# PRESSURE, POINCARÉ SERIES AND BOX DIMENSION OF THE BOUNDARY

GODOFREDO IOMMI AND ANÍBAL VELOZO

ABSTRACT. In this note we prove two related results. First, we show that for certain Markov interval maps with infinitely many branches the upper box dimension of the boundary can be read from the pressure of the geometrical potential. Secondly, we prove that the box dimension of the iterates of a point of the boundary of the hyperbolic space with respect to a parabolic isometry equals the critical exponent of the Poincaré series of the associated group.

## 1. Introduction

The relation between thermodynamic formalism and the dimension theory of dynamical systems has been studied at least since the work of Bowen [Bow]. He related the Hausdorff dimension of a dynamically defined set with the root of a certain pressure function. This idea has been generalized to a wide range of different settings, including non-compact systems or even non-conformal ones. The results in this note are part of these circle of ideas. We first consider a class of Markov interval maps with countably many branches and infinite topological entropy. It has been known for quite some time that the Hausdorff dimension of the corresponding non-compact repeller is essentially the root of an equation involving the pressure (see Sub-section 2.3). We prove that the box dimension (or lower bounds) of the boundary of the Markov partition corresponds to the critical value of the pressure function after which it becomes finite (see Theorem 2.12). Depending on the properties of the pressure function at this point we are able to prove certain stability results. For example, existence of measures of maximal dimension for perturbations of the original map. We also relate the box dimension of the boundary of the Markov partition with the behaviour of sequences of measures for which the mass escapes through the space. These measures are related to points having infinite Lyapunov exponents.

There is an interesting relation between the ergodic theory of countable Markov shifts (or related dynamical systems such as the interval maps we study) and the ergodic theory of the geodesic flow on non-compact negatively curved manifolds. In some cases Markov partitions can be constructed and then the relation is rather explicit (see [DP, IRV]). However, even if no such partition is available, results describing the ergodic theory of the geodesic flow have been proved in analogy to

*Date*: August 2, 2018.
2010 *Mathematics Subject Classification.* Primary 05C80; Secondary 05C70, 05C63.
G.I. was partially supported by CONICYT PIA ACT172001 and by Proyecto Fondecyt 1150058.

those obtained for countable Markov shifts (see for example [PPS, RV, V1, V2]). In Section 3, following this line of thought, we prove that the box dimension of the iterates of a point of the boundary of the hyperbolic space with respect to a parabolic isometry is the critical exponent of the corresponding Poincaré series (see Theorem 3.9). We emphasize that a parabolic group of isometries is elementary, and that the limit set is precisely its fixed point at infinity. In particular we can not recover significant information (e.g. its critical exponent) from the limit set. For non-elementary groups it is a completely different story: the critical exponent corresponds to the Hausdorff dimension of the radial limit set (see [BJ]).

## 2. Pressure and dimension

2.1. **Dimension theory.** In this sub-section we recall basic definitions and results from dimension theory that will be used in what follows (see [Fa1, Fa2] for details). Let $(M, d)$ be a metric space, a countable collection of sets $\{U_i\}_{i\in\mathbb{N}}$ is called a $\delta$-cover of $F \subset M$ if $F \subset \bigcup_{i\in\mathbb{N}} U_i$, and $U_i$ has diameter $|U_i|$ at most $\delta$ for every $i \in \mathbb{N}$. Letting $s > 0$, we define

$$\mathcal{H}^s(J) := \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ is a } \delta\text{-cover of } J \right\}.$$

The *Hausdorff dimension* of the set $J$ is defined by

$$\dim_H(J) := \inf \{s > 0 : \mathcal{H}^s(J) = 0\}.$$

The *Hausdorff dimension* of a Borel measure $\mu$ is defined by

$$\dim_H(\mu) := \inf \{\dim_H(Z) : \mu(Z) = 1\}.$$

For more properties of Hausdorff dimension see [Fa1, Chapter 2]. An alternative definition of dimension that will be central to our work is the *box dimension*. Let $F \subset M$ and $N_\delta(F)$ be the smallest number of sets of diameter at most $\delta$ which can cover $F$. The *lower* and *upper box dimensions* are defined by

$$\underline{\dim}_B(F) := \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \quad , \quad \overline{\dim}_B(F) := \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$$

If the limits above are equal then we call this common value the *box dimension* of the set $F$,

$$\dim_B(F) := \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$$

*Remark* 2.1. In order to clarify the difference between Hausdorff and box dimension it is useful to note that in the definition of box dimension it is possible to replace the number $N_\delta(F)$ by the largest number of disjoint balls of radius $\delta$ and centers in $F$ (see [Fa1, p.41]).

The following results will be of importance to us (see [Fa2, Proposition 3.6 and 3.7]).

**Proposition 2.2.** *Let $I_n = [a_n, b_n]$ be a sequence of intervals with disjoint interiors such that $[0, 1] = \bigcup_{n=1}^{\infty} I_n$ and let $F = \bigcup_{n=1}^{\infty} \{a_n . b_n\}$. Assume that $(b_n - a_n)_n$ is non-increasing. Then*

$$\left( -\liminf_{n\to\infty} \frac{\log(b_n - a_n)}{\log n} \right)^{-1} \leqslant \underline{\dim}_B(F) \leqslant \overline{\dim}_B(F) \leqslant \left( -\limsup_{n\to\infty} \frac{\log(b_n - a_n)}{\log n} \right)^{-1}.$$

2.2. **The class of maps.** Let $I = [0,1]$ be the unit interval. The class of EMR (expanding-Markov-Renyi) interval maps was considered by Pollicott and Weiss in [PW] and further studied in [IJ] when studying multifractal analysis.

**Definition 2.3.** Let $\{I_i\}_{i\in\mathbb{N}}$ be a countable collection of closed intervals where where $\text{int}(I_i) \cap \text{int}(I_j) = \emptyset$ for $i, j \in \mathbb{N}$ with $i \neq j$ and $[a_i, b_i] := I_i \subset I$ for every $i \in \mathbb{N}$. A map $T : \bigcup_{n=1}^{\infty} I_n \to I$ is an EMR map, if the following properties are satisfied

(a) We have that $\lim_{n\to\infty} a_n = 0$.
(b) The map is $C^2$ on $\bigcup_{i=1}^{\infty} \text{int } I_i$.
(c) There exists $\xi > 1$ and $N \in \mathbb{N}$ such that for every $x \in \bigcup_{i=1}^{\infty} I_i$ and $n \geqslant N$ we have $|(T^n)'(x)| > \xi^n$.
(d) The map $T$ is Markov and it can be coded by a full-shift on a countable alphabet.
(e) The map satisfies the Renyi condition, that is, there exists a positive number $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{x,y,z\in I_n} \frac{|T''(x)|}{|T'(y)||T'(z)|} \leqslant K.$$

The *repeller* of such a map is defined by

$$\Lambda := \left\{ x \in \bigcup_{i=1}^{\infty} I_i : T^n(x) \text{ is well defined for every } n \in \mathbb{N} \right\}.$$

*Example* 2.4. The Gauss map $G : (0,1] \to (0,1]$ defined by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right],$$

where $[\cdot]$ is the integer part, satisfies our assumptions. Also the Gauss map with restricted digits (that is the Gauss map with branches erased so that there are still infinitely many branches left) is a EMR map.

