that is increasingly more compliant as $\varepsilon \to 0$. A scaling law is then proposed for the Lamé moduli of the adhesive, of the form $(\lambda^\varepsilon, \mu^\varepsilon) = \varepsilon^q(\lambda_b, \mu_b)$, for some fixed $\lambda_b, \mu_b$ and some exponent $q$ to be determined. Calling $\varepsilon_b := \frac{h_b}{L}$ to the aspect ratio in $\Omega_b$ and using the rescaled displacements $u_\alpha(x', x_3) := v_\alpha(Lx', h_b x_3)$, $u_3(x', x_3) = h_b v_3(Lx', x_3)$, where $x' \in \omega/L$, $x_3 \in [-1, 0]$ are non-dimensional rescaled spatial variables, we are able to write the energy of the bonding layer in the form

$$J_\varepsilon(\boldsymbol{v}_\varepsilon, \Omega_b) = h_b \varepsilon^q \varepsilon_b^{-2} \tilde{J}_\varepsilon(\boldsymbol{v}_\varepsilon),$$

with

$$\tilde{J}_\varepsilon(\boldsymbol{v}) := \frac{1}{2}\int_{\frac{\omega}{L}\times(-1,0)} \Big\{\varepsilon_b^2\Big[e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u}) + 2\mu_b e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\Big]$$
$$+ \Big[2\lambda_b e_{\alpha\alpha}(\boldsymbol{u})e_{33}(\boldsymbol{u}) + 4\mu_b e_{\alpha 3}(\boldsymbol{u})e_{\alpha 3}(\boldsymbol{u})\Big] + \frac{1}{\varepsilon_b^2}(\lambda_b + 2\mu_b)e_{33}(\boldsymbol{u})e_{33}(\boldsymbol{u})\Big\}\,\mathrm{d}x.$$

If $\tilde{J}_\varepsilon$ remains bounded as $\varepsilon \to 0$, due to the $\varepsilon_b^{-2}$ in front of the third term, $u_3$ is expected to be planar in the limit; if the displacement of the substrate is planar, this means $u_3 \equiv 0$ outside the delamination zone. On the other hand, due to the $\varepsilon_b^2$ coefficient for the first term, we expect the in-plane gradient to be irrelevant. Thus, the bonding layer is expected to behave according to $\frac{\mu_b}{2}\int_{\omega\setminus\Delta}\partial_\alpha u_3 \partial_\alpha u_3$, which is minimized if the strain mismatch between the film and the substrate is accomodated by an affine transition in the $x_3$ variable, giving rise to the cohesive energy $E_c$. The assumption that $\tilde{J}_\varepsilon$ is bounded in the asymptotic analysis corresponds to the energy in the bonding layer being of the same order of magnitude as the elastic energy of the film, which scales as $h_f$. This yields the scaling

$$h_b\varepsilon^q\varepsilon_b^{-2} \sim h_f \quad\Leftrightarrow\quad \frac{h_b}{h_f}\varepsilon^q \sim \left(\varepsilon\frac{h_b}{h_f}\right)^2 \quad\Leftrightarrow\quad \frac{h_b}{h_f} \sim \varepsilon^{q-2}.$$

Without loss of generality, in this paper we consider the case when the thicknesses of the film and of the bonding layer have the same order of magnitude and $q = 2$, that is, $(\lambda^\varepsilon, \mu^\varepsilon) = \varepsilon^2(\lambda_b, \mu_b)$. For the effect of other scaling assumptions, we refer to [34].

The above heuristics were made rigorous in [33] in the simplified case of scalar displacements. In the anti-plane case, where the problem becomes scalar, a simple adaptation of that result shows the convergence to a model coupling transverse cracks, cohesive transitions as long as the in-plane displacement is below a precise threshold, and delamination when the threshold is overpassed (Theorem 6.1). In the full vectorial linearly elastic case, the reduced model was rigorously derived in [34] for the problem of Winkler foundations, that is, when the displacements are Sobolev maps so that neither the film nor the bonding layer are allowed to undergo fracture. In Theorem 4.1 below we give a simpler proof of the same result.

We have been unable to prove the convergence to $E(\boldsymbol{u}, \Gamma, \Delta)$ in the case of interest of a linearly elastic bonding layer which may undergo fracture. We limit ourselves to present some partial results which, in our opinion, ought to be considered in any attempt to establish the desired $\Gamma$-convergence. We prove an energy upper bound by constructing, for every admissible limit displacement, an optimal recovery sequence (Proposition 6.1). What remains open is to establish the optimality of the affine transitions in the $x_3$ variable in order to accommodate the mismatch between the film and the substrate. Indeed, the ability to break gives the bonding layer the opportunity to reduce its elastic energy by performing a periodic sequence of small rotations (Example 6.3). This implies that the delamination zone cannot be identified just by taking the orthogonal projections of the jump set of the displacement, as is done in the Sobolev and scalar cases. As a possible remedy, we consider instead, “almost vertical” projections. We are able to prove a surface energy lower bound (although with a bad multiplicative constant) and to show the validity of the desired bulk energy lower bound under the assumption that the minimizing sequence satisfies better a priori estimates than just the energy bound (Lemma 6.7).

We end this Introduction by mentioning [36], where a Griffith energy for the debonding at the interface is obtained as the limit elastic energy of a thin bonding layer in a problem involving a damage internal variable. The techniques of that paper may prove relevant in the derivation of the reduced model $E(\bar{\boldsymbol{u}}, \Gamma, \Delta)$ discussed in this paper.

The paper is organized as follows: Section 2 is devoted to introduce various notations used throughout this work. In Section 3, we precisely decribe the model and perform a scaling to make the problem more tractable from a mathematical point of view. Section 4 investigates the asymptotic analysis in the absence of cracks, and evidences the appearance of a debonding type limiting energy (Theorem 4.1). In Section 5, we carry out the analysis a linearly elastic thin film, and show the emergence of transverse cracks (Theorem 5.1). Finally, Section 6 discusses the interplay between transverse cracks, debonding, and delamination.

## 2. Notation and preliminaries

If $a$ and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open ball of center $x$ and radius $\varrho$ is denoted by $B_\varrho(x)$. If $x = 0$, we simply write $B_\varrho$ instead of $B_\varrho(0)$.

We denote by $\mathbb{M}^{m\times n}$ the set of real $m \times n$ matrices, and by $\mathbb{M}^{n\times n}_{\rm sym}$ the set of all real symmetric $n\times n$ matrices. Given two matrices $A$ and $B \in \mathbb{M}^{m\times n}$, we let $A : B := \mathrm{tr}(A^T B)$ for the Frobenius scalar product, and $|A| := \sqrt{\mathrm{tr}(A^T A)}$ for the associated norm ($A^T$ is the transpose of $A$, and $\mathrm{tr}(A)$ is its trace). We recall that for any two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{m\times n}$ stands for the tensor product, *i.e.*, $(a\otimes b)_{ij} = a_i b_j$ for all $1 \le i \le m$ and $1 \le j \le n$. If $m = n$, then $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a) \in \mathbb{M}^{n\times n}_{\rm sym}$ denotes the symmetric tensor product.

Given an open subset $U$ of $\mathbb{R}^n$ and a finite dimensional Euclidean space $X$. We use standard notations for Lebesgues spaces $L^p(U; X)$ and Sobolev spaces $H^1(U; X)$ or $W^{1,p}(U; X)$. We denote by $\mathcal{M}(U; X)$ the space of all $X$-valued Radon measures with finite total variation. If the target space $X = \mathbb{R}$, we omit to write it for simplicity. According to the Riesz representation Theorem, it is identified to the topological dual of $\mathcal{C}_0(U; X)$ (the space of all continuous functions $\varphi : U \to X$ such that $\{\varphi \ge \varepsilon\}$ is compact for every $\varepsilon > 0$), and a weak* topology is defined according to this duality. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $k$-dimensional Hausdorff measure by $\mathcal{H}^k$. Sometimes, the notation $\#$ will be used instead of $\mathcal{H}^0$ for the counting measure, and $|\cdot|$ instead of the Lebesgue measure $\mathcal{L}^n$. In dimension $n$, equality of inclusion of sets up to a $\mathcal{H}^{n-1}$-negligible set will be respectively denoted by $\cong$ and $\tilde{\subset}$.

Given a function $u \in L^1(U; \mathbb{R}^m)$ with $m \ge 1$. We say that $u$ has an approximate limit at $x \in U$ if there exists $\tilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{\varrho \to 0} \frac{1}{\varrho^n} \int_{B_\varrho(x)} |u(y) - \tilde{u}(x)| \, \mathrm{d}y = 0.$$

The set $S_u$ where this property fails is called the approximate discontinuity set.

We say that $u$ has one-sided Lebesgue limits $u^\pm(x) \in \mathbb{R}^m$ at $x \in U$ with respect to a direction $\nu_u(x) \in \mathbb{S}^{n-1} := \{\zeta \in \mathbb{R}^n : |\zeta| = 1\}$ if

$$\lim_{\varrho \to 0} \frac{1}{\varrho^n} \int_{B_\varrho^\pm(x,\nu_u(x))} |u(y) - u^\pm(x)| \, dy = 0,$$

where $B_\varrho^\pm(x, \nu_u(x)) := \{y \in B_\varrho(x) : \pm\nu_u(x) \cdot (y - x) \geq 0\}$. We will denote by $[u](x) := u^+(x) - u^-(x)$ the jump of $u$ at $x$. The jump set $J_u$ of $u$ is defined as the set of points $x \in U$ such that the one-sided Lebesgue limits with respect to a direction $\nu_u(x)$ exist, and in addition $u^+(x) \neq u^-(x)$. Clearly we have $J_u \subset S_u$.

2.1. **Functions of bounded variation.** The space $BV(U; \mathbb{R}^m)$ of functions of bounded variation in $U$ with values in $\mathbb{R}^m$ is made of all functions $u \in L^1(U; \mathbb{R}^m)$ such that the distributional derivative satisfies $Du \in \mathcal{M}(U; \mathbb{M}^{m \times n})$. The measure $Du$ can be decomposed as

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u + D^c u,$$

where $\nabla u$ is the Radon-Nikodým derivative of $Du$ with respect to the Lebesgue measure $\mathcal{L}^n$, which coincides with the approximate gradient of $u$. For any $1 \leq i \leq m$ and $1 \leq j \leq n$, we denote by $\partial_j u_i := (\nabla u)_{ij}$ the entries of $\nabla u$. The measure $D^c u$ is the Cantor part of $Du$ which has the property of vanishing on any $\sigma$-finite set with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. The jump set $J_u$ is a countably $\mathcal{H}^{n-1}$-rectifiable Borel set, $\nu_u$ is an approximate unit normal to $J_u$, and $u^\pm(x)$ are the one-sided Lebesgue limits of $u$ at $x \in U$ in the direction $\nu_u(x)$. In addition, we have $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$.

We say that $u$ is a special function of bounded variation, and we write $u \in SBV(U; \mathbb{R}^m)$, if $D^c u = 0$. If further $\nabla u \in L^p(U; \mathbb{R}^{m \times n})$ for some $p > 1$, and $\mathcal{H}^{n-1}(J_u) < \infty$, we write $u \in SBV^p(U; \mathbb{R}^m)$. We refer to [2] for general properties of $BV$-functions.

2.2. **Functions of bounded deformation.** The space $BD(U)$ of functions of bounded deformation is made of all vector fields $\boldsymbol{u} \in L^1(U; \mathbb{R}^n)$ whose distributional symmetric gradient satisfies

$$E\boldsymbol{u} = \frac{D\boldsymbol{u} + D\boldsymbol{u}^T}{2} \in \mathcal{M}(U; \mathbb{M}^{n \times n}_{\rm sym}).$$

This measure can be decomposed as

$$E\boldsymbol{u} = e(\boldsymbol{u})\mathcal{L}^n + (\boldsymbol{u}^+ - \boldsymbol{u}^-) \odot \nu_{\boldsymbol{u}} \mathcal{H}^{n-1} \llcorner J_{\boldsymbol{u}} + E^c \boldsymbol{u}. \tag{2.1}$$

In the previous expression, $e(\boldsymbol{u})$ denotes the absolutely continuous part of $E\boldsymbol{u}$ with respect to $\mathcal{L}^n$. For any $1 \leq i, j \leq n$, we denote by $e_{ij}(\boldsymbol{u}) = (e(\boldsymbol{u}))_{ij}$ the entries of $e(\boldsymbol{u})$. The measure $E^c\boldsymbol{u}$ is the Cantor part of $E\boldsymbol{u}$ which has the property to vanish on any $\sigma$-finite set with respect to $\mathcal{H}^{n-1}$. The jump set $J_{\boldsymbol{u}}$ of $\boldsymbol{u}$ is a countably $\mathcal{H}^{n-1}$-rectifiable Borel set, $\nu_{\boldsymbol{u}}$ is an approximate unit normal to $J_{\boldsymbol{u}}$, and $\boldsymbol{u}^\pm(x)$ are the one-sided Lebesgue limits of $\boldsymbol{u}$ at $x \in U$ in the direction $\nu_{\boldsymbol{u}}(x)$. If $E^c\boldsymbol{u} = 0$, we say that $\boldsymbol{u}$ is a special function of bounded deformation and we write $\boldsymbol{u} \in SBD(U)$. We refer to [40, 35, 38, 39, 5, 8, 1, 10, 17, 20] for general properties of $BD$-functions.

2.3. **General conventions.** In the sequel we will always work in dimensions 1, 2 or 3. Latin indices $i$, $j$, $k$, $l$, ... (except $f$ and $b$) take their values in the set $\{1, 2, 3\}$ unless otherwise indicated. Greek indices $\alpha$, $\beta$, $\gamma$, ... (except $\varepsilon$) take their values in the set $\{1, 2\}$. The repeated index summation convention is systematically used.

## 3. Description of the problem

3.1. **In the original configuration.** Let $\omega$ be a bounded and connected open subset of $\mathbb{R}^2$ with Lipschitz boundary which denotes the basis of a thin domain occupying the open set $\Omega^\varepsilon := \omega \times (-2\varepsilon, \varepsilon)$ in its reference configuration. We assume that this domain is made of the union of a film $\Omega^\varepsilon_f := \omega \times (0, \varepsilon)$, a bonding layer $\Omega^\varepsilon_b := \omega \times [-\varepsilon, 0]$, and a substrate $\Omega^\varepsilon_s := \omega \times (-2\varepsilon, -\varepsilon)$. Let us underline that the set $\Omega^\varepsilon_b$ is not open. Any kinematically admissible displacement $\boldsymbol{v} : \Omega^\varepsilon \to \mathbb{R}^3$ is required to satisfy the boundary condition $\boldsymbol{v} = 0$ in $\Omega^\varepsilon_s$. In the sequel we shall denote by $x' := (x_1, x_2)$ the in-plane variable.

The background behavior of this medium in that of an isotropic linearly elastic material whose Lamé coefficients are given by

$$(\lambda^\varepsilon, \mu^\varepsilon) = \begin{cases} (\lambda_f, \mu_f) & \text{in } \Omega^\varepsilon_f, \\ \varepsilon^2(\lambda_b, \mu_b) & \text{in } \Omega^\varepsilon_b. \end{cases}$$

The elastic energy associated to a displacement $\boldsymbol{v} \in H^1(\Omega^\varepsilon; \mathbb{R}^3)$ satisfying $\boldsymbol{v} = 0$ $\mathcal{L}^3$-a.e. in $\Omega^\varepsilon_s$ is given by

$$\frac{1}{2} \int_{\Omega^\varepsilon} \Big[\lambda^\varepsilon e_{ii}(\boldsymbol{v}) e_{jj}(\boldsymbol{v}) + 2\mu^\varepsilon e_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\Big] \, \mathrm{d}x. \tag{3.1}$$

If the body undergoes cracks, according to the variational approach to fracture (see [25, 13]), the presence of cracks is penalized by means of a surface energy of Griffith type where the toughness is given by

$$\kappa^\varepsilon = \begin{cases} \kappa_f & \text{in } \Omega^\varepsilon_f, \\ \varepsilon \kappa_b & \text{in } \Omega^\varepsilon_b. \end{cases}$$

In this case, Sobolev spaces cannot describe admissible displacements since they may jump across the cracks. The natural framework is to consider displacements which are special functions of bounded deformation. Identifying the cracks with the jump set of the displacement, denoted by $J_{\boldsymbol{v}}$, the surface energy is given by

$$\int_{J_{\boldsymbol{v}} \cap \Omega^\varepsilon} \kappa^\varepsilon \, \mathrm{d}\mathcal{H}^2. \tag{3.2}$$

The total energy is then given by the sum of the bulk energy, given by (3.1), where $e(\boldsymbol{v})$ is intended as the absolutely continuous part of the strain with respect to the Lebesgue measure, and the surface energy, given by (3.2). It is well defined for any displacements $\boldsymbol{v} \in SBD(\Omega^\varepsilon)$ satisfying the boundary condition $\boldsymbol{v} = 0$ $\mathcal{L}^3$-a.e. in the substrate $\Omega^\varepsilon_s$.

