

On the control of the improved Boussinesq equation

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January 2016

What is the plan?

- 1 The improved Boussinesq equation
- 2 Boundary control: lack of exact controllability
- 3 Boundary control: approximate controllability
- 4 Moving control: exact controllability

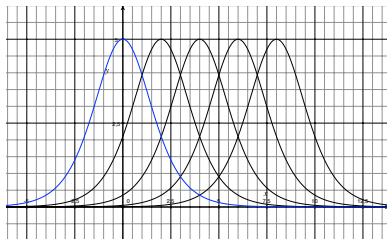
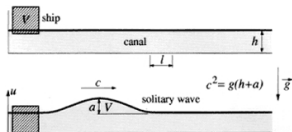
The improved Boussinesq equation – Origin

In [Boussinesq, 1872], it is derived the so called “bad” Boussinesq equation:

$$y_{tt} - y_{xx} - y_{xxxx} = (y^2)_{xx}. \quad (1)$$

This equation describes the flow of shallow water waves with small amplitudes in a flat bottom canal. It is called “bad” because its poor existence and uniqueness properties. There is no local well-posedness results for equation (1). Only solitons-like solutions are known:

$$y(x, t) = \frac{A}{\operatorname{sech}^2(kx - \omega t)}$$



The improved Boussinesq equation – Origin

The “good” Boussinesq equation is

$$y_{tt} - y_{xx} + y_{xxxx} = (y^2)_{xx}.$$

This equation, used to study nonlinear strings, is well-posed thanks to the “right” sign in front of the fourth order operator.

There are control results for the “good” Boussinesq equation:

- Internal controls with periodic boundary conditions by [Zhang, 1998]
- Same result in lower regularity framework by [Cerpa, Rivas, 2016].
- Exact controllability from the boundary by [Lions, 1988] and [Crépeau, 2003].

The improved Boussinesq equation – Origin

In [Makhankov, 1978] it is proved that the “bad” Boussinesq equation can be approached by the following one, called improved Boussinesq equation,

$$y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx}.$$

We have here a well-posed equation [Zhijian, 1998] !

... control properties must be studied !

Boundary control: lack of exact controllability

Boundary control: existence of solutions

Given a time $T > 0$, we consider the linear improved Boussinesq equation posed on a bounded domain $(0, 1)$ with a boundary control $h \in H^2(0, T)$ and initial conditions $y^0, y^1 \in L^2(0, 1)$:

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = 0, y(1, t) = h(t), & t \in (0, T), \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in (0, 1). \end{cases}$$

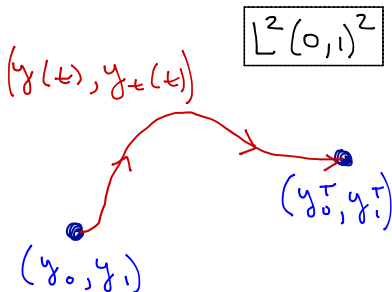
We can prove that the system is well-posed and that the solution satisfies

$$y \in C^1([0, T], L^2(0, 1) \times L^2(0, 1))$$

and in this way each solution can be seen as a continuous trajectory in the state space

$$(y(t), y_t(t)) \in L^2(0, 1) \times L^2(0, 1), \quad \forall t \in [0, T].$$

Boundary control: definition of exact controllability



Definition (Exact controllability)

Given a time $T > 0$, we say that system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = 0, y(1, t) = h(t), & t \in (0, T), \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in (0, 1), \end{cases}$$

is exactly controllable if $\forall (y^0, y^1), \forall (y_T^0, y_T^1), \exists h$ such that y satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y_t(T) = y_T^1.$$

Boundary control: spectral analysis

We can rewrite the homogeneous system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = 0, y(1, t) = 0, & t \in (0, T), \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in (0, 1), \end{cases}$$

as follows,

$$y_{tt} + Ay = 0,$$

where, for $D(A) = H^2 \cap H_0^1(0, 1)$, we define the operator

$$A : D(A) \subset L^2(0, 1) \longrightarrow L^2(0, 1)$$

$$w \longmapsto A(w) = (I - \partial_{xx})^{-1} \partial_{xx} w.$$

Proposition

There exists a basis of $L^2(0, 1)$ formed by eigenfunctions $\{f_k\}_{k \in \mathbb{N}^*}$ of the operator A . Moreover, this family is given by $f_k(x) = \sqrt{2} \sin(k\pi x)$ and the corresponding eigenvalues are

$$\lambda_k = \frac{k^2 \pi^2}{k^2 \pi^2 + 1},$$

for any $k \in \mathbb{N}^*$.