*Remark* 2.5. Let $\mathcal{O} := \bigcup_{k=0}^{\infty} T^{-k}(\cup_{i=0}^{\infty}\{a_i, b_i\})$. The fact that $T$ can be coded by a full-shift means that every point in $x \in \Lambda \setminus \mathcal{O}$ has a unique symbolic coding. That is, if $(\Sigma, \sigma)$ denotes the full-shift on a the alphabet $\mathbb{N}$ of natural numbers, there exists a homemorphism $\pi : \Lambda \setminus \mathcal{O} \to \Sigma$ that conjugates the dynamics of $T$ and $\sigma$. We denote by $I_{i_1\dots i_n} \subset I$ the projection of the symbolic cylinder $C_{i_1\dots i_n} := \{\omega \in \Sigma : \omega_j = i_j, j \in \{1, \dots, n\}\}$.

The following is a fundamental property of EMR maps, see [CFS, Chapter 7 Section 4] or [PW, p.149].

**Lemma 2.6.** *There exists a positive constant* $C > 0$ *such that for every* $x \in \Lambda \setminus \mathcal{O}$ *with* $x \in I_{i_1\dots i_n}$ *we have*

$$\frac{1}{C} \leqslant \sup_{n\geqslant 0} \sup_{y\in I_{i_1\dots i_n}} \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leqslant C.$$

2.3. **Thermodynamic formalism and Hausdorff dimension.** Thermodynamic formalism is a set of tools brought to ergodic theory from statistical mechanics in the 1960's that allows for the choice of relevant invariant probability measures. It has surprising and relevant applications in the dimension theory of dynamical systems. For EMR maps and regular potentials it has been extensively studied and it is fairly well understood (see for example [IJ, MU, PW, Sa]). We now summarize some of the results.

**Definition 2.7.** The *pressure* of $T$ at the point $t \in \mathbb{R}$ is defined by

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \left( \prod_{i=0}^{n-1} |T'(T^i x)|^{-t} \right).$$

It satisfies the following variational principle and approximation property.

**Proposition 2.8.** *For every $t \in \mathbb{R}$ we have*

$$\begin{aligned} P(t) &= \sup\left\{ h(\nu) - t\int \log|T'| \, d\nu : \nu \in \mathcal{M}_T \text{ and } \int \log|T'| \, d\nu < \infty \right\} \\ &= \sup\{P_K(t) : K \in \mathcal{K}\}, \end{aligned}$$

*where the space of $T-$invariant probability measures is denoted by $\mathcal{M}_T$, the entropy of the measure $\nu$ is denoted by $h(\nu)$ and the collection of sets*

$$\mathcal{K} := \{K \subset \Sigma : K \neq \emptyset \text{ compact and } T\text{-invariant}\}.$$

There is actually a precise description of the regularity properties of the pressure (see [IJ, Sub-sections 2.1 and 2.2])

**Proposition 2.9.** *There exists $s_\infty \in (0, \infty]$ such that pressure function $t \to P(t)$ has the following properties*

$$P(t) = \begin{cases} \infty & \text{if } t < s_\infty \\ \text{real analytic, strictly decreasing and strictly convex} & \text{if } t > s_\infty. \end{cases}$$

*Moreover, if $t > s_\infty$ then there exists a unique measure $\mu_t \in \mathcal{M}_T$, that we call* equilibrium measure for $t$*, such that $P(t) = h(\mu_t) - t\int \log|T'| \, d\mu_t$.*

It was noted by Bowen [Bow] in the finitely many branches setting (the compact case) that the pressure $P(t)$ captures a great deal of geometric information about $\Lambda$. This observation was first generalized to the EMR setting by Mauldin and Urbański in [MU, Theorems 3.15 and 3.21].

**Proposition 2.10.** *If $T$ is an EMR map then*

$$\dim_H(\Lambda) = \inf\{t \in \mathbb{R} : P(t) \leqslant 0\}.$$

*Moreover, if $s_\infty < \dim_H(\Lambda)$ there exists a unique ergodic measure $\nu \in \mathcal{M}_T$ such that $\dim_H \nu = \dim_H \Lambda$. This measure is called* measure of maximal dimension.

*Remark* 2.11. Note that, as opposite to the compact case, it is possible that $P(\dim_H(\Lambda)) < 0$, in this case there is no measure of maximal dimension. However, if $s_\infty < \dim_H(\Lambda)$ then $P(\dim_H(\Lambda)) = 0$ and there exists a measure of maximal dimension. Indeed, it follows from Proposition 2.8 that there exists a unique equilibrium measure $\nu$ for $-\dim_H(\Lambda)\log|T'|$. Since $s_\infty < \dim_H(\Lambda)$ we have that $\int \log|T'| d\nu < \infty$. It follows from results in [HR] that $\dim_H \nu = h(\nu)/\int \log|T'| d\nu$. But $\nu$ is an equilibrium measure for $-\dim_H(\Lambda)\log|T'|$ thus $h(\nu) - \dim_H(\Lambda)\int \log|T'| d\nu = 0$. Therefore

$$\dim_H \nu = \frac{h(\nu)}{\int \log|T'| d\nu} = \dim_H(\Lambda).$$

2.4. **Pressure and box dimension.** In this note we observe that the number $s_\infty$ has a dimension interpretation. It is a lower bound for the upper box dimension of the boundary points of the Markov partition. Actually, if the box dimension of such sets exists then it coincides with $s_\infty$.

**Theorem 2.12.** *Let $T$ be an EMR map then*

$$s_\infty \leqslant \overline{\dim}_B \left( \bigcup_{n=1}^{\infty} \{a_n, b_n\} \right).$$

*If the box dimension of $\bigcup_{n=1}^{\infty} \{a_n, b_n\}$ exists, then*

$$s_\infty = \dim_B \bigcup_{n=1}^{\infty} \{a_n, b_n\}.$$

*Proof.* Assume first that the map $T$ is piecewise linear. In this case the slope of $T$ restricted to the sub-interval $I_n = [a_n, b_n]$ satisfies $|T'| = (b_n - a_n)^{-1}$. The pressure function can be computed explicitly (see for example [BI, equation 9]). We have that

$$P(t) = \log \sum_{n=1}^{\infty} (b_n - a_n)^t. \tag{2.1}$$

Define

$$\underline{L} = \liminf_{n\to\infty} \frac{\log n}{-\log(b_n - a_n)} \quad \text{and} \quad \overline{L} = \limsup_{n\to\infty} \frac{\log n}{-\log(b_n - a_n)}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$ we have that

$$(b_n - a_n)^{\overline{L}+\varepsilon} < \frac{1}{n} < (b_n - a_n)^{\underline{L}-\varepsilon}.$$

