Control of higher order PDEs: Korteweg-de Vries and Kuramoto-Sivashinsky equations

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The talk plan

1. Introduction
2. The Korteweg-de Vries equation
3. The Kuramoto-Sivashinsky equation
4. Open Problems

Introduction

We consider two nonlinear one-dimensional PDEs.

- The Korteweg-de Vries equation models waves propagation in shallow water surfaces

$$\text{(KdV)} \quad u_t + u_x + u_{xxx} + uu_x = 0.$$

- The Kuramoto-Sivashinsky equation models front propagation in reaction-diffusion systems

$$\text{(KS)} \quad y_t + y_{xxxx} + \gamma y_{xx} + yy_x = 0.$$

Why presenting them together if they are in fact very different?

KdV is a third order dispersive equation
KS is a fourth order parabolic equation

Introduction

From a control point of view, a common characteristic arises:

When studying these equations on a bounded interval $[0, L]$, two boundary conditions have to be imposed at the same point ($x = L$)

$$\text{(KdV)} \quad u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = CONTROL$$

$$\text{(KS)} \quad y(t,0) = 0, \quad y_x(t,0) = 0, \quad y(t,L) = 0, \quad y_x(t,L) = CONTROL$$

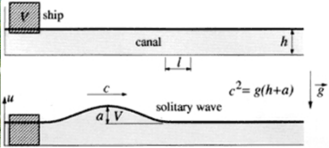
We obtain control systems where we control one boundary condition but not all the boundary data at one endpoint of the interval.

That never happens for the classical Wave or Heat equations

Other important fact: For both equations, control properties depend on some parameters!

Control of the Korteweg-de Vries equation

The Korteweg-de Vries equation (1895)



Wave amplitude: $u = u(t,x), x \in [0,L], t \in [0,T]$

(KdV)
$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = \kappa(t), \\ u(0,x) = u_0(x), \end{array}}$$

State $u(t,\cdot) : [0,L] \to \mathbb{R}$ and control $\kappa(t) \in \mathbb{R}$.

Well-posedness [Rosier 97]:

$$\begin{array}{l} u_0 \in L^2(0,L) \\ \kappa \in L^2(0,T) \end{array} \implies \left\{ \begin{array}{l} \exists! \text{ solution} \\ u \in C([0,T], L^2(0,L)) \end{array} \right.$$

Asymptotic behavior with no control

We are interested in the long-time behavior of the energy

$$E(t) = \int_0^L |u(t,x)|^2 \, dx.$$

Let us start considering the linear equation

$$\text{(LKdV)} \quad \boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0 \end{array}}$$

By performing integration by parts in the equation

$$\int_0^L (u_t + u_x + u_{xxx}) u \, dx = 0$$

we get

$$\frac{d}{dt} \int_0^L |u(t,x)|^2 \, dx = -|u_x(t,0)|^2 \leq 0.$$

The energy is non-increasing, but is it strictly decreasing?

Solutions with constant energy / Stabilization

The energy is not decreasing. In fact there are solutions with constant energy!

For instance, if $L = 2\pi$ and

$$u_0(x) = 1 - \cos(x),$$

the solution of the linear KdV $u_t + u_x + u_{xxx} = 0$ is stationary

$$u(t,x) = 1 - \cos(x),$$

which satisfies $u_x(t,0) = 0$ for any $t \geq 0$ and then

$$\boxed{\dot{E}(t) = \frac{d}{dt}\int_0^L |u(t,x)|^2 dx = 0}$$

Question: Can we add a control to exponentially stabilize the origin?
[Chu-Coron-Shang 15] shows KdV is locally stable in $L = 2\pi$.
[Doronin-Natali 14] shows KdV has large amplitude solutions with constant energy.

Critical domains and Controllability Break

For the linear KdV equation there exist constant energy solutions if and only if

$$L \in \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+k\ell+\ell^2}{3}};\ k,\ell \in \mathbb{N}^* \right\}.$$

This phenomenon is linked to the lack of controllability of the linear KdV.

