Stabilization of a cascade system of linear Korteweg-de Vries equations

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work in collaboration with
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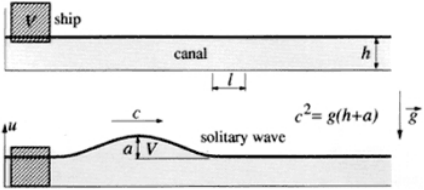

- Propagation of waters with small amplitude in closed channels, 1895
- Other applications: internal ocean waves, nonlinear acoustics waves, magma flow, spatially periodic cnoïdal waves in vehicle headway distance, etc.
- Coupled KdV equations: interactions of waves of instance but much more technical difficulties, from well-posedness to stability and control properties.

Cascade System of three equations

$$\begin{aligned} v^1_t + v^1_x + v^1_{xxx} + v^1 &= v^2, \text{ in } (0,+\infty)\times(0,R),\\ v^2_t + v^2_x + v^2_{xxx} + v^2 &= v^3, \text{ in } (0,+\infty)\times(0,R),\\ v^3_t + v^3_x + v^3_{xxx} - v^3 &= 0, \text{ in } (0,+\infty)\times(0,R), \end{aligned} \tag{1}$$

Boundary conditions first equations (v^1 and v^2):

$$\begin{aligned} v^i(t,0) &= 0, \quad i=1,2, \quad \forall t>0,\\ v^i(t,R) &= 0, \quad i=1,2, \quad \forall t>0,\\ v^i_x(t,R) &= 0, \quad i=1,2, \quad \forall t>0. \end{aligned} \tag{2}$$

Boundary conditions last equation (v^3):

$$\begin{aligned} v^3(t,0) &= u(t), \quad \forall t>0,\\ v^3(t,R) &= 0, \quad \forall t>0,\\ v^3_x(t,R) &= 0, \quad i=1,2,3 \quad \forall t>0. \end{aligned} \tag{3}$$

Stability and Problem

When no control is applied, $u(t) = 0$, the last equation in

$$\begin{aligned} v^1_t + v^1_x + v^1_{xxx} + v^1 &= v^2, \text{ in } (0,+\infty)\times(0,R), \\ v^2_t + v^2_x + v^2_{xxx} + v^2 &= v^3, \text{ in } (0,+\infty)\times(0,R), \\ v^3_t + v^3_x + v^3_{xxx} - v^3 &= 0, \text{ in } (0,+\infty)\times(0,R), \end{aligned} \tag{4}$$

is unstable and that implies the instability of the whole system.

Our Problem

Can we design a feedback control to avoid that instability?

$$u = u(v^1, v^2, v^3)$$

Can that feedback control depends only on v^1*?*

$$u = u(v^1)$$

The last one is called an output feedback control !

Full state feedback

Let $u = Kx$ be a feedback control stabilizing the system $\dot{x} = Ax + Bu$. Thus,

$$\dot{x} = (A + BK)x$$

is asymptotically stable: $x \to 0$ as time goes to ∞. This is equivalent to pick a matrix K such that $(A + BK)$ is Hurwitz.

Static output feedback

Now, assume our control has to depend only on a measure of the full solution $u = Ky$ where $y = Cx$. Even if (A, B) is controllable and (A, C) is observable, the existence of K is not guaranteed. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad K = k \in \mathbb{R}$$

but

$$(A + BKC) = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}$$

is never Hurwitz.

Dynamic output feedback

IDEA: to build an auxiliar system $\hat{x}$, the observer, that converge to the solution x:

$$\dot{\hat{x}} = Ax + Bu + L(C\hat{x} - y).$$

To study the closed-loop system

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = Ax + Bu + LC(\hat{x} - x)$$

$$u = K\hat{x}$$

we define $e = (\hat{x} - x)$ which is solution of

$$\dot{e} = (A + LC)e.$$

We have here a general result: if matrices K, L are such $(A + BK)$ and $(A + LC)$ are both Hurwitz, then the closed loop system converges to the origin.

Our strategy for the cascade system of linear KdV equations is the same. For infinite-dimensional systems, there is no general result and we have to prove the convergence. An important tool to do that is the Lyapunov approach.