In particular, for $r > 0$, we get that

$$\sum_{n=N}^{\infty} (b_n - a_n)^{r(\overline{L}+\varepsilon)} \leqslant \sum_{n=N}^{\infty} \frac{1}{n^r} \leqslant \sum_{n=N}^{\infty} (b_n - a_n)^{r(\underline{L}-\varepsilon)} \tag{2.2}$$

Combining equation (2.1) and inequality (2.2) we obtain that

$$P(r(\overline{L}+\varepsilon)) = \log \sum_{n=1}^{\infty} (b_n - a_n)^{r(\overline{L}+\varepsilon)} \leqslant \log \left( \sum_{n=1}^{N-1} (b_n - a_n)^{r(\overline{L}+\varepsilon)} + \sum_{n=N}^{\infty} \frac{1}{n^r} \right). \tag{2.3}$$

If $r > 1$, then the right hand side of (2.3) converges. It follows from the definition of $s_\infty$ (see Proposition 2.9) that $r(\overline{L} + \varepsilon) \geqslant s_\infty$. Since $r$ is an arbitrary number larger than 1, it follows that $\overline{L} + \varepsilon \geqslant s_\infty$. Since $\varepsilon$ is an arbitrary positive number, we conclude that $\overline{L} \geqslant s_\infty$. A similar argument using equation (2.1) and the right hand side of inequality (2.2) give us that $\underline{L} \leqslant s_\infty$. We obtained that

$$\underline{L} \leqslant s_\infty \leqslant \overline{L}. \tag{2.4}$$

Inequality (2.4) also holds in the general case; we can reduce it to the linear case. Indeed, by the Mean Value Theorem for every $n \in \mathbb{N}$ there exists $x \in [a_n, b_n]$ such that $|T'(x)| = (b_n - a_n)^{-1}$. By the Jacobian estimate (see Lemma 2.6) we have that if $y \in [a_n, b_n]$ then

$$\frac{(b_n - a_n)^{-1}}{C} \leqslant |T'(y)| \leqslant C(b_n - a_n)^{-1}.$$

Therefore

$$-t\log C+\log\sum_{n=1}^{\infty}(b_n-a_n)^t \leqslant P(t) \leqslant t\log C+\log\sum_{n=1}^{\infty}(b_n-a_n)^t.$$

Set $S:=\bigcup_{n=1}^{\infty}\{a_n,b_n\}$. Observe that by [Fa2, Proposition 3.6] and [Fa2, Proposition 3.7] we know that

$$\underline{L} \leqslant \underline{\dim}_B(S) \leqslant \overline{\dim}_B(S) \leqslant \overline{L},$$

and that

$$\frac{\underline{\dim}_B(S)(1-\overline{\dim}_B(S))}{(1-\underline{\dim}_B(S))} \leqslant \underline{L} \leqslant \overline{L} \leqslant \overline{\dim}_B(S).$$

In particular we have that $\overline{\dim}_B(S)=\overline{L}$. As mentioned in [Fa2, Corollary 3.8], the box dimension of $S$ exists if and only if $\underline{L}=\overline{L}$. It follows from this and inequality (2.4) that if $\underline{L}=\overline{L}$, then

$$s_\infty=\dim_B(S).$$

In general we only have the inequality

$$s_\infty \leqslant \overline{L}=\overline{\dim}_B(S).$$

□

*Example* 2.13. If $G$ is the Gauss map then the set of boundary points is $\{1/n : n\in\mathbb{N}\}$. It is well known that the box dimension of this set is equal to $1/2$ and that $s_\infty=1/2$ (see [IJ, PW]).

*Remark* 2.14. Note that if $T$ is and EMR map then, for any $k\geqslant 1$ the map $T^k$ satisfies all assumptions of an EMR map except for condition $(a)$. If we denote by $P_{T^k}(\cdot)$ the pressure associated to the dynamical system $T^k$ then a classical result relates it to the pressure of $T$ (see [Wa, Theorem 9.8 (i)]). Indeed, if $f:\Lambda\to\mathbb{R}$ is a regular potential then

$$P_{T^k}(S_kf)=kP_T(f),$$

where $S_kf$ is the Birkhoff sum of $f$. In particular, if $f=\log|T'|$ then

$$P_{T^k}(-t\log|(T^k)'|)=kP_T(-t\log|T'|).$$

Therefore the number $s_\infty(T)$ corresponding to $T$ coincides with $s_\infty(T^k)$ corresponding to $T^k$. Since condition $(a)$ in the definition of EMR map is not used in the proof of Theorem 2.12 we have $s_\infty$ is a lower bound for the upper box dimension of the boundary points of the Markov partition of $T^k$, for any $k\geqslant 1$.

2.5. **Compact perturbations and measures of maximal dimension.** The following is a consequence of Theorem 2.12 and Proposition 2.10.

**Corollary 2.15.** *Let $T$ be an EMR map. If $\overline{\dim}_B\left(\bigcup_{n=1}^{\infty}\{a_n,b_n\}\right)<\dim_H(\Lambda)$ then there exists a measure of maximal dimension.*

We now define compact perturbations of the map $T$.

**Definition 2.16.** Let $T$ be an EMR map defined on the sequence of closed intervals $(I_n)_n$. We say that $\tilde{T}$ is a *compact perturbation* of $T$ if the following two conditions are satisfied:

(a) There exists a compact subset $K\subset(0,1]$ with the property that if $\operatorname{int} I_n\cap K\neq\varnothing$ then $I_n\subset K$ and $T(x)=\tilde{T}(x)$ for every $x\in(\cup_{i=1}^{\infty}I_i)\setminus K$.

(b) The map $\tilde{T}$ is an EMR map.

The following result shows that the beahviour of pressure at $s_\infty$ not only determines the existence of measures of maximal dimension for the map $T$, but also for compact perturbations of it.

**Corollary 2.17.** *Let $T$ be an EMR map such that $P(s_\infty) = \infty$ then any compact perturbation $\tilde{T}$ of $T$ has a measure of maximal dimension.*

*Proof.* Note that for any compact perturbation $\tilde{T}$ the number $s_\infty$ corresponding to $\tilde{T}$ is equal to that of $T$. Moreover the pressure evaluated at that point is infinity in both cases. Therefore $\tilde{T}$ has a measure of maximal dimension. □

*Remark* 2.18. If $T$ is an EMR map with $s_\infty < \dim_H(\Lambda)$ but with $P(s_\infty) < \infty$ then there exists compact perturbations without measures of maximal dimension. An example can be constructed considering the partition of $[0, 1]$ given by the sequence defined by $a_n = 1/(n(\log n)^2)$, see [I, Example 3.1].

2.6. **Extremes of the multifractal spectrum.** The following result is a consequence of Theorem 2.12 and results obtained by Fan, Jordan, Liao and Rams [FJLR].

