Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**
Website: http://AIMsciences.org
pp. **X–XX**

MULTIFRACTAL ANALYSIS FOR THE EXPONENTIAL FAMILY

Godofredo Iommi†

Departamento de Matemática, Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Bartłomiej Skorulski‡

Institute of Mathematics, Polish Academy of Sciences
ul. Śniadeckich 8, 00-956 Warszawa, Poland

(Communicated by Aim Sciences)

Abstract. We study the multifractal spectrum for hyperbolic maps from the exponential family. We define a class of potentials for which we prove the existence of conformal measures. Next, we show that the multifractal spectrum of this conformal measure is the Legendre transform of the temperature function. We prove that the domain of the spectrum is unbounded and show that there are two possibilities for its shape.

1. **Introduction.** This note is devoted to the study of the multifractal analysis of conformal measures for the exponential family $E_\lambda(z) : \mathbb{C} \to \mathbb{C}$, where $E_\lambda(z) = \lambda \exp(z)$ and $\lambda \in (0, 1/e)$. Denote by $J(E_\lambda)$ the Julia set of E_λ. For the exponential map the Julia set has the following equivalent characterisations:

1. $J(E_\lambda)$ is the closure of the set of repelling periodic points of E_λ.
2. $J(E_\lambda)$ is the closure of the set of points whose orbit tend to infinity.

For the parameters that we consider the Julia set is a Cantor bouquet (see e.g. [2]).

There are two main difficulties when developing multifractal analysis for maps in this family. First, the Julia set is not compact and therefore the methods of the classical thermodynamic formalism do not apply. Secondly, the exponential map is not Markov over the Julia set, so the approach using symbolic dynamics has to be treated carefully and can not be applied directly. Our first tool to overcome these difficulties is a construction done by Urbański and Zdunik in [12]. They introduced a map F defined on a stripe of height 2π. This map can be used to study the dimension and ergodic properties of the map E_λ restricted to set of points of the Julia set that do not escape to infinity under iterates of E_λ. The construction goes along the following lines. Recall that E_λ is $2\pi i-$periodic. Consider the projection

$$\pi_0 : \mathbb{C} \to \mathcal{C} := \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) \leq \pi\}$$

2000 *Mathematics Subject Classification.* Primary 37F10, 37D35; Secondary 30D05, 37F35.

†Partially supported by Fundação para a Ciência e a Tecnologia by Program POCTI/FEDER, the grant SFRH/BPD/21927/2005 and by a Marie Curie fellowship while at the Banach Center, Warsaw.

‡ Partially supported by Polish KBN Grant No. 2PO3A03425 and Warsaw University of Technology Grant No. 504G11200023000. Currently MeceSup UCN0202 Postdoctoral position at Universidad Católica del Norte, Chile.

defined by $\pi_0(z) = w$ if, and only if, $w \in \mathcal{C}$ and $\exp(z) = \exp(w)$. The map $F = F_\lambda : \mathcal{C} \to \mathcal{C}$ is defined by $F(z) = \pi_0(E_\lambda(z))$. We will be interested in the dynamically relevant subset $J^r(F) \subset J(F) := \pi_0(J(E_\lambda))$, that it is defined by

$$J^r(F) = J^r := \pi_0(\{z \in J(E_\lambda) : \lim_{n\to\infty} E_\lambda^n(z) \neq \infty\})$$

and called *radial Julia set* of the map F. Urbański and Zdunik [12] constructed a sequence of finite iterated function systems that approximate the radial Julia set (see Section 2). We perform the multifractal analysis of measures supported on this set. Let $\log\phi : J(F) \to \mathbb{R}$ be a continuous potential. A finite Borel measure μ is called *ϕ-conformal* if it satisfies

$$\mu(F(A)) = \int_A \phi^{-1} d\mu$$

for any Borel set $A \subset \mathcal{C}$ where the map F is injective. In this note we will be interested in a family of conformal measures corresponding to a fixed class of potentials, denoted by $\mathcal{P}$ and defined in subsection (3.2). The *pointwise dimension* of a measure μ at the point $z \in \mathcal{C}$ is defined by

$$d_\mu(z) := \lim_{r\to 0} \frac{\log\mu(B(z,r))}{\log r},$$

provided the limit exists. The pointwise dimension of μ induces a decomposition of the space into level sets

$$\mathrm{Lev}(\alpha) := \{z \in J^r : d_\mu(z) = \alpha\} \text{ and}$$

$$\mathrm{Lev}' := \left\{z \in J^r : \text{ the limit } \lim_{r\to 0} \frac{\log\mu(B(z,r))}{\log r} \text{ does not exist}\right\}.$$

The decomposition of the radial Julia set

$$J^r = \left(\biguplus_\alpha \mathrm{Lev}(\alpha)\right) \biguplus \mathrm{Lev}'$$

is called *multifractal decomposition*. The *multifractal spectrum* of the pointwise dimension is defined by

$$f_\mu(\alpha) = \dim_H(\mathrm{Lev}(\alpha)),$$

where $\dim_H$ denotes the Hausdorff dimension. Our second tool is the multifractal analysis developed by Pesin and Weiss [8] (see also the work by Olsen [6]). They introduced techniques based on thermodynamic formalism that allow them to describe the multifractal spectrum. Their techniques hold for hyperbolic dynamical systems defined over compact spaces and for equilibrium measures corresponding to Hölder continuous potentials. They proved that the multifractal spectrum is real analytic, concave and that has bounded domain (the last part of this statement was proved by Schmeling in [10]). Moreover, Barreira and Schmeling [1] proved that the set Lev' has full Hausdorff dimension. Note that each finite iterated function system considered in the sequence constructed by Urbański and Zdunik (see section 2) satisfies the assumptions of the work by Pesin and Weiss.