Boundary control: homogeneous solutions

Remark

We can easily remark that the eigenvalues $\lambda_k \in \mathbb{R}^+$ are simple, and $\lim_{k \rightarrow +\infty} \lambda_k = 1$. Thus, the spectrum of A admits a finite point of accumulation.

Having in mind

$$H^s(0, 1) = \left\{ \sum_{k \geq 1} a_k f_k(x) / \sum_{k \geq 1} k^{2s} |a_k|^2 \leq \infty \right\},$$

we obtain, for $s \geq 0$, and $y^0 = \sum_{k \geq 1} a_k f_k \in H_0^s(0, 1)$, $y^1 = \sum_{k \geq 1} b_k f_k \in H_0^s(0, 1)$, that the solution y belongs to $C^1([0, +\infty[, H_0^s(0, 1))$ and

$$y(x, t) = \sum_{k \geq 1} \left(a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) f_k(x).$$

Remark

From the cosine and sine functions, we see that there is *no gain of regularity* for the improved Boussinesq equation. Moreover, from the asymptotic behavior of eigenvalues λ_k , we see the fact that the *amplitude and velocity have the same regularity*.

Boundary control: lack of exact controllability

Let y be the solution of the **direct** equation

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T) \\ y(0, t) = 0, y(1, t) = h(t), & t \in (0, T) \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in (0, 1). \end{cases} \quad (2)$$

Let z be the solution of the **adjoint** problem

$$\begin{cases} z_{tt} - z_{xx} - z_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T) \\ z(0, t) = z(1, t) = 0, & t \in (0, T) \\ z(x, T) = z_T^0(x), z_t(x, T) = z_T^1(x), & x \in (0, 1). \end{cases}$$

Suppose that the system (2) is exactly controllable. Then, there exists a boundary control h such that the solution of (2) with initial data

$$y^0 = \sum_{n \geq 1}^N a_n f_n, \quad y^1 = \sum_{n \geq 1}^N b_n f_n,$$

satisfies $y(T) = y_t(T) = 0$.

We obtain that if $y(T) = y_t(T) = 0$, then for any z_T^0, z_T^1 ,

$$\int_0^1 \left[y^1 \left(z(x, 0) - z_{xx}(x, 0) \right) - y^0 \left(z_t(x, 0) - z_{xxt}(x, 0) \right) \right] dx = \int_0^T \left(h(t) + \ddot{h}(t) \right) z_x(1, t) dt$$

Boundary control: lack of exact controllability

Using that for any z_T^0, z_T^1 ,

$$\int_0^1 \left[y^1 \left(z(x, 0) - z_{xx}(x, 0) \right) - y^0 \left(z_t(x, 0) - z_{xxt}(x, 0) \right) \right] dx = \int_0^T \left(h(t) + \ddot{h}(t) \right) z_x(1, t) dt$$

with appropriate trajectories:

- First $z(x, t) = e^{i\sqrt{\lambda_n}(T-t)} f_n(x)$
- Then $z(x, t) = e^{-i\sqrt{\lambda_n}(T-t)} f_n(x)$

We see that the control h has to satisfy, for any $n \geq 1$, the following equations

$$\begin{aligned} (1 + n^2 \pi^2)(i\sqrt{\lambda_n} a_n + b_n) &= \sqrt{2}(n\pi)(-1)^n \int_0^T (h(t) + \ddot{h}(t)) e^{-i\sqrt{\lambda_n} t} dt, \\ (1 + n^2 \pi^2)(-i\sqrt{\lambda_n} a_n + b_n) &= \sqrt{2}(n\pi)(-1)^n \int_0^T (h(t) + \ddot{h}(t)) e^{i\sqrt{\lambda_n} t} dt. \end{aligned} \tag{3}$$

