Control properties of some parabolic-elliptic systems

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What is the plan?

Introduction

Controllability

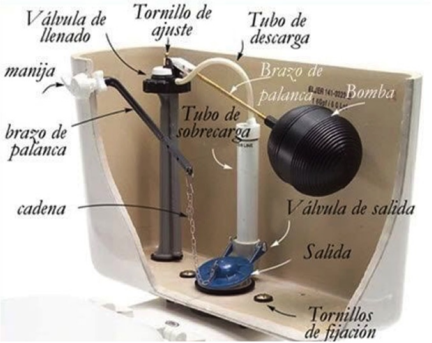
Very simple task: To fill a glass (recipient) with water!

Solution: By means of acting on the system (turning the faucet on/off) we can drive it from an initial state (empty) to a final state (full).

Stabilization

Problem: Very boring task, not robust with respect to phone calls!

Solution: To introduce a device (feedback control) in order to do the task in an automatic way!



Very Basic Example

Let us consider the very simple system

$$\dot{x}(t) = u(t), \quad x(0) = \alpha$$

where for any time the state is $x(t) \in \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$.

Definition

The system is said to be controllable in time $T > 0$ if for any $\alpha, \beta \in \mathbb{R}$ the exists a control u such that the solution satisfies

$$x(T) = \beta$$

Here, given $\alpha, \beta \in \mathbb{R}$, you can pick

$$u(t) = (\beta - \alpha)/T$$

and you get

$$x(T) = \alpha + \int_0^T u(t)dt = \beta.$$

Thus, this system is controllable for any time T.

Very Basic Example

Let us consider the very simple system

$$\dot{x}(t) = u(t), \quad x(0) = \alpha$$

where for any time the state is $x(t) \in \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$.

If $u = 0$, then the solution is $x(t) = \alpha$ for any time t.

Definition

The system is said to be stabilizable if there exists a control $u = K(x)$ such that for any $\alpha \in \mathbb{R}$ we have $x(t) \to 0$ as $t \to \infty$.

Here, we can take

$$u(t) = -x(t)$$

and then the solution is

$$x(t) = \alpha e^{-t} \to 0$$

for any $\alpha \in \mathbb{R}$.

We have stabilized the system making the origin asymptotically stable.

Attention: troubles from time delays

As seen before, the closed-loop system

$$\dot{x}(t) = -x(t), \quad x(0) = \alpha$$

is asymptotically stable with solutions $x(t) = \alpha e^{-t}$. BUT what happens with

$$\dot{x}(t) = -x(t - D)?$$

Answer: it becomes unstable if $D > \pi/2$.

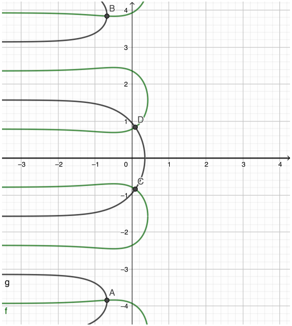
We get solutions like

$$e^{at}\cos(bt) \quad \text{and} \quad e^{at}\sin(bt)$$

with $a > 0$ coming from

$$a = -e^{-aD}\cos(bD)$$

$$b = e^{-aD}\sin(bD)$$



Null controllability of some parabolic-elliptic systems

Parabolic-elliptic system

Let us consider the system

$$\begin{cases} y_t - y_{xx} + q(x)y = z, & (t,x) \in (0,T) \times (0,L), \\ -z_{xx} + \gamma(x)z = y & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = u(t), \quad y(t,L) = 0, & t \in (0,T), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{1}$$

where $T > 0$, $L > 0$, the state is given by the couple (y,z), the space dependent parameter functions q, γ, are in $L^\infty(0,L)$ and the time-dependent function u is a boundary control acting on the parabolic equation.

Related papers with results for control of parabolic-elliptic systems [Chaves, Guerrero, 2015] [Fernandez-Cara, Limaco, Menezes, 2016]

Control Goal

We are interested in studying the null controllability of system (1) by the action of the scalar control u at the boundary.