In this note, following the ideas of Pesin and Weiss, we define an auxiliary function $T(q)$ in terms of a topological pressure (for precise definitions see Subsection (3.2) and Subsection (3.3)). We prove that for a conformal measure μ corresponding to a potential on the class $\mathcal{P}$ defined in Section (3.2)

Theorem. *The multifractal spectrum $f_\mu(\cdot)$ is the Legendre transform of $T(\cdot)$.*

We prove that the function $T(q)$ is infinite for negative values of q. In particular we obtain that, in strong contrast with the compact case, the multifractal spectrum has unbounded domain. Two possible shapes of the multifractal spectrum are described. Examples of each of these possibilities are provided.

Our proof is based on an approximation argument. We apply the techniques developed by Pesin and Weiss to each of the finite iterated function systems belonging to the sequence constructed by Urbański and Zdunik so that we obtain a mutlifractal spectrum restricted to each of the finite iterated function systems. We then prove that the limit of these functions corresponds to the multifractal spectrum. As a byproduct we also propose a definition of pressure for these type of systems and prove the existence of conformal measures.

We remark that the multifractal analysis on non compact settings has been studied in [5, 4, 9] and [3] among others. Also, in a recent preprint Urbański [11] studied the multifractal spectrum for the exponential family. However the approach and the class of potentials he consider are different from ours.

2. **Preliminaries.**

2.1. **Iterated Function System.**

Throughout the rest of the note we will assume $\lambda \in (0, 1/e)$ to be fixed. In this section we describe the construction done by Urbański and Zdunik (see [12] and [13]) of the sequence of iterated function systems that approximate the radial Julia set J^r. This construction allow them to give a geometrical description of this set. They prove that

$$\dim_H(J^r) < \dim_H(J(E_\lambda)) = 2$$

and the existence of a $\dim_H(J^r)-$conformal measure.

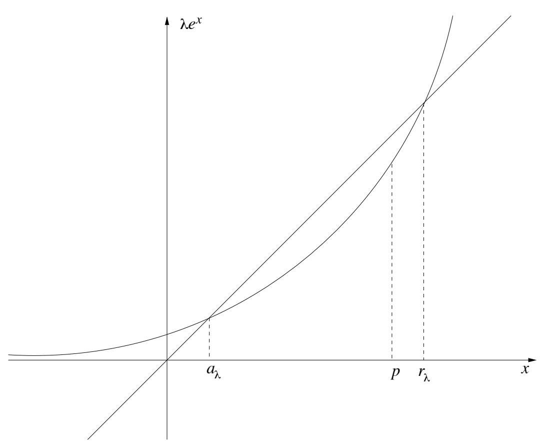

FIGURE 2.1. The graph of the function $E_\lambda|_{\mathbb{R}} : \mathbb{R} \ni x \mapsto \lambda e^x \in \mathbb{R}$.

The map E_λ has a unique attracting fixed point a_λ and $E_\lambda|_{\mathbb{R}}$ has a repelling fixed point r_λ (see Figure 2.1). Both are located on the real axis. Let p be a point in the open interval (a_λ, r_λ) such that $E'_\lambda(p) > 1$ and let us recall that $\mathcal{C} = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) \leq \pi\}$. Note that the set $\mathcal{C}_+ := \{z \in \mathcal{C} : -3\pi/4 \leq \mathrm{Im}(z) \leq 3\pi/4, \mathrm{Re}(z) \geq p\}$ is disjoint from the forward orbit of 0 under iterates of E_λ.

The function $F = F_\lambda : \mathcal{C} \to \mathcal{C}$ defined by $F(z) = \pi_0(E_\lambda(z))$ (see the introduction) has infinitely many inverse branches $\{L_j\}_{j\in\mathbb{Z}}$ such that $E_\lambda(L_j(z)) = z + 2j\pi i$ for each $z \in \mathcal{C}$. The set

$$J(F) := \bigcap_{n\in\mathbb{N}} \bigcup_{(j_1,\ldots,j_n)\in\mathbb{Z}^n} L_{j_1,\ldots,j_n}(\mathcal{C}_+), \quad \text{where } L_{j_1,\ldots,j_n} := L_{j_1} \circ \ldots \circ L_{j_n},$$

is such that $J(F) = \pi_0(J(E_\lambda))$.

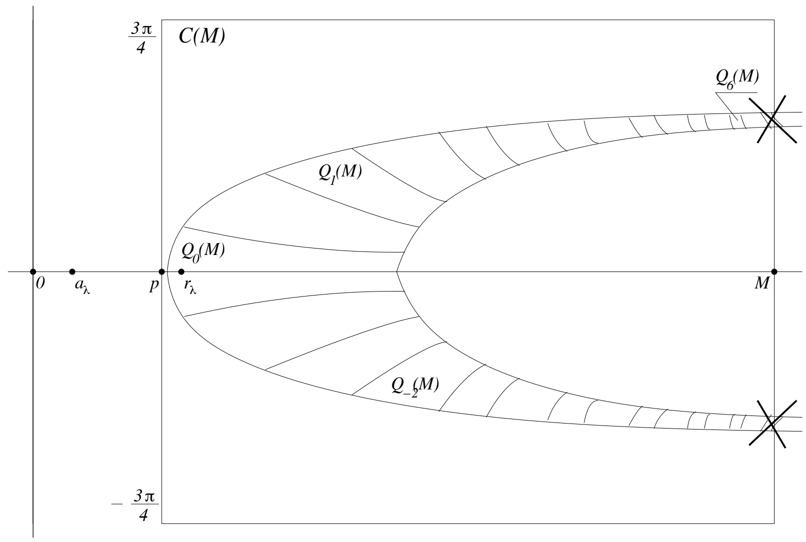

FIGURE 2.2. The construction of the finite iteration functions system. In this example $k(M) = 6$.