Boundary control: lack of exact controllability

Let us define the complex function

$$F(z) := \int_0^T (h(t) + \ddot{h}(t)) e^{izzt} dt.$$

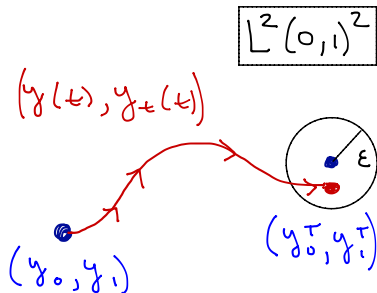
- F is an entire function and it satisfies $F(\pm\sqrt{\lambda_n}) = 0$, for all $n > N$.
- Remember that $\sqrt{\lambda_n} \rightarrow 1$ as $n \rightarrow \infty$
- Consequently, F vanishes on a set with a finite accumulation point.
- Therefore, we conclude that $F \equiv 0$.
- From (3), we easily obtain that $a_n = b_n = 0$ for each $n \geq 1$.
- That means that the trivial state is the only one which can be steered to zero.

Theorem (EC-Crépeau, 2015)

The linear improved Boussinesq equation is not spectrally controllable in $L^2(0, 1)$.

Boundary control: approximate controllability

Boundary control: definition of approximate controllability



Definition (Approximate controllability)

Given a time $T > 0$, we say that system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = 0, y(1, t) = h(t), & t \in (0, T), \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in (0, 1), \end{cases}$$

is approximately controllable if $\forall \varepsilon > 0, \forall (y^0, y^1), \forall (y_T^0, y_T^1), \exists h$ such that y satisfies

$$\|y(T) - y_T^0\|_{L^2(0,1)} \leq \varepsilon \quad \text{and} \quad \|y_t(T) - y_T^1\|_{L^2(0,1)} \leq \varepsilon.$$

Boundary control: approximate controllability

By linearity, it is enough to consider null initial data.

Let y be the solution of the **direct** equation

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = 0, & (x, t) \in (0, 1) \times (0, T) \\ y(0, t) = 0, y(1, t) = h(t), & t \in (0, T) \\ y(x, 0) = 0, y_t(x, 0) = 0, & x \in (0, 1). \end{cases} \quad (4)$$

Let us define the map

$$\Lambda : h \in H^2(0, T) \longmapsto (y(T), y_t(T)) \in L^2(0, 1)^2.$$

Remark

Notice that system (4) is approximately controllable if and only if the range of this linear operator Λ is dense in $L^2(0, 1)^2$.

Let us assume that it is not the case, i.e., there exist $w_T^0, w_T^1 \in L^2(0, 1) \setminus \{0\}$ such that

$$-\int_0^1 y_t(T) w_T^0 dx + \int_0^1 y(T) w_T^1 dx = 0. \quad (5)$$

Let us define $z_T^0, z_T^1 \in H^2 \times H_0^1(0, 1)$ such that

$$z_T^0 - \partial_{xx} z_T^0 = w_T^0, \quad \text{and} \quad z_T^1 - \partial_{xx} z_T^1 = w_T^1.$$

Boundary control: approximate controllability

Recall that we have null initial data $y(x, 0) = y_t(x, 0) = 0$.

Let us consider z as the solution of the **adjoint** equation with initial condition (at $t = T$) given by z_T^0, z_T^1 . As before, we use the direct and adjoint equations to get

$$\begin{aligned} \int_0^1 \left[-y_t(x, T) \left(z(x, T) - z_{xx}(x, T) \right) + y(x, T) \left(z_t(x, T) - z_{xxt}(x, T) \right) \right] dx \\ = \int_0^T \left(h(t) + \ddot{h}(t) \right) z_x(1, t) dt \end{aligned}$$

Using the orthogonality

$$- \int_0^1 y_t(T) w_T^0 dx + \int_0^1 y(T) w_T^1 dx = 0,$$

we obtain, for any $h \in H^2(0, T)$, the identity

$$\int_0^T \left(h(t) + \ddot{h}(t) \right) z_x(1, t) dt = 0.$$

We prove now that we necessary have $z = 0$. This would imply that $z_T^0 = z_T^1 = 0$ and consequently $w_T^0 = w_T^1 = 0$, which is a contradiction.