Definition

The above system is null controllable in time $T > 0$ in space X if for any initial condition $y_0 \in X$, there exists a boundary control u such that the solution to (1) with $y(0, \cdot) = y_0$ satisfies

$$(y(T, \cdot), z(T, \cdot)) = (0, 0)$$

Remark

Due to smoothing effect we do not expect to have a property of type:
For any initial state y_0, for any final state y_T, z_T there exists a control u such that the solution to (1) *with $y(0, \cdot) = y_0$ satisfies*

$$(y(T, \cdot), z(T, \cdot)) = (y_T, z_T)$$

Re-writing the system

Let us set the following operator

$$F_\gamma : g \in L^2(0,L) \longmapsto F_\gamma(g) = z \in H_0^1(0,L), \quad (2)$$

where z is the solution to

$$\begin{cases} -z_{xx} + \gamma(x) z = g, & x \in (0,L), \\ z(0) = 0, z(L) = 0, \end{cases} \quad (3)$$

with $\gamma \in L^\infty(0,L)$. We collect some properties of the operator F_γ in the following lemma

Lemma

Let $g \in L^2(0,L)$ and $\gamma \in L^\infty(0,L)$ such that $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$ for all $x \in [0,L]$. Then, the operator F_γ defined by (2) *is well defined, linear continuous and self-adjoint. Moreover, there exists a positive constant $C = C(\gamma_0, L)$ such that*

$$\|F_\gamma(g)\|_{H_0^1(0,L)} \leq C(\gamma_0, L)\|g\|_{L^2(0,L)}, \quad \forall g \in L^2(0,L),$$

where $C(\gamma_0, L) = \max\left\{\frac{L}{\pi}, \frac{L}{\pi^2 + \gamma_0 L^2}\right\}$.

Re-writing the system

Using that operator we are able to re-write the system (1) in a equivalent way as follows

$$\begin{cases} y_t - y_{xx} + q(x)y = F_\gamma(y), & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = u(t), \quad y(t,L) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{4}$$

which is a heat equation with a non-local term.

Definition

The above system is null controllable in time $T > 0$ in space $H^{-1}(0,L)$ if for any initial condition $y_0 \in H^{-1}(0,L)$, there exists a boundary control $u \in L^2(0,T)$ such that the solution to (1) with $y(0,\cdot) = y_0$ satisfies

$$y(T,\cdot) = 0$$

Well-posedness with no control

Let us consider

$$\begin{cases} y_t - y_{xx} + q(x)y - F_\gamma(y) = g, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0, \quad y(L,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{5}$$

where F_γ is the operator given previously and $g = g(x,t)$.

Proposition

(a) *The underlying spatial operator*

$$A : \phi \in D(A) \subset L^2(0,L) \longmapsto \phi_{xx} - q(x)\phi + F_\gamma(\phi) \ \in L^2(0,L), \tag{6}$$

with domain $D(A) := H^2(0,L) \cap H_0^1(0,L)$ *is self-adjoint and dissipative. Moreover,* A *is a m-dissipative operator.*

(b) *By the Hille-Yosida-Phillips Theorem we conclude that* A *is a generator of a contraction semigroup in* $L^2(0,L)$*. Then, if* $y_0 \in D(A)$ *and* $g \in C^1([0,T];L^2(0,L))$ *, then the solution to* (5) *satisfies*

$$y \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,L))$$

Well-posedness with no control

Using previous results and energy estimations we get well-posedness in a more regular context.