Let $\mathcal{C}(M) := \{z \in \mathcal{C}_+ : \mathrm{Re} z \le M\}$. The preimage $F^{-1}(\mathcal{C}(M))$ is the union of infinitely many disjoint topological disks $Q_j(M) := L_j(\mathcal{C}(M))$. Let $\mathcal{K}_M := \{j \in \mathbb{Z} : Q_j(M) \subset \mathrm{Int}(\mathcal{C}(M))\}$ and

$$k(M) := \max\{j \in \mathcal{K}_M\}. \tag{2.1}$$

Since the set $F^{-1}(\mathcal{C}(M))$ is symmetric we have that $k(M) = -\min\{j \in \mathcal{K}_M\}$. Hence $\mathcal{K}_M = \mathcal{K}(k(M))$, where $\mathcal{K}(k) := \{j \in \mathbb{Z} : |j| \le k\}$. Define

$$J_M := \bigcap_{n\in\mathbb{N}} \bigcup_{(j_1,\ldots,j_n)\in\mathcal{K}_M^n} L_{j_1,\ldots,j_n}(\mathcal{C}(M)).$$

Note that J_M is the limit set of a finite iterated function system (see Figure 2.2). Since $|F'(z)| = \lambda e^{\mathrm{Re}\, z}$, for sufficiently large values of M, we have that $J_M \subset J_{M+1}$ (see [12]). In this way we obtain a sequence of finite iterated function systems such that their limit sets form an increasing sequence. Moreover, they approximate the radial Julia set.

To conclude this section we give a lower and upper bound on $k(M)$.

Lemma 2.1.

$$\frac{\sqrt{(\lambda e^M)^2 - M^2} - \pi}{2\pi} \le k(M) \le \frac{\lambda e^M + \pi}{2\pi}.$$

Proof. Note that if $(2|j|-1)\pi \geq \lambda e^M$, then

$$\mathcal{C}(M) \cap L_j(\mathcal{C}_+) = \emptyset. \tag{2.2}$$

This follows combining the fact that if $\operatorname{Re} z \leq M$, then $|E_\lambda(z)| = \lambda e^{\operatorname{Re} z} \leq \lambda e^M$, and the estimate $|\operatorname{Im} E(z)| - 2|j|\pi \in [-\pi, \pi]$ for points $z \in L_j(\mathcal{C}_+)$. Therefore, from equation (2.2) we have the upper bound on $k(M)$, namely $k(M) \leq (\lambda e^M + \pi)/(2\pi)$.

To obtain a lower bound on $k(M)$ note that $L_0(C(M)) \subset C(M)$, for $M > r_\lambda$ (recall that r_λ denotes the real repelling fixed point). Moreover, if $E_\lambda(L_k(C(M))) \subset E_\lambda(C(M))$, then $|k| \leq k(M)$. Then, the lower bound follows. □

3. Multifractal Analysis.

3.1. **The Compact Case.** In this subsection we discuss some of the techniques introduced by Pesin and Weiss [8] to study the multifractal spectrum with respect to finite iterated function systems. Let $\log \phi : J_M \to \mathbb{R}$ be a Hölder continuous potential. Consider the dynamical system $F : J_M \to J_M$. The approach introduced in [8] is based on the study of an auxiliary function. For every $q \in \mathbb{R}$ consider the function $T_M(q)$ implicitly defined by

$$P_M(-T_M(q) \log |F'| + q \log \phi) = 0,$$

where P_M denotes the topological pressure (see [7, chapter 4] or [14, chapter 9]) defined over the set J_M. Using thermodynamic formalism it is possible to show that this function is analytic, decreasing and convex. Pesin and Weiss were able to relate this function to some multifractal spectrum in such a way that the properties of the function $T_M(\cdot)$ were inherited by the multifractal spectrum. They proved that both functions from a Legendre pair. See subsection 3.5 for details and precise statements.

Note that for fixed q the sequence $T_M(q)$ is increasing. Therefore, it has a pointwise limit (it might be infinity). For every $q \in \mathbb{R}$ define

$$T(q) := \lim_{M \to \infty} T_M(q).$$

3.2. **The Potential.** The family of potentials we consider, denoted by $\mathcal{P}$, is defined as follows. For every $k \in \mathbb{Z}$ the set $\mathcal{C}_k := L_k(J(F))$ is called a *cylinder*. Let $(c_k)_{k \in \mathbb{Z}}$ be a sequence of positive numbers such that,

$$\lim_{k \to \infty} \frac{\log c_k}{\log k} = -\infty, \tag{3.1}$$

$c_{-k} = c_k$ and let $t_1 > 0$. A potential $\log \phi : J(F) \to \mathbb{R}$ belongs to the class $\mathcal{P}$ if it is defined on cylinders by

$$\log \phi\big|_{\mathcal{C}_k} = \log(c_k |F'(z)|^{-t_1}).$$

For potentials belonging to $\mathcal{P}$ we define the *topological pressure* by

$$P(\log \phi) := \sup_{M \in \mathbb{N}} \{P_M(\log \phi)\}.$$

Lemma 3.1. *If $\log \phi \in \mathcal{P}$, then $P(\log \phi) < \infty$.*

Proof. Note that

$$P(\log \phi) \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} \prod_{i=0}^{n-1} \left(\phi(F^i x)\right).$$

Moreover,

$$\begin{aligned}\sum_{x\in F^{-n}(z)}\prod_{i=0}^{n-1}\big(\phi(F^i x)\big) &= \sum_{w\in F^{-(n-1)}(z)}\prod_{i=1}^{n-1}\big(\phi(F^i w)\big)\sum_{x\in F^{-1}(w)}\phi(x)\\ &\leq \sup_{z\in J^r}\Big\{\sum_{x\in F^{-1}z}\phi(x)\Big\}\sum_{w\in F^{-(n-1)}(z)}\prod_{i=1}^{n-1}\big(\phi(F^i w)\big).\end{aligned}$$

