On the determination of the principal coefficient from boundary measurements in a KdV equation

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Joint work with Lucie Baudouin, Emmanuelle Crépeau and Alberto Mercado.

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Presentation of the problem

The KdV equation with non-constant coefficient $a = a(x)$ is given as

$$\begin{cases} y_t + a(x)y_{xxx} + y_x + yy_x = g, & \forall (x,t) \in (0,L) \times (0,T), \\ y(0,t) = g_0(t), \quad y(L,t) = g_1(t), & \forall t \in (0,T), \\ \qquad\qquad\qquad\quad y_x(L,t) = g_2(t), & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,L), \end{cases}$$

where the initial data y_0, the source term g, and the functions g_0, g_1, g_2 are assumed to be known.

The KdV equation is used to describe approximately long waves in water of relatively shallow depth.

In this context, the principal coefficient $a = a(x)$ is related with the shape of the bottom of the channel where the water wave propagates. See Samer Israwi's PhD thesis under supervision of David Lannes (Bordeaux, 2010).

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In general, in an inverse problem, we want to get some information of a system (conductivity, diffusion, shape of the bottom of a channel, etc) from partial knowledge (observations/measurements) of the solution.

Inverse Problem

Can we recover $a = a(x)$ *from some partial knowldege of* $y = y(x,t)$*?*

Inverse Problem (Uniqueness)

Given some boundary measurements $meas(y)$*, is there a unique* $a = a(x)$ *associated?*

$$\text{i.e.} \quad meas(y) = meas(\tilde{y}) \implies a = \tilde{a}?$$

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$$\|a - \tilde{a}\|_X \leq C \|meas(y) - meas(\tilde{y})\|_Y ?$$

Inverse Problem (Reconstruction)

Given some measurement $meas(y)$, is it possible to reconstruct the coefficient $a = a(x)$?

In this talk, we are concerned with the stability problem (and consequently with the uniqueness problem!).

Remark: This kind of inverse problem is called a single-measurement IP

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We hope to get only boundary measurements:

$$\|y_x(0,t) - \tilde{y}_x(0,t)\|, \quad \|y_{xx}(0,t) - \tilde{y}_{xx}(0,t)\|$$

or

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The approach

1. The Bukhgeim-Klibanov-Malinsky method.
2. Carleman estimate for the linearized equation.

- Bukhgeim, Klibanov, 1981; Klibanov, Malinsky 1991: Inverse problems with Carleman estimates.
- Puel, Yamamoto 1996; Yamamoto, 1999; Imanuvilov, Yamamoto 2001: Wave equation.
- Imanuvilov, Yamamoto 1998, Benabdallah, Gaitan,Le Rousseau:2007 Parabolic equations.
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BMK method

We follow ideas of Bukhgeim, Klibanov (1981), and Klibanov, Malinsky (1991).

If we set:

- $u = y - \tilde{y}$ and
- $\sigma = \tilde{a} - a$

then u solves the following KdV equation:

$$\begin{cases} u_t + a(x)u_{xxx} + (1+\tilde{y})u_x + \tilde{y}_x u + uu_x = \sigma \tilde{y}_{xxx}, & \forall (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0 & \forall t \in (0,T), \\ u(x,0) = 0, & \forall x \in (0,L). \end{cases}$$

Then $z = u_t$ satisfies the following equation:

$$\begin{cases} z_t + a(x)z_{xxx} + (1+y)z_x + y_x z = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0, \quad z_x(L,t) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L), \end{cases}$$

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- We would like to have an estimate like

$$\|z(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)}$$

  where C can be chosen small !!
- We shall need $y_{0,xxx}(x)$ bounded by below by a positive constant!
- Here Carleman estimates will be used. Time-singular weights are required!

Remark: This kind of inequality is called observability in control theory

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BMK method

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$$\begin{cases} z_t + a(x) z_{xxx} + (1+y) z_x + y_x z = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0, \quad z_x(L,t) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L), \end{cases}$$

where

$$f_\sigma = \sigma(x) \tilde{y}_{xxxt} - \tilde{y}_{xt} u - \tilde{y}_t u_x.$$

- We would like to have an estimate like

  $$\|z(x,0)\|_X \leq C \|f_\sigma\|_Y + \text{ (boundary terms)}$$

  where C can be chosen small !!

- We shall need $y_{0,xxx}(x)$ bounded by below by a positive constant!

- Here Carleman estimates will be used. Time-singular weights are required!

