HIDDEN GIBBS MEASURES ON SHIFT SPACES OVER COUNTABLE ALPHABETS

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Abstract. We study the thermodynamic formalism for particular types of sub-additive sequences on a class of subshifts over countable alphabets. The subshifts we consider include factors of irreducible countable Markov shifts under certain conditions, which we call irreducible countable sofic shifts. We show the variational principle for topological pressure for some sub-additive sequences with tempered variation on irreducible countable sofic shifts. We also study conditions for the existence and uniqueness of invariant ergodic Gibbs measures and the uniqueness of equilibrium states. Applications are given to some dimension problems and study of factors of (generalized) Gibbs measures on certain subshifts over countable alphabets.

Keywords: Thermodynamic formalism, countable alphabets, sub-additive sequences, Hidden Gibbs measures, Gibbs measures.

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1. Introduction

Thermodynamic formalism is an area of ergodic theory which addresses the problem of choosing relevant invariant measures among the, sometimes very large, set of invariant probabilities. This theory was brought from statistical mechanics into dynamics in the early seventies by Ruelle and Sinai among others [41, 45]. The powerful formalism developed to study equilibrium of systems consisting of a large number of particles (e.g. gases) has been surprisingly efficient to describe certain dynamical systems that exhibit complicated behavior. The theory has been developed in several directions. Originally the dynamical system was assumed to be defined on a compact set and the observable was a continuous function. Both assumptions have been relaxed over the years. For example, Gurevich [23, 24, 25], Mauldin and Urbański [34, 35] and Sarig [42, 44, 44] have developed thermodynamic formalism in the non-compact setting of countable Markov shifts. Since there exists a wide range of relevant dynamical systems that can be coded with countable Markov shifts, this theory has had relevant applications. Other extension of thermodynamical formalism to non-compact settings was developed by Pesin and Pitskel [39]. In that case, the system is not assumed to have any Markov structure but it has to be the restriction of a continuous map defined on a compact set. Also, the observables have to have continuous extensions (therefore observables are assumed to be bounded). In a different direction, the theory was extended to consider not only a single observable but instead a sequence of them. Certain additivity assumptions were required on the sequence in order for the ergodic theorems to hold. This circle of ideas

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was called non-additive thermodynamic formalism. It was originally formulated by Falconer [14] with the purpose of applying it in the study of the dimension theory of non-conformal dynamical systems. Ever since, different additivity assumptions have been considered in the sequence. For example, Barreria [2, 3, 4] and Mummert [37] independently developed the theory assuming a strong additivity assumption called almost-additivity. Cao, Feng and Huang [8] studied the case in which the sequence was only assumed to be sub-additive. More generally, Feng and Huang [20] extended the theory to handle asymptotically sub-additive sequences. Over the last few years, thermodynamic formalism for non-compact dynamical systems and sequences of observables has been developed. Iommi and Yayama [26, 27] have studied thermodynamic formalism for almost-additive sequences on (non-compact) countable Markov shifts. Also, Käenmäki and Reeve [29] studied the formalism for sequences of potentials under weaker additivity assumptions but for the full shift over a countable alphabet.

In this paper, we further develop the theory. We consider particular types of sub-additive sequences on a fairly general class of subshifts. We call this class the class of *countable sofic shifts* (see Section 2.3). This class generalizes the concept of a sofic shift over a finite alphabet. We stress that this dynamical system is non-Markov and it is defined on a non-compact space. Even in the case of a single observable, several of our results are new, to the best of our knowledge. The types of sub-additive sequences we consider are generalizations of continuous functions with tempered variation on subshifts satisfying the weak specification property (see Section 2.2 for details). We propose a definition of the topological pressure and compare it with the Gurevich pressure in Section 2 and prove the variational principle in Section 4. Unlike the case of subshifts on finite alphabets, we need to study the structure of the space on which a sequence is defined as well as regularity conditions on sequences in order to establish a variational principle. Section 4.2 studies a variational principle for sequences defined on *finitely irreducible subshifts* (Definition 2.3) which preserve a certain finiteness property. The case when the sequences are defined on countable Markov shifts which are not necessarily finitely irreducible is studied in Section 4.1. In Section 5, we study the existence and uniqueness of Gibbs measures on finitely irreducible countable sofic shifts, together with uniqueness of the Gibbs equilibrium states (Theorem 5.1). Our results extend those in [29], encompassing more general classes of sequences and far more general dynamical systems.

Differences with the work in [26, 27] are discussed in Section 3.1. Not every almost-additive sequence studied in [26] is in the class of sequences we study here (Example 3.2). This phenomenon is different from what is observed in the compact case.