By induction we obtain that

$$\sum_{x\in F^{-n}(z)}\prod_{i=0}^{n-1}\big(\phi(F^i x)\big)\leq\bigg(\sup_{z\in J^r}\Big\{\sum_{x\in F^{-1}z}\phi(x)\Big\}\bigg)^n.$$

Note that since $\log\phi\in\mathcal{P}$ the following holds,

$$\sup_{z\in J(F)}\Big\{\sum_{x\in F^{-1}z}\phi(x)\Big\}=\sup_{z\in J(F)}\Big\{\sum_{k\in\mathbb{Z}}c_k|z+2\pi ik|^{-t_1}\Big\}\leq C+2\sum_{k>1}\frac{c_k}{k^{t_1}}<\infty$$

for some constant C. The result now follows. □

3.3. **The Temperature Function.** The next two propositions describe the behaviour of the function $T(q)$ for $q\leq 1$. In what follows we fix a potential $\log\phi\in\mathcal{P}$ such that $P(\log\phi)=0$. Let $c(z):=c_k$ for every $z\in\mathcal{C}_k$.

Proposition 3.1. *If $q<0$, then $T(q)=+\infty$.*

Proof. Note that

$$\begin{aligned}0=P_M(-T_M(q)\log|F'|+q\log(c(z)|F'|^{-t_1}))\\ \geq -T_M(q)\log(\lambda e^M)+P_M(q\log(c(z)|F'|^{-t_1})).\end{aligned}$$

Thus

$$T_M(q)\geq\frac{P_M(q\log(c(z)|F'|^{-t_1}))}{\log(\lambda e^M)}=\frac{\sup\{h_\mu+\int\log(c(z)^q|F'|^{-t_1 q}):\mu\in\mathcal{M}\}}{\log(\lambda e^M)},$$

where $\mathcal{M}$ denotes the set of invariant probability measures. Denote by $z_{k(M)}$ the fixed point of the cylinder $\mathcal{C}_{k(M)}$, where $k(M)$ was defined by equation (2.1). It follows from Lemma 2.1 that for M large enough

$$\lambda e^M\leq 4\pi k(M).$$

Therefore

$$T_M(q)\geq\frac{\log(c_{k(M)}^q|F'(z_{k(M)})|^{-t_1 q})}{\log(\lambda e^M)}\geq\frac{\log(c_{k(M)}^q)}{\log 4\pi k(M)}+\frac{\log(|F'(z_{k(M)})|^{-t_1 q})}{\log 4\pi k(M)}.$$

But $\lim_{M\to\infty}\frac{q\log(c_M)}{\log M}=+\infty$, therefore $T(q)=+\infty$. □

Its a consequence of our definition of topological pressure that

$$T(q)=\inf\{t\in\mathbb{R}:P(-t\log|F'|+q\log\phi)\leq 0\}.$$

Proposition 3.2. *For $q\in[0,1]$ there exists $T(q)$ such that*

$$P(-T(q)\log|F'|+q\log(c(z)|F'|^{-t_1}))=0.$$

Proof. Note that for $q \in (0,1]$ we have that $P(q \log(c(z)|F'|^{-t_1})) < \infty$. Indeed, by equation (3.1) we have
$$\lim_{k\to\infty} \frac{q \log c_k}{\log k} = -\infty.$$
Therefore, the finiteness of the pressure follows directly from lemma 3.1. Moreover, for $q \in (0,1)$ the function
$$t \mapsto P(-t \log|F'| + q \log(c(z)|F'|^{-t_1}))$$
is convex and decreasing (being the supremum of such functions). If $t = 0$ then $0 < P(q\log(c(z)|F'|^{-t_1})) < \infty$. For $t = \dim_H(J^r)$,
$$P(-\dim_H(J^r)\log|F'| + q\log(c(z)|F'|^{-t_1})) \leq P(-\dim_H(J^r)\log|F'|) \leq 0.$$
The continuity of the pressure implies the existence of $T(q)$. □

Therefore, the function $T(q)$ is such that,
$$T(q) = \begin{cases} \text{Finite} & \text{if } q \in [0,1], \\ \infty & \text{if } q < 0. \end{cases}$$

3.4. **Conformal Measure.** We prove the existence of a conformal measure corresponding to $\log\phi$. Denote by μ_M the conformal measure corresponding to $\log\phi|J_M$. Recall that J_M is a compact space and that the function $\log\phi$ restricted to it is a regular potential.

Proposition 3.3. *The sequence $(\mu_M)_M$ is tight.*

Proof. We prove that for every $\epsilon > 0$ there exists $M > 0$ such that for every $N \in \mathbb{N}$
$$\mu_N(A(M)) < \epsilon,$$
where $A(M) := \{z \in J(F) : \operatorname{Re} z > M\}$. Fix $M \in \mathbb{R}$. Note that for $B(M) := \{z \in A(M) : \operatorname{Re} F(z) > M\}$ we have that
$$\begin{aligned} 1 \geq \mu_N(A(M)) \geq \mu_N(F(B(M)\cap\mathcal{C}_k)) &= \exp(P_N(\log\phi)) \int_{B(M)\cap\mathcal{C}_k} \phi^{-1} d\mu_N \\ &\geq \exp(P_N(\log\phi))\Big(\inf\big\{(c_k|F'(z)|^{-t_1})^{-1} : z \in B(M)\cap\mathcal{C}_k\big\}\Big)\mu_N\Big(B(M)\cap\mathcal{C}_k\Big). \end{aligned}$$
Moreover, if $z \in B(M)\cap\mathcal{C}_k$ then $|F'(z)| \geq (M + \pi|k|)/2 \geq (M+|k|)/2$. Therefore,
$$1 \geq \exp(P_N(\log\phi))c_k^{-1}\Big((M+|k|)/2\Big)^{t_1}\mu_N(B(M,k)).$$
That is $\mu_N(B(M)\cap\mathcal{C}_k) \leq c_k^{-1}(2/(M+|k|))^{t_1}\exp(-P_N(\log\phi))$. Hence,
$$\begin{aligned} \mu_N(B(M)) \leq \sum_{k\in\mathbb{Z}} \mu_N(B(M)\cap\mathcal{C}_k) &\leq 2\sum_{k=0}^{\infty}\mu_N(B(M)\cap\mathcal{C}_k) \\ &\leq 2^{t_1+1}\exp(-P_N(\log\phi))\sum_{k=0}^{\infty}\frac{c_k}{(M+k)^{t_1}} < o(1) \end{aligned}$$
as M tends to infinity.

