ERGODIC OPTIMIZATION FOR RENEWAL TYPE SHIFTS

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Abstract. We consider a class of countable Markov shifts $\mathcal{R}$ and a locally Hölder potential ϕ. We prove that the existence of $\phi-$optimal measures is closely related to the behaviour of the pressure function $t \to P(t\phi)$. Using a Theorem by Sarig it is possible to prove that there exists a critical value $t_c \in (0, \infty]$ such that for $t < t_c$ the pressure is analytic and for $t > t_c$ is linear. We prove that if t_c is finite then there are no $\phi-$optimal measures and if it is infinite then $\phi-$optimal measures do exist.

1. Introduction

We consider a special class of countable Markov shifts $\mathcal{R}$ (for the definition see Section 2) that includes the symbolic models for certain tower extensions [8] and the symbolic model for expanding interval maps with a parabolic fixed point [6]. For $(\Sigma, \sigma) \in \mathcal{R}$ we study the following ergodic optimization problem. Let $\phi : \Sigma \to \mathbb{R}$ be a regular potential (see Section 2 for precise statements), denote by $\mathcal{M}$ the set of shift invariant probability measures. Let

$$\alpha(\phi) := \sup \left\{ \int \phi d\mu : \mu \in \mathcal{M} \right\}.$$

A measure $m \in \mathcal{M}$ is called $\phi-$*optimal* if $\int \phi dm = \alpha(\phi)$. When considering Markov shifts defined on finite alphabets the compactness of the space ensures the existence of $\phi-$optimal measures. This is not the case when considering countable Markov shifts. Jenkinson, Mauldin and Urbański [3, 4] have proved that for a certain class of countable Markov shifts, where the thermodynamic formalism is similar to the one observed in finite state Markov shifts, there is always an optimal measure. The class $\mathcal{R}$ has no intersection with the class considered by them. We have at our disposal the thermodynamic formalism developed by Sarig [5]. For a regular potential ϕ defined on a Markov shift belonging to the class $\mathcal{R}$, it is possible to show using results from [6] that there exists a critical value $t_c \in (0, \infty]$ such that the pressure function $P(t\phi)$ is real analytic on $(0, t_c)$ and linear on (t_c, ∞). In this note we tie together results from Jenkinson-Mauldin-Urbański [3] and Sarig [6] to prove the following

Theorem 1.1. *Let* $(\Sigma, \sigma) \in \mathcal{R}$ *and* $\phi : \Sigma \to \mathbb{R}$ *a locally Hölder potential such that* $\sup \phi < \infty$*, then*

(1) *If* $t_c = \infty$ *then there exist* $\phi-$*optimal measures.*
(2) *If* $t_c < \infty$ *then there are no* $\phi-$*optimal measures and* $\alpha(\phi) = M$*, where* M *is the slope of the linear part of the pressure function* $P(t\phi)$*.*

2000 *Mathematics Subject Classification.* Primary 37A99, 37D35 Secondary 37B10
Key words and phrases. Optimal measures, thermodynamic formalism, renewal shift.
The author was partially supported by FCT/POCTI/FEDER.

If $t_c = \infty$ some of the $\phi-$optimal measures can be described as a limit of equilibrium sates.

2. Preliminaries

Denote by $\mathbb{N}$ the set of positive integers and let $T = (t_{ij})_{\mathbb{N}\times\mathbb{N}}$ be a matrix of zeroes and ones. Let $\Sigma_T := \{x \in \mathbb{N}^{\mathbb{N}\cup\{0\}} : t_{x_i x_{i+1}} = 1 \text{ for every } i \geq 0\}$ be the corresponding Markov shift, equipped with the topology generated by the base of cylinder sets $C_{i_0 i_1 \dots i_{n-1}} := \{x \in \Sigma_T : x_j = i_j \text{ for } 0 \leq j \leq n-1\}$. The function $\sigma : \Sigma_T \to \Sigma_T$ defined by $(\sigma x)_i = x_{i+1}$ is called the *shift map.* An *admissible word* is an element $\underline{a} \in \mathbb{N}^n$ such that $C_{\underline{a}} \neq \emptyset$. Throughout this note we will always assume a Markov shift (Σ_T, σ) to be topologically mixing. Denote by $\mathcal{R}$ (from renewal) the class of topologically mixing countable Markov shifts (Σ_T, σ) such that for each n there exists at most one admissible word $a_0 \dots a_{n-1}$ of length n such that $t_{a_{n-1}1} = 1$ and $a_i = 1$ if and only if $i = 0$ (we assume that 1 is the renewal vertex).