**Corollary 2.19.** *Let $T$ be an EMR map for which the limit $\lim_{n\to\infty} \frac{\log n}{-\log(b_n - a_n)}$ exist. Let $\varphi : \Lambda \to \mathbb{R}$ a bounded Hölder potential. Then for every*

$$\alpha \in \left[ \inf \left\{ \int \varphi d\mu : \mu \in \mathcal{M}_T \right\}, \sup \left\{ \int \varphi d\mu : \mu \in \mathcal{M}_T \right\} \right]$$

*we have that*

$$\dim_H \left( \left\{ x \in \Lambda : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \alpha \right\} \right) \geqslant \dim_B \left( \bigcup_{n=1}^{\infty} \{a_n, b_n\} \right).$$

The next result is a consequence of Theorem 2.12 and [IJ, Theorem 7.1].

**Corollary 2.20.** *Let $T$ be an EMR map for which the limit $\lim_{n\to\infty} \frac{\log n}{-\log(b_n - a_n)}$ exist. Then*

$$\dim_H \left( \left\{ x \in \Lambda : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |T^i x| = \infty \right\} \right) = \dim_B \left( \bigcup_{n=1}^{\infty} \{a_n, b_n\} \right).$$

## 3. Geodesic flow on a negatively curved manifold

In this section we will relate the box dimension of points at the boundary at infinity of Hadamard manifolds with some critical exponents. A good reference for the notation and language used in this section is [BH]. Let $(X, g)$ be a pinched negatively curved Hadamard manifold. We will moreover assume that the sectional curvature of $X$ is bounded above by $-1$, in other words, that $X$ is a Cat$(-1)$ space. The main example of a Cat$(-1)$ space is hyperbolic space $\mathbb{H}^n$, where the sectional curvature is constant equal to $-1$. The Gromov boundary of $X$ is denoted by $\partial X$. The Gromov product of $\xi \in \partial X$ and $\xi' \in \partial X$ with respect to the point $x \in X$ is the number

$$(\xi|\xi')_x = \frac{B_\xi(x, z) + B_{\xi'}(x, z)}{2},$$

where $B_\xi(p, q)$ is the Busemann function and $z$ is a point in the geodesic connecting $\xi$ and $\xi'$. The Gromov product is independent of the choice of the point $z$. For now on we fix a reference point $o \in X$ and we use the notation $(\xi|\xi') := (\xi|\xi')_o$. Define $d_{\partial X} : \partial X \times \partial X \to \mathbb{R}_{\geqslant 0}$, by the formula

$$d_{\partial X}(\xi, \xi') = \begin{cases} e^{-(\xi|\xi')} & \text{if } \xi \neq \xi', \\ 0 & \text{if } \xi = \xi'. \end{cases}$$

**Proposition 3.1.** [Bou] *The function $d_{\partial X} : \partial X \times \partial X \to \mathbb{R}_{\geqslant 0}$ is a metric on $\partial X$.*

The metric $d_{\partial X}$ was introduced by Bourdon in [Bou] and defines a canonical conformal structure on $\partial X$ (which is independent of the reference point $o$). Given a geodesic triangle with vertices $x$, $y$ and $z$ we associate a comparison hyperbolic triangle with vertices $A$, $B$ and $C$, such that $d_{\mathbb{H}^2}(A, B) = d(x, y)$, $d_{\mathbb{H}^2}(B, C) = d(y, z)$ and $d_{\mathbb{H}^2}(C, A) = d(z, x)$, where $d$ is the Riemannian distance of $X$ and $d_{\mathbb{H}^2}$ the Riemannian distance of $\mathbb{H}^2$. The angle at $x$ of the geodesic triangle $xyz$ is defined as the angle at $A$ of the hyperbolic triangle $ABC$ and it is denoted by $\angle_x(y, z)$. The following lemma follows directly from the hyperbolic law of cosines.

**Lemma 3.2.** *Given $D > 0$, there exists a constant $C = C(D) > 0$, such that for every geodesic triangle with vertices $x, y, z$, and angle at $z$ bigger than $D$, then*

$$d(x, y) \geqslant d(z, x) + d(z, y) - C.$$

To a group of isometries $G$ we can associate a non-negative number, the so called critical exponent of $G$. A fundamental property of the critical exponent is the following: if $G$ acts free and properly discontinuous on $X$ and $G$ is non-elementary, then the topological entropy of the geodesic flow on $X/G$ is equal to the critical exponent of $G$ (see [OP]).

**Definition 3.3.** Let $G \subset Iso(X)$ be a group of isometries. The *Poincaré series* of G is defined by

$$\mathcal{P}(s) = \sum_{g \in G} e^{-sd(o, go)}.$$

The *critical exponent* $\delta_G$ is defined by

$$\delta_G := \inf \{s \in \mathbb{R} : \mathcal{P}(s) < \infty\}.$$

We remark that the critical exponent of $G$ is independent of the reference point $o \in X$. A group $G \subset Iso(X)$ is called *parabolic* if there exists a point $w \in \partial X$ that is fixed by the action of $G$. We will now verify that the same box dimension interpretation obtained for EMR maps in Theorem 2.12 holds for the action of parabolic groups on the Gromov boundary $\partial X$.

**Proposition 3.4.** *Let $(X, g)$ be a $Cat(-1)$ surface. Let $p \in Iso(X)$ be parabolic isometry and $\xi \in \partial X$ a point that is not fixed by $p$. Then*

$$\liminf_{k \to \infty} \frac{\log k}{-\log d_{\partial X}(p^k \xi, p^{k+1} \xi)} \leqslant \delta_{\langle p \rangle} \leqslant \limsup_{k \to \infty} \frac{\log k}{-\log d_{\partial X}(p^k \xi, p^{k+1} \xi)}.$$

*If the sequence $\left(\frac{\log k}{-\log d_{\partial X}(p^k \xi, p^{k+1} \xi)}\right)_k$ converges as $k$ goes to infinity, then*

$$\lim_{k \to \infty} \frac{\log k}{-\log d_{\partial X}(p^k \xi, p^{k+1} \xi)} = \delta_{\langle p \rangle}.$$

*Proof.* We denote by $\eta$ the fixed point of $p$. Choose a point $z$ in the geodesic connecting $\xi$ and $p\xi$. Observe that $p^k z$ belongs to the geodesic connecting $p^k\xi$ and $p^{k+1}\xi$. Note that

$$(p^k\xi|p^{k+1}\xi) = \frac{1}{2}\left(B_{p^k\xi}(o,p^kz) + B_{p^{k+1}\xi}(o,p^kz)\right) = \frac{1}{2}\left(B_\xi(p^{-k}o,z) + B_{p\xi}(p^{-k}o,z)\right).$$

Since $B_q(x,y) \leqslant d(x,y)$, for all $q \in \partial X$ and $x, y \in X$ we conclude that

$$(p^k\xi|p^{k+1}\xi) \leqslant d(p^{-k}o,z) \leqslant d(p^{-k}o,o) + d(o,z) = d(o,p^ko) + d(o,z). \tag{3.1}$$