We estimate now the measure μ_N of $C(M) := A(M)\setminus B(M)$. As in the previous estimations we have
$$1 \geq \exp(P_N(\log\phi))\mu_N(C(M)\cap\mathcal{C}_k) \inf_{z\in C(M)\cap\mathcal{C}_k}(c_k|F'(z)|^{-t_1})^{-1}.$$

Let $z \in C(M)$ that is $z \in J_F$, $\operatorname{Re} F(z) \leq M$ and $\operatorname{Re} z > M$. Then $|E_\lambda(z)| = \sqrt{(\operatorname{Im} E_\lambda(z))^2 + (\operatorname{Re} E_\lambda(z))^2} > \lambda \exp(M)$ and

$$|\operatorname{Im} E_\lambda(z)| > \sqrt{(\lambda \exp(M))^2 - M^2}.$$

Hence,

$$\inf_{z \in C(M) \cap \mathcal{C}_k} \{(c_k |F'(z)|^{-t_1})^{-1}\} \inf_{z \in C(M) \cap \mathcal{C}_k} \{c_k^{-1} |\operatorname{Im} E_\lambda(z)|^{t_1}\} \geq (2\pi k)^{t_1}/c_k.$$

Therefore, $\mu_N(C(M) \cap \mathcal{C}_k) \leq \exp(-P_N(\log \phi)) c_k/(2\pi k)^{t_1}$. Thus,

$$\mu_N(C(M)) \leq 2 \sum_{k \geq \left((\lambda \exp(M))^2 - M^2\right)^{1/2}/(2\pi)} \frac{c_k}{(2\pi k)^{t_1}} < o(1)$$

as M tends to infinity. Since $\mu_N(A(M)) = \mu_N(B(M)) + \mu_N(C(M))$ the proof is completed. □

¿From this proposition it follows that the sequence of measures (μ_M) has a week* limit, which does not have a singularity at infinity. Moreover, from the proof of Theorem 3.5 from [12] we obtain the following proposition.

Proposition 3.4. *Any weak* limit of $(\mu_M)_M$ is $\phi-$conformal.*

The conformal measure satisfies the following Ball Lemma.

Lemma 3.2. *Let μ be a ϕ-conformal measure on J_F. Then for all $x \in J^r(F)$, there exists a constant $C(x)$ and a sequence (n_k), with $\lim_{k\to\infty} n_k = \infty$, such that*

$$\frac{1}{C(x)} \prod_{i=0}^{n_k-1} \phi(F^i(x)) \leq \mu(B(x, \delta/4|(F^{n_k})'(x)|)) \leq C(x) \prod_{i=0}^{n_k-1} \phi(F^i(x))$$

where $\delta := (r_\lambda - p)/2$.

Proof. Let $n_k \to_{k\to\infty} \infty$ be a sequence satisfying $\lim_{k\to\infty} F^{n_k}(x) = z$ for some $z \in J(F)$. Applying Koebe Theorem to the disk $B\big(x, \delta/(4|(F^{n_k})'(x)|)\big)$ we obtain

$$B(z, \delta/32) \subset B(F^{n_k}(x), \delta/16) \subset F^{n_k}\big(B(x, \delta/|4(F^{n_k})'(x)|)\big) \subset B(F^{n_k}(x), \delta) \subset B(z, 2\delta).$$

Since the measure μ is conformal, it follows that

$$\mu(B(z, \delta/32)) \leq \mu\Big(F^{n_k}\big(B(x, \delta/|4(F^{n_k})'(x)|)\big)\Big) \leq \mu(B(x, \delta/|4(F^{n_k})'(x)|)) K \prod_{i=0}^{n_k-1} \phi^{-1}(F^i(x)),$$

and

$$1 \geq \mu\big(F^{n_k}(B(x, \delta/|4(F^{n_k})'(x)|))\big) \geq \mu(B(x, \delta/|4(F^{n_k})'(x)|)) \frac{1}{K} \prod_{i=0}^{n_k-1} \phi^{-1}(F^i(x)),$$

where K is a constant from Koebe Theorem. Hence

$$\frac{\mu(B(z, \delta/32))}{K} \prod_{i=0}^{n_k-1} \phi(F^i(x)) \leq \mu(B(x, \delta/|4(F^{n_k})'(x)|)) \leq K \prod_{i=0}^{n_k-1} \phi(F^i(x)).$$

□

Let $q \in [0,1]$ and denote by ν_q the conformal measure corresponding to the potential

$$-T(q)\log|F'| + q\log\phi.$$

Note that this potential belongs to the class $\mathcal{P}$ that was defined in subsection 3.2, therefore it has a corresponding conformal measure. We now discuss the integrability of the potentials with respect to the conformal measures considered.

Proposition 3.5.

1. $\log|F'| \in L^1_{\nu_q}$ *for* $q \in [0,1]$.
2. $\log\phi \in L^1_{\nu_q}$ *for* $q \in (0,1]$.