Example 2.1. *Let $(d_i)_{i\geq 1}$ be an increasing sequence of positive integers. Consider the transition matrix $T = (t_{ij})_{\mathbb{N}\times\mathbb{N}}$ with entries $t_{11}, t_{i+1,i}, t_{1,d_i}$ equal to one and the rest of the entries equal to zero. The Markov shifts (Σ_T, σ) belong to the class $\mathcal{R}$.*

We will make use of the thermodynamic formalism developed by Sarig [5]. Let $\phi : \Sigma \to \mathbb{R}$ be a real function. The *variations* of ϕ are defined by

$$V_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma, x_i = y_i, 0 \leq i \leq n-1\}.$$

We say that ϕ is *locally Hölder continuous* (with parameter θ) if there exists $B > 0$ and $\theta \in (0,1)$ such that for all $n \geq 1$, $V_n(\phi) \leq B\theta^n$. Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder potential. The *Gurevich Pressure* of ϕ is defined by

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\Big(\sum_{i=0}^{n-1} \phi(\sigma^i x)\Big) 1_{C_{i_0}}(x),$$

where $1_{C_i}(x)$ is the indicator function of the cylinder C_i. It can be shown that the limit exists and is independent of the choice of i_0. The first return time map to C_i is defined by $\varphi_i(x) := 1_{C_i}(x) \inf\{n \geq 1 : \sigma^n x \in C_i\}$. Let $[\varphi_1 = n] := \{x \in \Sigma : \varphi_1(x) = n\}$.

For Markov shifts belonging to $\mathcal{R}$ the behaviour of the pressure function, $t \to P(t\phi)$, can be described.

Theorem 2.1 (Sarig)**.** *Let $\Sigma \in \mathcal{R}$ and $\phi : \Sigma \to \mathbb{R}$ a locally Hölder potential such that* $\sup \phi < \infty$. *There exists a constant $t_c \in (0, \infty]$ such that,*

1. *For $0 < t < t_c$ there exists an equilibrium probability measure μ_t corresponding to $t\phi$. For $t > t_c$ there is no conformal conservative measure corresponding to $t\phi$.*
2. *$P(t\phi)$ is real analytic on $(0, t_c)$ and linear on (t_c, ∞). At t_c it is continuous but not analytic.*

Sarig proved the above theorem for the renewal shift [6, Theorem 5]. His proof relies on the Discriminant theorem [6] and on the identity $p_0^*[t\phi] = tp_0^*[\phi]$ for all t, where

$$p_0^*[\phi] := -\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\Big(\sum_{i=0}^{n-1} \phi(\sigma^i x)\Big) 1_{[\varphi_1 = n]}(x).$$

This identity is also valid for Markov shifts belonging to class $\mathcal{R}$, therefore the result also holds for this class.

Let $\Sigma \in \mathcal{R}$, the *induced system* on the symbol 1 will be denoted by $(\Sigma_i, \overline{\sigma})$. It is defined as the full-shift on the new alphabet, $\{C_{\overline{a}} : a_i = 1 \text{ iff } i = 0, C_{\overline{a}1} \neq \emptyset\}$. For every locally Hölder potential $\phi : \Sigma \to \mathbb{R}$ set

$$\overline{\phi} := \Big(\sum_{k=0}^{\varphi_1 - 1} \phi \circ \sigma^k \Big) \circ \pi,$$

where $\pi : \Sigma_i \to C_1$ is defined by $\pi(C_{\underline{a}_0}, C_{\underline{a}_1}, \dots) = (\underline{a}_0, \underline{a}_1, \dots)$. The pair $(\Sigma_i, \overline{\phi})$ is called the induced system and $\overline{\phi}$ is called the induced potential. Note that if the potential ϕ is locally Hölder then $\overline{\phi}$ is locally Hölder. Since the system is topologically mixing our results do not depend on the symbol we do induce.