Take a look at the linear control system

$$\text{(LKdV)} \quad \boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0 \\ u(t,0) = u(t,L) = 0, \quad u_x(t,L) = \kappa(t), \\ u(0,\cdot) = 0 \end{array}}$$

Definition (Exactly controllable)

This linear system (LKdV) is said to be exactly controllable in time $T > 0$ if $\forall u_T \in L^2(0,L)$, $\exists\ \kappa \in L^2(0,T)$ such that

$$u(T,\cdot) = u_T(\cdot)$$

Critical domains and Controllability Break

- (Definition) Linear KdV is controllable $\Leftrightarrow$ the following map is onto

$$B : \kappa \in L^2(0,T) \mapsto u(T,\cdot) \in L^2(0,L)\,.$$

- (Duality) The map B is onto $\Leftrightarrow$ the following inequality holds

$$\text{(Obs)} \qquad \|B^*(\phi_T)\|_{L^2(0,T)} \geq C\|\phi_T\|_{L^2(0,L)}$$

  (Remark that (Obs) implies the operator B^* is one-to-one)

- (Compute of B^*) The map B is onto $\Leftrightarrow$ the following inequality holds

$$\text{(Obs)} \qquad \|\underbrace{\phi_x(\cdot,L)}_{B^*(\phi_T)}\|_{L^2(0,T)} \geq C\|\phi_T\|_{L^2(0,L)}$$

  where $\phi = \phi(t,x)$ satisfies,

$$\text{(Adj)} \quad \begin{cases} \phi_t + \phi_x + \phi_{xxx} = 0, \\ \phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0, \\ \phi(T,\cdot) = \phi_T. \end{cases}$$

Critical domains and Controllability Break

Theorem (Rosier 97)

- *The linear KdV system is exactly controllable* $\Leftrightarrow L \notin \mathcal{N}$.
- *If* $L \notin \mathcal{N}$, *the nonlinear system (KdV) is locally exactly controllable.*

Theorem (Coron-Crépeau 04, C 07, C-Crépeau 09, C 14)

$\forall\ L \in \mathcal{N}$, *there exists* $T_L \geq 0$ *such that (KdV) is locally exactly controllable in* $L^2(0,L)$ *if* $T \geq T_L$.

Other control results when other boundary controls are considered: [Rosier 04], [Chapouly 09], [Glass-Guerrero 09,10]

Null controllable from the left and exactly controllable from the right!

Open Problem

- *Is the minimal control time* T_L *really necessary?*
- *Local stability of the nonlinear system in critical lengths. See the paper [Chu-Coron-Shang, 2015] concerning the case* $L = 2\pi k$

Back to Stabilization

We will design some feedback control laws in order to get

$$E(t) \leq Ce^{-\omega t} E(0), \quad \forall t \geq 0.$$

Full state boundary feedback control from the right:

$$\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \; u(0,\cdot) = u_0, \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = F_\omega(u), \end{array}$$

Output state boundary feedback control from the left:

$$\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \; u(0,\cdot) = u_0, \\ u(t,0) = K_\omega(u), \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}$$

Saturating internal feedback control:

$$\begin{array}{l} u_t + u_x + u_{xxx} + uu_x + sat(au) = 0, \; u(0,\cdot) = u_0, \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}$$

Boundary stabilization from the right

Gramian-based approach to build a feedback control as in [Urquiza, 05].