3.2. **In the rescaled configuration.** As usual in dimension reduction, we rescale the problem on a fixed domain of unit thickness (see [18]). We denote by $\Omega := \Omega^1$, $\Omega_f := \Omega^1_f$, $\Omega_b := \Omega^1_b$, and $\Omega_s := \Omega^1_s$. For every original displacement $\boldsymbol{v} \in H^1(\Omega^\varepsilon; \mathbb{R}^3)$ (resp. $\boldsymbol{v} \in SBD(\Omega^\varepsilon)$) such that $\boldsymbol{v} = 0$ $\mathcal{L}^3$-a.e. in $\Omega^\varepsilon_s$, we define the rescaled displacement $\boldsymbol{u}$ in the rescaled configuration by

$$\begin{cases} u_\alpha(x', x_3) = v_\alpha(x', \varepsilon x_3), \\ u_3(x', x_3) = \varepsilon v_3(x', \varepsilon x_3), \end{cases} \qquad \text{for all } x = (x', x_3) \in \Omega.$$

Replacing $\boldsymbol{v}$ by this expression in the energy (3.1), and dividing the resulting expression by $\varepsilon$ yields the following rescaled elastic energy (see [18])

$$J_\varepsilon(\boldsymbol{u}) = J_\varepsilon(\boldsymbol{u}, \Omega_f) + J_\varepsilon(\boldsymbol{u}, \Omega_b),$$

where

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}, \Omega_f) := {} & \frac{1}{2}\int_{\Omega_f} \Big[\lambda_f e_{\alpha\alpha}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{u}) + 2\mu_f e_{\alpha\beta}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{u})\Big]\, \mathrm{d}x \\ & + \frac{1}{2\varepsilon^2}\int_{\Omega_f} \Big[2\lambda_f e_{\alpha\alpha}(\boldsymbol{u}) e_{33}(\boldsymbol{u}) + 4\mu_f e_{\alpha 3}(\boldsymbol{u}) e_{\alpha 3}(\boldsymbol{u})\Big]\, \mathrm{d}x \\ & + \frac{1}{2\varepsilon^4}\int_{\Omega_f} (\lambda_f + 2\mu_f) e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{u})\, \mathrm{d}x, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}, \Omega_b) := {} & \frac{\varepsilon^2}{2}\int_{\Omega_b} \Big[\lambda_b e_{\alpha\alpha}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{u}) + 2\mu_b e_{\alpha\beta}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{u})\Big]\, \mathrm{d}x \\ & + \frac{1}{2}\int_{\Omega_b} \Big[2\lambda_b e_{\alpha\alpha}(\boldsymbol{u}) e_{33}(\boldsymbol{u}) + 4\mu_b e_{\alpha 3}(\boldsymbol{u}) e_{\alpha 3}(\boldsymbol{u})\Big]\, \mathrm{d}x \\ & + \frac{1}{2\varepsilon^2}\int_{\Omega_b} (\lambda_b + 2\mu_b) e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{u})\, \mathrm{d}x. \end{aligned} \tag{3.4}$$

In the case of cracks, the total energy is obtained by adding the surface energy. In the rescaled configuration, it is given by (see [14, 12, 6, 7])

$$E_\varepsilon(\boldsymbol{u}) = E_\varepsilon(\boldsymbol{u}, \Omega_f) + E_\varepsilon(\boldsymbol{u}, \Omega_b),$$

where

$$E_\varepsilon(\boldsymbol{u}, \Omega_f) = J_\varepsilon(\boldsymbol{u}, \Omega_f) + \kappa_f \int_{J_{\boldsymbol{u}} \cap \Omega_f} \left| \left( (\nu_{\boldsymbol{u}})', \frac{1}{\varepsilon}(\nu_{\boldsymbol{u}})_3 \right) \right| \, \mathrm{d}\mathcal{H}^2,$$

and

$$E_\varepsilon(\boldsymbol{u}, \Omega_b) = J_\varepsilon(\boldsymbol{u}, \Omega_b) + \kappa_b \int_{J_{\boldsymbol{u}} \cap \Omega_b} \big| \big( \varepsilon (\nu_{\boldsymbol{u}})', (\nu_{\boldsymbol{u}})_3 \big) \big| \, \mathrm{d}\mathcal{H}^2.$$

## 4. Debonding of thin films

In this section, we assume that the body is purely elastic, *i.e.*, no cracks are allowed. Through an asymptotic analysis as the thickness $\varepsilon$ tends to zero, we rigorously recover a reduced two-dimensional model of a thin film system as an elastic membrane on an in-plane elastic foundation. A similar model has been derived in [34, Theorem 2.1] by means of a different method. The original three-dimensional energy $J_\varepsilon : L^2(\Omega; \mathbb{R}^3) \to [0, +\infty]$ is defined by

$$J_\varepsilon(\boldsymbol{u}) := \begin{cases} J_\varepsilon(\boldsymbol{u}, \Omega_f) + J_\varepsilon(\boldsymbol{u}, \Omega_b) & \text{if } \boldsymbol{u} \in H^1(\Omega; \mathbb{R}^3) \text{ and } \boldsymbol{u} = 0\ \mathcal{L}^3\text{-a.e. in } \Omega_s, \\ +\infty & \text{otherwise,} \end{cases}$$

while the reduced two dimensional energy $J_0 : L^2(\Omega;\mathbb{R}^3) \to [0,+\infty]$ is given by

$$J_0(\boldsymbol{u}) := \begin{cases} \displaystyle\int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\right] \mathrm{d}x' \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{\mu_b}{2}\displaystyle\int_\omega |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x' & \text{if } \begin{cases} \boldsymbol{u} = (\bar{\boldsymbol{u}},0), \\ \bar{\boldsymbol{u}} \in H^1(\omega;\mathbb{R}^2), \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

Our first main result in the following Γ-convergence type result.

**Theorem 4.1.** *Let $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$, then*

- *for any sequence $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ with $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, then*
$$J_0(\boldsymbol{u}) \le \liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon);$$
- *there exists a recovery sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{u}^*_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, and*
$$J_0(\boldsymbol{u}) \ge \limsup_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}^*_\varepsilon).$$

*Proof.* Although some parts of the proof are already well known (see [18, Theorem 1.11.2]), it will be convenient for us to reproduce the entire argument.

**Step 1. Compactness.** Let $(\boldsymbol{u}_\varepsilon) \subset L^2(\Omega;\mathbb{R}^3)$ be such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$. If $\liminf_\varepsilon J_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$, there is nothing to prove. We therefore assume that $\liminf_\varepsilon J_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty$. Up to a subsequence, there is no loss of generality to suppose that

$$J_\varepsilon(\boldsymbol{u}_\varepsilon) = J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_f) + J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) \le C,$$

for some constant $C > 0$ independent of $\varepsilon$. The expression (3.3) of the energy in the film $\Omega_f$ combined with Korn's inequality implies that $(\boldsymbol{u}_\varepsilon)$ is actually bounded in $H^1(\Omega_f;\mathbb{R}^3)$, and that $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ weakly in $H^1(\Omega_f;\mathbb{R}^3)$ with $\boldsymbol{u} \in H^1(\Omega_f;\mathbb{R}^3)$. Contrary to the case of a standard linearly elastic plate model (see [18]), we will show that, thanks to the Dirichlet condition in the substrate, the limit displacement $\boldsymbol{u}$ is planar instead of just Kirchhoff-Love type. Indeed, using also the expression of the energy (3.3)–(3.4), the fact that $\boldsymbol{u}_\varepsilon = 0$ $\mathcal{L}^3$-a.e. in $\Omega_s$, and Poincaré's inequality, we get that

$$\int_{\Omega_f} |(u_\varepsilon)_3|^2\,\mathrm{d}x \le \int_{\Omega_f\cup\Omega_b} |e_{33}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x \le C\varepsilon^2 \to 0,$$

so that $u_3 = 0$. Thanks again to the bound of the energy in the film (3.3), we have

$$\|e_{\alpha 3}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} \le C\varepsilon \to 0,$$

which shows that $e_{\alpha 3}(\boldsymbol{u}) = 0$. It thus follows that $\partial_3 u_\alpha = -\partial_\alpha u_3 = 0$ which implies that $u_\alpha(x',x_3) = \bar{u}_\alpha(x')$ for $\mathcal{L}^3$-a.e. $x \in \Omega_f$, for some $\bar{\boldsymbol{u}} \in H^1(\omega;\mathbb{R}^2)$. We have thus identified the right limit space.

**Step 2. Lower bound.** We next derive the lower bound. Up to a further subsequence, we may assume that

$$\begin{cases} \varepsilon^{-2} e_{33}(\boldsymbol{u}_\varepsilon) \rightharpoonup \zeta_3 \\ \varepsilon^{-1} e_{\alpha 3}(\boldsymbol{u}_\varepsilon) \rightharpoonup \zeta_\alpha \end{cases} \quad \text{weakly in } L^2(\Omega_f),$$

for some functions $\zeta_1$, $\zeta_2$ and $\zeta_3 \in L^2(\Omega_f)$. Then, by lower semicontinuity of the norm with respect to weak convergence, we get that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_f) \geq \frac{1}{2}\int_{\Omega_f}\left[\lambda_f(e_{\alpha\alpha}(\bar{\boldsymbol{u}})+\zeta_3)^2+2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})+4\mu_f\zeta_\alpha\zeta_\alpha+2\mu_f\zeta_3\zeta_3\right]\mathrm{d}x.$$

Minimizing with respect to $(\zeta_1,\zeta_2,\zeta_3)$, we find that the minimal value is attained when $\zeta_\alpha=0$ and $\zeta_3=-\frac{\lambda_f}{\lambda_f+2\mu_f}e_{\alpha\alpha}(\bar{\boldsymbol{u}})$, and thus

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_f) \geq \int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f}e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}})+\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\right]\mathrm{d}x.$$

We now examine the contribution of the bonding layer. To this aim, according to (3.4), isolating the only term of order 1 leads to

$$\begin{aligned}J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) &\geq 2\mu_b\int_{\Omega_b} e_{\alpha 3}(\boldsymbol{u})e_{\alpha 3}(\boldsymbol{u})\,\mathrm{d}x\\ &\geq \frac{\mu_b}{2}\int_\omega\left|\int_{-1}^0[\partial_3(u_\varepsilon)_1+\partial_1(u_\varepsilon)_3]\,\mathrm{d}x_3\right|^2\mathrm{d}x'+\frac{\mu_b}{2}\int_\omega\left|\int_{-1}^0[\partial_3(u_\varepsilon)_2+\partial_2(u_\varepsilon)_3]\,\mathrm{d}x_3\right|^2\mathrm{d}x',\end{aligned}$$

thanks to the Cauchy-Schwarz inequality with respect to the $x_3$ variable. Since $\boldsymbol{u}_\varepsilon=0$ $\mathcal{L}^3$-a.e. in $\Omega_s$, then

$$\int_{-1}^0 \partial_3\boldsymbol{u}_\varepsilon(x',x_3)\,\mathrm{d}x_3=\boldsymbol{u}_\varepsilon(x',0)\qquad\text{for } \mathcal{L}^2\text{-a.e. } x'\in\omega,$$

where $\boldsymbol{u}_\varepsilon(\cdot,0)$ denotes the trace of $\boldsymbol{u}_\varepsilon$ on $\{x_3=0\}$. On the other hand, setting $\bar{u}_3^\varepsilon=\int_{-1}^0(u_\varepsilon)_3(\cdot,x_3)\,\mathrm{d}x_3\in H^1(\omega)$, we have

$$\int_{-1}^0\partial_\alpha(u_\varepsilon)_3(x',x_3)\,\mathrm{d}x_3=\partial_\alpha\bar{u}_3^\varepsilon(x')\qquad\text{for } \mathcal{L}^2\text{-a.e. } x'\in\omega.$$

Gathering everything, we infer that

$$J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b)\geq\frac{\mu_b}{2}\int_\omega|(u_\varepsilon)_1(x',0)+\partial_1\bar{u}_3^\varepsilon(x')|^2\,\mathrm{d}x'+\frac{\mu_b}{2}\int_\omega|(u_\varepsilon)_2(x',0)+\partial_2\bar{u}_3^\varepsilon(x')|^2\,\mathrm{d}x'. \tag{4.1}$$

According to the trace theorem, and since $\bar{u}_\alpha$ is independent of $x_3$, we have $(u_\varepsilon)_\alpha(\cdot,0)\to\bar{u}_\alpha$ strongly in $L^2(\omega)$. On the other hand, the energy in the bonding layer (3.4) together with the Cauchy-Schwarz and Poincaré inequalities yield

$$\int_\omega|\bar{u}_3^\varepsilon|^2\,\mathrm{d}x'\leq\int_{\Omega_b}|e_{33}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x\leq C\varepsilon^2\to 0,$$

while (4.1) shows that the sequence $(\nabla\bar{u}_3^\varepsilon)$ in bounded in $L^2(\omega;\mathbb{R}^2)$. Consequently, $\nabla\bar{u}_3^\varepsilon\rightharpoonup 0$ weakly in $L^2(\omega;\mathbb{R}^2)$, and combining all the convergences established so far, we deduce that

$$\liminf_{\varepsilon\to 0}J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b)\geq\frac{\mu_b}{2}\int_\omega|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x',$$

which completes the proof of the lower bound.

**Step 3. Upper bound**. We assume without loss of generality that $\boldsymbol{u} = (\bar{\boldsymbol{u}}, 0)$ for some $\bar{\boldsymbol{u}} \in H^1(\omega; \mathbb{R}^2)$, otherwise the limit energy is infinite. We now define a recovery sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0}$. For all $\varepsilon > 0$, let

$$\boldsymbol{u}^*_\varepsilon(x', x_3) = \begin{cases} \big(\bar{\boldsymbol{u}}(x'), \varepsilon^2 x_3 h_\varepsilon(x')\big) & \text{if } x \in \Omega_f, \\ (x_3+1)(\bar{\boldsymbol{u}}(x'), 0) & \text{if } x \in \Omega_b, \\ 0 & \text{if } x \in \Omega_s, \end{cases}$$

where $(h_\varepsilon)_{\varepsilon>0}$ is a sequence in $\mathcal{C}^\infty_c(\omega)$ such that

$$h_\varepsilon \to -\frac{\lambda_f}{\lambda_f + 2\mu_f} \boldsymbol{e}_{\alpha\alpha}(\bar{\boldsymbol{u}}) \text{ in } L^2(\omega), \quad \lim_{\varepsilon\to 0} \varepsilon \|\nabla h_\varepsilon\|_{L^2(\omega)} = 0. \tag{4.2}$$

Clearly, $\boldsymbol{u}^*_\varepsilon \in H^1(\Omega; \mathbb{R}^3)$ and $\boldsymbol{u}^*_\varepsilon = 0$ $\mathcal{L}^3$-a.e. in $\Omega_s$. Using (3.3) we have that

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}^*_\varepsilon, \Omega_f) = {} & \frac{1}{2} \int_{\Omega_f} \big[\lambda_f e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\big] \, \mathrm{d}x \\ & + \frac{1}{2\varepsilon^2} \int_{\Omega_f} \big[2\lambda_f e_{\alpha\alpha}(\bar{\boldsymbol{u}}) \varepsilon^2 h_\varepsilon + \mu_f \varepsilon^4 x_3^2 |\nabla h_\varepsilon|^2\big] \, \mathrm{d}x \\ & + \frac{1}{2\varepsilon^4} \int_{\Omega_f} (\lambda_f + 2\mu_f) \varepsilon^4 |h_\varepsilon|^2 \, \mathrm{d}x, \end{aligned}$$

and according to the convergence properties (4.2), we get that

$$\begin{aligned} \lim_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}^*_\varepsilon, \Omega_f) = {} & \frac{1}{2} \int_\omega \big[\lambda_f e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\big] \, \mathrm{d}x' \\ & - \frac{1}{2} \int_\omega \frac{2\lambda_f^2}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) \, \mathrm{d}x' + \frac{1}{2} \int_\omega \frac{\lambda_f^2}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) \, \mathrm{d}x' \\ = {} & \frac{1}{2} \int_\omega \left[\frac{2\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\right] \, \mathrm{d}x'. \end{aligned}$$

On the other hand, (3.4) yields

$$J_\varepsilon(\boldsymbol{u}^*_\varepsilon, \Omega_b) = \frac{\varepsilon^2}{2} \int_{\Omega_b} (x_3+1)^2 \big[\lambda_b e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + 2\mu_b e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}})\big] \, \mathrm{d}x + \frac{\mu_b}{2} \int_\omega \bar{u}_\alpha \bar{u}_\alpha \, \mathrm{d}x',$$

and thus

$$\lim_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}^*_\varepsilon, \Omega_f) = \frac{\mu_b}{2} \int_\omega |\bar{\boldsymbol{u}}|^2 \, \mathrm{d}x',$$

which completes the proof of the upper bound. □

## 5. Transverse cracks in thin films

In this section, we assume that the body can fracture. We first only address the analysis of the film $\Omega_f$ in order to highlight the appearance of transverse cracks in the reduced model. This property is already known in the framework of nonlinear elasticity where energies depend on the deformation gradient [6, 7, 12, 14]. The difficulty here is to consider a linearly elastic material outside the crack so that the energy depends on the elastic strain.