Proposition

(c) *Consider* G *being either* $L^2(0,T;L^2(0,L))$ *or* $L^1(0,T;H_0^1(0,L))$. *Let* $g \in G$ *and* γ, $q \in L^\infty(0,L)$. *If* $y_0 \in H_0^1(0,L)$, *then* (5) *has a unique solution* $y \in C([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L) \cap H_0^1(0,L))$. *Moreover, there exists* $C > 0$ *such that*

$$\|y\|_{L^\infty(0,T;H_0^1(0,L)) \cap L^2(0,T;H^2(0,L) \cap H_0^1(0,L))} \leq C \left(\|g\|_G + \|y_0\|_{H_0^1(0,L)} \right).$$

(d) *In addition, there exists* $C > 0$ *such that the solution* y *to* (5) *satisfies*

$$\|y_x(\cdot,0)\|_{L^2(0,T)} \leq C \left(\|g\|_G + \|y_0\|_{H_0^1(0,L)} \right).$$

Well-posedness with control

Definition

Let $y_0 \in H^{-1}(0,L)$, $g \in L^1(0,T;H^{-1}(0,L))$, $u \in L^2(0,T)$ and γ, $q \in L^\infty(0,L)$. A solution to (5) defined by transposition is a function $y \in L^2(0,T;L^2(0,L))$ such that for any $h \in L^2(0,T;L^2(0,L))$,

$$\int_0^T \int_0^L y(t,x)h(t,x)\mathrm{d}x\mathrm{d}t = \langle y_0, w(0,x)\rangle_{H^{-1}(0,L),H_0^1(0,L)} + \langle g, w\rangle_{L^1(0,T;H^{-1}(0,L)),L^\infty(0,T;H_0^1(0,L))} + \int_0^T u(t)w_x(t,0)\mathrm{d}t, \quad (7)$$

where w is solution to

$$\begin{cases} -w_t - w_{xx} + q(x)w - F_\gamma(w) = h, & (t,x) \in (0,T)\times(0,L), \\ w(t,0) = 0, \quad w(L,t) = 0, & t \in (0,T), \\ w(T,x) = 0, & x \in (0,L). \end{cases} \quad (8)$$

Well-posedness with control

The following proposition establish the existence and uniqueness of the solutions to (5) defined by transposition.

Theorem

Let $y_0 \in H^{-1}(0,L)$, $g \in L^1(0,T;H^{-1}(0,L))$, $u \in L^2(0,T)$ *and* $\gamma, q \in L^\infty(0,L)$. *Then, there is a unique*

$$y \in C([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$$

solution to (5). *Furthermore, there exists* $C > 0$ *such that*

$$\|y\|_{L^\infty(0,T;H^{-1}(0,L))} \leq C\left(\|y_0\|_{H^{-1}(0,L)} + \|g\|_{L^1(0,T;H^{-1}(0,L))} + \|u\|_{L^2(0,T)}\right).$$

IDEA: Fix y_0, g, u and see h as the variable. The right-hand side of (7) defines a linear bounded functional from $L^2(0,T;L^2(0,L))$ to $\mathbb{R}$. The Riesz representation theorem gives the existence and uniqueness of $y \in L^2(0,T;L^2(0,L))$. Energy estimations give the extra regularity $y \in C([0,T];H^{-1}(0,L))$.

Null Controllability Results

Theorem (Hernández-Prieur-C)

Let $T>0$, $L>0$, $\gamma,\ q \in L^\infty(0,L)$. *Then, system* (4) *is null controllable. That is, for all* $y_0 \in H^{-1}(0,L)$, *there exists* $u \in L^2(0,T)$ *such that* $y \in C([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$ *solution to* (4) *satisfies* $y(T,\cdot)=0$.

Corollary

Under same assumptions , system (1) *is null controllable. In other words, for every* $y_0 \in H^{-1}(0,L)$, *there exists* $u \in L^2(0,T)$ *such that the unique solution* (y,z) *belonging to*

$$C([0,T];H^{-1}(0,L)\times H_0^1(0,L)) \bigcap L^2(0,T;L^2(0,L)\times H^2(0,L)\cap H_0^1(0,L))$$

to (1) *satisfies* $(y(T,\cdot),z(T,\cdot))=0$.