Let $\alpha(t)$ be a parametrization of the geodesic ray starting at $z$ and converging to $\xi$. If $|k|$ is sufficiently large, then $p^{-k}o$ will be in a small neighborhood of $\eta$ in $X \cup \partial X$. In particular if $|k|$ and $t$ are sufficiently large, then the angle at $z$ of the geodesic triangle with vertices $z$, $p^{-k}o$, and $\alpha(t)$ is uniformly bounded below (it will be close to the angle at $z$ between the geodesic rays $[z,\xi)$ and $[z,\eta)$). It follows from Lemma 3.2 that for sufficiently large values of $t$ and $|k|$ we have

$$d(\alpha(t),p^{-k}o) - d(\alpha(t),z) \geqslant d(p^{-k}o,z) - C,$$

for some positive constant $C = C(\xi,\eta)$. By definition of the Busemann function we obtain that

$$\begin{aligned} B_\xi(p^{-k}o,z) = \lim_{t\to\infty} d(\alpha(t),p^{-k}o) - d(\alpha(t),z) &\geqslant d(p^{-k}o,z) - C \\ &\geqslant d(p^{-k}o,o) - d(o,z) - C \\ &= d(o,p^ko) - d(o,z) - C. \end{aligned}$$

An analogous argument gives that there exists a constant $C' > 0$ (independent of $k$) such that for $|k|$ sufficiently large we have

$$B_{p\xi}(p^{-k}o,z) \geqslant d(p^{-k}o,z) - C' \geqslant d(o,p^ko) - d(o,z) - C'.$$

We conclude that for $|k|$ sufficiently large we have

$$(p^k\xi|p^{k+1}\xi) \geqslant d(o,p^ko) - c, \tag{3.2}$$

for some constant $c > 0$ independent of $k$. In order to conclude the proof we will need the following fact.

**Lemma 3.5.** *The following inequality holds*

$$\liminf_{k\to\infty} \frac{\log k}{d(o,p^ko)} \leqslant \delta_{\langle p\rangle} \leqslant \limsup_{k\to\infty} \frac{\log k}{d(o,p^ko)}. \tag{3.3}$$

*If the sequence* $\left\{\frac{\log k}{d(o,p^ko)}\right\}_k$ *converges as $k$ goes to infinity, then*

$$\delta_{\langle p\rangle} = \lim_{k\to\infty} \frac{\log k}{d(o,p^ko)}.$$

*Proof.* Observe that if $\limsup_{n\to\infty} \frac{\log n}{\log a_n^{-1}} < 1$, then the series $\sum_n a_n$ converges. Applying this fact to $a_n = e^{-sd(o,p^no)}$ we obtain that if

$$\limsup_{n\to\infty} \frac{\log n}{d(o,p^no)} < s,$$

then $\mathcal{P}(s)$ converges. This immediately implies that $\delta_{\langle p\rangle} \leqslant \limsup_{n\to\infty} \frac{\log n}{d(o,p^n o)}$. Similarly, if $\liminf_{n\to\infty} \frac{\log n}{\log a_n^{-1}} > 1$, then the series $\sum a_n$ diverges. Applying this to $a_n = e^{-sd(o,p^n o)}$ we obtain that

$$\delta_{\langle p\rangle} \geqslant \liminf_{n\to\infty} \frac{\log n}{d(o,p^n o)}.$$

□

Inequalities (3.1) and (3.2) imply that in inequality (3.3) we can replace $d(o, p^k o)$ by $-\log d_{\partial X}(p^k\xi, p^{k+1}\xi) = (p^k\xi|p^{k+1}\xi)$. □

If $(X, g)$ is the hyperbolic disk, then $d_{\partial X}$ is related to the angle at $o$ between the geodesic rays $[o, \xi)$ and $[o, \xi')$. For now on we assume that $o$ is the origin of the hyperbolic disk. Bourdon [Bou] proved that

$$d_{\partial\mathbb{H}}(\xi, \xi') = \sin \frac{1}{2}\angle_o(\xi, \xi'),$$

where $\angle_o(\xi, \xi')$ is the angle (using radians) between the (straight) rays $[o, \xi)$ and $[o, \xi')$. The euclidean metric on $\mathbb{R}^2$ induces a metric on $\partial\mathbb{H}$. We denote such metric by $d_1$, and it is given by the formula

$$d_1(\xi, \xi') = \angle_o(\xi, \xi').$$

**Corollary 3.6.** *Let $\xi \in \partial\mathbb{H}$ and $p \in Iso(\mathbb{H})$ a parabolic isometry. Assume that $\xi$ is not the fixed point of $p$. Then*

$$\delta_{\langle p\rangle} = \dim_B \left(\bigcup_{k\in\mathbb{Z}} p^k\xi\right) = \frac{1}{2},$$

*where the box dimension is computed using the spherical metric on $\partial\mathbb{H}$.*

*Proof.* Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and $\lim_{k\to\infty} \angle_o(p^k\xi, p^{k+1}\xi) = 0$, one can easily check that in Proposition 3.4 it is possible to replace $d_{\partial X}$ by $d_1$. In the hyperbolic disk it is known that the sequence $(d(o, p^n o) - 2\log n)_n$ is bounded (see proof of Lemma 3.11). In particular we know that $\lim_{k\to\infty} \frac{\log k}{d(o,p^k o)} = \frac{1}{2}$. This implies that equality holds in inequality (3.3). Since the metric $d_1$ is equivalent to the flat metric on $\mathbb{R}$ we conclude the desired result. □

We can generalize Corollary 3.6 to higher dimensions. Fix a reference point $o \in \mathbb{H}^n$. For a discrete, torsion free group of isometries $\Gamma \subset Iso(\mathbb{H}^n)$ we define the *limit set of* $\Gamma$ as the set of accumulation points of $\Gamma.o$, and denoted it by $\Lambda(\Gamma)$. The limit set of $\Gamma$ is a subset of $\partial\mathbb{H}^n$ and it is independent of the base point $o$. We say that $\Gamma$ is *non-elementary* if it is not a parabolic subgroup, nor generated by one hyperbolic element of $Iso(\mathbb{H}^n)$. If $\Gamma$ is non-elementary we have a very nice characterization of $\Lambda(\Gamma)$: it is the minimal $\Gamma$-invariant closed subset of $\partial\mathbb{H}^n$ (for instance see [BH]). We will need the following result of Stratmann and Urbánski (see [SU]).