Proof. Assume $k \in \mathbb{Z}$ to be fixed. Let $I_M := \{z \in J_F : M < \mathrm{Re} z \le M+1\}$ and $t := T(q) + qt_1$ with $q \in [0,1]$. Since ν_q is a conformal measure we obtain

$$1 \ge \nu_q(F(\mathcal{C}_k \cap I_M)) = \int_{\mathcal{C}_k \cap I_M} |F'|^t c_k^{-q} d\nu_q \ge \lambda^t e^{Mt} c_k^{-q} \nu_q(\mathcal{C}_k \cap I_M).$$

Hence $\nu_q(\mathcal{C}_k \cap I_M) \le c_k^q/(\lambda^t e^{Mt})$. It is possible to obtain bounds for the measure of the entire cylinder as well, in fact

$$\nu_q(F(\mathcal{C}_k)) = \int_{\mathcal{C}_k} |F'|^t c_k^{-q} d\nu_q \ge \frac{(\lambda\sqrt{p^2 + (2\pi k)^2})^t}{c_k^q} \nu_q(\mathcal{C}_k).$$

Hence $\nu_q(\mathcal{C}_k) \le c_k^q/(\lambda\sqrt{p^2+(2\pi k)^2})^t)$.

In what follows we prove that $\log|F'| \in L^1_{\nu_q}$ for $q \in (0,1]$ (the case $q = 0$ was proved in [12]).

$$\begin{aligned}\int_{\mathcal{C}_k} \log|F'| d\nu_q &= \sum_{M\in\mathbb{N}} \int_{\mathcal{C}_k \cap I_M} \log|F'| d\nu_q \\ &\le \sum_{M\in\mathbb{N}} (\log\lambda + (M+1)) \frac{1}{\lambda^t} \frac{c_k^q}{e^{Mt}} = \frac{c_k^q}{\lambda^t} \sum_{M\in\mathbb{N}} \Big(\frac{\log\lambda + 1}{e^{Mt}} + \frac{M}{e^{Mt}} \Big) \le C(q) c_k^q,\end{aligned}$$

where $C(q)$ does not depend on k. Hence

$$\int \log|F'| d\nu_q = \sum_{k\in\mathbb{Z}} \int_{\mathcal{C}_k} \log|F'| d\nu_q \le \sum_{k\in\mathbb{Z}} C(q) c_k^q = C(q) \sum_{k\in\mathbb{Z}} c_k^q < \infty.$$

Next, we show that $\log\phi \in L^1_{\nu_q}$ for $q \in (0,1]$. Observe that

$$\int_{\mathcal{C}_k} \log(|F'|^t c_k^{-1}) d\nu_q = t \int_{\mathcal{C}_k} \log|F'| d\nu_q + \int_{\mathcal{C}_k} \log c_k^{-1} d\nu_q.$$

Note that we already proved the integrability of $\log|F'|$. Therefore, in order to prove the integrability of $\log\phi$ we only need to prove that $\log c(z)^{-1} \in L^1_{\nu_q}$. But,

$$\int_{\mathcal{C}_k} \log c_k^{-1} d\nu_q = \log c_k^{-1} \nu_q(\mathcal{C}_k) \le \frac{\log c_k^{-1}}{c_k^{-1}} \frac{1}{(\sqrt{p^2 + (2\pi k)^2})^t} \le C \frac{\log c_k^{-1}}{c_k^{-1}},$$

where the constant C does not depend on k. Hence

$$\int_{\mathcal{C}} \log c(z)^{-1} d\nu_q \le C \sum_{k\in\mathbb{Z}} \frac{\log c_k^{-1}}{c_k^{-1}} < \infty.$$

□

3.5. **The multifractal spectrum.**

Theorem. *The multifractal spectrum $f_\mu(\cdot)$ is the Legendre transform of $T(\cdot)$.*

Proof. Denote by L the Legendre transform of T and by $f_{\mu,M}$ the Legendre transform of T_M. Since the sequence $f_{\mu,M}$ is non-decreasing on M and is bounded above by $\dim_H J^r$ there exists a pointwise limit that we denote by f_*. In order to prove that $f_\mu = L$, we first show that $f_\mu(\alpha) \le L(\alpha)$, then $f_*(\alpha) \le f_\mu(\alpha)$ and finally $L(\alpha) \le f_*(\alpha)$ for $\alpha \in \mathrm{domain}(f_\mu)$.

Let $\alpha \in \mathrm{domain}(f_\mu)$ and consider $z \in \mathrm{Lev}(\alpha)$. Let ν_q be the conformal measure corresponding to $-T(q)\log|F'| + q\log\phi$. Note that

$$\underline{d}_{\nu_q}(z) = \liminf_{r\to 0}\frac{\log \nu_q(B(z,r))}{\log r} \le \lim_{k\to\infty}\frac{\log\nu_q\big(B(z,\delta/|4(F^{n_k})'(z)|)\big)}{\log(\delta/|4(F^{n_k})'(z)|)}$$
$$= T(q) + q\lim_{k\to\infty}\frac{\sum_{i=0}^{n_k-1}\log\phi(F^i(z))}{\log(\delta/|4(F^{n_k})'(z)|)},$$

where the sequence (n_k) is defined as in Lemma 3.2. Since $z \in \mathrm{Lev}(\alpha)$,

$$\alpha = d_\mu(z) = \lim_{r\to 0}\frac{\log\mu(B(z,r))}{\log r} = \lim_{k\to\infty}\frac{\log\mu\big(B(z,\delta/|4(F^{n_k})'(z)|)\big)}{\log(\delta/|4(F^{n_k})'(z)|)}$$
$$= \lim_{k\to\infty}\frac{\sum_{i=0}^{n_k-1}\log\phi(F^i(z))}{\log(\delta/|4(F^{n_k})'(z)|)}.$$