Boundary control: approximate controllability

Indeed, let us choose $h(t) = e^{i\frac{2\pi n}{T}t}$, for $n \in \mathbb{Z}$. Then

$$0 = \int_0^T (h(t) + \ddot{h}(t))z_x(1, t)dt = \left(1 - \left(\frac{2\pi n}{T}\right)^2\right) \int_0^T e^{i\frac{2\pi n}{T}t} z_x(1, t)dt, \forall n \in \mathbb{Z}.$$

Thus, $z_x(1, \cdot) \in \text{SPAN}\{e^{it}, e^{-it}\}$. As

$$z(x, t) = \sqrt{2} \sum_{n \geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sin(n\pi x),$$

we see that there exist $\alpha, \beta \in \mathbb{C}$ such that we have,

$$z_x(1, t) = \sum_{n \geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n = \alpha e^{it} + \beta e^{-it}.$$

From this, we see that

$$\sum_{n \geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it} = 0, \quad (6)$$

for all $t \in (0, T)$. As this function is analytic, it vanishes for any $t \in \mathbb{R}$.

Boundary control: approximate controllability

By using (6), we obtain for any $m \geq 1$ the following

$$\frac{1}{2S} \int_{-S}^S \left(\sum_{n \geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it} \right) e^{i\sqrt{\lambda_m}t} dt$$
$$\rightarrow_{S \rightarrow \infty} \sqrt{2}(m\pi)c_m e^{i\sqrt{\lambda_m}T} = 0,$$

and

$$\frac{1}{2S} \int_{-S}^S \left(\sum_{n \geq 1} \left[c_n e^{i\sqrt{\lambda_n}(T-t)} + d_n e^{-i\sqrt{\lambda_n}(T-t)} \right] \sqrt{2}(n\pi)(-1)^n - \alpha e^{it} - \beta e^{-it} \right) e^{-i\sqrt{\lambda_m}t} dt$$
$$\rightarrow_{S \rightarrow \infty} \sqrt{2}(m\pi)d_m e^{-i\sqrt{\lambda_m}T} = 0.$$

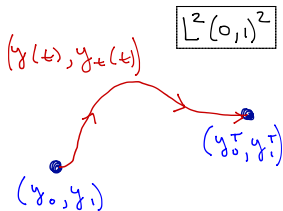
In consequence, for any $m \geq 1$, we obtain that $c_m = d_m = 0$. This fact implies that $z = 0$, which ends the proof.

Theorem (EC-Crépeau, 2015)

The linear improved Boussinesq equation is approximately controllable in $L^2(0, 1)$.

Moving control: exact controllability

Moving control: definition of exact controllability



Definition (Exact controllability)

Given a time $T > 0$, we say that system

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(t), & x \in \mathbb{T}, t > 0, \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

is exactly controllable if $\forall (y^0, y^1), \forall (y_T^0, y_T^1), \exists h$ such that y satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y_t(T) = y_T^1.$$

Remark

We consider b such that $\int_{\mathbb{T}} b(x) dx = 0$.

Moving control: main ideas

KEY IDEAS:

- Linearize around the origin, i.e., drop the nonlinear term $(y^2)_{xx}$.
- Make the following change of variables

$$v(x, t) = y(x - ct, t).$$

Thus, y solves

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x + ct)h(t), & x \in \mathbb{T}, t > 0, \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

if and only if v solves the following equation

$$\begin{cases} v_{tt} + (c^2 - 1)v_{xx} + 2cv_{xt} - v_{xxtt} - c^2 v_{xxxx} - 2cv_{txxx} = b(x)h(t), & x \in \mathbb{T}, t > 0 \\ v(x, 0) = y^0(x), v_t(x, 0) = -cy_x^0(x) + y^1(x), & x \in \mathbb{T}. \end{cases}$$

The equation for v has much better spectral properties!