One of the main applications of the thermodynamic formalism studied in this article is to develop the theory of factors of Gibbs measures on shift spaces over countable alphabets. An important question in the area is to determine under which conditions the (generalized) Gibbs property is preserved under a one-block factor map. For Gibbs measures for continuous functions on subshifts over finite alphabets, this problem has been studied widely, for example, by Chazotte and Ugalde [9, 10], Kempton and Pollicott [40], Kempton [30], Piranio [38], Jung [28], Verbitskiy [47] and Yoo [48]. For generalized Gibbs measures for sequences on subshifts over finite alphabets, this type of question has been addressed by Barral and Feng [1], Feng [18] and Yayama [49, 50], especially in connection with dimension problems on non-conformal repellers. In Section 6, applying the results of Sections 4 and 5, we address this question in the (non-compact and non Markov) context of finitely irreducible countable sofic shifts (Theorems 6.1 and 6.2). Finally, in Section 7, applications are given to the study of some problems in dimension theory, in particular, product of matrices and the singular value function.

2. Background

2.1. **Subshifts on countable alphabets and specification properties.** This section is devoted to recall basic notions of symbolic dynamics. We discuss countable Markov shifts, factor maps and different specification properties in this setting. For more details we refer the reader to [33, 7]. Let $(t_{ij})_{\mathbb{N}\times\mathbb{N}}$ be a transition matrix of zeros and ones (with no row and no column made entirely of zeros). The associated *(one-sided) countable Markov shift* (Σ,σ) is the set

$$\Sigma := \left\{(x_n)_{n\in\mathbb{N}} : t_{x_n,x_{n+1}} = 1 \text{ for every } n\in\mathbb{N}\right\},$$

together with the shift map $\sigma:\Sigma\to\Sigma$ defined by $\sigma(x)=x'$, for $x=(x_n)_{n=1}^{\infty}, x'=(x'_n)_{n=1}^{\infty}$ with $x'_n=x_{n+1}$ for all $n\in\mathbb{N}$. If for every $(i,j)\in\mathbb{N}^2$ the transition matrix satisfies $t_{ij}=1$, then we say that the corresponding countable Markov shift is the *full shift on a countable alphabet*.

An *allowable word* of length $n\in\mathbb{N}$ for Σ is a string $i_1\dots i_n$ where $t_{i_j,i_{j+1}}=1$ for every $j\in\{1,\dots,n-1\}$. For each $n\in\mathbb{N}$, denote by $B_n(\Sigma)$ the set of allowable words of length n of Σ. For $i_1\dots i_n\in B_n(\Sigma)$, we define a cylinder set $[i_1\dots i_n]$ of length n by

$$[i_1\dots i_n] = \{x\in\Sigma : x_j=i_j \text{ for } 1\le j\le n\}.$$

We endow Σ with the topology generated by cylinder sets. This is a metrizable space. The following metric generates the cylinder topology. Let d be the metric on Σ by $d(x,x')=1/2^k$ if $x_i={x'}_i$ for all $1\le i\le k$ and $x_{k+1}\neq {x'}_{k+1}$, $d(x,x')=1$ if $x_1\neq x'_1$, and $d(x,x')=0$ otherwise. We stress that, in general, Σ is a non-compact space.

We can drop the Markov structure and define subshifts on countable alphabets. Let X be a closed subset of the full shift Σ. If X is σ-invariant, that is $\sigma(X)\subseteq X$, then we say that $(X,\sigma|_X)$ is a *subshift* and we write σ_X instead of $\sigma|_X$. In particular, if X is not a subset of the full shift on a finite alphabet, then we say that (X,σ_X) is a *subshift on a countable alphabet*. We also write (X,σ) for simplicity. The set X is endowed with the topology induced by Σ. In this context the set of allowable words of length n of X is defined by

$$B_n(X) := \{i_1\dots i_n\in B_n(\Sigma) : [i_1\dots i_n]\cap X\neq\emptyset\}.$$

For an allowable word $w=i_1\dots i_n$ we denote by $|w|$ its length, $|i_1\dots i_n|=n$. Given a subshift (X,σ) on a countable alphabet, we now define the language of X. The word of length $n=0$ of X is called the empty word and it is denoted by ε. The language of X is the set $B(X)=\bigcup_{n=0}^{\infty}B_n(X)$, i.e., the union of all allowable words of X and the empty word ε.

We now define several notions of specification that generalize the one first introduced by Bowen [6] with the purpose of proving that there exists a unique measure of maximal entropy for a large class of compact subshifts. Our definitions are given in terms of the language of X.

Definition 2.1. A subshift (X, σ) on a countable alphabet is *irreducible* if for any allowable words $u, v \in B(X)$, there exists an allowable word $w \in B(X)$ such that $uwv \in B(X)$.