5.1. **Compactness.** From a mathematical point of view, the natural functional setting is to consider displacement fields $\boldsymbol{u} \in SBD(\Omega_f)$. For technical reasons, we also assume that all the deformations take place in a fixed container $K$ which is a compact subset of $\mathbb{R}^3$. Therefore, we assume that any displacement is uniformly bounded by some fixed positive constant $M > 0$.

Throughout this section, we assume that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset SBD(\Omega_f)$ is a sequence of displacements in the film such that $\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f)} \leq M$, and

$$\sup_{\varepsilon>0} E_\varepsilon(\boldsymbol{u}_\varepsilon, \Omega_f) < \infty.$$

We establish that any admissible sequence of displacements with uniformly bounded energy converges to some limit displacement having a Kirchhoff-Love type structure.

**Proposition 5.1.** *Up to a subsequence, there exists $\boldsymbol{u} \in SBD(\Omega_f) \cap L^\infty(\Omega_f;\mathbb{R}^3)$ such that*

*i)* $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ *strongly in* $L^2(\Omega_f;\mathbb{R}^3)$ *and* $\boldsymbol{u}_\varepsilon \xrightharpoonup{*} \boldsymbol{u}$ *weakly* in* $L^\infty(\Omega_f;\mathbb{R}^3)$*;*
*ii)* $e(\boldsymbol{u}_\varepsilon) \rightharpoonup e(\boldsymbol{u})$ *weakly in* $L^2(\Omega_f;\mathbb{M}^{3\times3}_{\rm sym})$*;*
*iii)* $e_{\alpha3}(\boldsymbol{u}) = e_{33}(\boldsymbol{u}) = 0$ $\mathcal{L}^3$*-a.e. in* $\Omega_f$ *and* $(\nu_{\boldsymbol{u}})_3 = 0$ $\mathcal{H}^2$*-a.e. on* $J_{\boldsymbol{u}} \cap \Omega_f$.

*Proof.* From the hypotheses and the definition of $E_\varepsilon(\cdot,\Omega_f)$, we have that

$$\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f)} + \|e(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} + \mathcal{H}^2(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f) \leq C,$$

for some constant $C > 0$ independent of $\varepsilon$. According to the compactness theorem in $SBD$ [10, Theorem 1.1], we deduce the existence of a subsequence (not relabeled) and a function $\boldsymbol{u} \in SBD(\Omega_f)$ such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, $\boldsymbol{u}_\varepsilon \xrightharpoonup{*} \boldsymbol{u}$ weakly* in $L^\infty(\Omega_f;\mathbb{R}^3)$, $e(\boldsymbol{u}_\varepsilon) \rightharpoonup e(\boldsymbol{u})$ weakly in $L^2(\Omega_f;\mathbb{M}^{3\times3}_{\rm sym})$, and

$$\begin{aligned}\mathcal{H}^2(J_{\boldsymbol{u}} \cap \Omega_f) &\leq \liminf_{\varepsilon\to0} \mathcal{H}^2(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f)\\ &\leq \liminf_{\varepsilon\to0} \int_{\Omega_f\cap J_{\boldsymbol{u}_\varepsilon}} \left|\left((\nu_{\boldsymbol{u}_\varepsilon})', \frac{1}{\varepsilon}(\nu_{\boldsymbol{u}_\varepsilon})_3\right)\right| \,\mathrm{d}\mathcal{H}^2. \end{aligned} \tag{5.1}$$

Using the expression of the energy in the film, we deduce that

$$\|e_{\alpha3}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} + \int_{\Omega_f\cap J_{\boldsymbol{u}_\varepsilon}} |(\nu_{\boldsymbol{u}_\varepsilon})_3| \,\mathrm{d}\mathcal{H}^2 \leq C\varepsilon \tag{5.2}$$

and

$$\|e_{33}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} \leq C\varepsilon^2$$

for some $C$ independent of $\varepsilon$. Using the lower semicontinuity of the left hand side of both equations with respect to the convergences established for $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ (see [10, Corollary 1.2]) we conclude that $e_{\alpha3}(\boldsymbol{u}) = e_{33}(\boldsymbol{u}) = 0$ $\mathcal{L}^3$-a.e. in $\Omega_f$, and that $(\nu_{\boldsymbol{u}})_3 = 0$ $\mathcal{H}^2$-a.e. on $J_{\boldsymbol{u}} \cap \Omega_f$. $\square$

In the sequel, $\boldsymbol{u}$ denotes a displacement as in the conclusion of Proposition 5.1. Our next goal is to get a more precise structure of such displacements. Contrary to the case of linear elasticity (see [18]) or linearly elastic-perfectly plastic plates (see [22]), they in general are not of Kirchhoff-Love type (*i.e.* such that $E_{i3}\boldsymbol{u} = 0$) since we do not control the full distributional strain $E\boldsymbol{u}$. In particular, the singular part of the shearing strain $E_{\alpha3}\boldsymbol{u}$ is given by $\frac{[\boldsymbol{u}]_3\nu_\alpha}{2}\mathcal{H}^2 \llcorner J_{\boldsymbol{u}}$ which might not vanish. However, we shall prove below that they have the same structure in the sense that the transverse displacement $u_3$ only depends on the planar variable $x'$, while the in-plane displacement $(u_1, u_2)$ is affine with respect to the transverse variable $x_3$.

**Proposition 5.2.** *Let $\boldsymbol{u} \in SBD(\Omega_f) \cap L^\infty(\Omega_f;\mathbb{R}^3)$ be such that $e_{i3}(\boldsymbol{u}) = 0$ $\mathcal{L}^3$-a.e. in $\Omega_f$, and $(\nu_{\boldsymbol{u}})_3 = 0$ $\mathcal{H}^2$-a.e. on $J_{\boldsymbol{u}} \cap \Omega_f$. Then the following properties hold:*

- *the function $u_3$ is independent of $x_3$ and it (is identified to a function which) belongs to $SBV(\omega) \cap L^\infty(\omega)$. In addition, its approximate gradient $\nabla u_3 = (\partial_1 u_3, \partial_2 u_3) \in SBD(\omega) \cap L^\infty(\omega;\mathbb{R}^2)$;*
- *for $\mathcal{L}^3$-a.e. $(x', x_3) \in \Omega_f$,*

$$u_\alpha(x', x_3) = \bar{u}_\alpha(x') + \left(\frac{1}{2} - x_3\right)\partial_\alpha u_3(x'), \tag{5.3}$$

  *where $\bar{u}_\alpha := \int_0^1 u_\alpha(\cdot, x_3)\,\mathrm{d}x_3$, and $\bar{\boldsymbol{u}} := (\bar{u}_1, \bar{u}_2) \in SBD(\omega) \cap L^\infty(\omega;\mathbb{R}^2)$;*
- *$J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0,1)$;*

*Proof.* **Step 1.** First of all, by virtue of (2.1), the distributional derivative of $u_3$ with respect to $x_3$ satisfies

$$D_3 u_3 = E_{33}\boldsymbol{u} = e_{33}(\boldsymbol{u})\mathcal{L}^3 + [\boldsymbol{u}]_3(\nu_{\boldsymbol{u}})_3\mathcal{H}^2 \llcorner J_{\boldsymbol{u}} = 0.$$

This implies that $u_3$ is independent of $x_3$, and that it can be identified to a function defined on $\omega$.

**Step 2.** We next show that $u_3 \in SBV(\omega)$ and that formula (5.3) holds. This will be obtained thanks to a suitable mollification of $\boldsymbol{u}$. We first extend $\boldsymbol{u}$ to the whole space in the following way: since the trace of an $SBD(\Omega_f)$ function belongs to $L^1(\partial\Omega_f;\mathbb{R}^3)$ (see [8, Theorem 3.2]), according to Gagliardo's Theorem, $\boldsymbol{u}$ may be extended to $\mathbb{R}^3$ by a function, still denoted by $\boldsymbol{u}$, that is compactly supported in $\mathbb{R}^3$ and such that $\boldsymbol{u} \in W^{1,1}(\mathbb{R}^3 \setminus \Omega_f;\mathbb{R}^3)$ with $|E\boldsymbol{u}|(\partial\Omega_f) = 0$.

Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ be an even and non negative function such that $\int_{\mathbb{R}} \chi(t)\,\mathrm{d}t = 1$ and $\mathrm{Supp}\chi \subset (-1,1)$. For all $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we define $\bar{\rho}(x') := \chi(x_1)\chi(x_2)$ and $\rho(x) := \chi(x_1)\chi(x_2)\chi(x_3)$. We then denote by $\bar{\rho}_\delta(x') = \delta^{-2}\bar{\rho}(x'/\delta)$ a sequence of two-dimensional mollifiers, and by $\rho_\delta(x) = \delta^{-3}\rho(x/\delta)$ a sequence of three-dimensional mollifiers. Since $\boldsymbol{u} * \rho_\delta \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R}^3)$ and

$$\partial_3(\boldsymbol{u} * \rho_\delta)_\alpha = 2e_{\alpha 3}(\boldsymbol{u} * \rho_\delta) - \partial_\alpha(\boldsymbol{u} * \rho_\delta)_3,$$

it follows from the fundamental theorem of calculus that for each $(x', x_3) \in \Omega_f$,

$$\begin{aligned} u_\alpha * \rho_\delta(x', x_3) = u_\alpha * \rho_\delta(x', 0) + 2\int_0^{x_3} e_{\alpha 3}(\boldsymbol{u} * \rho_\delta)(x', s)\,\mathrm{d}s \\ - \int_0^{x_3} \partial_\alpha(u_3 * \rho_\delta)(x', s)\,\mathrm{d}s. \end{aligned} \tag{5.4}$$

Let us study each of the above terms separately. The term in the left hand side of (5.4) clearly satisfies $u_\alpha * \rho_\delta \to u_\alpha$ strongly in $L^2(\Omega_f)$, and thus (for a suitable subsequence)

$$u_\alpha * \rho_\delta \to u_\alpha \quad \mathcal{L}^3\text{-a.e. in } \Omega_f. \tag{5.5}$$

Concerning the first term on the right-hand side of (5.4), standard properties of convolution of measures ensure that $E\boldsymbol{u}_\delta \xrightharpoonup{*} E\boldsymbol{u}$ weakly* in $\mathcal{M}(\mathbb{R}^3)$ and $|E\boldsymbol{u}_\delta|(\mathbb{R}^3) \to |E\boldsymbol{u}|(\mathbb{R}^3)$.

Therefore, since $|E\boldsymbol{u}|(\partial\Omega_f)=0$, we deduce that $|E\boldsymbol{u}_\delta|(\Omega_f)\to|E\boldsymbol{u}|(\Omega_f)$ which implies, by the continuity property of the trace (see [8, Proposition 3.4]) that $u_\alpha*\rho_\delta\to u_\alpha$ strongly in $L^1(\partial\Omega_f)$. Thus, denoting by $u_\alpha^+(\cdot,0)$ the upper trace of $u_\alpha$ on $\omega\times\{0\}$, there is a subsequence such that

$$u_\alpha*\rho_\delta(\cdot,0)\to u_\alpha^+(\cdot,0)\quad \mathcal{L}^2\text{-a.e. in } \omega. \tag{5.6}$$

Regarding the second term on the right-hand side of (5.4), we have $e_{\alpha 3}(\boldsymbol{u}*\rho_\delta)=(E_{\alpha 3}\boldsymbol{u})*\rho_\delta$ with $E_{\alpha 3}\boldsymbol{u}=\frac{[\boldsymbol{u}]_3(\nu_{\boldsymbol{u}})_\alpha}{2}\mathcal{H}^2\llcorner J_{\boldsymbol{u}}$, and thus

$$\begin{aligned}\mathcal{E}(x',x_3) &:= \int_0^{x_3} e_{\alpha 3}(\boldsymbol{u}*\rho_\delta)(x',s)\,\mathrm{d}s\\ &= \frac{1}{2}\int_0^{x_3}\int_{J_{\boldsymbol{u}}}\rho_\delta(x'-y',s-y_3)[\boldsymbol{u}]_3(y)(\nu_{\boldsymbol{u}})_\alpha(y)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}s.\end{aligned}$$

Since $\boldsymbol{u}\in L^\infty(\Omega_f;\mathbb{R}^3)$ with $\|u\|_{L^\infty(\Omega_f)}\le M$, then $|[\boldsymbol{u}]|\le 2M$ which leads to

$$\begin{aligned}|\mathcal{E}(x',x_3)| &\le M\int_0^1\int_{J_{\boldsymbol{u}}}\rho_\delta(x'-y',s-y_3)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}s\\ &= M\int_{J_{\boldsymbol{u}}}\int_0^1\rho_\delta(x'-y',s-y_3)\,\mathrm{d}s\,\mathrm{d}\mathcal{H}^2(y),\end{aligned}$$

where we used Fubini's Theorem in the last equality. We next denote by $Q'(x',\delta):=x'+(-\delta,\delta)^2$ the open square of $\mathbb{R}^2$ (parallel to the coordinate axis) centered at $x'$ and of edge length $2\delta$. Observing that $\rho_\delta(x'-y',s-y_3)=0$ if $y'\notin Q'(x',\delta)$ and that $\rho_\delta(x'-y',s-y_3)=\bar\rho_\delta(x'-y')\delta^{-1}\chi((s-y_3)/\delta)$ with $\int_{\mathbb{R}}\chi(t)\,\mathrm{d}t=1$, we get that

$$\begin{aligned}|\mathcal{E}(x',x_3)|\le M\int_{J_{\boldsymbol{u}}\cap[Q'(x',\delta)\times(0,1)]}\bar\rho_\delta(x'-y')\left(\int_{\mathbb{R}}\delta^{-1}\chi((s-y_3)/\delta)\,\mathrm{d}s\right)\mathrm{d}\mathcal{H}^2(y)\\ = M\int_{J_{\boldsymbol{u}}\cap[Q'(x',\delta)\times(0,1)]}\bar\rho_\delta(x'-y')\,\mathrm{d}\mathcal{H}^2(y).\end{aligned}$$

For any Borel set $B\subset\omega$, let us define the measure $\mu(B):=\mathcal{H}^2(J_{\boldsymbol{u}}\cap(B\times(0,1)))$ which is nothing but the push-forward of $\mathcal{H}^2\llcorner J_{\boldsymbol{u}}$ by the orthogonal projection $\pi:\mathbb{R}^3\to\mathbb{R}^2\times\{0\}$. Note that $\mu$ is concentrated on $\pi(J_{\boldsymbol{u}})$ since $\mu(\omega\setminus\pi(J_{\boldsymbol{u}}))=\mathcal{H}^2(J_{\boldsymbol{u}}\cap[(\omega\setminus\pi(J_{\boldsymbol{u}}))\times(0,1)])=0$. On the other hand, the generalized coarea formula (see [2, Theorem 293]) yields

$$\mathcal{L}^2(\pi(J_{\boldsymbol{u}}))\le\int_{\pi(J_{\boldsymbol{u}})}\mathcal{H}^0(J_{\boldsymbol{u}}\cap\pi^{-1}(x'))\,dx'=\int_{J_{\boldsymbol{u}}}|(\nu_{\boldsymbol{u}})_3|\,\mathrm{d}\mathcal{H}^2=0.$$

Therefore, $\mu$ and $\mathcal{L}^2$ are mutually singular which ensures that the Radon-Nikodým derivative $\frac{d\mu}{d\mathcal{L}^2}(x')=0$ at $\mathcal{L}^2$-a.e. $x'\in\omega$. It follows that for $\mathcal{L}^2$-a.e. $x'\in\omega$,

$$\sup_{x_3\in(0,1)}|\mathcal{E}(x',x_3)|\le M\|\chi\|^2_{L^\infty(\mathbb{R})}\frac{\mu(Q'(x',\delta))}{\delta^2}\to 0,$$

and thus, in particular,

$$\int_0^{x_3}e_{\alpha 3}(\boldsymbol{u}*\rho_\delta)(x',s)\,\mathrm{d}s\to 0 \text{ for } \mathcal{L}^3\text{-a.e. } (x',x_3)\in\Omega_f. \tag{5.7}$$

For what concerns the last term on the right-hand side of (5.4), since $u_3$ is independent of $x_3$, we infer that $u_3 * \rho_\delta$ is independent of $x_3$ as well since $u_3 * \rho_\delta(x) = u_3 * \bar\rho_\delta(x')$ for all $x \in \mathbb{R}^3$. Therefore,

$$\int_0^{x_3} \partial_\alpha(u_3 * \rho_\delta)(x', s)\,\mathrm{d}s = x_3 \partial_\alpha(u_3 * \bar\rho_\delta)(x'),$$

and (5.4) – (5.7) thus imply that

$$\partial_\alpha(u_3 * \bar\rho_\delta)(x') \to \frac{u^+_\alpha(x', 0) - u_\alpha(x', x_3)}{x_3} := \psi_\alpha(x') \quad \text{for } \mathcal{L}^3\text{-a.e. } (x', x_3) \in \Omega_f.$$