Back to Our System, now with n equations

$$v_t + v_x + v_{xxx} = (A - B)v, \text{ in } (0, +\infty) \times (0, R), \tag{5}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}, \qquad B = \text{diag}\,(1, 1, \ldots, 1, -1)\,.$$

Boundary conditions:

$$\begin{aligned} v^i(t,0) &= 0, i = 1, \ldots, n-1, \forall t > 0, \\ v^n(t,0) &= u(t), \forall t > 0, \\ v(t,R) &= 0, v_x(t,R) = 0, \text{ for all } t > 0. \end{aligned} \tag{6}$$

Output:

$$y(t,x) = v^1(t,x), \quad t \geq 0, x \in [0,R];$$
$$y = Cv, \text{ with } C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

Full state feedback (looking for K)

The instability comes from the last equation v^n, on which the control u acts directly. Thus, we expect to have an easy case when

$$u = u(v^n).$$

Indeed, we can apply existing results of the Backstepping method for a single KdV equation [C-Coron, 2013]:

$$u(t) = \int_0^R p(0,y) v^n(t,y) \mathrm{d}y,$$

where p is the unique solution in $\Pi := \Big\{(x,y); x \in [0,R], y \in [x,R]\Big\}$ of

$$\begin{cases} p_{xxx} + p_{yyy} + p_x + p_y + (\omega+1)p = 0, & (x,y) \in \Pi \\ p(x,x) = p(x,L) = 0, & x \in [0,L] \\ p_x(x,x) = \frac{\omega+1}{3}(L-x), & x \in [0,L] \end{cases}$$

We need now an observer to define

$$u(t) = \int_0^R p(0,y) \hat{v}^n(t,y) \mathrm{d}y.$$

We follow [Kitsos, Besançon, Prieur, 2021] for hyperbolic systems.

- The matrix $L = \begin{pmatrix} \ell_1 & \cdots \ell_n \end{pmatrix}^\top$ such that $(A + LC)$ is Hurwitz.
- Let $\theta > 0$. We define $\Theta := \text{diag}\left(\theta, \theta^2, \ldots, \theta^n\right).$
- System

$$v_t + v_x + v_{xxx} = (A - B)v$$

$$\text{(BC):} \quad \begin{array}{l} v^i(t,0) = 0, i = 1, \ldots, n-1, \\ v^n(t,0) = u(t), \\ v(t,R) = v_x(t,R) = 0, \end{array}$$

- Observer

$$\hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = (A - B)\hat{v} + \Theta L \left(\hat{v}^1 - v^1\right)$$

$$\text{(BC):} \quad \begin{array}{l} \hat{v}^i(t,0) = 0, i = 1, \ldots, n-1, \\ \hat{v}^n(t,0) = u(t), \\ \hat{v}(t,R) = \hat{v}_x(t,R) = 0, \end{array}$$

Convergence of the observer

Theorem (Kitsos, C, Besançon, Prieur, 2021)

Assume $v^0 \in L^2(0,R)^n, u \in L^2_{loc}(0,\infty)$*. Then, for every* $\kappa > 0$*, there exist* $\theta_0 > 0$*, such that for every* $\theta > \theta_0$*, the following holds for all* $\hat{v}_0 \in L^2(0,R)^n$*,* $t \geq 0$*:*

$$\|\hat{v}(t,\cdot) - v(t,\cdot)\|_{L^2(0,R)^n} \leq C\theta^{n-1}e^{-\kappa t}\|\hat{v}_0(\cdot) - v_0(\cdot)\|_{L^2(0,R)^n} \tag{7}$$

with $C > 0$*.*

PROOF: Let us notice that we will prove that $\hat{v}$ converges to v... the behavior of v depends on the control u which is not necessarily a stabilizing control.

Let $\varepsilon := \Theta^{-1}(\hat{v} - v)$, which is solution of

$$\varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A + LC)\varepsilon - B\varepsilon$$

(BC): $\varepsilon(t,0) = \varepsilon(t,R) = \varepsilon_x(t,R) = 0,$

We have to prove the exponential stability of ε.

Convergence of the observer

Let $P \succ 0$ be the unique solution of

$$P(A+LC) + (A+LC)^\top P = -I_n$$

and define the Lyapunov function

$$V(t) = \int_0^R \varepsilon^\top(x) P \varepsilon(x) \mathrm{d}x,$$

By choosing θ large enough, we are able to prove

$$\dot{V}(t) \leq -2\kappa V(t)$$

which gives the exponential decay desired. In fact, we need

$$\theta > \theta_0 = 2\frac{\|P\|^2}{\lambda_{min}(P)}$$

and κ increases with θ.