**Theorem 3.7.** *Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite manifold. Then*

$$\delta_\Gamma = \dim_H \Lambda(\Gamma) = \dim_B \Lambda(\Gamma),$$

*where the box and Hausdorff dimension are computed using the spherical metric on $\partial\mathbb{H}^n$.*

We remark that for hyperbolic geometrically finite manifolds the equality $\delta_\Gamma = \dim_H \Lambda(\Gamma)$, was proved by Sullivan in [Sul]. This result was latter generalized by Bishop and Jones to cover all hyperbolic manifolds (see [BJ]). More precisely they proved that $\delta_\Gamma = \dim_H \Lambda_{rad}(\Gamma)$, where $\Lambda_{rad}(\Gamma)$ is the radial limit set of $\Gamma$ (for precise definitions we refer the reader to [BJ]). If $G$ is a group we denote by $G^*$ to $G \smallsetminus \{id\}$. We will need the following definition.

**Definition 3.8.** Let $F_1$ and $F_2$ be discrete, torsion free subgroups of $Iso(\mathbb{H}^n)$. We say that $F_1$ and $F_2$ are in Schottky position if there exist disjoint closed subsets $U_{F_1}$ and $U_{F_2}$ of $\partial\mathbb{H}^n$ such that $F_1^*(\partial\mathbb{H}^n \smallsetminus U_{F_1}) \subset U_{F_1}$ and $F_2^*(\partial\mathbb{H}^n \smallsetminus U_{F_2}) \subset U_{F_2}$.

We now state one of the main results of this section.

**Theorem 3.9.** *Let $\mathcal{P} \subset Iso(\mathbb{H}^n)$ be a parabolic subgroup and $\xi \in \partial\mathbb{H}^n$ a point not fixed by $\mathcal{P}$. Then*

$$\delta_{\mathcal{P}} = \dim_B \left( \bigcup_{p \in \mathcal{P}} p\xi \right),$$

*where the box dimension is computed using the spherical metric on $\partial\mathbb{H}^n$.*

*Remark* 3.10. Theorem 3.9 does not immediately follow from Theorem 3.7 since the group $\mathcal{P}$ is not geometrically finite. Moreover, the group $\mathcal{P}$ is elementary and $\Lambda(\mathcal{P}) = \{\eta\}$, where $\eta$ is the point fixed by $\mathcal{P}$. Nevertheless, using box dimension we can recover the critical exponent from the orbit of $\xi$ under $\mathcal{P}$.

*Proof.* We first prove the inequality $\overline{\dim}_B \mathcal{P}\xi \leqslant \delta_{\mathcal{P}}$. There exists a hyperbolic isometry $h$ which fixes $\xi$ and such that $\langle h \rangle$ and $\mathcal{P}$ are in Schottky position. Indeed, let $\mathcal{V}_{\mathcal{P}}$ be a fundamental domain of the action of $\mathcal{P}$ on $\partial\mathbb{H}^n$ such that $\xi \in \operatorname{int} \mathcal{V}_{\mathcal{P}}$, and define $\mathcal{U}_{\mathcal{P}} = \partial\mathbb{H}^n \smallsetminus \mathcal{V}_{\mathcal{P}}$. Observe that for every $p \in \mathcal{P} \smallsetminus \{id\}$ we have that $p(\partial\mathbb{H}^n \smallsetminus \mathcal{U}_{\mathcal{P}}) \subset \mathcal{U}_{\mathcal{P}}$. Now choose a hyperbolic isometry $w$ such that $w(\xi) = \xi$, and such that the other fixed point of $w$, say $\xi_0$, also belongs to $\operatorname{int} \mathcal{V}_{\mathcal{P}}$. If one choose $k$ large enough, then $h = w^k$ will satisfy the properties stated above: if $k$ is large enough, then we can find a neighborhood $\mathcal{U}_h$ of $\{\xi, \xi_0\}$ such that $\mathcal{U}_h \subset \mathcal{V}_{\mathcal{P}} = \partial\mathbb{H}^n \smallsetminus \mathcal{U}_{\mathcal{P}}$ and $h^s(\partial\mathbb{H}^n \smallsetminus \mathcal{U}_h) \subset \mathcal{U}_h$, for every $s \neq 0$. This implies that $\langle h \rangle$ and $\mathcal{P}$ are in Schottky position.

Define $\Gamma_k = \mathcal{P} * \langle h^k \rangle$. We can now use [IRV, Proposition 5.3] to conclude that $\lim_{k\to\infty} \delta_{\Gamma_k} = \delta_{\mathcal{P}}$. Moreover, the manifold $\mathbb{H}^n/\Gamma_k$ is geometrically finite (see [DP] for a proof when $\mathcal{P}$ has rank one, but the same argument applies to $\Gamma_k$). By construction $\xi \in \Lambda(\Gamma_k)$, therefore $\Gamma_k\xi \subset \Lambda(\Gamma_k)$; in particular $\overline{\Gamma_k\xi} \subset \Lambda(\Gamma_k)$. Observe that $\overline{\Gamma_k\xi}$ is a closed $\Gamma_k$-invariant subset of $\partial\mathbb{H}^n$. Since $\Gamma_k$ is non-elementary we conclude that $\overline{\Gamma_k\xi} = \Lambda(\Gamma_k)$. We obtained that $\mathcal{P}\xi \subset \Gamma_k\xi \subset \overline{\Gamma_k\xi} = \Lambda(\Gamma_k)$. Using Theorem 3.7 and this inclusion we conclude that

$$\overline{\dim}_B \mathcal{P}\xi \leqslant \overline{\dim}_B \Lambda(\Gamma_k) = \dim_B \Lambda(\Gamma_k) = \delta_{\Gamma_k},$$

for every $k \in \mathbb{N}$. Therefore

$$\overline{\dim}_B \mathcal{P}\xi \leqslant \lim_{k\to\infty} \delta_{\Gamma_k} = \delta_{\mathcal{P}}.$$

We now prove the inequality $\underline{\dim}_B \mathcal{P}\xi \geqslant \delta_{\mathcal{P}}$. It will be convenient to consider the upper half-space model of hyperbolic space. We identify $\mathbb{H}^n$ with $\{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$, and the fixed point of $\mathcal{P}$ with the point of $\partial\mathbb{H}^n$ whose $x_n$–coordinate is $\infty$ (the rest of $\partial\mathbb{H}^n$ is the plane $x_n = 0$). Our reference point will be $o = (0, ..., 0, 1)$ (this point corresponds to the origin in the ball model). Via the action of $\mathcal{P}$ on the horosphere $x_n = 1$, we can identify $\mathcal{P}$ with a subgroup of $\text{Aff}(\mathbb{R}^{n-1})$, where $\text{Aff}(\mathbb{R}^{n-1})$ is the space of affine transformations of $\mathbb{R}^{n-1}$. It follows from Bieberbach's theorem that there exists a finite index subgroup $\mathcal{P}_0$ of $\mathcal{P}$ such that $\mathcal{P}_0$ acts on $x_n = 1$ as a group of translations. In particular $\mathcal{P}_0 \cong \mathbb{Z}^k$, for some $k \in \{1, ..., n-1\}$. The number $k$ is called the rank of $\mathcal{P}$. It is well known that $\delta_{\mathcal{P}} = \delta_{\mathcal{P}_0} = \frac{k}{2}$ (see for instance [DOP, Section 3]). There exists a sequence $\{g_1, ..., g_m\} \subset \mathcal{P}$ such that $\mathcal{P} = \bigcup_{i=1}^m \mathcal{P}_0 g_i$, in particular $\mathcal{P}\xi = \bigcup_{i=1}^m \mathcal{P}_0 g_i \xi$. We will prove that