That $\psi_\alpha$ only depends on $x'$ is due to the fact that the left-hand side only depends on $x'$. Moreover, since $u^+_\alpha(\cdot, 0) \in L^1(\omega)$ and $u_\alpha(\cdot, x_3) \in L^2(\omega)$ for a.e. $x_3 \in (0, 1)$, we deduce that $\psi_\alpha \in L^1(\omega)$. From the last formula we get that

$$u_\alpha(x', x_3) = u^+_\alpha(x', 0) - x_3\psi_\alpha(x'), \tag{5.8}$$

which in particular implies that $D_3u_\alpha = -\psi_\alpha\mathcal{L}^3$, and

$$D_\alpha u_3 = -D_3u_\alpha + 2E_{\alpha 3}\boldsymbol{u} = \psi_\alpha\mathcal{L}^3 + [\boldsymbol{u}]_3(\nu_{\boldsymbol{u}})_\alpha\mathcal{H}^2 \llcorner J_{\boldsymbol{u}}.$$

As a consequence, the distributional derivative in $\Omega_f$ of $u_3$ is a bounded Radon measure in $\Omega_f$, and therefore $u_3 \in BV(\Omega_f)$. Since the singular part of the above measure is concentrated on $J_{\boldsymbol{u}}$ which is $\sigma$-finite with respect to $\mathcal{H}^2$, we deduce thanks to [2, Proposition 3.92] that $u_3 \in SBV(\Omega_f)$. Finally, since $u_3$ is independent of $x_3$, we actually infer that $u_3 \in SBV(\omega)$. In addition, by uniqueness of the Lebesgue decomposition, it follows that

$$\psi_\alpha = \partial_\alpha u_3, \quad [\boldsymbol{u}]_3(\nu_{\boldsymbol{u}})_\alpha\mathcal{H}^2 \llcorner J_{\boldsymbol{u}} = [u_3](\nu_{u_3})_\alpha\mathcal{H}^2 \llcorner [J_{u_3} \times (0, 1)]$$

so that

$$J_{u_3} \times (0, 1) \,\tilde{\subset}\, J_{\boldsymbol{u}}. \tag{5.9}$$

Integrating relation (5.8) with respect to $x_3$ yields

$$\bar u_\alpha(x') := \int_0^1 u_\alpha(x', x_3)\,\mathrm{d}x_3 = u^+_\alpha(x', 0) - \frac{1}{2}\partial_\alpha u_3(x') \quad \text{for } \mathcal{L}^2\text{-a.e. } x' \in \omega,$$

from where (5.3) follows.

**Step 3.** Let us prove that the approximate gradient of $u_3$, denoted by $\nabla u_3 := (\partial_1 u_3, \partial_2 u_3)$, and the averaged planar displacement $\bar{\boldsymbol{u}} := (\bar u_1, \bar u_2)$ belong to $BD(\omega)$. For any $\varphi \in \mathcal{C}^\infty_c(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, according to the integration by parts formula in $BD$ (see [8, Theorem 3.2]), we infer that

$$-\int_\omega \partial_\beta\varphi_{\alpha\beta}\bar u_\alpha\,\mathrm{d}x' = -\int_{\Omega_f} \partial_\beta\varphi_{\alpha\beta}u_\alpha\,\mathrm{d}x = \int_{\Omega_f} \varphi_{\alpha\beta}\,\mathrm{d}E_{\alpha\beta}\boldsymbol{u} - \int_{\partial\Omega_f} \varphi_{\alpha\beta}u_\alpha\nu_\beta\,\mathrm{d}\mathcal{H}^2.$$

Since $\varphi = 0$ in a neighborhood of $\partial\omega \times (0, 1)$ and $\nu = \pm e_3$ on $\omega \times \{0, 1\}$, we get that the boundary term in the previous expression is zero. Therefore

$$-\int_\omega \partial_\beta\varphi_{\alpha\beta}\bar u_\alpha\,\mathrm{d}x' = \int_{\Omega_f} \varphi_{\alpha\beta}e_{\alpha\beta}(\boldsymbol{u})\,\mathrm{d}x + \int_{J_{\boldsymbol{u}}} \varphi_{\alpha\beta}([\boldsymbol{u}] \odot \nu_{\boldsymbol{u}})_{\alpha\beta}\,\mathrm{d}\mathcal{H}^2 \tag{5.10}$$

which shows that $\bar{\boldsymbol{u}} \in BD(\omega)$. According to slicing properties of $BD$ functions (see [1, Proposition 3.4]), for $\mathcal{L}^1$-a.e. $x_3 \in (0, 1)$, the function $(u_1(\cdot, x_3), u_2(\cdot, x_3)) \in BD(\omega)$ so that relation (5.3) yields in turn that $\nabla u_3 \in BD(\omega)$.

**Step 4.** We next establish that $J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0,1)$. To this aim, let us define the functions $\boldsymbol{v} := (\bar{u}_1, \bar{u}_2, u_3)$ and $\boldsymbol{g} := (\partial_1 u_3, \partial_2 u_3, 0)$. Since $u_3 \in SBV(\omega)$, $\bar{\boldsymbol{u}} \in BD(\omega)$ and $\nabla u_3 \in BD(\omega)$, then clearly both $\boldsymbol{v}$, $\boldsymbol{g} \in BD(\Omega_f)$, and

$$J_{\boldsymbol{g}} = J_{\nabla u_3} \times (0,1). \tag{5.11}$$

Moreover [2, Proposition 3.92 (b)] and [1, Proposition 3.5] imply that

$$J_{\bar{\boldsymbol{u}}} \cong \left\{ x' \in \omega : \limsup_{\varrho \to 0} \frac{|E\bar{\boldsymbol{u}}|(B'_\varrho(x'))}{\varrho} > 0 \right\}, \; J_{u_3} \cong \left\{ x' \in \omega : \limsup_{\varrho \to 0} \frac{|Du_3|(B'_\varrho(x'))}{\varrho} > 0 \right\},$$

and

$$J_{\boldsymbol{v}} \cong \Theta_{\boldsymbol{v}} := \left\{ x \in \Omega_f : \limsup_{\varrho \to 0} \frac{|E\boldsymbol{v}|(B_\varrho(x))}{\varrho^2} > 0 \right\},$$

where $B'_\varrho(x')$ stands for the two-dimensional open ball of center $x'$ and radius $\varrho$, while $B_\varrho(x)$ stands for the three-dimensional open ball of center $x$ and radius $\varrho$. Since $\boldsymbol{v}$ is independent of $x_3$, then

$$J_{\boldsymbol{v}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3}) \times (0,1). \tag{5.12}$$

According to (5.11) and (5.12), it is thus enough to show that $J_{\boldsymbol{u}} \cong J_{\boldsymbol{v}} \cup J_{\boldsymbol{g}}$.

Let us also define the sets

$$\Theta_{\boldsymbol{u}} := \left\{ x \in \Omega_f : \limsup_{\varrho \to 0} \frac{|E\boldsymbol{u}|(B_\varrho(x))}{\varrho^2} > 0 \right\},$$
$$\Theta_{\boldsymbol{g}} := \left\{ x \in \Omega_f : \limsup_{\varrho \to 0} \frac{|E\boldsymbol{g}|(B_\varrho(x))}{\varrho^2} > 0 \right\},$$

and recall that, according again to [1, Proposition 3.5], $\Theta_{\boldsymbol{u}} \cong J_{\boldsymbol{u}}$ and $\Theta_{\boldsymbol{g}} \cong J_{\boldsymbol{g}}$. Using the expression of the displacement (5.3), we have $\boldsymbol{u} = \boldsymbol{v} + (\frac{1}{2} - x_3)\boldsymbol{g}$. Since $\boldsymbol{u} \in L^\infty(\Omega; \mathbb{R}^3)$, then $\boldsymbol{v} \in L^\infty(\omega; \mathbb{R}^3)$ as well, and the previous relation yields $\boldsymbol{g} \in L^\infty(\omega; \mathbb{R}^3)$ with

$$\lim_{\varrho \to 0} \frac{1}{\varrho^2} \int_{B_\varrho(x)} |\boldsymbol{g}| \, \mathrm{d}y = 0 \quad \text{for all } x \in \Omega_f.$$

Consequently since $E\boldsymbol{u} = E\boldsymbol{v} + (\frac{1}{2} - x_3)E\boldsymbol{g} - e_3 \odot \boldsymbol{g}$, we deduce that $\Omega_f \setminus (\Theta_{\boldsymbol{v}} \cup \Theta_{\boldsymbol{g}}) \subset \Omega_f \setminus \Theta_{\boldsymbol{u}}$ *i.e.* $\Theta_{\boldsymbol{u}} \subset \Theta_{\boldsymbol{v}} \cup \Theta_{\boldsymbol{g}}$ and

$$J_{\boldsymbol{u}} \tilde{\subset} J_{\boldsymbol{v}} \cup J_{\boldsymbol{g}}. \tag{5.13}$$

We now prove the converse inclusion. From the relations $\boldsymbol{v} = \boldsymbol{u} + (x_3 - \frac{1}{2})\boldsymbol{g}$ and $(\frac{1}{2} - x_3)\boldsymbol{g} = \boldsymbol{u} - \boldsymbol{v}$, and the fact that $\boldsymbol{g}$ is independent of $x_3$, we similarly obtain that $\Theta_{\boldsymbol{v}} \subset \Theta_{\boldsymbol{u}} \cup \Theta_{\boldsymbol{g}}$ and $\Theta_{\boldsymbol{g}} \subset \Theta_{\boldsymbol{u}} \cup \Theta_{\boldsymbol{v}}$ which imply that

$$J_{\boldsymbol{v}} \setminus J_{\boldsymbol{g}} \tilde{\subset} J_{\boldsymbol{u}}, \quad J_{\boldsymbol{g}} \setminus J_{\boldsymbol{v}} \tilde{\subset} J_{\boldsymbol{u}}. \tag{5.14}$$

It thus remains to prove that

$$J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}} \tilde{\subset} J_{\boldsymbol{u}}. \tag{5.15}$$

According to (5.9), (5.11) and (5.12), we get

$$(J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}} \cong ([(J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \times (0,1)] \setminus J_{\boldsymbol{u}} \tilde{\subset} [(J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \setminus S_{u_3}] \times (0,1),$$

where we used that, since $u_3 \in SBV(\omega)$, then $J_{u_3} \cong S_{u_3}$. Assume by contradiction that

$$\mathcal{H}^2((J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}}) > 0, \tag{5.16}$$

then there is some $x = (x', x_3) \in (J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}}$ with $x' \in (J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \setminus S_{u_3}$ such that $\nu_{\bar{\boldsymbol{u}}}(x') = \pm \nu_{\nabla u_3}(x')$. Let us assume without loss of generality that $\nu_{\bar{\boldsymbol{u}}}(x') = \nu_{\nabla u_3}(x') =: \nu(x')$, the other case can be dealt with similarly. Since $x'$ is a Lebesgue point of $u_3$, then the one-sided Lebesgue limits of $u_3$ at $x'$ in the direction $\nu(x')$ are equal and coincide with its approximate limit. On the other hand, since $x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$, then the functions $\bar{\boldsymbol{u}}$ and $\nabla u_3$ admit one-sided Lebesgue limits at $x'$ in the direction $\nu(x')$. Next, from the expression (5.3) of the displacement, we deduce that for all $\alpha \in \{1, 2\}$, the functions $u_\alpha$ admit as well one-sided Lebesgue limits at $x$ in the direction $(\nu(x'), 0)$. Gathering all previous informations, we get that the full displacement $\boldsymbol{u}$ admits one-sided Lebesgue limits at $x$ in the direction $(\nu(x'), 0)$. Using the fact that $x \notin J_{\boldsymbol{u}}$, we infer that necessarily $[\boldsymbol{u}](x) = 0$, and thus, using again (5.3) yields

$$[\bar{u}_\alpha](x') + \left(\frac{1}{2} - x_3\right) [\partial_\alpha u_3](x') = 0 \quad \text{for all } \alpha \in \{1, 2\}. \tag{5.17}$$

We observe that, by (5.11) and (5.12), the sets $J_{\boldsymbol{v}}$ and $J_{\boldsymbol{g}}$ are invariant in the transverse direction, and consequently $(x', y_3) \in J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}$ for any $y_3 \in (0, 1)$. Therefore if $(x', y_3) \notin J_{\boldsymbol{u}}$ for some $y_3 \neq x_3$, then reproducing the same argument than above implies that

$$[\bar{u}_\alpha](x') + \left(\frac{1}{2} - y_3\right) [\partial_\alpha u_3](x') = 0 \quad \text{for all } \alpha \in \{1, 2\}.$$

Subtracting the previous relation to (5.17) yields $[\bar{u}_\alpha](x') = [\partial_\alpha u_3](x') = 0$ for all $\alpha \in \{1, 2\}$, which is against the fact that $x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$. As a consequence, $(x', y_3) \in J_{\boldsymbol{u}}$ for all $y_3 \in (0, 1)$ with $y_3 \neq x_3$. In addition, since $x' \in J_{\nabla u_3}$, there is some $\alpha \in \{1, 2\}$ such that $[\partial_\alpha u_3](x') \neq 0$, and $x_3$ is therefore given by

$$x_3 = \frac{1}{2} + \frac{[\bar{u}_\alpha](x')}{[\partial_\alpha u_3](x')}.$$

Consequently, we have proved that

$$(J_{\boldsymbol{v}} \cap J_{\boldsymbol{g}}) \setminus J_{\boldsymbol{u}} \,\tilde{\subset}\, \bigcup_{\alpha=1}^{2} \left\{ (x', x_3) : x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3},\ [\partial_\alpha u_3](x') \neq 0,\ x_3 = \frac{1}{2} + \frac{[\bar{u}_\alpha](x')}{[\partial_\alpha u_3](x')} \right\} =: A.$$

The set $A$ is Borel measurable, and, for each $x' \in J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$, its transverse section passing through $x'$, denoted by $A^{x'} := \{x_3 \in (0, 1) : (x', x_3) \in A\}$ is reduced to at most two points. Since the set $J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}$ is countably $\mathcal{H}^1$-rectifiable, [23, Theorem 3.2.23] ensures that $\mathcal{H}^2 \llcorner ((J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}) \times (0, 1)) = (\mathcal{H}^1 \llcorner (J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3})) \otimes (\mathcal{L}^1 \llcorner (0, 1))$, and Fubini's Theorem yields

$$\mathcal{H}^2(A) = \int_{J_{\bar{\boldsymbol{u}}} \cap J_{\nabla u_3}} \mathcal{L}^1(A^{x'}) \, \mathrm{d}\mathcal{H}^1(x') = 0,$$

which is against (5.16), and therefore completes the proof of (5.15). Gathering (5.13) – (5.15) leads to $J_{\boldsymbol{u}} \cong J_{\boldsymbol{v}} \cup J_{\boldsymbol{g}}$, and thus $J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0, 1)$.