(... much better for control purposes!)

Moving control: spectral analysis

We study the spectral properties of the equation

$$v_{tt} - v_{xxtt} + 2cv_{xt} - 2cv_{txxx} + (c^2 - 1)v_{xx} - c^2 v_{xxxx} = 0. \quad (7)$$

Let us denote $w = v_t$ to write equation (7) as

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = A \begin{pmatrix} v \\ w \end{pmatrix} := \begin{pmatrix} w \\ (I - \partial_{xx})^{-1} [(1 - c^2)v_{xx} + c^2 v_{xxxx} + 2c(w_{xxx} - w_x)] \end{pmatrix}$$

Thus the eigenvalues $i\lambda$ of the linear operator A are solutions of

$$\begin{cases} w = i\lambda v, \\ (1 - c^2)v_{xx} + c^2 v_{xxxx} + 2c(w_{xxx} - w_x) = i\lambda(w - w_x), \end{cases}$$

and we get two families of solutions, for $k \in \mathbb{Z}$,

$$\lambda_k^+ = \left(ck + \frac{|k|}{\sqrt{1+k^2}} \right) \text{ and } \lambda_k^- = \left(ck - \frac{|k|}{\sqrt{1+k^2}} \right).$$

If $c > 2$, all the eigenvalues are different and there is a gap Δ between them!

If $c = 4$, the gap is $\Delta = 2$.

Moving control: to a moment problem

Direct equation:

$$v_{tt} - v_{xxtt} + 2cv_{xt} - 2cv_{txxx} + (c^2 - 1)v_{xx} - c^2v_{xxxx} = b(x)h(t). \quad (8)$$

Adjoint equation:

$$\varphi_{tt} - \varphi_{xxtt} + 2c\varphi_{xt} - 2c\varphi_{txxx} + (c^2 - 1)\varphi_{xx} - c^2\varphi_{xxxx} = 0.$$

Let us multiply equation (8) by $\bar{\varphi}$ and integrate by parts on $[0, T] \times \mathbb{T}$ to obtain

$$\int_{\mathbb{T}} [v_t(\bar{\varphi} - \bar{\varphi}_{xx}) + v(-\bar{\varphi}_t - 2c\bar{\varphi}_x + \bar{\varphi}_{xxt} + 2c\bar{\varphi}_{xxx})]_0^T dx = \int_0^T \int_{\mathbb{T}} hb\bar{\varphi} dx dt \quad (9)$$

Using, for $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, $\varphi(x, t) = e^{i\lambda_{-k}^\mp(T-t)} e^{ikx} = e^{-i\lambda_k^\pm(T-t)} e^{ikx}$, (9) becomes

$$\begin{aligned} \langle v_t(T) - e^{i\lambda_k^\pm T} v_t(0), e^{-ikx} \rangle + i(\lambda_k^\pm + 2ck) \langle v(T) - e^{i\lambda_k^\pm T} v(0), e^{-ikx} \rangle \\ = \frac{1}{1+k^2} \int_0^T h(t) e^{i\lambda_k^\pm(T-t)} dt \int_{\mathbb{T}} b(x) e^{-ikx} dx. \end{aligned}$$

where $\langle f, e^{ikx} \rangle$ stands for the coordinate f_k in the decomposition $f = \sum_{k \in \mathbb{Z}} f_k e^{ikx}$.

Moving control: to a moment problem

For $k = 0$, we take first $\varphi(x, t) = (T - t)$ and then $\tilde{\varphi}(x, t) = 1$, to obtain

$$\langle v(T) - v(0), 1 \rangle - T \langle v_t(0), 1 \rangle = \int_0^T h(t)(T - t) dt \underbrace{\int_{\mathbb{T}} b(x) dx}_{=0} = 0.$$

$$\langle v_t(T) - v_t(0), 1 \rangle = \int_0^T h(t) dt \underbrace{\int_{\mathbb{T}} b(x) dx}_{=0} = 0.$$