Remark: This kind of inequality is called observability in control theory

BMK method - Heat equations

$$\begin{cases} z_t - a(x)z_{xx} = f_\sigma, & \forall (x,t) \in (0,L)\times(0,T), \\ z(0,t) = 0, \quad z(L,t) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xx}(x), & \forall x \in (0,L), \end{cases}$$

What happens for the heat equation?

- Observability

$$\|z(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)},$$

  can not be proved for parabolic equation.
- Instead, one gets $\|z(x,T_0)\|_X \leq C\|f_\sigma\|_Y +$ (boundary terms).
- We use the equation

$$\|z(x,T_0)\| = \|u_t(x,T_0)\| = \|\sigma R(x,T_0) + a(x)u_{xx}(x,T_0)\|$$
$$\geq \|\sigma R(x,T_0)\| - \|a(x)u_{xx}(x,T_0)\|$$

and we have to add an observation like $\|y_{xx}(x,T_0) - \tilde{y}_{xx}(x,T_0)\|$!

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BMK method - Wave equation

$$\begin{cases} z_{tt} - a(x) z_{xx} = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xx}(x), & \forall x \in (0,L), \end{cases}$$

What happens for wave equation?

- Extend the solution to $(-T, T)$ by using the symmetry under the change of variable $t \to (T - t)$.
- Use Carleman inequalities on $(-T, T)$.
- The time $t = 0$ is not singular and you get

$$\|z(x,0)\|_X \leq C \|f_\sigma\|_Y + \text{ (boundary terms)},$$

with C small.

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What happens for the Schrödinger equation?

- The Carleman weight is singular at $t = 0$ and $t = T$.
- Extend the solution to $(-T,T)$ by defining the solution for negative time as

$$z(x,t) := -\bar{z}(x,-t), \quad f_\sigma(x,t) = -\bar{f}_\sigma(x,-t)$$

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- The time $t = 0$ is not singular anymore and you get

$$\|z(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)},$$

with C small.

Remark: The method needs $\mathrm{Re}(y_0(x))$ to be zero.

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BMK method - KdV equation

$$\begin{cases} z_t + a(x) z_{xxx} + (1+y) z_x + y_x z = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(0,t) = 0, \quad z(L,t) = 0, \quad z_x(L,t) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L), \end{cases}$$

What happens for KdV equation?

- Not parabolic neither hyperbolic.
- From a control point of view, in some cases it is parabolic and in others hyperbolic.
- KdV has only one time-derivative and so the change $t \to T - t$ is not adequate.
- But it has the symmetry $t \to T - t$ and $x \to L - x$, which allows to define the solution for negative times.
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Remark: Some symmetry conditions have to be imposed.

BMK method - Extension for negative time

Symmetric extension to $(0, L) \times (-T, T)$ of g defined on $(0, L) \times (0, T)$:

$$g^s(x,t) = \begin{cases} g(x,t) & \text{if } x \in [0,L],\ t \in [0,T], \\ g(L-x,-t) & \text{if } x \in [0,L],\ t \in [-T,0). \end{cases}$$

Anti-symmetric extension to $(0, L) \times (-T, T)$ of g defined on $(0, L) \times (0, T)$:

$$g^a(x,t) = \begin{cases} g(x,t) & \text{if } x \in [0,L],\ t \in [0,T], \\ -g(L-x,-t) & \text{if } x \in [0,L],\ t \in [-T,0). \end{cases}$$

Defining $v = z^s$, we obtain:

$$\begin{cases} v_t + a(x)v_{xxx} + (1+y^s)v_x + (y_x)^a v = f_\sigma^a, & \forall x \in (0,L), t \in (-T,T), \\ v(0,t) = 0, \quad v(L,t) = 0, & \forall t \in (-T,T), \\ v_x(L,t) = \begin{cases} 0, & \forall t \in (0,T), \\ -z_x(0,-t), & \forall t \in (-T,0). \end{cases} & \\ v(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L). \end{cases}$$

BMK method - Extension for negative time

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Anti-symmetric extension to $(0, L) \times (-T, T)$ of g defined on $(0, L) \times (0, T)$:

$$g^a(x,t) = \begin{cases} g(x,t) & \text{if } x \in [0,L],\ t \in [0,T], \\ -g(L-x,-t) & \text{if } x \in [0,L],\ t \in [-T,0). \end{cases}$$

Defining $v = z^s$, we obtain:

$$\begin{cases} v_t + a(x)v_{xxx} + (1+y^s)v_x + (y_x)^a v = f_\sigma^a, & \forall x \in (0,L),\ t \in (-T,T), \\ v(0,t) = 0, \quad v(L,t) = 0, & \forall t \in (-T,T), \\ v_x(L,t) = \begin{cases} 0, & \forall t \in (0,T), \\ -z_x(0,-t), & \forall t \in (-T,0). \end{cases} & \\ v(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L). \end{cases}$$