**Step 5.** We complete the proof of the proposition by establishing that $\bar{\boldsymbol{u}}$ and $\nabla u_3$ are actually $SBD(\omega)$ functions. Indeed, since we know that $J_{\boldsymbol{u}} \cong \Gamma \times (0, 1)$ for some countably $\mathcal{H}^1$-rectifiable set $\Gamma \subset \omega$, equation (5.10) reads

$$-\int_\omega \partial_\beta \varphi_{\alpha\beta} \bar{u}_\alpha \, \mathrm{d}x' = \int_\omega \varphi_{\alpha\beta} \left( \int_0^1 e_{\alpha\beta}(\boldsymbol{u}) \, \mathrm{d}x_3 \right) \mathrm{d}x' + \int_\Gamma \varphi_{\alpha\beta} \left( \int_0^1 ([\boldsymbol{u}] \odot \nu_\Gamma)_{\alpha\beta} \, \mathrm{d}x_3 \right) \mathrm{d}\mathcal{H}^1,$$

which implies that $e_{\alpha\beta}(\bar{\boldsymbol{u}}) = \int_0^1 e_{\alpha\beta}(\boldsymbol{u})(\cdot, x_3)\,\mathrm{d}x_3$ by uniqueness of the Lebesgue decomposition, and that the singular part of $E\bar{\boldsymbol{u}}$ is concentrated on a countably $\mathcal{H}^1$-rectifiable set. It follows from [1, Proposition 4.7] that $\bar{\boldsymbol{u}} \in SBD(\omega)$ and the same can be said, therefore, first for $\nabla u_3$ and then for $(u_1^+(\cdot, 0), u_2^+(\cdot, 0))$. □

Propositions 5.1 and 5.2 suggest one to define the limiting space of all kinematically admissible displacements by

$$\begin{aligned}
\mathcal{A}_{KL} \;\; := \;\; \Big\{ & \boldsymbol{u} \in SBD(\Omega_f) : \|\boldsymbol{u}\|_{L^\infty(\Omega_f)} \leq M,\ u_3 \in SBV(\omega) \cap L^\infty(\omega) \\
& \text{with } \nabla u_3 \in SBD(\omega) \cap L^\infty(\omega;\mathbb{R}^2), \\
& u_\alpha(x', x_3) = \bar{u}_\alpha(x') + \Big(\frac{1}{2} - x_3\Big)\partial_\alpha u_3(x') \text{ for } \mathcal{L}^3\text{-a.e. } x = (x', x_3) \in \Omega_f, \qquad (5.18) \\
& \text{where } \bar{\boldsymbol{u}} := (\bar{u}_1, \bar{u}_2) \in SBD(\omega) \cap L^\infty(\omega;\mathbb{R}^2), \\
& \text{and } J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0,1) \Big\}. \qquad (5.19)
\end{aligned}$$

5.2. **Γ-limit in the film.** For each $\varepsilon > 0$, let us define the functionals $\mathcal{E}^f_\varepsilon$ and $\mathcal{E}^f_0$ : $L^2(\Omega_f;\mathbb{R}^3) \to [0, +\infty]$ by

$$\mathcal{E}^f_\varepsilon(\boldsymbol{u}) := \begin{cases} E_\varepsilon(\boldsymbol{u}, \Omega_f) & \text{if } \boldsymbol{u} \in SBD(\Omega_f) \text{ and } \|\boldsymbol{u}\|_{L^\infty(\Omega_f)} \leq M, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\mathcal{E}^f_0(\boldsymbol{u}) := \begin{cases} \displaystyle\int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\right] \mathrm{d}x' & \\ \quad + \displaystyle\frac{1}{12}\int_\omega \left[\frac{\lambda_f\mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\nabla u_3)e_{\beta\beta}(\nabla u_3) + \mu_f e_{\alpha\beta}(\nabla u_3)e_{\alpha\beta}(\nabla u_3)\right] \mathrm{d}x' & \\ \quad + \kappa_f \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) & \text{if } \boldsymbol{u} \in \mathcal{A}_{KL}, \\ +\infty & \text{otherwise.} \end{cases}$$

**Theorem 5.1.** *The sequence of functionals* $(\mathcal{E}^f_\varepsilon)_{\varepsilon>0}$ *Γ-converges to* $\mathcal{E}^f_0$ *with respect to the strong* $L^2(\Omega_f;\mathbb{R}^3)$*-topology.*

*Proof.* **Step 1.** We start by deriving a lower bound inequality, *i.e.*, for any $\boldsymbol{u} \in L^2(\Omega_f;\mathbb{R}^3)$ and any sequence $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega_f;\mathbb{R}^3)$ such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, then

$$\liminf_{\varepsilon\to 0} \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \mathcal{E}^f_0(\boldsymbol{u}).$$

If $\liminf_\varepsilon \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$, the result is obvious. Otherwise, up to a subsequence, we can assume that

$$\lim_{\varepsilon\to 0} \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{E}^f_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty.$$

By virtue of the above energy bound, we can assume without loss of generality that the conclusions of Propositions 5.1 and 5.2 hold so that $\boldsymbol{u} \in \mathcal{A}_{KL}$. Using a very similar argument than that used in the proof of the lower bound in Theorem 4.1, combined with the lower semicontinuity of the surface energy established in (5.1), we obtain that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_f) \geq \int_{\Omega_f} \left[\frac{\lambda_f\mu_f}{\lambda_f+2\mu_f} e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u}) + \mu_f e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\right] \mathrm{d}x + \kappa_f \mathcal{H}^2(J_{\boldsymbol{u}}\cap\Omega_f).$$

According to (5.18), we get that

$$\begin{aligned}\int_{\Omega_f} e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{u})\,\mathrm{d}x &= \int_{\Omega_f} \Big[e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}}) + 2\left(\frac{1}{2}-x_3\right) e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\nabla u_3) \\ &\qquad + \left(\frac{1}{2}-x_3\right)^2 e_{\alpha\beta}(\nabla u_3)e_{\alpha\beta}(\nabla u_3)\Big]\,\mathrm{d}x \\ &= \int_\omega e_{\alpha\beta}(\bar{\boldsymbol{u}})e_{\alpha\beta}(\bar{\boldsymbol{u}})\,\mathrm{d}x' + \frac{1}{12}\int_\omega e_{\alpha\beta}(\nabla u_3)e_{\alpha\beta}(\nabla u_3)\,\mathrm{d}x',\end{aligned}$$

and similarly for the other term

$$\int_{\Omega_f} e_{\alpha\alpha}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{u})\,\mathrm{d}x = \int_\omega e_{\alpha\alpha}(\bar{\boldsymbol{u}})e_{\beta\beta}(\bar{\boldsymbol{u}})\,\mathrm{d}x' + \frac{1}{12}\int_\omega e_{\alpha\alpha}(\nabla u_3)e_{\beta\beta}(\nabla u_3)\,\mathrm{d}x'.$$

Therefore (5.19) yields the announced energy lower bound.

**Step 2.** We next derive an upper bound through the construction of a recovery sequence, *i.e.*, for every $\boldsymbol{u}\in L^2(\Omega_f;\mathbb{R}^3)$, there exists a recovery sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0}\subset L^2(\Omega_f;\mathbb{R}^3)$ such that $\boldsymbol{u}^*_\varepsilon\to\boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$, and

$$\limsup_{\varepsilon\to 0}\mathcal{E}^f_\varepsilon(\boldsymbol{u}^*_\varepsilon)\leq\mathcal{E}^f_0(\boldsymbol{u}).$$

If $\boldsymbol{u}\notin\mathcal{A}_{KL}$, then $\mathcal{E}^f_0(\boldsymbol{u})=+\infty$ and the result is obvious. It therefore suffices to assume that $\boldsymbol{u}\in\mathcal{A}_{KL}$. We now define a recovery sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0}$. For $\mathcal{L}^3$-a.e. $x=(x',x_3)\in\Omega_f$ and all $\varepsilon>0$, let

$$\boldsymbol{u}^*_\varepsilon(x',x_3) = c_\varepsilon\big(\boldsymbol{u}(x)+(0,0,\varepsilon^2 x_3 h_\varepsilon(x'))\big),$$

where $(h_\varepsilon)_{\varepsilon>0}$ is a sequence in $\mathcal{C}^\infty_c(\omega)$ such that

$$h_\varepsilon\to-\frac{\lambda_f}{\lambda_f+2\mu_f}\boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u}) \text{ in } L^2(\omega),\quad \lim_{\varepsilon\to 0}\varepsilon\|\nabla h_\varepsilon\|_{L^2(\omega)}=\lim_{\varepsilon\to 0}\varepsilon\|h_\varepsilon\|_{L^\infty(\omega)}=0, \tag{5.20}$$

and $c_\varepsilon := M/(M+\varepsilon^2\|h_\varepsilon\|_{L^\infty(\omega)})$. Clearly, $\boldsymbol{u}^*_\varepsilon\in SBD(\Omega_f)$ and $\|\boldsymbol{u}^*_\varepsilon\|_{L^\infty(\Omega_f)}\leq M$. Using (3.3) we get that

$$\begin{aligned}J_\varepsilon(\boldsymbol{u}^*_\varepsilon,\Omega_f) &= \frac{c_\varepsilon^2}{2}\int_{\Omega_f}\left[\lambda_f\boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u})\boldsymbol{e}_{\beta\beta}(\boldsymbol{u})+2\mu_f\boldsymbol{e}_{\alpha\beta}(\boldsymbol{u})\boldsymbol{e}_{\alpha\beta}(\boldsymbol{u})\right]\mathrm{d}x \\ &\quad+\frac{c_\varepsilon^2}{2\varepsilon^2}\int_{\Omega_f}\left[2\lambda_f\boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u})\varepsilon^2 h_\varepsilon+\mu_f\varepsilon^4 x_3^2|\nabla h_\varepsilon|^2\right]\mathrm{d}x \\ &\quad+\frac{c_\varepsilon^2}{2\varepsilon^4}\int_{\Omega_f}(\lambda_f+2\mu_f)\varepsilon^4|h_\varepsilon|^2\,\mathrm{d}x.\end{aligned}$$

Thus, since $c_\varepsilon\to 1$ and according to the convergence properties (5.20), we get that

$$\lim_{\varepsilon\to 0}J_\varepsilon(\boldsymbol{u}^*_\varepsilon,\Omega_f)=\frac{1}{2}\int_{\Omega_f}\left[\frac{2\lambda_f\mu_f}{\lambda_f+2\mu_f}\boldsymbol{e}_{\alpha\alpha}(\boldsymbol{u})\boldsymbol{e}_{\beta\beta}(\boldsymbol{u})+2\mu_f\boldsymbol{e}_{\alpha\beta}(\boldsymbol{u})\boldsymbol{e}_{\alpha\beta}(\boldsymbol{u})\right]\mathrm{d}x.$$

Concerning the surface energy, since $J_{\boldsymbol{u}^*_\varepsilon} = J_{\boldsymbol{u}} \cong (J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) \times (0,1)$ it follows that

$$\int_{\Omega_f \cap J_{\boldsymbol{u}^*_\varepsilon}} \left| \left( (\boldsymbol{\nu}_{\boldsymbol{u}^*_\varepsilon})', \frac{1}{\varepsilon} (\boldsymbol{\nu}_{\boldsymbol{u}^*_\varepsilon})_3 \right) \right| \mathrm{d}\mathcal{H}^2 = \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}),$$

which completes the proof of the upper bound. □

## 6. Multifissuration: debonding and delamination vs transverse cracks

In this section, we consider the full model of a film $\Omega_f$ deposited on a substrate $\Omega_s$ through a bonding layer $\Omega_b$, and we assume that both $\Omega_f$ and $\Omega_b$ can crack.

6.1. **The anti-plane case.** Following [32], it is assumed that the geometry is invariant in the direction $\boldsymbol{e}_2$, *i.e.*, $\omega = I \times \mathbb{R}$, where $I$ is a bounded open interval, and that the admissible displacements take the form

$$\boldsymbol{u}(x) = u(x_1, x_3)\boldsymbol{e}_2.$$

In this case the elastic energy reduces to

$$\tilde{J}_\varepsilon(u) = \frac{\mu_f}{2} \int_{I \times (0,1)} (|\partial_1 u|^2 + \varepsilon^{-2} |\partial_3 u|^2) \, \mathrm{d}x_1 \, \mathrm{d}x_3 \\ + \frac{\mu_b}{2} \int_{I \times (-1,0)} (\varepsilon^2 |\partial_1 u|^2 + |\partial_3 u|^2) \, \mathrm{d}x_1 \, \mathrm{d}x_3,$$

and the total energy is given by

$$\tilde{E}_\varepsilon(u) := \tilde{J}_\varepsilon(u) + \kappa_f \int_{J_u \cap [I \times (0,1)]} \left| \left( (\nu_u)_1, \varepsilon^{-1} (\nu_u)_3 \right) \right| \mathrm{d}\mathcal{H}^1 \\ + \kappa_b \int_{J_u \cap [I \times [-1,0]]} |(\varepsilon (\nu_u)_1, (\nu_u)_3)| \, \mathrm{d}\mathcal{H}^1.$$

The natural functional setting is to consider (scalar) displacements in the class

$$\tilde{\mathcal{A}} := \{u \in SBV(I \times (-2,1)) : u = 0 \ \mathcal{L}^2\text{-a.e. in } I \times (-2,-1) \text{ and } \|u\|_{L^\infty(I \times (0,1))} \le M\},$$

where $M > 0$ is an arbitrary fixed constant.

In [32], the following one-dimensional energy, defined for all $u \in SBV(I)$, was proposed as an approximation of the previous two-dimensional energy

$$\tilde{E}_0(u) := \frac{\mu_f}{2} \int_I |u'|^2 \, \mathrm{d}x_1 + \frac{\mu_b}{2} \int_{I \setminus \Delta_u} |u|^2 \, \mathrm{d}x_1 + \kappa_f \#(J_u) + \kappa_b \mathcal{L}^1(\Delta_u),$$

where $\Delta_u := \{|u| > \sqrt{2\kappa_b/\mu_b}\}$ is the delamination set. An easy adaptation of the proof of [33, Theorem A.1] justifies rigorously this conjecture through the following Γ-convergence type result.

**Theorem 6.1.** *Let $u \in SBV(I)$, then*

- *for any sequence $(u_\varepsilon)_{\varepsilon > 0} \subset \tilde{\mathcal{A}}$ satisfying $u_\varepsilon \to u$ strongly in $L^2(I \times (0,1))$, then*

$$\tilde{E}_0(u) \le \liminf_{\varepsilon \to 0} \tilde{E}_\varepsilon(u_\varepsilon);$$

- *there exists a recovery sequence* $(u_\varepsilon^*)_{\varepsilon>0} \subset \tilde{A}$ *such that* $u_\varepsilon^* \to u$ *strongly in* $L^2(I \times (0,1))$*, and*

$$\tilde{E}_0(u) \geq \liminf_{\varepsilon\to 0} \tilde{E}_\varepsilon(u_\varepsilon^*).$$

Let us observe that if $u_\varepsilon$ is a sequence of minimizers of $\tilde{E}_\varepsilon$ (under suitable loadings), the (characteristic function of the) delamination set $\Delta_u$ is constructed as the $L^1$-limit of the orthogonal projection of the jump sets $J_{u_\varepsilon}$ onto the mid-surface $\{x_3 = 0\}$. In particular, the vertical cracks in the bonding layer do not contribute to delamination.

6.2. **The general case.** We conjecture that Theorem 6.1 can be extended to the general three-dimensional vectorial case. In this situation, the space of kinematically admissible displacements is given by

$$\mathcal{A} := \Big\{ \boldsymbol{u} \in SBD(\Omega) : \boldsymbol{u} = 0 \ \mathcal{L}^3\text{-a.e. on } \Omega_s, \text{ and } \|\boldsymbol{u}\|_{L^\infty(\Omega_f)} \leq M \Big\}.$$

Let us define the energy functionals $\mathcal{E}_\varepsilon$ and $\mathcal{E}_0 : L^2(\Omega;\mathbb{R}^3) \to [0,+\infty]$ by

$$\mathcal{E}_\varepsilon(\boldsymbol{u}) := \begin{cases} E_\varepsilon(\boldsymbol{u}) & \text{if } \boldsymbol{u} \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{6.1}$$

and

$$\mathcal{E}_0(\boldsymbol{u}) := \begin{cases} \displaystyle\int_\omega \left[ \frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\bar{\boldsymbol{u}}) e_{\beta\beta}(\bar{\boldsymbol{u}}) + \mu_f e_{\alpha\beta}(\bar{\boldsymbol{u}}) e_{\alpha\beta}(\bar{\boldsymbol{u}}) \right] \mathrm{d}x' \\ \displaystyle\quad + \frac{1}{12} \int_\omega \left[ \frac{\lambda_f \mu_f}{\lambda_f + 2\mu_f} e_{\alpha\alpha}(\nabla u_3) e_{\beta\beta}(\nabla u_3) + \mu_f e_{\alpha\beta}(\nabla u_3) e_{\alpha\beta}(\nabla u_3) \right] \mathrm{d}x' \\ \displaystyle\quad + \frac{\mu_b}{2} \int_{\omega\setminus\Delta} |\bar{\boldsymbol{u}}|^2 \,\mathrm{d}x' + \kappa_f \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \cup J_{u_3} \cup J_{\nabla u_3}) + \kappa_b \mathcal{L}^2(\Delta) \text{ if } \boldsymbol{u} \in \mathcal{A}_{KL}, \\ +\infty \qquad \text{otherwise,} \end{cases}$$

where the delamination set is defined by

$$\Delta := \left\{ x' \in \omega : |\bar{\boldsymbol{u}}(x')| > \sqrt{\frac{2\kappa_b}{\mu_b}} \right\} \cup \{ x' \in \omega : u_3 \neq 0 \}. \tag{6.2}$$

We expect $\mathcal{E}_0$ to be the $\Gamma$-limit of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$, but have been unable to prove the corresponding lower bound inequality:

**Conjecture 6.1.** *If* $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$ *and* $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ *is any sequence converging strongly to* $\boldsymbol{u}$ *in* $L^2(\Omega_f;\mathbb{R}^3)$*, then*

$$\mathcal{E}_0(\boldsymbol{u}) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\boldsymbol{u}_\varepsilon).$$

Our aim here is only to prove the $\Gamma$-lim sup inequality and to present some partial results and techniques which could be relevant in future investigations of this problem.

**Proposition 6.1.** *For every* $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$*, there exists a sequence* $(\boldsymbol{u}_\varepsilon^*)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ *such that* $\boldsymbol{u}_\varepsilon^* \to \boldsymbol{u}$ *strongly in* $L^2(\Omega_f;\mathbb{R}^3)$*, and*

$$\mathcal{E}_0(\boldsymbol{u}) \geq \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\boldsymbol{u}_\varepsilon^*).$$

*Proof.* If $\boldsymbol{u} \notin \mathcal{A}_{KL}$, then $\mathcal{E}_0(\boldsymbol{u}) = +\infty$ and there is nothing to prove. Therefore, we assume from now on that $\boldsymbol{u} \in \mathcal{A}_{KL}$.