Observability Inequality

We get the null controllability property of

$$\begin{cases} y_t - y_{xx} + q(x)y = F_\gamma(y), & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = u(t), \quad y(t,L) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{9}$$

by proving (Carleman estimates) the existence of a positive constant C such that

$$\|w(0,\cdot)\|^2_{H^1_0(0,L)} \leq C \int_0^T w_x^2(t,0)\mathrm{d}t, \tag{10}$$

for every solution $w = w(t,x)$ to the adjoint equation

$$\begin{cases} -w_t - w_{xx} + q(x)w = F_\gamma(w), & (t,x) \in (0,T) \times (0,L), \\ w(t,0) = 0, \quad w(t,L) = 0, & t \in (0,T), \\ w(T,x) = w_T(x), & x \in (0,L), \end{cases} \tag{11}$$

with $w_T \in H^1_0(0,L)$. Inequality (10) is called observability inequality.

Why this inequality implies the null controllability?

Let us consider the functional $J_\varepsilon : H_0^1(0,L) \longrightarrow \mathbb{R}$, given by

$$J_\varepsilon(w_T) = \frac{1}{2}\int_0^T w_x^2(t,0)\mathrm{d}t - \langle y_0(\cdot), w(0,\cdot)\rangle_{H^{-1}(0,L),H_0^1(0,L)} + \varepsilon\|w_T\|_{H_0^1(0,L)} \quad (12)$$

where w is the solution to the adjoint equation with final data w_T.

Lemma

Let $\varepsilon > 0$. The functional J_ε is continuous, strictly convex and coercive. Thus, J_ε has a unique minimum.

Approximate controls

The previously introduced functionals give a kind of controls called approximate controls because they bring the solutions as close to the origin as we want (playing with ε).

Proposition

Let $\varepsilon > 0$, $\hat{w}_T^\varepsilon \in H_0^1(0, L)$ a minimum of J_ε and $\hat{w}^\varepsilon$ be the solution to the adjoint equation with final data $\hat{w}_T^\varepsilon$. Then, $u^\varepsilon = \hat{w}_x^\varepsilon(\cdot, 0) \in L^2(0, T)$ is a control such that the solution y to (9) *satisfies*

$$\|y(T, \cdot)\|_{H^{-1}(0,L)} \leq \varepsilon. \tag{13}$$

Limit argument - Null controls

Proposition

Let $\{u^\varepsilon\} \in L^2(0,T)$ *a sequence of controls given by previous Proposition when* $\varepsilon \to 0$. *Then, there exists* $u \in L^2(0,T)$ *such that* $u^\varepsilon \rightharpoonup u$ *weakly in* $L^2(0,T)$ *when* $\varepsilon \to 0$ *and* u *is a control such that the solution* y *to* (9) *satisfies*

$$y(T,\cdot) = 0$$

The key point is that the sequence $\{u^\varepsilon\}$ is uniformly bounded in $L^2(0,T)$. Indeed, fix $\varepsilon > 0$ and consider $\hat{w}_T^\varepsilon$ given by previous Proposition. Then

$$J_\varepsilon(\hat{w}_T^\varepsilon) \leq J_\varepsilon(0) = 0. \tag{14}$$

This implies that

$$\frac{1}{2}\int_0^T (\hat{w}_x^\varepsilon)^2(t,0)\mathrm{d}t \leq \langle y_0(\cdot), \hat{w}^\varepsilon(0,\cdot)\rangle_{H^{-1}(0,L),H_0^1(0,L)} - \varepsilon\|\hat{w}_T^\varepsilon\|_{H_0^1(0,L)}, \tag{15}$$

and we get $\int_0^T (\hat{w}_x^\varepsilon)^2(t,0)\mathrm{d}t \leq 2\|y_0(\cdot)\|_{H^{-1}(0,L)}\|\hat{w}^\varepsilon(0,\cdot)\|_{H_0^1(0,L)}$.