Therefore $\underline{d}_{\nu_q}(z) \le T(q) + q\alpha$ for every $z \in \mathrm{Lev}(\alpha)$. This implies that

$$f_\mu(\alpha) \le L(\alpha).$$

Next, we prove that $f_*(\alpha) \le f_\mu(\alpha)$. Consider the set

$$\mathrm{Lev}_M(\alpha) := \Big\{ z \in J_M : \lim_{k\to\infty}\frac{\sum_{i=0}^{k-1}\log\phi(F^i(z))}{\sum_{i=1}^{k-1}\log(|F'(F^i(z))|^{-1})} = \alpha\Big\}.$$

Note that $\mathrm{Lev}_M(\alpha) \subset \mathrm{Lev}(\alpha)$. In fact, since the measure μ is ϕ-conformal, the measure of the ball $B(z,\delta/4|(F^k)'(x)|)$ is, for $z \in J_M$, comparable with $\prod_{i=0}^{k-1}\phi(F^i(z))$ (see the proof of Lemma 3.2). Therefore, in order to complete the proof of the second inequality, it is enough to show that the Legendre transform $f_{\mu,M}(\cdot)$ of T_M is equal to $\dim_H(\mathrm{Lev}_M(\cdot))$. But this follows from standard techniques in multifractal analysis. Let $\mu_{M,q}$ be the Gibbs measure on J_M corresponding to the potential $-T_M(q)\log|F'| + q\log\phi$. Then

$$\mu_{M,q}(B(z,|(F^k)'(z)|^{-1})) \asymp \prod_{i=0}^{k-1}|F'(F^i(z))|^{-T_M(q)}\phi^q(F^i(z)),$$

where $A_k \asymp B_k$ if, and only if, there exists a constant $C > 0$, independent of k, such that $C^{-1}B_k \le A_k \le CB_k$. Hence

$$\frac{\log\mu_{M,q}(B(z,|(F^k)'(z)|^{-1}))}{\log|(F^k)'(z)|^{-1}} \asymp T_M(q) + q\frac{\sum_{i=0}^{k-1}\log\phi(F^i(z))}{\sum_{i=1}^{k-1}\log(|F'(F^i(z))|^{-1})}.$$

Since

$$-T_M'(q) = \frac{\int\log\phi\, d\mu_{M,q}}{\int\log|F'|d\mu_{M,q}}$$

and $\mu_{M,q}(\mathrm{Lev}_M(\alpha)) = 1$ (let $\alpha = \int\log\phi\, d\mu_{M,q}/\int\log|F'|d\mu_{M,q}$), it follows that the Legendre transform of T_M is equal to $\dim_H(\mathrm{Lev}_M(\alpha))$.

Finally, by results of Wijsman from [15] we have that $L(\alpha) \le f_*(\alpha)$. Therefore, $f_\mu(\alpha) = L(\alpha)$. □

4. **Examples: fast vs. faster.** Recall that the function $T(q)$ is infinite for negative values of q (Proposition 3.1). Hence, the Legendre transform L has unbounded domain. By the theorem proved in the previous subsection this implies that, in strong contrast with the compact case, the multifractal spectrum has unbounded domain. Moreover, two possible behaviours can occur:

1. There exists a constant α^* such that, for every $\alpha > \alpha^*$ we have $f_\mu(\alpha) = \dim_H(J^r)$ and for $\alpha < \alpha^*$ the multifractal spectrum is strictly increasing.
2. The multifractal spectrum f_μ is strictly increasing.

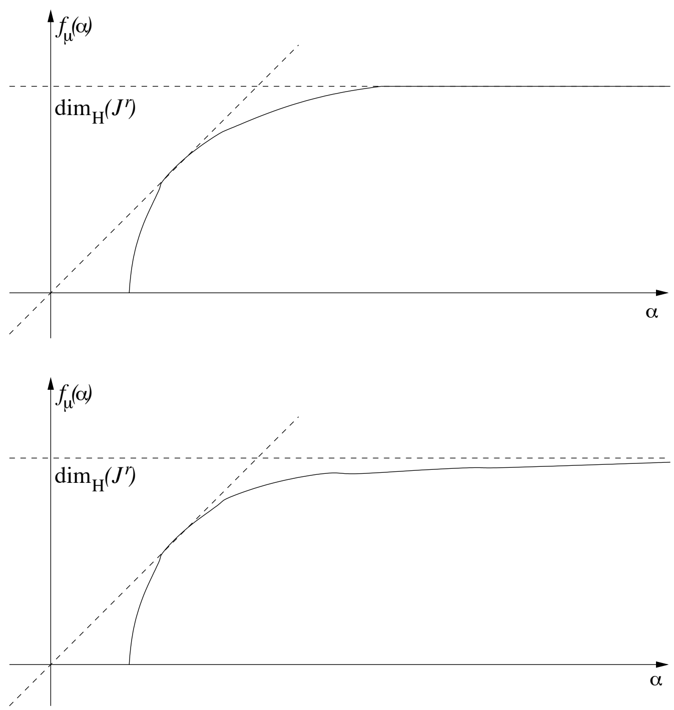

Figure 4.1. The graphs of the multifractal spectrum for the possibility (1) and (2).

In the next two subsections examples of these type of behaviour are provided.

4.1. **Possibility (1).** Denote by μ_0 the d-conformal measure for F, where $d := \dim_H(J^r)$, and by m the corresponding invariant measure (see [12]). Let m_M be the $F|_{J_M}$-invariant measure equivalent to the $\dim_H(J_M)$-conformal measure $\mu_{0,M}$. Since $\frac{dm_M}{d\mu_{0,M}} \le \frac{dm}{d\mu_0}$ and $\int \log|F'| d\mu_0$ is finite, there exist constants C_1, C_2 such that

$$0 < C_1 \le \int \log|F'| dm_M \le C_2 < \infty.$$

Recall that (see [7, p. 211–220]) $f_M(-T_M'(0)) = \dim_H J_M$ and

$$-T_M'(0) = \frac{\int \log \phi dm_M}{\int \log|F'| dm_M}.$$

Therefore, if $\lim_{M\to\infty}\int \log\phi dm_M < \infty$, then $\lim_{M\to\infty} -T'_M(0) < \infty$. Since f_μ is concave and has unbounded domain, we obtain the following lemma.