3. Proof of Theorem 1.1

3.1. **Proof of statement (1).** This result follows from a tightness argument from Jenkinson *et al* [3, Lemma 2], when applied to the induced system.

Assume that $t_c = \infty$. In virtue of Theorem 2.1 and of [1, Theorem 1.1], for every $t > 0$ there exists a unique equilibrium (probability) measure μ_t corresponding to $t\phi$ and the pressure function is analytic. Denote by $\overline{t\phi}$ the induced potential and by $\overline{\mu_t}$ the unique corresponding Gibbs measure (such a measure exists by [7, Theorem 1]). Note that $\overline{\mu_t}$ is the restriction of μ_t to C_1,

$$\int \overline{\psi} d\overline{\mu_t} = \frac{1}{\mu(C_1)} \int_{C_1} \psi d\mu_t. \tag{1}$$

Lemma 3.1 (Jenkinson-Mauldin-Urbański (see Lemma 2 in [3]))**.** *The sequence of Gibbs measures $\{\overline{\mu_t}\}$ is tight.*

Since the system Σ is topologically mixing and because of the relation (1) the above Lemma implies that the sequence $\{\mu_t\}$ is tight. Therefore, the sequence has a weak* accumulation point. The following result appears in [2, Lemma 2] in the context of Markov shifts defined on finite alphabets. Since the pressure function is analytic the same proof holds in this case.

Lemma 3.2 (Jenkinson)**.** *Any weak* accumulation point μ of $\{\mu_t\}$ is a $\phi-$optimal measure.*

3.2. **Proof of statement (2).** This result is a slight generalisation of a result by Sarig [6, Theorem 5].

Assume that $t_c < \infty$. In virtue of Theorem 2.1 and of [6, Theorem 2] we have that for $t > t_c$ the pressure function is linear, $P(t\phi) = tM$, where $M = -p_0^*[\phi]$.

Lemma 3.3.

$$\sup \Big\{ \int \phi d\mu : \mu \in \mathcal{M} \Big\} = M.$$

Proof. Let $\mu \in \mathcal{M}$. For $t > t_c$ we have, $tM = P(t\phi) \geq h_\mu + t \int \phi d\mu \geq t \int \phi d\mu$. Therefore $M \geq \int \phi d\mu$. In order to prove the other inequality, note that for renewal type graphs there is at the most one cylinder of length n on the induced system. So we have that either

$$\{x \in [\varphi_1 = n] : \sigma^n x = x\} = \{x^n\},$$

where x^n is the periodic point of prime period n such that the first coordinate is equal to one, or

$$\{x \in [\varphi_1 = n] : \sigma^n x = x\} = \emptyset.$$

Let $\delta_n := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k x^n}$ be the measure supported on the periodic orbit ($\delta_{\sigma^k x^n}$ denotes the atomic measure supported on the point $\sigma^k x^n$). Passing to a subsequence we have that,

$$M = -p_0^*[\phi] = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp(\sum_{i=0}^{n-1} \phi(\sigma^i x)) 1_{[\varphi_1 = n]}(x) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp(n \int \phi d\delta_n) 1_{[\varphi_1 = n]}(x) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \exp(n \int \phi d\delta_n) = \lim_{n\to\infty} \int \phi d\delta_n.$$

Therefore, $\sup \left\{ \int \phi d\delta_n : n \in \mathbb{N} \right\} = M.$ □

Recall that for $t > t_c$ the potential $t\phi$ has no equilibrium probability measures. In particular, this implies that there are no $\phi-$optimal measures.

Lemma 3.4. *If the critical value t_c of the pressure function $t \to P(t\phi)$ is finite, then there are no $\phi-$optimal measures.*

Proof. Assume by way of contradiction that there exists a $\phi-$optimal measure μ. That is $\int \phi d\mu = M$. Thus, for $t > t_c$

$$P(t\phi) = tM \geq h_\mu + t \int \phi d\mu.$$

In particular we have that $h_\mu = 0$ and that for every $t > t_c$ the measure μ is an equilibrium measure for $t\phi$. This contradiction proves the statement. □

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