Limit argument - Null controls

Finally, using

$$\int_0^T (\hat{w}_x^\varepsilon)^2(t,0)\mathrm{d}t \leq 2\|y_0(\cdot)\|_{H^{-1}(0,L)}\|\hat{w}^\varepsilon(0,\cdot)\|_{H_0^1(0,L)}$$

and the observability inequality

$$\|w(0,\cdot)\|_{H_0^1(0,L)}^2 \leq C\int_0^T w_x^2(t,0)\mathrm{d}t,$$

we get

$$\|\hat{w}_x^\varepsilon(\cdot,0)\|_{L^2(0,T)} \leq 2C\|y_0(\cdot)\|_{H^{-1}(0,L)}. \tag{16}$$

which implies that $\|u^\varepsilon\|_{L^2(0,T)} \leq 2C\|y_0\|_{H^{-1}(0,L)}$, for all $\varepsilon > 0$. Thus, $\{u^\varepsilon\}$ is uniformly bounded in $L^2(0,T)$ and therefore weakly convergent to a control u which can be proven to be a null control.

Null control

Theorem (Hernández-Prieur-C)

Let $T > 0$, $L > 0$, $\gamma,\ q \in L^\infty(0,L)$. *Then, system*

$$\begin{cases} y_t - y_{xx} + q(x)y = F_\gamma(y), & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = u(t), \quad y(t,L) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases}$$

is null controllable. That is, for all $y_0 \in H^{-1}(0,L)$, *there exist* $u \in L^2(0,T)$ *and* $y \in C([0,T]; H^{-1}(0,L)) \cap L^2(0,T; L^2(0,L))$ *such that* $y(T,\cdot) = 0$.

Stabilization
of some parabolic-elliptic systems

Parabolic-Elliptic System

The system is

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \lambda u(x,t) = \alpha v(x,t), & x \in (0,L),\ t > 0, \\ -v_{xx}(x,t) + \gamma v(x,t) = \beta u(x,t), & x \in (0,L),\ t > 0, \\ u(0,t) = 0,\ u(L,t) = h(t-D), & t > 0, \\ v(0,t) = 0,\ v(L,t) = 0, & t > 0. \end{cases}$$

When no control is applied ($h = 0$) the system is unstable under the condition

$$\lambda \leq \frac{\beta\alpha}{\frac{\pi^2}{L^2} + \gamma} - \frac{\pi^2}{L^2}$$

Our purpose is to consider these unstable cases and to design feedback laws to exponentially stabilize the system.

We apply the same strategy as in [Prieur-Trélat, 2019] (heat equation).

See also [BreschPietri-Prieur-Trélat, 2018] (finite-dimensional systems) and [Guzmán-Marx-C, 2019] (fourth-order parabolic equation), [Hernández-Prieur-C] (null controllability).

Solutions when no control is applied

The eigenvalues (with $h = 0$) are

$$\sigma_n = \frac{\beta\alpha}{\left(\frac{n\pi}{L}\right)^2 + \gamma} - \lambda - \left(\frac{n\pi}{L}\right)^2$$

and the solutions u, v are

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{\sigma_n t} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$v(x,t) = \frac{2\beta}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \frac{e^{\sigma_n t}}{\left(\frac{n\pi}{L}\right)^2 + \gamma} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

System rewritten as a single PDE

Let us define operator $(-\partial_{xx} + \gamma I_d)^{-1}$ as

$$F_\gamma : L^2(0,L) \longmapsto H^2(0,L) \cap H_0^1(0,L) \subset L^2(0,L)$$

by $F_\gamma(u) = v$ where v is the solution of

$$\begin{cases} -v_{xx} + \gamma v = u, \\ v(0) = v(L) = 0. \end{cases}$$

We obtain that our system can be written as a single PDE

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \lambda u(x,t) = \alpha\beta F_\gamma(u)(x,t), & x \in (0,L),\, t > 0, \\ u(0,t) = 0, \quad u(L,t) = h(t-D) = h_D(t), & t > 0, \\ v(x,t) = \beta F_\gamma(u)(x,t), & x \in (0,L),\, t > 0, \end{cases}$$