Theorem (C-Crépeau 09)

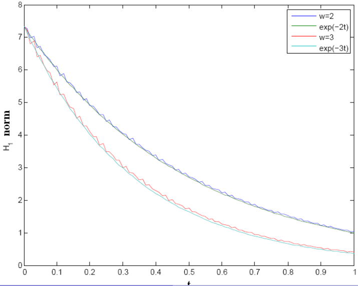
Let $\omega > 0$ and $L \notin \mathcal{N}$ (critical set). The solutions of

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0,\ u(0,\cdot) = u_0, \\ u(t,0) = u(t,L) = 0,\ u_x(t,L) = F_\omega(u(t)), \end{array}}$$

satisfy $\|u(t,\cdot)\|_{H_0^1} \leq Ce^{-2\omega t}\|u_0\|_{H_0^1}$ *with* "$F_\omega = -B^*(\int_0^\infty e^{-tA}BB^*e^{-tA^*}dt)^{-1}$"

Simulation:

Time-evolution of the norm $\|u\|_{H1}$ compared with $e^{-\omega t}\|u_0\|_{H^1}$ for $\omega = 2$ and $\omega = 3$.



Boundary stabilization from the left

Backstepping method (by Krstic et al) to build an output feedback control.

Theorem (C-Coron 13, Marx-C 15)

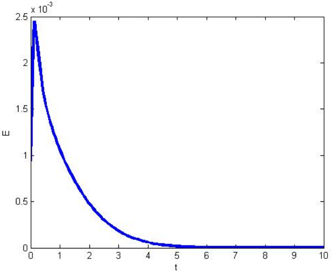
Let $\omega > 0$ *and* $L > 0$. *The solutions of*

$$\begin{aligned} &u_t + u_x + u_{xxx} + uu_x = 0, \ u(0,\cdot) = u_0, \\ &u(t,0) = K_\omega(u_{xx}(t,L)), \quad u(t,L) = 0, \ u_x(t,L) = 0, \end{aligned}$$

satisfy $\|u(t,\cdot)\|_{L^2} \leq Ce^{-\omega t}\|u_0\|_{H^3}$ *for* u_0 *small enough.*

Simulation:

Time evolution of the L^2-energy with $\omega = 1$, $u_0(x) = 0.01(1 - \cos(x))$ on the interval $[0, 2\pi]$



Internal stabilization – Saturating feedback law

Case without saturation but with localized damping by [Perla-Vasconcellos-Zuazua 02], [Pazoto 05], [Rosier-Zhang 06]

Theorem (Marx-C-Prieur-Andrieu, 15)

There exist a positive value ω and a nondecreasing continuous function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that the solutions of

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x + sat(au) = 0,\ \ u(0,\cdot) = u_0, \\ u(t,0) = 0, \quad u(t,L) = 0,\ \ u_x(t,L) = 0, \end{array}}$$

satisfy $\quad \|u(t,\cdot)\|_{L^2} \leq e^{-\omega t}\beta(\|u_0\|_{L^2})$.

Here, we use a nonlocal saturation based on the L^2-norm in space as in [Slemrod 89], [Seidman-Li 01], [Lasiecka-Seidman 03]:

$$sat(s) = \begin{cases} s, & \text{if } \|s\|_{L^2(0,L)} \leq s_0 \\ (ss_0)/\|s\|_{L^2(0,L)}, & \text{if } \|s\|_{L^2(0,L)} > s_0 \end{cases}$$

Other Results - The KdV equation with other BC

Theorem (C-Rivas-Zhang, 13)

Let $T, L > 0$ *given. The system*

$$\begin{aligned} & y_t + y_x + y_{xxx} + yy_x = 0, \\ & y(t,0) = 0,\ y_x(t,L) = h_1(t),\ y_{xx}(t,L) = h_2(t), \end{aligned}$$

is locally exactly controllable.

Well-posedness results in [Colin-Ghidaglia, 01], [Kramer-Rivas-Zhang, 13]

Null controllability from the left boundary condition in [Guilleron, 14]

Critical lengths appear when only one boundary control is considered!

Open Problem

Control of the nonlinear system in critical cases with one control only

Other Results - A KdV-KdV system

Control of a KdV-KdV system with two control inputs.

Multiplier technique gives results for some L.