□

Stabilization of the closed-loop system

Theorem (Kitsos, C, Besançon, Prieur, 2021)

Let $R > 0$. There exist $C, \kappa > 0$, such that for every initial conditions $v_0, \hat{v}_0$ we have

$$\|\hat{v} - v\|_{L^2(0,R)^n} + \|\hat{v}\|_{L^2(0,R)^n} \leq Ce^{-\kappa t}\left(\|\hat{v}^0 - v^0\|_{L^2(0,R)^n} + \|\hat{v}^0\|_{L^2(0,R)^n}\right),$$

for $(v, \hat{v})$ solutions of

$$v_t + v_x + v_{xxx} = (A - B)v$$

$$\textit{(BC):}\quad \begin{array}{l} v^i(t,0) = 0, i = 1, \ldots, n-1, \\ v^n(t,0) = \int_0^R p(0,y)\hat{v}^n(t,y)\mathrm{d}y, \\ v(t,R) = v_x(t,R) = 0, \end{array}$$

$$\hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = (A - B)\hat{v} + \Theta L\left(\hat{v}^1 - v^1\right)$$

$$\textit{(BC):}\quad \begin{array}{l} \hat{v}^i(t,0) = 0, i = 1, \ldots, n-1, \\ \hat{v}^n(t,0) = \int_0^R p(0,y)\hat{v}^n(t,y)\mathrm{d}y, \\ \hat{v}(t,R) = \hat{v}_x(t,R) = 0, \end{array}$$

Remark

Now the full system converges to the origin!

Sketch of the proof

The observer error $\varepsilon = \hat{v} - v$ and observer $\hat{v}$ satisfy

$$\begin{cases} \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = & \theta(A + LC)\varepsilon - B\varepsilon, \\ \hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = & (A - B)\hat{v} + \theta\Theta L\varepsilon_1, \end{cases} \tag{8}$$

The last equation of the observer is

$$\hat{v}^n_t + \hat{v}^n_x + \hat{v}^n_{xxx} = \hat{v}^n + \ell_n \theta^{n+1}\varepsilon_1. \tag{9}$$

We apply a Volterra transformation to this last equation and $q = T(\hat{v}^n)$ satisfies

$$q_t + q_x + q_{xxx} = -\omega q + \varepsilon_1 - \int_x^R p(x,y)\varepsilon_1(t,y)\mathrm{d}y, \tag{10}$$

with boundary conditions

$$q(t,0) = q(t,L) = q_x(t,L) = 0$$

(if the kernel of the transformation is chosen correctly.)

Sketch of the proof

We use the Lyapunov function

$$U_1(t) = \int_0^R \varepsilon^\top(x) P \varepsilon(x) \mathrm{d}x + \int_0^R q^2(x) \mathrm{d}x$$

and prove, after some lines of computations, that

$$\dot{U}_1(t) \leq -2\mu U_1(t)$$

Thus, we obtain the decay $\forall t \geq 0$

$$\|\hat{v} - v\|_{L^2(0,R)^n} + \|q\|_{L^2(0,R)} \leq C e^{-\mu t} \left(\|\hat{v}^0 - v^0\|_{L^2(0,R)^n} + \|q(0,\cdot)\|_{L^2(0,R)} \right)$$

As the Volterra transformation is invertible we get the decay $\forall t \geq 0$

$$\|\hat{v} - v\|_{L^2(0,R)^n} + \|\hat{v}^n\|_{L^2(0,R)} \leq C e^{-\mu t} \left(\|\hat{v}^0 - v^0\|_{L^2(0,R)^n} + \|\hat{v}^n(0,\cdot)\|_{L^2(0,R)} \right)$$

Finally, by playing with $U_2(t) = \int_0^R |(\hat{v}^1, \cdots, \hat{v}^{n-1})|^2 \mathrm{d}x$ we can propagate the damping to the other equations in $\hat{v}$ to obtain the decay $\forall t \geq 0$

$$\|\hat{v} - v\|_{L^2(0,R)^n} + \|\hat{v}\|_{L^2(0,R)} \leq C e^{-\mu t} \left(\|\hat{v}^0 - v^0\|_{L^2(0,R)^n} + \|\hat{v}(0,\cdot)\|_{L^2(0,R)} \right)$$

□

A more difficult case

We have also dealt with the system

$$v_t + v_x + v_{xxx} = (A - B)v$$

$$\text{(BC)}: \begin{array}{l} v^i_{xx}(t,0) = 0, i = 1, \ldots, n-1, \\ v^n_{xx}(t,0) = u(t), \\ v(t,R) = v_x(t,R) = 0, \end{array}$$

Here the approach is the same as before but we need a new Lyapunov function

$$V(t) = \int_0^R \pi(x) v^\top(t,x) P v(t,x) \mathrm{d}x,$$

with π the solution of a particular ordinary differential equation. The required changes impose conditions to have the same result:

- If $n = 2$, then the same holds for any domain $(0, R)$.
- if $n > 2$, then the results only holds for small enough R.

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