We suppose that the data satisfy

$$\langle v(0), 1 \rangle = \langle v(T), 1 \rangle = \langle v_t(0), 1 \rangle = \langle v_t(T), 1 \rangle = 0,$$

and that for any $k \in \mathbb{Z}^*$, we have $b_k := \int_{\mathbb{T}} b(x) e^{ikx} \neq 0$, we can state the problem of finding a control function $h \in L^2(0, T)$ such that

$$\begin{aligned} \int_0^T h(t) e^{i\lambda_k^\pm (T-t)} dt &= \frac{1 + k^2}{b_k} \langle v_T^1 - e^{i\lambda_k^\pm T} v^1, e^{-ikx} \rangle \\ &+ i(\lambda_k^\pm + 2ck) \frac{1 + k^2}{b_k} \langle v_T^0 - e^{-i\lambda_k^\pm T} v^0, e^{-ikx} \rangle, \quad \forall k \in \mathbb{Z}^*, \end{aligned}$$

Moving control: exact controllability, linear case

The previous trigonometric moment problem can be solved thanks to the behavior of the eigenvalues λ_k^\pm , by imposing the right hand side to define a sequence in $l^2(\mathbb{C})$.

To do that, we have to impose an asymptotic behavior for b_k and regularity!

Theorem (EC-Crépeau, 2015)

Let $s \geq 0$ and c be such that $|c| > 2$. Let $b = b(x)$ such that

$$b_k := \int_{\mathbb{T}} b(x) e^{ikx} dx \neq 0, \quad \forall k \in \mathbb{Z}^*, \quad b_0 = 0, \quad \text{and} \quad \left| \frac{k^{2-s}}{b_k} \right| \text{ bounded as } |k| \rightarrow \infty. \quad (10)$$

Then, for all $T > \frac{2\pi}{\Delta}$ and all $(v^0, v^1), (v_T^0, v_T^1) \in H^{s+1}(\mathbb{T}) \times H^s(\mathbb{T})$, such that

$$\int_{\mathbb{T}} v^0(x) dx = \int_{\mathbb{T}} v^1(x) dx = \int_{\mathbb{T}} v_T^0(x) dx = \int_{\mathbb{T}} v_T^1(x) dx = 0, \quad (11)$$

there exists a control $h \in L^2(0, T)$ such that the linear problem

$$\begin{cases} v_{tt} + (c^2 - 1)v_{xx} + 2cv_{xt} - v_{xxtt} - c^2v_{xxxx} - 2cv_{txxx} = b(x)h(t), & x \in \mathbb{T}, t > 0 \\ v(x, 0) = v^0(x), v_t(x, 0) = v^1(x), & x \in \mathbb{T}. \end{cases}$$

admits a unique solution $v \in C([0, T], H^{s+1}(\mathbb{T})) \cap C^1([0, T], H^s(\mathbb{T}))$ such that $v(x, T) = v_T^0(x)$ and $v_t(x, T) = v_T^1(x)$.

Moving control: exact controllability, nonlinear case

We decompose the solution of the nonlinear problem,

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x+ct)h(t), & x \in \mathbb{T}, t > 0, \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in \mathbb{T}, \end{cases} \quad (12)$$

in the following way, $y = \alpha + \beta + \gamma$, where α, β, γ satisfy

$$\begin{cases} \alpha_{tt} - \alpha_{xx} - \alpha_{xxtt} = 0, & x \in \mathbb{T}, t > 0, \\ \alpha(x, 0) = y^0(x), \alpha_t(x, 0) = y^1(x), & x \in \mathbb{T}, \end{cases} \quad (13)$$

$$\begin{cases} \beta_{tt} - \beta_{xx} - \beta_{xxtt} = b(x+ct)h(t), & x \in \mathbb{T}, t > 0, \\ \beta(x, 0) = 0, \beta_t(x, 0) = 0, & x \in \mathbb{T}, \end{cases} \quad (14)$$

$$\begin{cases} \gamma_{tt} - \gamma_{xx} - \gamma_{xxtt} = F, & x \in \mathbb{T}, t > 0, \\ \gamma(x, 0) = 0, \gamma_t(x, 0) = 0, & x \in \mathbb{T}. \end{cases} \quad (15)$$

with $F = (y^2)_{xx}$.