Theorem (C-Pazoto, 2011)

Let $T > 0$ and $L > 0$ small enough. The system

$$\boxed{\begin{array}{l} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x = 0, \\ c v_t + r v_x + v v_x + b a_3 u_{xxx} + v_{xxx} + b a_1 (uv)_x = 0, \\ u(t,0) = 0,\ u(t,L) = 0,\ u_x(t,L) = h_1(t), \\ v(t,0) = 0,\ v(t,L) = 0,\ v_x(t,L) = h_2(t), \end{array}}$$

is locally exactly controllable.

Previous result by [Micu-Ortega-Pazoto, 09] with 4 controls.

With 2 controls and bigger $L > 0$, they should appear critical lengths!

Other Results - A Linear KdV-Schrödinger system

Carleman estimates approach and duality.

Theorem (Araruna-C-Mercado-Santos, under review)

Let $T > 0$, $L > 0$. *Let suppose that*

$$|Im(a_3)| \geq \delta, \text{ in } \omega, \text{ for some } \delta > 0.$$

$\forall (w_0, y_0) \in \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)$, $\exists\, h\mathbf{1}_\omega \in L^2(0,T; \boldsymbol{H}^{-1}(0,1))$ *such that the unique solution* $(w,y) \in C([0,T], \boldsymbol{H}^{-1}(0,1) \times L^2(0,1))$ *of*

$$\begin{cases} iw_t + w_{xx} = a_1 w + a_2 y + h\mathbf{1}_\omega & \text{in} \quad Q, \\ y_t + y_{xxx} + (My)_x = Re(a_3 w) + a_4 y & \text{in} \quad Q, \\ w(0,t) = w(1,t) = 0 & \text{in} \quad (0,T), \\ y(0,t) = y(1,t) = y_x(1,t) = 0 & \text{in} \quad (0,T), \\ w(x,0) = w_0(x), \quad y(x,0) = y_0(x) & \text{in} \quad (0,1), \end{cases}$$

satisfies

$$w(T,\cdot) = 0, \quad y(T,\cdot) = 0.$$

Related to the internal control of KdV [Capistrano-Filho, Pazoto, Rosier, 2015]

Control of the Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation

Let $\gamma > 0$.

$$
\begin{aligned}
y_t + y_{xxxx} + \gamma y_{xx} + y y_x &= 0, \\
y(t,0) = h_1(t), \quad y(t,1) &= 0, \\
y_x(t,0) = h_2(t), \quad y_x(t,1) &= 0, \\
y(0,x) &= y_0(x).
\end{aligned}
$$

State: $y(t,.) : [0,1] \to \mathbb{R}$

Controls: $h_1(t), h_2(t) \in \mathbb{R}$

Well-posedness:

$$
\begin{array}{l} y_0 \in H^{-2}(0,1) \\ h_1, h_2 \in L^2(0,T) \end{array} \Longrightarrow \left\{ \begin{array}{l} \exists! \text{ solution} \\ y \in C([0,T], H^{-2}(0,1)) \end{array} \right.
$$

No control situation

Question: What happens if $h_1 = h_2 = 0$?

Let us consider the linear equation (drop the term yy_x).

$$A : w \in D(A) \subset L^2(0,1) \longmapsto -w'''' - \gamma w'' \in L^2(0,1),$$
$$D(A) := H^4(0,1) \cap H_0^2(0,1).$$

A is self-adjoint with compact resolvent. The spectrum of A is a discrete subset $\{\sigma_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}$ satisfying $\lim_{k\to\infty} \sigma_k = -\infty$
The eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$ form a basis of $L^2(0,1)$.

$$\begin{cases} -\gamma\phi_k'' - \phi_k'''' = \sigma_k\phi_k, \\ \phi_k(0) = \phi_k(1) = \phi_k'(0) = \phi_k'(1) = 0. \end{cases}$$

Unstable for some γ – Stabilization

The operator A has some positive eigenvalues IFF $\gamma > 4\pi^2$.