$$\underline{\dim}_B \mathcal{P}_0 \eta \geqslant \frac{k}{2}, \tag{3.4}$$

for every $\eta \in \partial \mathbb{H}^n$. Assume that inequality (3.4) holds. As mentioned before $\delta_{\mathcal{P}_0} = \frac{k}{2}$. Combining inequality (3.4) with the inequality $\overline{\dim}_B \mathcal{P}_0 \eta \leqslant \delta_{\mathcal{P}_0}$, we obtain that $\dim_B \mathcal{P}_0 \eta = \frac{k}{2} = \delta_{\mathcal{P}_0} = \delta_{\mathcal{P}}$, for every $\eta \in \partial \mathbb{H}^n$. Since the box dimension is finitely additive we can conclude that $\dim_B \mathcal{P}\xi = \frac{k}{2} = \delta_{\mathcal{P}}$ (recall that $\mathcal{P}\xi = \bigcup_{i=1}^m \mathcal{P}_0 g_i \xi$).

We now prove inequality (3.4). We will identify the group $\mathcal{P}_0$ with $\mathbb{Z}^k$, and use the notation $n.\xi$ to denote the translate of $\xi$ under $n \in \mathbb{Z}^k$. The standard generators of $\mathbb{Z}^k$ will be denoted by $\{e_1, ..., e_k\}$. Denote by $\rho$ the fixed point of $\mathcal{P}_0$. Since $\lim_{k \to \infty} k.\eta = \rho$, we can conclude that there exists $C > 0$ such that the geodesic connecting $\eta$ and $k.\eta$ is at distance at most $C$ from $o$. We choose $z_k \in \mathbb{H}^n$ in the geodesic connecting $\eta$ and $k.\eta$ such that $d(o, z_k) \leqslant C$. Observe that $N.z_k$ belongs to the geodesic connecting $N.\eta$ and $(N+k).\eta$, therefore

$$\begin{aligned}
((N+k).\eta | N.\eta) =& \frac{1}{2}(B_{(N+k).\eta}(o, N.z_k) + B_{N.\eta}(o, N.z_k)) \\
=& \frac{1}{2}(B_{k.\eta}((-N).o, z_k) + B_{\eta}((-N).o, z_k)) \\
\leqslant& d((-N).o, z_k) \\
\leqslant& d(o, N.o) + d(o.z_k) \\
\leqslant& d(o, N.o) + C.
\end{aligned}$$

We conclude that

$$d_{\partial \mathbb{H}^n}((N+k).\eta, N.\eta) \geqslant e^{-d(o,N.o)-C}.$$

In order to conclude the proof we will need the following result.

**Lemma 3.11.** *Using the notation above we have*

$$\lim_{t \to \infty} \frac{\# \log\{N \in \mathbb{Z}^k : d(o, N.o) \leqslant t\}}{t} = \frac{k}{2}.$$

*Proof.* Recall that $o = (0, ..., 0, 1)$ is the origin of hyperbolic space. Let $\alpha_i \in \mathbb{R}^{n-1}$ be the translation vector of $e_i$ while acting on the horosphere $x_n = 1$. If follows from the definition of the hyperbolic metric on $\mathbb{H}^n$ that

$$d(o, N.o) = 2\text{arcsinh}\left(\frac{1}{2}|N_1\alpha_1 + ... + N_k\alpha_k|\right),$$

where $|.|$ stands for the euclidean metric on $\mathbb{R}^{n-1}$. It follows that the set

$$\{d(o, N.o) - \log(|N_1\alpha_1 + ... + N_k\alpha_k|^2) : N = (N_1, ..., N_k) \in \mathbb{Z}^k\},$$

is bounded. It is enough to prove that

$$\lim_{t \to \infty} \frac{\# \log\{N \in \mathbb{Z}^k : \log(|N_1\alpha_1 + ... + N_k\alpha_k|^2) \leqslant t\}}{t} = \frac{k}{2}. \tag{3.5}$$

There exists $D = D(\alpha_1, ..., \alpha_n) > 0$ such that the set $A_t := \{N \in \mathbb{R}^n : |N_1\alpha_1 + ... + N_k\alpha_k| \leqslant e^{t/2}\}$, contains a cube centered at the origin of radius $D^{-1}e^{\frac{t}{2}}$, and it is contained in a cube centered at the origin of radius $De^{\frac{t}{2}}$. For $t$ large the number of integral points in $A_t$ will be of the order $e^{tk/2}$. This immediately implies the equality (3.5). □

It follows from Lemma 3.11 that given $\varepsilon > 0$, there exists $E = E(\varepsilon)$ such that for $t \geqslant E$ we have

$$\#\{N \in \mathbb{Z}^k : d(o, N.o) \leqslant t\} \geqslant e^{(\frac{k}{2}-\varepsilon)t}.$$

Assume that $r$ is sufficiently small in order to have $\log(e^{-C}/r) \geqslant E$, and define $A_r := \{N \in \mathbb{Z}^k : d(o, N.o) \leqslant \log(e^{-C}/r)\}$. We conclude that

$$|A_r| = \#\{N \in \mathbb{Z}^k : d(o, N.o) \leqslant \log(e^{-C}/r)\} \geqslant \frac{e^{-C(\frac{k}{2}-\varepsilon)}}{r^{\frac{k}{2}-\varepsilon}}.$$

Observe that inequality $d(o, N.o) \leqslant \log(e^{-C}/r)$, is equivalent to $r \leqslant e^{-d(o,N.o)-C}$. Therefore $N \in A_r$ implies that for all $k \in \mathbb{Z}^k$ we have

$$\begin{aligned} r \leqslant e^{-d(o,N.o)-C} &\leqslant d_{\partial\mathbb{H}^n}((N+k).\eta, N.\eta) = \sin\frac{1}{2}d_1((N+k).\eta, N.\eta) \\ &\leqslant \frac{1}{2}d_1((N+k).\eta, N.\eta) \end{aligned}$$

where $d_1$ is the spherical metric on $\partial\mathbb{H}^n$ (coming from the natural embedding of the ball model into $\mathbb{R}^n$). In other words, to cover $\mathcal{P}_0.\eta$ with $d_1$-balls of radius $r$ we need at least $|A_r|$ balls (at least one ball for each $N.\eta$, where $N \in A_r$). We conclude that

$$\liminf_{r\to 0} \frac{\log N(r)}{\log(\frac{1}{r})} \geqslant \liminf_{t\to 0} \frac{\log|A_r|}{\log(\frac{1}{r})} \geqslant \liminf_{r\to 0} \frac{\log\left(\frac{e^{-C(\frac{k}{2}-\varepsilon)}}{r^{\frac{k}{2}-\varepsilon}}\right)}{\log\left(\frac{1}{2r}\right)} = \frac{k}{2} - \varepsilon.$$

Since $\varepsilon$ was arbitrary we conclude that $\underline{\dim}_B \mathcal{P}_0\eta \geqslant \frac{k}{2}$.