**Step 1.** In order to define the recovery sequence, we need several approximation steps. We start by approximating the delamination set defined in (6.2) by a sequence of sets of finite perimeter. Let $(\rho_m)_{m\in\mathbb{N}}$ be a standard sequence of mollifiers in $\mathbb{R}^2$, and set $\chi_m := \rho_m * \chi_\Delta$. We know that $\chi_m \to \chi_\Delta$ strongly in $L^1(\omega)$. Set

$$\delta_m := \sqrt{\|\chi_m - \chi_\Delta\|_{L^1(\omega)}} \to 0.$$

By the coarea formula [2, Theorem 3.40], for every $m \in \mathbb{N}$ large enough, there exists $\frac{1}{2} \le t_m \le 1 - \delta_m$ such that

$$\Delta_m := \{x' \in \omega : \chi_m(x') > t_m\}$$

has finite perimeter. We claim that

$$\chi_{\Delta_m} \to \chi_\Delta \text{ in } L^1(\omega). \tag{6.3}$$

Indeed,

$$\mathcal{L}^2(\Delta_m \setminus \Delta) \le \frac{1}{t_m} \int_{\Delta_m \setminus \Delta} \chi_m(x')\,\mathrm{d}x' \le \frac{1}{t_m} \int_{\Delta_m \setminus \Delta} |\chi_m - \chi_\Delta|\,\mathrm{d}x' \to 0,$$

and

$$\begin{aligned} \mathcal{L}^2(\Delta \setminus \Delta_m) &\le \mathcal{L}^2(\{x' \in \Delta : \chi_\Delta(x') = 1 \text{ and } \chi_m(x') \le 1 - \delta_m\}) \\ &\le \frac{1}{\delta_m} \int_\Delta |\chi_\Delta(x') - \chi_m(x')|\,\mathrm{d}x' \le \delta_m \to 0, \end{aligned}$$

hence $\|\chi_\Delta - \chi_{\Delta_m}\|_{L^1(\omega)} = \mathcal{L}^2(\Delta_m \setminus \Delta) + \mathcal{L}^2(\Delta \setminus \Delta_m) \to 0$. In addition, it is possible to find a sequence $\varepsilon_m \overset{m\to\infty}{\longrightarrow} 0$ such that $\varepsilon_m \mathcal{H}^1(\partial^*\Delta_m) \overset{m\to\infty}{\longrightarrow} 0$. With a slight abuse of notation, we refer to the sequences $(\varepsilon_m)$ and $(\Delta_m)$ simply as $(\varepsilon)$ and $(\Delta_\varepsilon)$ and henceforth assume that

$$\lim_{\varepsilon\to 0} \varepsilon \mathcal{H}^1(\partial^*\Delta_\varepsilon) = 0. \tag{6.4}$$

We next approximate the displacement $\boldsymbol{u}$. Indeed, according to [17, Theorem 3] (see also [30, Theorem 3]), there exists a sequence $(\bar{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0} \in SBV(\omega;\mathbb{R}^2)$ such that $\bar{\boldsymbol{u}}_\varepsilon \to \bar{\boldsymbol{u}}$ strongly in $L^2(\omega;\mathbb{R}^3)$, $e(\bar{\boldsymbol{u}}_\varepsilon) \to e(\bar{\boldsymbol{u}})$ strongly in $L^2(\omega;\mathbb{M}^{2\times 2}_{\rm sym})$, $\mathcal{H}^1(J_{\bar{\boldsymbol{u}}_\varepsilon} \setminus J_{\bar{\boldsymbol{u}}}) + \mathcal{H}^1(J_{\bar{\boldsymbol{u}}} \setminus J_{\bar{\boldsymbol{u}}_\varepsilon}) \to 0$, and $\|\bar{\boldsymbol{u}}_\varepsilon\|_{L^\infty(\omega)} \le \|\bar{\boldsymbol{u}}\|_{L^\infty(\omega)}$. Let us define for a.e. $x' \in \omega$ and all $x_3 \in (0,1)$,

$$(u_\varepsilon)_\alpha(x', x_3) := (\bar{u}_\varepsilon)_\alpha(x') + \left(\frac{1}{2} - x_3\right) \partial_\alpha u_3(x'), \quad (u_\varepsilon)_3(x', x_3) := u_3(x')$$

so that $\boldsymbol{u}_\varepsilon \in SBD(\Omega_f)$, and $\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f)} \le \|\boldsymbol{u}\|_{L^\infty(\Omega_f)} \le M$.

As in the proof of Theorem 5.1, we consider a sequence $(h_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty_c(\omega)$ satisfying (5.20).

We now define the recovery sequence by setting, for all $\varepsilon > 0$ and for $\mathcal{L}^3$-a.e. $x = (x', x_3) \in \Omega$,

$$\boldsymbol{u}^*_\varepsilon(x', x_3) = \begin{cases} c_\varepsilon\big(\boldsymbol{u}_\varepsilon(x) + \big(0, 0, \varepsilon^2 x_3 h_\varepsilon(x')\big)\big) & \text{if } (x', x_3) \in \Omega_f, \\ c_\varepsilon(x_3 + 1)(\bar{\boldsymbol{u}}_\varepsilon(x'), 0) & \text{if } (x', x_3) \in (\omega \setminus \Delta_\varepsilon) \times [-1, 0], \\ 0 & \text{if } (x', x_3) \in (\Delta_\varepsilon \times [-1, 0]) \cup \Omega_s, \end{cases}$$

where $c_\varepsilon = \frac{M}{M+\varepsilon^2\|h_\varepsilon\|_{L^\infty(\omega)}}$. Since the set $\Delta_\varepsilon$ has finite perimeter in $\omega$ and $\bar{\boldsymbol{u}}_\varepsilon \in SBV(\omega;\mathbb{R}^2)$, then $\bar{\boldsymbol{u}}_\varepsilon \chi_{\omega\setminus\Delta_\varepsilon} \in SBV(\omega;\mathbb{R}^2)$, and thus $\boldsymbol{u}^*_\varepsilon \in SBD(\Omega)$ and $\boldsymbol{u}^*_\varepsilon = 0$ $\mathcal{L}^3$-a.e. in $\Omega_s$. In addition, the fact that $\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f)} \le M$ yields $\|\boldsymbol{u}^*_\varepsilon\|_{L^\infty(\Omega_f)} \le M$ as well so that $\boldsymbol{u}^*_\varepsilon \in \mathcal{A}$. The sequence $(\boldsymbol{u}^*_\varepsilon)_{\varepsilon>0}$ is thus admissible, and clearly $\boldsymbol{u}^*_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$.

**Step 2.** Using the convergence properties of $\bar{\boldsymbol{u}}_\varepsilon$, a similar argument than in the proof of Theorem 5.1 leads to

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}^*_\varepsilon, \Omega_f) \le \mathcal{E}^f_0(\boldsymbol{u}).$$

It thus remains to compute the energy associated to this sequence in the bonding layer. First, the bulk energy in the bonding layer gives

$$J_\varepsilon(\boldsymbol{u}^*_\varepsilon, \Omega_b) = \frac{c_\varepsilon^2\varepsilon^2}{2}\int_{(\omega\setminus\Delta_\varepsilon)\times(-1,0)} (x_3+1)^2\Big[\lambda_b e_{\alpha\alpha}(\bar{\boldsymbol{u}}_\varepsilon)e_{\beta\beta}(\bar{\boldsymbol{u}}_\varepsilon) + 2\mu_b e_{\alpha\beta}(\bar{\boldsymbol{u}}_\varepsilon)e_{\alpha\beta}(\bar{\boldsymbol{u}}_\varepsilon)\Big]\,\mathrm{d}x$$
$$+ \frac{c_\varepsilon^2\mu_b}{2}\int_{\omega\setminus\Delta_\varepsilon} |\bar{\boldsymbol{u}}_\varepsilon|^2\,\mathrm{d}x' \to \frac{\mu_b}{2}\int_{\omega\setminus\Delta} |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x'.$$

Concerning the surface energy in the bonding layer, we first observe that for each $\varepsilon > 0$,

$$J_{\boldsymbol{u}^*_\varepsilon}\cap\Omega_b \subset \Big[J_{\bar{\boldsymbol{u}}_\varepsilon}\times[-1,0]\Big] \cup \Big[\Delta_\varepsilon\times\{0\}\Big] \cup \Big[(\{(u_3,\nabla u_3)\neq 0\}\setminus\Delta_\varepsilon)\times\{0\}\Big] \cup \Big[\partial^*\Delta_\varepsilon\times[-1,0]\Big],$$

where $\partial^*\Delta_\varepsilon$ stands for the reduced boundary of $\Delta_\varepsilon$ [2, Definition 3.54]. Let us observe that $\omega\setminus\Delta \subset \{u_3 = 0\} \tilde{\subset} \{(u_3,\nabla u_3) = 0\}$ since, by locality of the approximate gradient, $\nabla u_3 = 0$ $\mathcal{L}^2$-a.e. in $\{u_3 = 0\}$ (see [2, Proposition 3.73 (c)]). Then

$$\limsup_{\varepsilon\to 0}\int_{J_{\boldsymbol{u}^*_\varepsilon}\cap\Omega_b} \big|\big(\varepsilon(\boldsymbol{\nu}_{\boldsymbol{u}^*_\varepsilon})', (\boldsymbol{\nu}_{\boldsymbol{u}^*_\varepsilon})_3\big)\big|\,\mathrm{d}\mathcal{H}^2$$
$$\le \limsup_{\varepsilon\to 0}\Big[\varepsilon\mathcal{H}^1(J_{\bar{\boldsymbol{u}}_\varepsilon}) + \mathcal{L}^2(\Delta_\varepsilon) + \mathcal{L}^2(\{(u_3,\nabla u_3)\neq 0\}\setminus\Delta_\varepsilon) + \varepsilon\mathcal{H}^1(\partial^*\Delta_\varepsilon)\Big] = \mathcal{L}^2(\Delta),$$

thanks to (6.2), (6.3) and (6.4). □

6.2.1. *Partial results for the lower bound.* Let $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3)$, and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega;\mathbb{R}^3)$ be a sequence such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(\Omega_f;\mathbb{R}^3)$. If $\liminf_\varepsilon \mathcal{E}_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$ there is nothing to prove. Otherwise by (6.1), up to a subsequence, we can assume without loss of generality that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subset \mathcal{A}$, and that

$$\sup_{\varepsilon>0} E_\varepsilon(\boldsymbol{u}_\varepsilon) < +\infty. \tag{6.5}$$

As a consequence, all the compactness results in the film $\Omega_f$ established in section 5.1 hold. In particular, Propositions 5.1 and 5.2 show that $\boldsymbol{u} \in \mathcal{A}_{KL}$, and the lower bound established in Theorem 5.1 yields the terms in $\mathcal{E}_0(\boldsymbol{u})$ corresponding to the energy in $\Omega_f$. The main problem is to deal with the bonding layer. Following the scalar case treated in [33], it is enough to show that the energy in $\Omega_b$ is bounded from below by some functional where the delamination set is replaced by a function $\theta \in L^\infty(\omega;[0,1])$, which can be interpreted as a delamination volume fraction density. On $\{\theta = 1\}$, the film is entirely debonded from the substrate, while on $\{\theta = 0\}$ it continuously accommodates the prescribed zero displacement on the substrate exactly as in the Sobolev case (Theorem 4.1). All intermediate states are contained in the set $\{0 < \theta < 1\}$.

**Proposition 6.2.** *Assume there exists $\theta \in L^\infty(\omega;[0,1])$ such that $(1-\theta)u_3 = 0$ $\mathcal{L}^2$-a.e. in $\omega$, and*

$$\frac{\mu_b}{2}\int_\omega (1-\theta)|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x' + \kappa_b \int_\omega \theta\,\mathrm{d}x' \le \liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b). \tag{6.6}$$

*Then*

$$\frac{\mu_b}{2}\int_{\omega\setminus\Delta} |\bar{\boldsymbol{u}}|^2\,\mathrm{d}x' + \kappa_b\mathcal{L}^2(\Delta) \le \liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b),$$

*where $\Delta$ be the delamination set defined in* (6.2)

*Proof.* By assumption, we have that

$$\int_\omega \min_{\{\eta\in[0,1]:(1-\eta)u_3(x')=0\}} \left(\frac{\mu_b}{2}(1-\eta)|\bar{\boldsymbol{u}}(x')|^2 + \kappa_b\eta\right)\,\mathrm{d}x' \le \liminf_{\varepsilon\to 0} E_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b).$$

The result follows by solving the pointwise minimization problem explicitly. □

The main point is to construct such a function $\theta$. As in the scalar case [33], $\theta$ is supposed to be obtained as the $L^\infty(\omega)$-weak* limit of a sequence $(\chi_{\Delta_\varepsilon})_{\varepsilon>0}$ of suitable measurable sets $\Delta_\varepsilon \subset \omega$. However, it is unclear what is the right notion of an $\varepsilon$-delamination set $\Delta_\varepsilon$ in the vectorial case. In particular, the following example shows that vertical cracks in the bonding layer cannot be neglected, so it is not enough to define $\Delta_\varepsilon$ as the orthogonal projection of $J_{\boldsymbol{u}_\varepsilon}$ onto the mid-plane $\omega\times\{0\}$, as in the anti-plane and in the Sobolev case (Thm. 6.1, [33, Prop. B.2], and Thm 4.1).

**Example 6.3** (Microstructure example)**.** Suppose that $\omega = (0,1)^2$ and $\varepsilon = \frac{1}{2N}$ for some $N\in\mathbb{N}$. In the film, set

$$\boldsymbol{u}_\varepsilon(x) = \boldsymbol{u}(x) = (0,\ell,0) \quad \text{ for all } x\in\Omega_f.$$

In $\Omega_b$ set, for each $i = 0,\dots,N-1$ and all $2i\varepsilon \le x_2 \le (2i+2)\varepsilon$, $-1\le x_3\le 0 < x_1 < 1$,

$$u_\varepsilon(x_1,x_2,x_3) = \left(0, \ell(1+x_3), \ell\varepsilon v\left(\frac{x_2-2i\varepsilon}{\varepsilon}, 1+x_3\right)\right),$$

where $v\in H^1\Big((0,2)\times(0,1)\Big)$ is any function such that $v(s,0) = v(s,1) = 0$ $\forall s\in[0,1]$ and

$$q := \fint_{s=0}^{2}\int_{t=0}^{1} \left((1+\partial_s v)^2 + 2\partial_t v^2\right)\mathrm{d}s\,\mathrm{d}t < 1.$$

If $\Delta_\varepsilon$ is defined as $\pi(J_{\boldsymbol{u}_\varepsilon}\cap\Omega)$, then

$$\int_{\Omega_b} \left(2\mu_b e_{\alpha 3}(\boldsymbol{u}_\varepsilon)e_{\alpha 3}(\boldsymbol{u}_\varepsilon) + \varepsilon^{-2}\mu_b e_{33}(\boldsymbol{u}_\varepsilon)e_{33}(\boldsymbol{u}_\varepsilon)\right)\mathrm{d}x + \kappa_b\mathcal{L}^2(\Delta_\varepsilon) = \frac{q\mu_b\ell^2}{2}. \tag{6.7}$$

On the other hand, if $\Delta = \{|\bar{\boldsymbol{u}}| > \sqrt{2\kappa_b/\mu_b}\}$ is the expected limit delamination set, then

$$\int_{\omega\setminus\Delta} \frac{\mu_b}{2} u_\alpha u_\alpha\,\mathrm{d}x' + \kappa_b\mathcal{L}^2(\Delta) = \begin{cases} \frac{\mu_b\ell^2}{2} & \text{if } \quad \ell \le \sqrt{\frac{2\kappa_b}{\mu_b}}, \\ \kappa_b & \text{if } \quad \ell > \sqrt{\frac{2\kappa_b}{\mu_b}}. \end{cases} \tag{6.8}$$

Choosing $\ell \in \left(\sqrt{\dfrac{2\kappa_b}{\mu_b}}, \sqrt{\dfrac{2\kappa_b}{q\mu_b}}\right)$ shows that (6.8) would not be a lower bound for (6.7). □

Regardless of the notion of an $\varepsilon$-delamination set $\Delta_\varepsilon$ one tries to define, it is convenient to impose that it should contain the set

$$P_\varepsilon := \pi(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f),$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(x) := x'$, is the orthogonal projection onto $\mathbb{R}^2 \times \{0\}$. On the one hand, there is no loss of generality in doing this, since it converges to a Lebesgue negligible set. Indeed, according to the coarea formula (see [2, Theorem 2.93]) and the surface energy bound (6.5) in the film, we have

$$\mathcal{L}^2(P_\varepsilon) \leq \int_{\mathbb{R}^2} \mathcal{H}^0(J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f \cap \pi^{-1}(x'))\,\mathrm{d}x' = \int_{J_{\boldsymbol{u}_\varepsilon} \cap \Omega_f} |(\nu_{\boldsymbol{u}_\varepsilon})_3|\,\mathrm{d}\mathcal{H}^2 \leq C\varepsilon \to 0.$$

On the other hand, excluding $P_\varepsilon$ enables one to slightly improve the convergences in the film, as in the following lemma which proves the convergence of the planar gradient of the anti-plane displacement.