- If $\gamma < 4\pi^2$, then all the eigenvalues are negative and therefore the linear system is asymptotically stable. The nonlinear KS equation is also asymptotically stable [Liu-Krstic, 2001].

- We are interested here in the unstable case $\gamma > 4\pi^2$.

Question: Which controls do we use to stabilize the system?

Null controllability - Linear KS equation

Definition (Null controllability)

Let $T>0$. The linear KS equation is null-controllable if for all $y_0 \in H^{-2}(0,1)$, there exist controls $h_1, h_2 \in L^2(0,T)$ *driving the solution from* y_0 *to* 0.

Let $z_T \in H_0^2(0,1)$. Let z be the solution of the adjoint equation

$$\begin{cases} -z_t + z_{xxxx} + \gamma z_{xx} = 0, \\ z(t,0) = 0, \quad z(t,1) = 0, \\ z_x(t,0) = 0, \quad z_x(t,1) = 0, \quad z(T,\cdot) = z_T. \end{cases}$$

We get

$$\langle y_0, z(0)\rangle_{-2,2} = \langle y(T), z_T\rangle_{-2,2} + \int_0^T z_{xx}(t,0) h_2(t) - \int_0^T z_{xxx}(t,0) h_1(t)$$

By duality, the linear KS equation is null-controllable iff we prove

$$\forall z_T,\ \|z(0,\cdot)\|_{H_0^2(0,1)} \leq C \left\{ \|z_{xx}(\cdot,0)\|_{L^2(0,T)} + \|z_{xxx}(\cdot,0)\|_{L^2(0,T)} \right\}$$

where z is the solution of the adjoint equation. IT HOLDS!

Null controllability - Linear KS equation

What happens with a single control?

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y(t,0) = 0, \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

The inequality to prove now is

$$\forall z_T, \ \|z(0,\cdot)\|_{H_0^2(0,1)} \leq C \left\{ \|z_{xx}(\cdot,0)\|_{L^2(0,T)} \right\}$$

Counterexample

If $\gamma = 20\pi^2$, we find out $\quad \phi = -2\sin(2\pi x) + \sin(4\pi x), \quad \sigma = 64\pi^4.$
This eigenfunction satisfies $\phi''(0) = 0$.

Theorem (C, 2010)

The linear system is null controllable with one control h_2 if and only if

$$\gamma \notin \{\pi^2(k^2 + l^2);\ k, l \in \mathbb{N}, 1 \leq k < l,\ (k+l) \text{ is even}\}$$

Null controllability - Linear KS equation

What happens with a single control?

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = 0, \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

The inequality to prove now is

$$\forall z_T, \ \|z(0,\cdot)\|_{H_0^2(0,1)} \leq C \left\{ \|z_{xxx}(\cdot,0)\|_{L^2(0,T)} \right\}$$

Counterexample

If $\gamma = 4\pi^2$*, we find out* $\quad \phi = 1 - \cos(2\pi x), \quad \sigma = 0.$
This eigenfunction satisfies $\phi'''(0) = 0.$

Theorem (C-Guzmán-Mercado, under review)

The linear system is null controllable with one control h_1 *if and only if*

$\lambda \notin \left\{(j^2+k^2)\pi^2 \ / \ (j,k) \in \mathbb{N}^2 \right.$ *with the same parity and* $\left. j < k\right\} \cup \left\{4l^2\pi^2 \ / \ l \in \mathbb{N}\right\}.$

Stabilization - Linear KS equation

Pole shifting method to move a finite number of unstable eigenvalues.

Theorem (C, 2010)

Let γ *be non critical. There exist a feedback operator* K *and two constants* $C, \nu > 0$ *such that for any* $y_0 \in L^2(0,1)$*, the solution of*

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, & \\ y(t,0) = 0, & y(t,1) = 0, \\ y_x(t,0) = K(y(t,\cdot)), & y_x(t,1) = 0, \\ y(0,\cdot) = y_0. & \end{cases}$$

satisfies

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\nu t}\|y_0\|_{L^2(0,L)}$$

Other result by [Coron-Lu, 2015] by using a Fredholm transform (similar to Backstepping by Krstic). Method already applied for the KdV equation by the same authors.