Move the control from the boundary to the interior

To get ride of the boundary control, we re-re-write the system for

$$w(t,x) = u(t,x) - \frac{x}{L} h_D(t)$$

and obtain

$$w_t = Aw + a(\cdot) h_D(t) + b(\cdot) h_D'(t)$$

where

$$A := \partial_{xx} + \alpha\beta F_\gamma(\cdot) - \lambda I_d(\cdot)$$

with $D(A) = H^2(0,L) \cap H_0^1(0,L)$ and

$$a(x) = \left(-\lambda \frac{x}{L} + \frac{\alpha\beta}{L} F_\gamma(x) \right) \qquad b(x) = -\frac{x}{L}$$

Remark

Note that A is self-adjoint and with compact inverse.

Rewritting the system by coordinates

Let $\{e_j\}_j \subset L^2(0,L)$ be a Hilbert basis formed by eigenfunctions of A and $\{\lambda_j\}_j$ be the eigenvalues that satisfies:

$$-\infty < \cdots < \lambda_j < \cdots < \lambda_1, \quad \text{with} \quad \lambda_j \to -\infty$$

With this, all the solutions $w(t,\cdot) \in H^2(0,L) \cap H_0^1(0,L)$ look as

$$w(t,x) = \sum_{j=1}^{\infty} w_j(t) \underbrace{e_j(x)}_{\sin(j\pi x/L)}$$

where the coordinates are given by an infinite dimensional ODE system:

$$w_j'(t) = \lambda_j w_j(t) + a_j h_D(t) + b_j h_D'(t) \qquad \forall j \in \mathbb{N}\backslash\{0\}$$

with:

$$a_j = \langle a(\cdot), e_j(\cdot)\rangle_{L^2} = \frac{1}{L}\int_0^L (-\lambda x + \alpha\beta F_\gamma(x))e_j(x)dx$$

$$b_j = \langle b(\cdot), e_j(\cdot)\rangle_{L^2} = -\frac{1}{L}\int_0^L x e_j(x)dx$$

Unstable finite-dimensional part

We define $\nu_D(t) = h_D^{'}(t)$ and consider that there are n eigenvalues positive. We use Π_n the orthogonal projection to $\langle\{e_1, \cdots, e_n\}\rangle$ in $L^2(0, L)$ and call

$$w^n = \Pi_n w = \sum_{j=1}^{n} w_j(t) e_j(\cdot)$$

Using the matrices

$$X_n(t) = \begin{pmatrix} h_D(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix} \quad B_n = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} \quad A_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix}$$

we can construct the next unstable finite dimensional system

$$X_n^{'}(t) = A_n X_n(t) + B_n \nu_D(t)$$

with a delayed control

Artstein transformation

In order to eliminate the delay we use the Artstein transformation

$$Z_n(t) = X_n(t) + \int_{t-D}^{t} e^{(t-s-D)A_n} B_n \nu(s) ds$$

and thus our finite-dimensional system becomes

$$Z_n'(t) = A_n Z_n(t) + e^{-DA_n} B_n \nu(t)$$

To apply a Pole Placement method we need this system to satisfy the Kalman condition and this holds if and only if

$$0 \neq \det(e^{-DA_n} B_n, A_n e^{-DA_n} B_n, \cdots, A_n^n e^{-DA_n} B_n)$$

$$= \det(B_n, A_n B_n, \cdots, A_n^n B_n) = \prod_{j=1}^{n} (a_j + \lambda_j b_j) Vdm(\lambda_1, \cdots, \lambda_n)$$

$$= \prod_{j=1}^{n} \underbrace{(-e_j'(L))}_{\propto \cos(j\pi)} Vdm(\lambda_1, \cdots, \lambda_n)$$

where $Vdm(\lambda_1, \cdots, \lambda_n)$ is the Vandermonde determinant and is never zero because all the eigenvalues are different.