Lemma 4.1 (The multifractal spectrum is of type (1))**.** *If $\int \log\phi d\mu_0 < \infty$, then there exists $\alpha^* > 0$ such that $f_\mu(\alpha) = \dim_H J^r$ for every $\alpha > \alpha^*$.*

We now construct a potential $\log\phi|_{\mathcal{C}_k} = \log c_k|F'|^{-t_1}$ satisfying the hypothesis of the previous lemma. Let $c_k := (e^{|k|})^{(1-d)/2}$. Note that for $k \neq 0$ we have

$$\mu_0(F(\mathcal{C}_k)) = \int_{\mathcal{C}_k} |F'|^d d\mu_0 \geq \mu(\mathcal{C}_k)(k)^d.$$

Therefore, $\mu_0(\mathcal{C}_k) \leq \frac{1}{k^d}$. Moreover,

$$\int_{\mathcal{C}_k} \log c_k^{-1} d\mu_0 \leq \mu_0(\mathcal{C}_k)\log c_k^{-1}.$$

Hence, by Proposition 3.5 we obtain

$$-\int_{\mathcal{C}} \log\phi d\mu_0 \leq 2\sum_{k=1}^{\infty}\int_{\mathcal{C}_k} \log c_k^{-1} d\mu_0 + O(1) \leq 2\sum_{k=1}^{\infty}\frac{1}{k^{(1+d)/2}} < \infty.$$

4.2. **Possibility (2).** First, observe that the following lemma is a consequence of the fact that $\dim_H J_M \to \dim_H J^r$.

Lemma 4.2 (The multifractal spectrum is of type (2))**.** *If $\lim_{M\to\infty}\int \log\phi dm_M = \infty$, then $f_\mu(\alpha) < \dim_H J^r$ for every $\alpha \in \operatorname{domain} f_\mu$.*

We construct a potential $\log\phi \in \mathcal{P}$ satisfying the hypothesis of the above lemma. The sequence (c_k) is obtained as follows.

Let $I_n := \{z \in J_F : r_\lambda + n \leq \operatorname{Re} z < r_\lambda + n + 1\}$ for $n \in \{0, 1, \ldots\}$ and $H_n := \{k \in \mathbb{Z} : \mathcal{C}_k \cap I_n \neq \emptyset \text{ and } \mathcal{C}_k \cap I_m = \emptyset \text{ for } m < n\}$. For every $n \in \{0, 1, \ldots\}$ and $k \in H_n$, let $c_k \in \mathbb{R}$ be chosen so that

$$\sum_{k\in H_n}\int_{I_n\cap\mathcal{C}_k} \log c_k^{-1}|F'|^{t_1} dm_{r_\lambda+n+1} \geq n,$$

$c_{-k} = c_k$ and $\log c_k^{-1}/\log k \to \infty$ for $k \to \infty$. Therefore, the potential defined by

$$\log\phi(z)|_{\mathcal{C}_k} = \log c_k|F'|^{-t_1},$$

where the sequence (c_k) is defined as above, is such that the multifractal spectrum of the corresponding conformal measure has unbounded domain and it is strictly increasing.

REFERENCES

[1] L. Barreira, J. Schmeling, *Sets of "Non-Typical" points have full topological entropy and full Hausdorff dimension,* Israel Journal of Maths. **116** (2000), 29–70.

[2] R. L. Devaney, *Cantor Bouquets, Explosions, and Knaster Continua: Dynamics of Complex Exponentials*, Publicacions Matematiques **43** (1999), 27–54.

[3] P. Hanus, D. Mauldin, M. Urbański, *Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems,* Acta. Math. Hungarica **96** (2002), 27–98.

[4] G. Iommi, *Multifractal Analysis for countable Markov shifts,* Ergodic Theory & Dynam. Systems **25** (2005), 1881–1907.

[5] K. Nakaishi, *Multifractal formalism for some parabolic maps,* Ergodic Theory & Dynam. Systems **20** (2000), 843–857.

[6] L. Olsen, *A Multifractal Formalism,* Adv. in Math. 116, 82-196 (1995)

[7] Y. Pesin, *Dimension Theory in Dynamical Systems,* CUP (1997).

[8] Y. Pesin, H. Weiss, *A multifractal Analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions,* J.Stat.Phys. **86** (1997), 233-275.

[9] M. Pollicott, H. Weiss, *Multifractal analysis of Lyapunov exponent for continued fraction and Manneville Pomeau transformations and applications to Diophantine approximations,* Comm.Math.Phys. **207** (1999), 145-171.

[10] J. Schmeling, *On the completeness of the multifractal spectra,* Ergodic Theory & Dynam. Systems **19** (1999), 1–22.

[11] M. Urbański, *Thermodynamic formalism and multifractal analysis of finer Julia sets of exponential family,* Preprint 2005.

[12] M. Urbański, A. Zdunik, *The finer geometry and dynamics of the hyperbolic exponential family,* Michigan Math. J. **51** (2003), 227-250.

[13] M. Urbański, A. Zdunik, *Real Analyticity of Hausdorff Dimension of Finer Julia Sets of Exponential Family,* Ergodic Theory & Dynam. Systems **24** (2004), 279-315.

[14] P. Walters, *An Introduction to Ergodic Theory,* GTM Springer (1982).

[15] R. A. Wijsman, *Convergence of sequence of convex sets, cones and functions II,* Trans. Amer. Math. Soc. **123** (1966), 32-45.

E-mail address: giommi@math.ist.utl.pt

E-mail address: bmariusz@ucn.cl