BMK method - Extension for negative time

The solution of

$$\begin{cases} v_t + a(x)v_{xxx} + (1+y^s)v_x + (y_x)^a v = f_\sigma^a, & \forall x \in (0,L),\ t \in (-T,T), \\ v(0,t) = 0, \quad v(L,t) = 0, & \forall t \in (-T,T), \\ v_x(L,t) = \begin{cases} 0, & \forall t \in (0,T), \\ -z_x(0,-t), & \forall t \in (-T,0). \end{cases} & \\ v(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L). \end{cases}$$

satisfies a Carleman estimate which allows to prove

$$\|v(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)}$$

with C small.

Carleman estimates - $Lv = v_t + av_{xxx} = f$

Any $v \in L^2(-T,T;H^3 \cap H_0^1(0,L))$ and a weight function $\phi(x,t) = \frac{\beta(x)}{(T+t)(T-t)}$.

$$w = e^{-\lambda\phi}v, \quad \text{and } L_\phi w = e^{-\lambda\phi}L(e^{\lambda\phi}w)$$

where λ is a large parameter to be chosen later.

The obtained Carleman estimate is an inequality like

$$\lambda^5\|w\|^2_{L^2_\phi} + \lambda^3\|w_x\|^2_{L^2_\phi} + \lambda\|w_{xx}\|^2_{L^2_\phi} + \frac{1}{\lambda}\|w_t\|^2_{L^2_\phi} \leq C\,\|L_\phi w\|^2_{L^2_\phi} + B.D.(w)$$

Note that $w(-T,0) = 0$, and therefore

$$\|w(0,x)\|^2_{L^2_\phi} = 2\int_{-T}^{0}\int ww_t \leq \left(\lambda\int\int|w|^2\right)^{1/2}\left(\frac{1}{\lambda}\int\int|w_t|^2\right)^{1/2}$$

$$\leq \frac{1}{\lambda^2}\left(\lambda^5\int\int|w|^2\right)^{1/2}\left(\|L_\phi w\|^2_{L^2_\phi} + B.D.(w)\right)^{1/2}$$

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Carleman estimates for KdV.

- Rosier [2004]. Null control of the surface of a water wave by means of a wavemaker at the left end-point.
- Glass-Guerrero [2008]. Cost of the null control of KdV by means of a control at the left end-point.
- Both papers prove Carleman estimates with one parameter $\lambda > 0$.
- For us, it is important a second parameter. Look at one dominating term:

$$\lambda^5 \iint \phi_x^4(-a_x\phi_x - 5a\phi_{xx} + 4a^2\phi_{xx})|w|^2$$

  This impose bad conditions of kind $\|a_x/a\|_{L^\infty} \leq M$.
- Solution is to choose ϕ such that $\phi_{xx} \approx s^2\varphi$ with a second parameter $s > 0$.

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Main Result.

$$\begin{cases} y_t + a(x) y_{xxx} + y_x + y y_x = g, & \forall (x,t) \in (0,L) \times (0,T), \\ y(0,t) = g_0(t), \quad y(L,t) = g_1(t), & \forall t \in (0,T), \\ \qquad\qquad\qquad y_x(L,t) = g_2(t), & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,L). \end{cases}$$

Data (g, g_k, y_0) fixed and regular enough!

Theorem

Let $|y_{0,xxx}(x)| \geq \delta > 0$, also symmetric wrt $L/2$. Let

$$\Sigma = \left\{ a \text{ symmetric wrt } L/2 \middle/ a \geq a_0 > 0, \|a\|_{W^{3,\infty}} \leq M_1, \text{ and } \|y(a)\|_{W^{1,\infty}(Q)} \leq M_2 \right\}$$

There exists a constant $C = C(L, T, a_0, M_1, M_2, \delta) > 0$ such that for any $a, \tilde{a} \in \Sigma$:

$$C\|a - \tilde{a}\|_{L^2(0,L)} \leq \|y_x(0,t) - \tilde{y}_x(0,t)\|_{H^1(0,T)} + \|y_{xx}(0,t) - \tilde{y}_{xx}(0,t)\|_{H^1(0,T)} + \|y_{xx}(L,t) - \tilde{y}_{xx}(L,t)\|_{H^1(0,T)}$$

Thank you for your attention!