Remark

We study the nonlinear problem in the following regularity framework:

$$(y_0, y_1) \in H_0^4(\mathbb{T}) \times H_0^3(\mathbb{T}), \quad h \in L^2(0, T), \quad F \in L^1(0, T; L_0^2(\mathbb{T})) \quad \text{and} \quad b \in L^2(\mathbb{T}).$$

Moving control: exact controllability, nonlinear case

Let us consider the following maps, which are well-defined, linear and continuous

- $\psi_0 : (y^0, y^1) \in H_0^4(\mathbb{T}) \times H_0^3(\mathbb{T}) \mapsto \alpha \in C([0, T], H_0^4(\mathbb{T})) \cap C^1([0, T], H_0^3(\mathbb{T})).$
- $\psi_1 : h \in L^2(0, T) \mapsto \beta \in C([0, T], H_0^4(\mathbb{T})) \cap C^1([0, T], H_0^3(\mathbb{T})).$
- $\psi_2 : F \in L^1(0, T; L_0^2(\mathbb{T})) \mapsto \gamma \in C([0, T], H_0^4(\mathbb{T})) \cap C^1([0, T], H_0^3(\mathbb{T})).$
- $\Gamma : (y_T^0, y_T^1) \in H_0^4(\mathbb{T}) \times H_0^3(\mathbb{T}) \mapsto h \in L^2(0, T)$ where h is the control such that β satisfies $(\beta(T), \beta_t(T)) = (y_T^0, y_T^1).$

We see that a trajectory $y = y(x, t)$ going from (y^0, y^1) at time $t = 0$ to (y_T^0, y_T^1) at time $t = T$ is a fixed point of

$$\Pi : y \in C([0, T], H_0^4(\mathbb{T})) \cap C^1([0, T], H_0^3(\mathbb{T})) \mapsto \Pi(y) \in C([0, T], H_0^4(\mathbb{T})) \cap C^1([0, T], H_0^3(\mathbb{T}))$$

where $\Pi(y)$ is

$$\psi_0(y^0, y^1) + \psi_1 \circ \Gamma \left((y_T^0, y_T^1) - \psi_0(y^0, y^1)(T) - (\psi_2((y^2)_{xx})(T), \psi_{2t}((y^2)_{xx})(T)) \right) + \psi_2((y^2)_{xx})$$

We conclude by applying the Banach Fixed Point Theorem!

Moving control: exact controllability, nonlinear case

Theorem (EC-Crépeau, 2015)

Let c be such that $|c| > 2$. Let $b = b(x)$ be such that (10) holds with $s = 3$. Then, for all $T > \frac{2\pi}{\Delta}$, there exists $\epsilon > 0$ such that for any $(y^0, y^1), (y_T^0, y_T^1) \in H^4(\mathbb{T}) \times H^3(\mathbb{T})$ satisfying (11) and

$$\|(y^0, y^1)\|_{H^4(\mathbb{T}) \times H^3(\mathbb{T})} \leq \epsilon, \quad \|(y_T^0, y_T^1)\|_{H^4(\mathbb{T}) \times H^3(\mathbb{T})} \leq \epsilon,$$

there exists a control $h \in L^2(0, T)$ such that

$$\begin{cases} y_{tt} - y_{xx} - y_{xxtt} = (y^2)_{xx} + b(x + ct)h(t), & x \in \mathbb{T}, t > 0, \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & x \in \mathbb{T}, \end{cases}$$

admits a unique solution $y \in C([0, T], H^4(\mathbb{T})) \cap C^1([0, T], H^3(\mathbb{T}))$ such that

$$y(x, T) = y_T^0(x), \quad \text{and} \quad y_t(x, T) = y_T^1(x).$$

Related papers:

[Micu 2001], [Rosier, Zhang 2013] Benjamin-Bona-Mahony equation (1d)

[Martin, Rosier, Rouchon 2013] Wave equation with structural damping (1d)

[Chaves-Silva, Rosier, Zuazua 2014] Multi-d wave equation with viscous Kelvin–Voigt and frictional damping

... that is all...

Thank you for your attention!