Stabilization with the delay

Due to previous developments, $\forall D \geq 0$, $\exists K_n(D) \in \mathbb{R}^{1\times(n+1)}$ such that

$$\tilde{A}_n(D) = A_n + e^{-DA_n} B_n K_n(D)$$

admits -1 has an eigenvalue of order $n+1$.

Furthermore, there exists a symmetric positive definite matrix $P(D)$ such that:

$$P(D)\tilde{A}_n(D) + \tilde{A}_n(D)P(D) = -I_{n+1}$$

Thus, the function

$$V(Z_n) = \frac{1}{2} Z_n^T P(D) Z_n$$

is a Lyapunov function for the Z_n system. So the feedback control

$$\nu(t) = \begin{cases} 0 & \text{if } t \leq D \\ K_n Z_n & \text{if } t > D \end{cases}$$

stabilizes

$$Z_n'(t) = A_n Z_n(t) + e^{-DA_n} B_n \nu(t)$$

Stabilization with the delay

Going back to the original finite-dimensional system

$$X_n'(t) = A_n X_n(t) + \chi_{(D,\infty)} B_n K_n(D) Z_n(t-D)$$

we get the feedback law (implicitly defined)

$$\nu(t) = \begin{cases} 0 & \text{if } t \leq D \\ \underbrace{K_n(D)X_n(t)}_{\text{State}} + \underbrace{K_n(D)\int_{\max(D,t-D)}^{t} e^{(t-D-s)A_n} B_n \nu(s)ds}_{\text{Control in the past}} & \text{if } t > D \end{cases}$$

The stability of the full infinite-dimensional system is proven using the Lyapunov function

$$V_D(t) = M(D)V(Z_n(t)) + M(D)\int_{(t-D,t)\cap(D,\infty)} V(Z_n(s))ds - \frac{1}{2}\langle w(t), Aw(t)\rangle_{L^2(0,L)}$$

where $M(D)$ is sufficiently large.

Parabolic-Elliptic System

Theorem (C-Morris-Parada)

Let $\gamma > 0$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$ *such that* $\alpha\beta > 0$.
System

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \lambda u(x,t) = \alpha v(x,t), & x \in (0,L),\ t > 0, \\ -v_{xx}(x,t) + \gamma v(x,t) = \beta u(x,t), & x \in (0,L),\ t > 0, \\ u(0,t) = 0,\ u(L,t) = h(t-D), & t > 0, \\ v(0,t) = 0,\ v(L,t) = 0, & t > 0, \end{cases}$$

is exponentially stabilizable, that is there exist a feedback control h, $\mu > 0$ *and* $C > 0$ *such that, for all* $u_0 \in H^1_L(0,L)$,

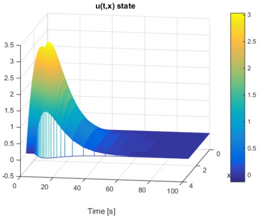
$$|h(t-D)| + \|(u,v)\|_{H^1_0(0,L)\times H^3(0,L)} \le Ce^{-\mu t}\|u_0\|_{H^1_0(0,L)}$$

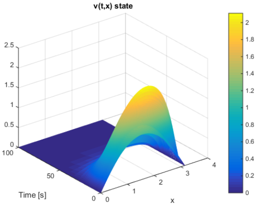
Numerical Simulations

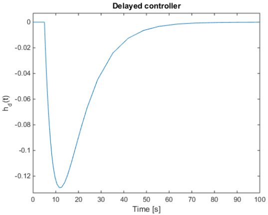
For $\alpha = \beta = 5$, $\gamma = 10$, $\lambda = 1.2720$, $L = \pi$ and $d = 1.5$, there exists only one positive eigenvalue, $\sigma_1 \approx 0.0007$ and thus we have only one unstable eigenmode.

We use a classical *MatLab* pole placement for the finite dimensional unstable system with -0.1 -0.2. The numerical implementation is based on the first 10 eigenmodes.

The initial condition is $u_0(x) = x(L - x)$.







Thank you very much!