It will be assumed henceforth that $u_\varepsilon \in SBV^2(\Omega;\mathbb{R}^3)$ and that $J_{\boldsymbol{u}_\varepsilon}$ is closed in $\Omega$ and contained in a finite union of closed connected pieces of $\mathcal{C}^1$ hypersurfaces. In doing this no generality is lost, thanks to the density result in $SBD$ of [17, Thm. 1]. In particular, we have that $\boldsymbol{u}_\varepsilon \in H^1((\omega \setminus P_\varepsilon) \times (0,1);\mathbb{R}^3)$.

**Lemma 6.4.** *Let $(\Delta_\varepsilon)_{\varepsilon>0}$ be a sequence of closed sets be such that $P_\varepsilon \subset \Delta_\varepsilon$ for each $\varepsilon > 0$. Assume that there exists a function $\theta \in L^\infty(\omega;[0,1])$ such that $\chi_{\Delta_\varepsilon} \xrightharpoonup{*} \theta$ weakly* in $L^\infty(\omega)$, and $(1-\theta)u_3 = 0$ $\mathcal{L}^2$-a.e. in $\omega$. Then*

$$\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3 \xrightharpoonup{*} 0 \qquad \text{weakly* in } L^2(\omega;H^{-1}(0,1)).$$

*Proof.* First note that for $\mathcal{L}^2$-a.e. $x' \notin P_\varepsilon$ and $\mathcal{L}^1$-a.e. $x_3 \in (0,1)$

$$\begin{aligned}
\zeta^\varepsilon_\alpha(x) &:= \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s + (u_\varepsilon)_\alpha(x) - (u_\varepsilon)^+_\alpha(x',0)\\
&= \int_0^{x_3} [\partial_\alpha(u_\varepsilon)_3(x',s) + \partial_3(u_\varepsilon)_\alpha(x',s)]\,\mathrm{d}s = 2\int_0^{x_3} e_{\alpha 3}(\boldsymbol{u}_\varepsilon)(x',s)\,\mathrm{d}s.
\end{aligned}$$

Thanks to the bulk energy bound (6.5) in the film (see also (5.2)), we have that

$$\|\zeta^\varepsilon_\alpha\|_{L^2((\omega\setminus\Delta_\varepsilon)\times(0,1))} \leq 2\|e_{\alpha 3}(\boldsymbol{u}_\varepsilon)\|_{L^2(\Omega_f)} \leq C\varepsilon \to 0. \tag{6.9}$$

Integrating (6.9) we obtain that also $\|\bar\zeta^\varepsilon_\alpha\|_{L^2(\omega\setminus\Delta_\varepsilon)} \to 0$, where

$$\bar\zeta^\varepsilon_\alpha(x') := \int_0^1 \zeta^\varepsilon_\alpha(x',x_3)\,\mathrm{d}x_3 = \int_0^1\int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s\,\mathrm{d}x_3 + (\bar u_\varepsilon)_\alpha(x') - (u_\varepsilon)^+_\alpha(x',0)$$

and $(\bar u_\varepsilon)_\alpha(x') := \int_0^1 (u_\varepsilon)_\alpha(x',x_3)\,\mathrm{d}x_3$. As a consequence,

$$\begin{aligned}
(u_\varepsilon)_\alpha(x) &= (u_\varepsilon)^+_\alpha(x',0) - \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s + \zeta^\varepsilon_\alpha(x)\\
&= (\bar u_\varepsilon)_\alpha(x') + \int_0^1\int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s\,\mathrm{d}x_3 - \int_0^{x_3} \partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s + \eta^\varepsilon_\alpha(x), \qquad (6.10)
\end{aligned}$$

where $\|\eta^\varepsilon_\alpha\|_{L^2((\omega\setminus\Delta_\varepsilon)\times(0,1))} \to 0$.

On the other hand, for $\mathcal{L}^3$-a.e. $x \in \Omega_f$, let us define the sequences

$$\begin{aligned} g^\varepsilon_\alpha(x', x_3) &:= \chi_{\omega\setminus\Delta_\varepsilon}(x') \int_0^{x_3} \partial_\alpha (u_\varepsilon)_3(x', s)\,\mathrm{d}s, \\ \bar g^\varepsilon_\alpha(x') &:= \chi_{\omega\setminus\Delta_\varepsilon}(x') \int_0^1 \int_0^{x_3} \partial_\alpha (u_\varepsilon)_3(x', s)\,\mathrm{d}s\,\mathrm{d}x_3. \end{aligned}$$

From (6.9) and the *a priori* bound $\|\boldsymbol{u}_\varepsilon\|_{L^\infty(\Omega_f)} \le M$, we get $\|g^\varepsilon_\alpha\|_{L^2(\Omega_f)} \le C$ for some constant $C > 0$ independent of $\varepsilon$. Therefore, up to a subsequence, $g^\varepsilon_\alpha \rightharpoonup g_\alpha$ weakly in $L^2(\Omega_f)$ for some $g_\alpha \in L^2(\Omega_f)$. In addition, $\bar g^\varepsilon_\alpha \rightharpoonup \bar g_\alpha$ weakly in $L^2(\omega)$, where $\bar g_\alpha(x') := \int_0^1 g_\alpha(x', x_3)\,\mathrm{d}x_3$.

Multiplying (6.10) by $\chi_{\omega\setminus\Delta_\varepsilon}$ leads to

$$(u_\varepsilon)_\alpha(x)\chi_{\omega\setminus\Delta_\varepsilon}(x') = (\bar u_\varepsilon)_\alpha(x')\chi_{\omega\setminus\Delta_\varepsilon}(x') + \bar g^\varepsilon_\alpha(x') - g^\varepsilon_\alpha(x) + \tilde\eta^\varepsilon_\alpha(x),$$

where $\|\tilde\eta^\varepsilon_\alpha\|_{L^2(\Omega_f)} \to 0$. Passing to the limit as $\varepsilon \to 0$ finally yields

$$(1-\theta(x'))(u_\alpha(x) - \bar u_\alpha(x')) = \bar g_\alpha(x') - g_\alpha(x),$$

and according to the structure (5.18) of planar displacements, we deduce that

$$\left(\frac{1}{2} - x_3\right)(1-\theta(x'))\partial_\alpha u_3(x') = \bar g_\alpha(x') - g_\alpha(x).$$

Since by assumption $u_3 = 0$ $\mathcal{L}^2$-a.e. in $\{\theta < 1\}$, we get by locality of approximate gradients of $SBV$ functions (see [2, Proposition 3.73 (c)]), that $\nabla u_3 = 0$ $\mathcal{L}^2$-a.e. in $\{\theta < 1\}$, hence $g_\alpha(x) = \bar g_\alpha(x')$. As a consequence, $\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3 = D_3 g^\varepsilon_\alpha \overset{*}{\rightharpoonup} D_3 g_\alpha = 0$ weakly* in $L^2(\omega; H^{-1}(0,1))$. ☐

An alternative to the definition of $\Delta_\varepsilon$ as the orthogonal projection of $J_{\boldsymbol{u}_\varepsilon}$ onto $\omega \times \{0\}$ is to consider its projection along certain almost-vertical oblique directions. Define the unit vectors

$$\boldsymbol{\xi}^\pm = \frac{1}{\sqrt{2}}(\pm 1, 0, 1), \quad \boldsymbol{\eta}^\pm = \frac{1}{\sqrt{2}}(0, \pm 1, 1),$$

and their rescaled versions

$$\boldsymbol{\xi}^\pm_\varepsilon := \frac{1}{\sqrt{2}}\left(\pm 1, 0, \varepsilon^{-1}\right), \quad \boldsymbol{\eta}^\pm_\varepsilon := \frac{1}{\sqrt{2}}\left(0, \pm 1, \varepsilon^{-1}\right).$$

Denote by $\pi_{\boldsymbol{\xi}^\pm_\varepsilon}$ (resp. $\pi_{\boldsymbol{\eta}^\pm_\varepsilon}$) $: \mathbb{R}^3 \to \mathbb{R}^2$ the projection onto $\{x_3 = 0\}$ parallel to the vector $\boldsymbol{\xi}^\pm_\varepsilon$ (resp. $\boldsymbol{\eta}^\pm_\varepsilon$), *i.e.*, for $x := (x', 0) + t\boldsymbol{\xi}^\pm_\varepsilon$ (resp. $x := (x', 0) + t\boldsymbol{\eta}^\pm_\varepsilon$), then $\pi_{\boldsymbol{\xi}^\pm_\varepsilon}(x) := x'$ (resp. $\pi_{\boldsymbol{\eta}^\pm_\varepsilon}(x) := x'$). Finally, consider the set

$$\begin{aligned} \Delta_\varepsilon := \; & \pi_{\boldsymbol{\xi}^+_\varepsilon}\big(J_{\boldsymbol{u}_\varepsilon} \cap (\omega_\varepsilon \times (-2,1))\big) \cup \pi_{\boldsymbol{\eta}^+_\varepsilon}\big(J_{\boldsymbol{u}_\varepsilon} \cap (\omega_\varepsilon \times (-2,1))\big) \\ & \cup \pi_{\boldsymbol{\xi}^-_\varepsilon}\big(J_{\boldsymbol{u}_\varepsilon} \cap (\omega_\varepsilon \times (-2,1))\big) \cup \pi_{\boldsymbol{\eta}^-_\varepsilon}\big(J_{\boldsymbol{u}_\varepsilon} \cap (\omega_\varepsilon \times (-2,1))\big) \cup P_\varepsilon. \end{aligned}$$

where $\omega_\varepsilon := \{x' \in \omega : \mathrm{dist}(x', \partial\omega) > 2\varepsilon\}$. Up to a subsequence, it can be assumed that

$$\chi_{\Delta_\varepsilon} \overset{*}{\rightharpoonup} \theta \quad \text{weakly* in } L^\infty(\omega) \quad \text{for some } \theta \in L^\infty(\omega; [0,1]).$$

Using the decomposition

$$\begin{aligned}\big|\big(\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})', (\nu_{\boldsymbol{u}_\varepsilon})_3\big)\big|^2 &= \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_1 + (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_1 - (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \varepsilon^2|(\nu_{\boldsymbol{u}_\varepsilon})_2|^2 \\ &= \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_2 + (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \frac{1}{2}|\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})_2 - (\nu_{\boldsymbol{u}_\varepsilon})_3|^2 + \varepsilon^2|(\nu_{\boldsymbol{u}_\varepsilon})_1|^2,\end{aligned}$$

it is possible to prove that

$$\liminf_{\varepsilon\to 0} \int_{J_{\boldsymbol{u}_\varepsilon}\cap\Omega_b} \big|\big(\varepsilon(\nu_{\boldsymbol{u}_\varepsilon})', (\nu_{\boldsymbol{u}_\varepsilon})_3\big)\big| \,\mathrm{d}\mathcal{H}^2 \geq \frac{1}{8}\int_\omega \theta\,\mathrm{d}x',$$

which shows that $\|\theta\|_{L^1(\omega)}$ is controlled (up to a multiplicative constant) by the fracture energy in the bonding layer. The constant $1/8$, however, is not optimal, since in order to obtain (6.6) that prefactor should not be present. In most situations (*e.g.* if the sets $\pi(J_{\boldsymbol{u}_\varepsilon}\cap\Omega_b)$ have uniformly bounded perimeters) it should be possible to obtain the optimal lower bound, but there are pathological cases (such as the microstructure Example 6.3) where $\int_\omega \theta\,\mathrm{d}x'$ is larger than the fracture energy on the left-hand side (because each vertical crack is counted twice in $\Delta_\varepsilon$, which is defined as the union of all the oblique projections).

Be it as it may, by including in $\Delta_\varepsilon$ the oblique projections of the cracks inside the bonding layer, one is able to obtain an optimal estimate for the elastic energy required by the body to accomodate the strain mismatch between the deformations in the film and in the rigid substrate. Before proving this final estimate, we need two preliminary technical results concerning sections of $BD$-functions along the oblique directions defined above. For $\mathcal{L}^2$-a.e. $x'\in\omega_\varepsilon$ and $\mathcal{L}^1$-a.e. $t\in(-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$, define the functions

$$(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}^\pm_\varepsilon}(t) := \boldsymbol{u}_\varepsilon\Big((x',0) + t\boldsymbol{\xi}^\pm_\varepsilon\Big)\cdot\boldsymbol{\xi}^\pm_\varepsilon, \quad (\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\eta}^\pm_\varepsilon}(t) := \boldsymbol{u}_\varepsilon\Big((x',0) + t\boldsymbol{\eta}^\pm_\varepsilon\Big)\cdot\boldsymbol{\eta}^\pm_\varepsilon.$$

**Lemma 6.5.** *For $\mathcal{L}^2$-a.e. $x'\in\omega_\varepsilon\setminus\Delta_\varepsilon$, we have*

$$(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}^\pm_\varepsilon} \in H^1(-\sqrt{2}\varepsilon, \sqrt{2}\varepsilon) \ \textit{and}\ (\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\eta}^\pm_\varepsilon} \in H^1(-\sqrt{2}\varepsilon, \sqrt{2}\varepsilon),$$

*with $(\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\xi}^\pm_\varepsilon}(-\sqrt{2}\varepsilon) = (\boldsymbol{u}_\varepsilon)^{x'}_{\boldsymbol{\eta}^\pm_\varepsilon}(-\sqrt{2}\varepsilon) = 0$, and*

$$x_3 \mapsto (u_\varepsilon)_3(x', x_3) \in H^1(0,1).$$

*Proof.* Let us denote by

$$\Pi_{\boldsymbol{\xi}^\pm_\varepsilon} := \{\zeta\in\mathbb{R}^3 :\ \zeta\cdot\boldsymbol{\xi}^\pm_\varepsilon = 0\}$$

the plane orthogonal to $\boldsymbol{\xi}^\pm_\varepsilon$ passing through the origin, and, for $y\in\Pi_{\boldsymbol{\xi}^\pm_\varepsilon}$, we define

$$\Omega^y_{\boldsymbol{\xi}^\pm_\varepsilon} := \{t\in\mathbb{R} : y + t\boldsymbol{\xi}^\pm_\varepsilon \in \Omega_f\}.$$

According to slicing properties of functions of bounded deformations (see [1, Theorem 4.5]), we know that for $\mathcal{H}^2$-a.e. $y\in\Pi_{\boldsymbol{\xi}^\pm_\varepsilon}$, the function

$$t\mapsto \boldsymbol{u}_\varepsilon\big(y + t\boldsymbol{\xi}^\pm_\varepsilon\big)\cdot\boldsymbol{\xi}^\pm_\varepsilon \text{ belongs to } SBV^2\big(\Omega^y_{\boldsymbol{\xi}^\pm_\varepsilon}\big),$$

and its jump set is contained in

$$\{t\in\Omega^y_{\boldsymbol{\xi}^\pm_\varepsilon} : y + t\boldsymbol{\xi}^\pm_\varepsilon \in J_{\boldsymbol{u}_\varepsilon}\}.$$

Let us denote by $N_{\boldsymbol{\xi}^\pm_\varepsilon}\subset\Pi_{\boldsymbol{\xi}^\pm_\varepsilon}$ the exceptional set of zero $\mathcal{H}^2$ measure on which the previous properties fail. Since $\pi_{\boldsymbol{\xi}^\pm_\varepsilon}$ are Lipschitz functions, it follows that the sets $Z_{\boldsymbol{\xi}^\pm_\varepsilon} :=$

$\pi_{\boldsymbol{\xi}_\varepsilon^\pm}(N_{\boldsymbol{\xi}_\varepsilon^\pm}) \subset \omega$ are $\mathcal{L}^2$-negligible as well. Consequently, for all $x' \in \omega_\varepsilon \setminus Z_{\boldsymbol{\xi}_\varepsilon^\pm}$ (and thus for $\mathcal{L}^2$-a.e. $x' \in \omega_\varepsilon$), we have that

$$(\boldsymbol{u}_\varepsilon)^{x'}_{\xi_\varepsilon^\pm} \in SBV^2(-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon),$$

and its jump set is contained in

$$\{t \in (-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon) : (x', 0) + t\boldsymbol{\xi}_\varepsilon^\pm \in J_{\boldsymbol{u}_\varepsilon}\}.$$

By definition of the set $\Delta_\varepsilon$, if $x' \in \omega_\varepsilon \setminus \Delta_\varepsilon$ then $(x', 0) + t\boldsymbol{\xi}_\varepsilon^\pm \notin J_{\boldsymbol{u}_\varepsilon} \cap [\omega_\varepsilon \times (-2, 1)]$ for all $t \in (-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$, and therefore $(\boldsymbol{u}_\varepsilon)^{x'}_{\xi_\varepsilon^\pm} \in H^1(-2\sqrt{2}\varepsilon, \sqrt{2}\varepsilon)$ for $\mathcal{L}^2$-a.e. $x' \in \omega_\varepsilon \setminus \Delta_\varepsilon$. In addition since $(\boldsymbol{u}_\varepsilon)^{x'}_{\xi_\varepsilon^\pm} = 0$ $\mathcal{L}^1$-a.e. in $(-2\sqrt{2}\varepsilon, -\sqrt{2}\varepsilon)$, it follows that $(\boldsymbol{u}_\varepsilon)^{x'}_{\xi_\varepsilon^\pm}(-\sqrt{2}\varepsilon) = 0$. The statement concerning the vectors $\boldsymbol{\eta}_\varepsilon^\pm$ can be proved in an analogous way.