Null controllability - Nonlinear KS equation

Carleman estimates approach to prove observability inequality.

Theorem (C-Mercado, 2011)

The KS equation is locally null controllable, i.e., for any $y_0 \in H^{-2}(0,1)$ *small enough (in norm), there exist controls*

$$h_1 \in L^2_t\left(e^{\frac{2k_1}{T-t}}(T-t)\right), \; h_2 \in L^2_t\left(e^{\frac{2k_1}{T-t}}(T-t)^3\right)$$

such that the solution of

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} + y y_x = 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, \\ y(0,\cdot) = y_0. \end{cases}$$

satisfies $y(T) = 0$.

Neumann BC - Linear KS equation

$$\begin{cases} y_t + y_{xxxx} + \gamma y_{xx} = 0, \\ y_{xx}(t,0) = h_1(t), \; y_{xx}(t,1) = 0, \\ y_{xxx}(t,0) = h_2(t), \; y_{xxx}(t,1) = 0, \\ y(0,x) = y_0(x) \; . \end{cases}$$

Theorem (C-Guzmán-Mercado, under review)

(a) *If* $h_2 = 0$ *and the control is* h_1*, then you cannot control any state* $y_0 \in L^2(0,1)$ *such that*

$$\int_0^1 y_0(x) \cos\left(\sqrt{\gamma}x\right) dx \neq 0.$$

(b) *If* $h_1 = 0$ *and the control is* h_2*, then you cannot control any state* $y_0 \in L^2(0,1)$ *such that*

$$\int_0^1 y_0(x) \sin\left(\sqrt{\gamma}x\right) dx \neq 0.$$

(c) *With two controls* h_1, h_2*, the system is null-controllable.*

Stabilized Kuramoto-Sivashinsky system

Coupling fourth order and second order equations. Internal control.

$$\text{(KS system)} \quad \begin{cases} u_t + u_{xxxx} + uu_x = v_x + h_1 \mathbf{1}_\omega, \\ v_t - v_{xx} = u_x + h_2 \mathbf{1}_\omega, \end{cases}$$

$$\text{(BC)} \quad \begin{cases} u(0,t) = 0, \quad u(1,t) = 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \\ v(0,t) = 0, \quad v(1,t) = 0, \end{cases}$$

$$\text{(IC)} \quad u(x,0) = u_0(x), \quad v(x,0) = v_0(x).$$

Theorem (C-Mercado-Pazoto, 2015; Carreño-C, under review)

Let $T > 0$ *and* ω *any nonempty open subset of* $(0,1)$. *For any* $(u_0, v_0) \in H^{-2}(0,1) \times H^{-1}(0,1)$ *small enough, we can find controls*

- $h_1 = 0$ *and* $h_2 \in L^2(0,T;L^2(\omega))$ *or*
- $h_2 = 0$ *and* $h_1 \in L^2(0,T;L^2(\omega))$

such that the solution of Stabilized KS system satisfies

$$u(\cdot,T) = v(\cdot,T) = 0.$$

Main Open Problems

- KdV: Minimal time for the controllability in critical lengths
- KdV: Stability of the nonlinear system in critical lengths.
  See [Chu-Coron-Shang, 2015] concerning the case $L = 2\pi k$
- KdV: Control of the nonlinear system in critical cases with Colin-Ghidaglia boundary conditions
- KdV systems: Almost everything!
- KS: Nonlinear case with only one control (h_1 or h_2). Much harder if γ is critical !
- KS: discontinuous main coefficient, singular optimal control, cost of control, degenerated coefficients, ...
- Stabilized KS system: Boundary control with less than 3 controls
- KS and Stabilized KS system: Dimension higher than 1