□

*Remark* 3.12. Roblin proved in [Rob] that if $\Gamma$ is non-elementary, then

$$\lim_{t\to\infty} \frac{\#\log\{\gamma \in \Gamma : d(o, \gamma o) \leqslant t\}}{t} = \delta_\Gamma.$$

Since $\mathcal{P}_0$ is elementary we can not use Roblin's result. The proof of Roblin uses in an essential way that the group is non-elementary, it uses the Patterson-Sullivan conformal density at infinity associated to $\Gamma$.

## References


[BI] Barreira, Luis; Iommi, Godofredo. *Suspension flows over countable Markov shifts.* J. Stat. Phys. 124 (2006), no. 1, 207–230. (Cited on page 5.)

[BJ] Bishop, Christopher; Jones, Peter. *Hausdorff dimension and Kleinian groups.* Acta Math. 179 (1997), no. 1, 1–39. (Cited on pages 2 and 11.)

[Bou] Bourdon, Marc. *Structure conforme au bord et flot géodésique d'un CAT(-1)-espace.* (French) Enseign. Math. (2) **41** (1995), no. 1-2, 63–102. (Cited on pages 8 and 10.)

[Bow] Bowen, Rufus. *Hausdorff dimension of quasicircles.* Inst. Hautes Etudes Sci. Publ. Math. No. 50 (1979), 11–25. (Cited on pages 1 and 4.)

[BH] Bridson, Martin; Haefliger, André. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. xxii+643 pp. (Cited on pages 7 and 10.)

[CFS] Cornfeld, I.P. ; Fomin, S. V.; Sinai, Ya. G. *Ergodic theory.* Grundlehren der Mathematischen Wissenschaften, 245. Springer-Verlag, New York, 1982. x+486 pp. (Cited on page 3.)

[DP] Dal'bo, Francoise; Peigné, Marc. *Some negatively curved manifolds with cusps, mixing and counting.* J. Reine Angew. Math. 497 (1998), 141–169. (Cited on pages 1 and 11.)

[DOP] Dal'bo, Francoise; Otal, Jean-Pierre; Peigné, Marc. *Séries de Poincaré des groupes géométriquement finis.* Israel J. Math. 118 (2000), 109–124. (Cited on page 12.)

[Fa1] Falconer, Kenneth. *Fractal geometry. Mathematical foundations and applications.* John Wiley and Sons, Ltd., Chichester, 1990. xxii+288 pp. (Cited on page 2.)

[Fa2] Falconer, Kenneth. *Techniques in fractal geometry.* John Wiley & Sons, Ltd., Chichester, 1997. xviii+256 pp. (Cited on pages 2 and 6.)

[FJLR] Fan, Ai-Hua; Jordan, Thomas; Liao, Lingmin; Rams, Michal. *Multifractal analysis for expanding interval maps with infinitely many branches.* Trans. Amer. Math. Soc. 367 (2015), no. 3, 1847–1870. (Cited on page 7.)

[HR] Hofbauer, Franz; Raith, Peter *The Hausdorff dimension of an ergodic invariant measure for a piecewise monotonic map of the interval.* Canad. Math. Bull. 35 (1992), no. 1, 84–98. (Cited on page 4.)

[I] Iommi, Godofredo *Multifractal analysis for countable Markov shifts.* Ergodic Theory Dynam. Systems 25 (2005), no. 6, 1881–1907. (Cited on page 7.)

[IJ] Iommi, Godofredo; Jordan, Thomas. *Multifractal analysis of Birkhoff averages for countable Markov maps.* Ergodic Theory Dynam. Systems 35 (2015), no. 8, 2559–2586. (Cited on pages 3, 4, 6, and 7.)

[IRV] Iommi, Godofredo; Riquelme, Felipe; Velozo, Anibal. *Entropy in the cusp and phase transitions for geodesic flows* Israel J. Math. 225 (2018), no. 2, 609–659. (Cited on pages 1 and 11.)

[MU] Mauldin, R. Daniel; Urbański, Mariusz. *Dimensions and measures in infinite iterated function systems.* Proc. London Math. Soc. (3) 73 (1996), no. 1, 105–154. (Cited on page 4.)

[OP] Otal, Jean-Pierre; Peigné, Marc. *Principe variationnel et groupes kleiniens.* Duke Math. J. 125 (2004), no. 1, 15–44. (Cited on page 8.)

[PPS] Paulin, Frédéric; Pollicott, Mark; Schapira, Barbara. *Equilibrium states in negative curvature.* Asterisque No. 373 (2015), viii+281 pp. (Cited on page 2.)

[PW] Pollicott, Mark; Weiss, Howard. *Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation.* Comm. Math. Phys. 207 (1999), no. 1, 145–171. (Cited on pages 3, 4, and 6.)

[RV] Riquelme, Felipe; Velozo, Aníbal. *Escape of mass and entropy for geodesic flows.* To appear in Ergodic Theory Dynam. Systems. (Cited on page 2.)

[Rob] Roblin, Thomas. *Sur la fonction orbitale des groupes discrets en courbure négative.* Ann. Inst. Fourier (Grenoble) 52 (2002), no. 1, 145–151. (Cited on page 13.)

[Sa] Sarig, Omri. *Existence of Gibbs measures for countable Markov shifts.* Proc. Amer. Math. Soc. 131 (2003), no. 6, 1751–1758. (Cited on page 4.)

[SU] Stratmann, Bernd Otto; Urbański, Mariusz. *The box-counting dimension for geometrically finite Kleinian groups.* Fund. Math. 149 (1996), no. 1, 83–93. (Cited on page 10.)

[Sul] Sullivan, Dennis. *The density at infinity of a discrete group of hyperbolic motions.* Inst. Hautes tudes Sci. Publ. Math. No. 50 (1979), 171–202. (Cited on page 11.)

[Wa] Walters, Peter. *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics 79, Springer, 1981. (Cited on page 6.)

[V1] Velozo, Anibal. *Phase transitions for geodesic flows and the geometric potential.* arXiv:1704.02562 (Cited on page 2.)

[V2] Velozo, Anibal. *Entropy theory of geodesic flows*, arXiv:1711.06796. (Cited on page 2.)

Facultad de Matemáticas, Pontificia Universidad Católica de Chile (PUC), Avenida Vicuña Mackenna 4860, Santiago, Chile

*E-mail address*: giommi@mat.puc.cl

*URL*: http://http://www.mat.uc.cl/~giommi/

Princeton University, Princeton NJ 08544-1000, USA.

*E-mail address*: avelozo@math.princeton.edu

*URL*: https://web.math.princeton.edu/~avelozo/