According again to slicing properties of functions of bounded deformations, we have that for $\mathcal{L}^2$-a.e. $x' \in \omega$, the function $x_3 \mapsto (u_\varepsilon)_3(x', x_3)$ belongs to $SBV^2(0, 1)$, and its jump set is contained in $\{x_3 \in (0, 1) : (x', x_3) \in J_{\boldsymbol{u}_\varepsilon}\}$. As a consequence, for $\mathcal{L}^2$-a.e. $x \in \omega \setminus \Delta_\varepsilon$, the function $x_3 \mapsto (u_\varepsilon)_3(x', x_3)$ belongs to $H^1(0, 1)$. □

The following technical result will be useful in the argument leading to a partial bulk energy lower bound.

**Lemma 6.6.** *Let $\vartheta \in [0, 2\pi)$, $p := \cos\vartheta$, $q := \sin\vartheta$, and define the unit vectors*

$$\boldsymbol{\xi}^\pm := \frac{1}{\sqrt{2}}(\pm p, \pm q, 1), \quad \boldsymbol{\eta}^\pm := \frac{1}{\sqrt{2}}(\mp q, \pm p, 1).$$

*For any matrix $\boldsymbol{A} = (a_{ij})_{1 \le i,j \le 3} \in \mathbb{M}^{3\times3}_{sym}$, we have the decomposition*

$$\begin{aligned} |\boldsymbol{A}|^2 \;=\; & |\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2 + |\boldsymbol{A}\boldsymbol{\xi}^- \cdot \boldsymbol{\xi}^-|^2 + |\boldsymbol{A}\boldsymbol{\eta}^+ \cdot \boldsymbol{\eta}^+|^2 + |\boldsymbol{A}\boldsymbol{\eta}^- \cdot \boldsymbol{\eta}^-|^2 - \frac{1}{2}(\mathrm{tr}\boldsymbol{A})^2 \\ & + \frac{1}{2}|q^2 a_{11} + p^2 a_{22} - 2pq a_{12}|^2 + \frac{1}{2}|p^2 a_{11} + q^2 a_{22} + 2pq a_{12}|^2 \\ & + 2|(p^2 - q^2) a_{12} + pq(a_{22} - a_{11})|^2 + \frac{1}{2}(a_{33}^2 + (a_{11} + a_{22})^2). \end{aligned}$$

*Proof.* Let us define $\boldsymbol{\xi}_0 := \boldsymbol{\xi}^+ \wedge \boldsymbol{\xi}^- = (q, -p, 0)$ so that $\{\boldsymbol{\xi}^+, \boldsymbol{\xi}^-, \boldsymbol{\xi}_0\}$ is an orthonormal basis of $\mathbb{R}^3$. Then the family

$$\{\boldsymbol{\xi}^+ \otimes \boldsymbol{\xi}^+, \boldsymbol{\xi}^- \otimes \boldsymbol{\xi}^-, \boldsymbol{\xi}_0 \otimes \boldsymbol{\xi}_0, \sqrt{2}(\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}_0), \sqrt{2}(\boldsymbol{\xi}^- \odot \boldsymbol{\xi}_0), \sqrt{2}(\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}^-)\}$$

defines an orthonormal basis of the set $\mathbb{M}^{3\times3}_{sym}$, and Pythagoras Theorem ensures that

$$\begin{aligned} |\boldsymbol{A}|^2 &= |\boldsymbol{A} : (\boldsymbol{\xi}^+ \otimes \boldsymbol{\xi}^+)|^2 + |\boldsymbol{A} : (\boldsymbol{\xi}^- \otimes \boldsymbol{\xi}^-)|^2 + |\boldsymbol{A} : (\boldsymbol{\xi}_0 \otimes \boldsymbol{\xi}_0)|^2 \\ &\qquad + 2|\boldsymbol{A} : (\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}_0)|^2 + 2|\boldsymbol{A} : (\boldsymbol{\xi}^- \odot \boldsymbol{\xi}_0)|^2 + 2|\boldsymbol{A} : (\boldsymbol{\xi}^+ \odot \boldsymbol{\xi}^-)|^2 \\ &= |\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^+|^2 + |\boldsymbol{A}\boldsymbol{\xi}^- \cdot \boldsymbol{\xi}^-|^2 + |\boldsymbol{A}\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}_0|^2 \\ &\qquad + 2|\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}_0|^2 + 2|\boldsymbol{A}\boldsymbol{\xi}^- \cdot \boldsymbol{\xi}_0|^2 + 2|\boldsymbol{A}\boldsymbol{\xi}^+ \cdot \boldsymbol{\xi}^-|^2. \end{aligned}$$

The conclusion follows from a straightfoward computation of each term. □

We now prove a partial bulk energy lower bound.

**Lemma 6.7.** *Assume that $\lambda_b \geq \mu_b$. Then $(1-\theta)u_3 = 0$ $\mathcal{L}^2$-a.e. in $\omega$, and*

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) \geq \frac{\mu_b}{2}\liminf_{\varepsilon\to 0}\int_{(\omega\setminus\Delta_\varepsilon)\times(0,1)}\left|u_\alpha(x)+\int_0^{x_3}\partial_\alpha(u_\varepsilon)_3(x',s)\,\mathrm{d}s\right|^2\,\mathrm{d}x. \tag{6.11}$$

*If in addition the sequences $(\partial_\alpha(u_\varepsilon)_3)_{\varepsilon>0}$ are bounded in $L^2(\Omega_f)$, then*

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) \geq \frac{\mu_b}{2}\int_\omega (1-\theta)|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x'.$$

*Proof.* Let us denote by

$$\boldsymbol{A}_\varepsilon := \begin{pmatrix} \varepsilon e_{11}(\boldsymbol{u}_\varepsilon) & \varepsilon e_{12}(\boldsymbol{u}_\varepsilon) & e_{13}(\boldsymbol{u}_\varepsilon) \\ \varepsilon e_{12}(\boldsymbol{u}_\varepsilon) & \varepsilon e_{22}(\boldsymbol{u}_\varepsilon) & e_{23}(\boldsymbol{u}_\varepsilon) \\ e_{13}(\boldsymbol{u}_\varepsilon) & e_{23}(\boldsymbol{u}_\varepsilon) & \varepsilon^{-1}e_{33}(\boldsymbol{u}_\varepsilon)\end{pmatrix}.$$

the scaled strain so that

$$J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) = \frac{\lambda_b}{2}\int_{\Omega_b}\mathrm{tr}(\boldsymbol{A}_\varepsilon)^2\,\mathrm{d}x + \mu_b\int_{\Omega_b}|\boldsymbol{A}_\varepsilon|^2\,\mathrm{d}x.$$

According to Lemma 6.6 with the angle $\vartheta = 0$, we get that

$$\begin{aligned} J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b) \geq{}& \frac{\lambda_b-\mu_b}{2}\int_{\Omega_b}\mathrm{tr}(\boldsymbol{A}_\varepsilon)^2\,\mathrm{d}x \\ &+ \mu_b\int_{\Omega_b}\left[|\boldsymbol{A}_\varepsilon\xi^+\cdot\xi^+|^2+|\boldsymbol{A}_\varepsilon\xi^-\cdot\xi^-|^2+|\boldsymbol{A}_\varepsilon\eta^+\cdot\eta^+|^2+|\boldsymbol{A}_\varepsilon\eta^-\cdot\eta^-|^2\right]\mathrm{d}x \\ &\geq \mu_b\int_{\Omega_b}\left[|\boldsymbol{A}_\varepsilon\xi^+\cdot\xi^+|^2+|\boldsymbol{A}_\varepsilon\xi^-\cdot\xi^-|^2+|\boldsymbol{A}_\varepsilon\eta^+\cdot\eta^+|^2+|\boldsymbol{A}_\varepsilon\eta^-\cdot\eta^-|^2\right]\mathrm{d}x, \end{aligned} \tag{6.12}$$

since $\lambda_b \geq \mu_b$. It remains to compute each of the four terms in the right hand side of the previous expression. Let us start with the first term. Changing variable $x = (y',0)+s\boldsymbol{\xi}^+_\varepsilon$ (with $\mathrm{d}x = (\sqrt{2}\varepsilon)^{-1}\,\mathrm{d}y'\,\mathrm{d}s$), and using Fubini's Theorem, we get that

$$\begin{aligned} \int_{\Omega_b}|\boldsymbol{A}_\varepsilon\boldsymbol{\xi}^+\cdot\boldsymbol{\xi}^+|^2\,\mathrm{d}x &\geq \varepsilon^2\int_{(\omega_\varepsilon\setminus\Delta_\varepsilon)\times(-1,0)}|\nabla\boldsymbol{u}_\varepsilon\boldsymbol{\xi}^+_\varepsilon\cdot\boldsymbol{\xi}^+_\varepsilon|^2\,\mathrm{d}x \\ &\geq \varepsilon^2\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\fint_{-\sqrt{2}\varepsilon}^0|\nabla\boldsymbol{u}_\varepsilon((y',0)+s\boldsymbol{\xi}^+_\varepsilon)\boldsymbol{\xi}^+_\varepsilon\cdot\boldsymbol{\xi}^+_\varepsilon|^2\,\mathrm{d}s\,\mathrm{d}y'. \end{aligned}$$

According to Lemma 6.5, since $(\boldsymbol{u}_\varepsilon)^{y'}_{\boldsymbol{\xi}^+_\varepsilon} \in H^1(-\sqrt{2}\varepsilon,\sqrt{2}\varepsilon)$ and $(\boldsymbol{u}_\varepsilon)^{y'}_{\xi^\pm_\varepsilon}(-\sqrt{2}\varepsilon) = 0$ for $\mathcal{L}^2$-a.e. $y' \in \omega\setminus\Delta_\varepsilon$, we get that

$$\begin{aligned} \int_{\Omega_b}|\boldsymbol{A}_\varepsilon\boldsymbol{\xi}^+\cdot\boldsymbol{\xi}^+|^2\,\mathrm{d}x &\geq \varepsilon^2\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\fint_{-\sqrt{2}\varepsilon}^0\left|\frac{d}{ds}[\boldsymbol{u}_\varepsilon((y',0)+s\boldsymbol{\xi}^+_\varepsilon)\cdot\boldsymbol{\xi}^+_\varepsilon]\right|^2\mathrm{d}s\,\mathrm{d}y' \\ &\geq \varepsilon^2\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\left|\fint_{-\sqrt{2}\varepsilon}^0\frac{d}{ds}[\boldsymbol{u}_\varepsilon((y',0)+s\boldsymbol{\xi}^+_\varepsilon)\cdot\boldsymbol{\xi}^+_\varepsilon]\,\mathrm{d}s\right|^2\mathrm{d}y' \\ &= \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\left|(u_\varepsilon)^-_1(y',0)+\frac{1}{\varepsilon}(u_\varepsilon)^-_3(y',0)\right|^2\mathrm{d}y', \end{aligned}$$

where $\boldsymbol{u}_\varepsilon^-(\cdot,0)$ denotes the lower trace of $\boldsymbol{u}_\varepsilon$ on $\omega\times\{0\}$. Using again Lemma 6.5, the function $(\boldsymbol{u}_\varepsilon)^{y'}_{\boldsymbol{\xi}^+\varepsilon}\in H^1(-\sqrt{2}\varepsilon,\sqrt{2}\varepsilon)$ does not jump at $t=0$. Thus according to [1, Theorem 4.5 (iv)], it follows that

$$(u_\varepsilon)_1^- + \varepsilon^{-1}(u_\varepsilon)_3^- = (u_\varepsilon)_1^+ + \varepsilon^{-1}(u_\varepsilon)_3^+ \quad \mathcal{H}^2\text{-a.e. on } \omega\times\{0\},$$

and therefore,

$$\int_{\Omega_b}|\boldsymbol{A}_\varepsilon\boldsymbol{\xi}^+\cdot\boldsymbol{\xi}^+|^2\,\mathrm{d}x \geq \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\left|(u_\varepsilon)_1^+(y',0)-\frac{1}{\varepsilon}(u_\varepsilon)_3^+(y',0)\right|^2\,\mathrm{d}y'. \tag{6.13}$$

Analogously, we can show that

$$\int_{\Omega_b}|\boldsymbol{A}_\varepsilon\boldsymbol{\xi}^-\cdot\boldsymbol{\xi}^-|^2\,\mathrm{d}x \geq \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\left|(u_\varepsilon)_1^+(y',0)-\frac{1}{\varepsilon}(u_\varepsilon)_3^+(y',0)\right|^2\,\mathrm{d}y', \tag{6.14}$$

$$\int_{\Omega_b}|\boldsymbol{A}_\varepsilon\boldsymbol{\eta}^+\cdot\boldsymbol{\eta}^+|^2\,\mathrm{d}x \geq \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\left|(u_\varepsilon)_2^+(y',0)+\frac{1}{\varepsilon}(u_\varepsilon)_3^+(y',0)\right|^2\,\mathrm{d}y', \tag{6.15}$$

$$\int_{\Omega_b}|\boldsymbol{A}_\varepsilon\boldsymbol{\eta}^-\cdot\boldsymbol{\eta}^-|^2\,\mathrm{d}x \geq \frac{1}{4}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}\left|(u_\varepsilon)_2^+(y',0)-\frac{1}{\varepsilon}(u_\varepsilon)_3^+(y',0)\right|^2\,\mathrm{d}y'. \tag{6.16}$$

Summing up (6.13), (6.14), (6.15), (6.16) and using (6.12) leads to

$$J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b)\geq\frac{\mu_b}{2}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}(u_\varepsilon)_\alpha^+(y',0)(u_\varepsilon)_\alpha^+(y',0)\,\mathrm{d}y' + \frac{\mu_b}{\varepsilon^2}\int_{\omega_\varepsilon\setminus\Delta_\varepsilon}|(u_\varepsilon)_3^+(y',0)|^2\,\mathrm{d}y'. \tag{6.17}$$

Since $P_\varepsilon\subset\Delta_\varepsilon$, Lemma 6.5 together with the fundamental Theorem of calculus yields,

$$\int_{(\omega\setminus\Delta_\varepsilon)\times(0,1)}\left|(u_\varepsilon)_3(x',x_3)-(u_\varepsilon)_3^+(x',0)\right|^2\,\mathrm{d}x\leq 4\int_{\Omega_f}|e_{33}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x\leq C\varepsilon^4. \tag{6.18}$$

In particular, (6.17), (6.18) and the energy bound (6.5) ensure that

$$\int_{(\omega_\varepsilon\setminus\Delta_\varepsilon)\times(0,1)}|(u_\varepsilon)_3|^2\,\mathrm{d}x\leq C\varepsilon^2,$$

which implies, letting $\varepsilon\to 0$, that $(1-\theta)u_3=0$ $\mathcal{L}^2$-a.e. in $\omega$. Therefore Lemma 6.4 shows that $\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3\xrightharpoonup{*}0$ weakly* in $L^2(\omega;H^{-1}(0,1))$. In addition, since $P_\varepsilon\subset\Delta_\varepsilon$, we can use (6.9) and the fact that $(u_\varepsilon)_\alpha\to u_\alpha$ strongly in $L^2(\Omega_f)$, to obtain (6.11).

Assume now that the sequences $(\partial_\alpha(u_\varepsilon)_3)_{\varepsilon>0}$ are bounded in $L^2(\Omega_f)$. Then the convergence of the planar gradient improves to $\chi_{\omega\setminus\Delta_\varepsilon}\partial_\alpha(u_\varepsilon)_3\rightharpoonup 0$ weakly in $L^2(\Omega_f)$, and thus (6.11) gives

$$\liminf_{\varepsilon\to 0}J_\varepsilon(\boldsymbol{u}_\varepsilon,\Omega_b)\geq\frac{\mu_b}{2}\int_{\Omega_f}(1-\theta)|\boldsymbol{u}|^2\,\mathrm{d}x=\frac{\mu_b}{2}\int_\omega(1-\theta)|\bar{\boldsymbol{u}}|^2\,\mathrm{d}x',$$

since $\theta$ is independent of $x_3$. $\square$

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