Definition 2.2. A subshift (X, σ) on a countable alphabet has the *weak specification property* if there exists $p \in \mathbb{N}$ such that for any allowable words $u, v \in B(X)$, there exist $0 \le k \le p$ and $w \in B_k(X)$ such that $uwv \in B(X)$. If in addition, $k = p$ for any u and v, then X has the *strong specification property.* We call such p a weak (strong, respectively) specification number.

Definition 2.3. A subshift (X, σ) is *finitely irreducible* if there exist $p \in \mathbb{N}$ and a finite subset $W_1 \subset \bigcup_{n=0}^{p} B_n(X)$ such that for every $u, v \in B(X)$, there exists $w \in W_1$ such that $uwv \in B(X)$. It is *finitely primitive* if there exist $p \in \mathbb{N}$ and a finite subset $W_1 \subset B_p(X)$ such that for every $u, v \in B(X)$, there exists $w \in W_1$ such that $uwv \in B(X)$.

The notion of finitely primitive (see Definition 2.3) is essentially the same as that of specification introduced by Bowen [6] in a non-compact symbolic setting. There is a closely related class of countable Markov shifts studied by Sarig [44]. A countable Markov shift is *topologically mixing*, i.e., for each pair $x, y \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every $n > N$ there is an allowable word $i_1 \dots i_n \in B_n(\Sigma)$ such that $i_1 = x, i_n = y$.

Definition 2.4. A countable Markov shift (Σ, σ) is said to satisfy the *big images and preimages property (BIP property)* if there exists $\{b_1, b_2, \dots, b_n\}$ in the alphabet S such that for every $a \in S$ there exist $i, j \in \{1, \dots, n\}$ such that $t_{b_i a} t_{a b_j} = 1$.

Lemma 2.1. [44] *If (Σ, σ) is a topologically mixing countable Markov shift with the BIP property, then it is finitely primitive.*

If a countable Markov shift (Σ, σ) satisfies the strong specification property then it is topologically mixing and has infinite entropy. On the other hand, if (Σ, σ) satisfies the weak specification property then it is irreducible and has infinite entropy (see Section 4). The weak specification property does not imply topologically mixing.

2.2. **Pressure for a class of sequences of continuous functions.** In this section, we provide two definitions of pressure of sequences of continuous functions defined on non-compact subshifts. We prove that under fairly general assumptions both coincide. Let (X, σ) be a subshift on a countable alphabet. For each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}^+$ be a continuous function.

Definition 2.5. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ of continuous functions on X is called *almost-additive* if there exists a constant $C \ge 0$ such that for every $n, m \in \mathbb{N}, x \in X$, $\mathcal{F}$ satisfies

$$f_{n+m}(x) \le f_n(x) f_m(\sigma^n x) e^C \tag{2.1}$$

and

$$f_n(x) f_m(\sigma^n x) e^{-C} \le f_{n+m}(x). \tag{2.2}$$

In particular, $\mathcal{F}$ is called *sub-additive* if $\mathcal{F}$ satisfies (2.1) with $C = 0$. The sequence $\mathcal{F} + C = \{\log(e^C f_n)\}_{n=1}^{\infty}$ is sub-additive if and only if (2.1) holds.

Definition 2.6. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ of continuous functions on X is called a *Bowen* sequence if there exists $M \in \mathbb{R}^+$ such that $\sup\{M_n : n \in \mathbb{N}\} \leq M$, where

$$M_n = \sup\left\{\frac{f_n(x)}{f_n(y)} : x, y \in X, x_i = y_i \text{ for } 1 \leq i \leq n\right\}.$$

More generally, if $M_n < \infty$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(1/n)\log M_n = 0$, then we say that $\mathcal{F}$ has *tempered variation*. Without loss of generality, we assume $M_n \leq M_{n+1}$ for all $n \in \mathbb{N}$.

Definition 2.6 extends a notion introduced by Walters [46]. A continuous function $f : X \to \mathbb{R}$ belongs to the *Bowen class* if the sequence $\{\log e^{S_n f}\}_{n=1}^{\infty}$, where $(S_n f)(x) = f(x) + f(\sigma(x)) + \cdots + f(\sigma^{n-1}(x))$ for each $x \in X$ is a Bowen sequence. The functions of summable variations belong to the Bowen class and the Bowen sequences are a generalization of functions in the Bowen class (see [3, 26]).

We now list several assumptions we will use throughout the paper. These are hypothesis on both the system (X, σ) and the sequence $\mathcal{F}$.

(C1) The sequence $\mathcal{F} + C$ is sub-additive for some $C \geq 0$.

(C2) There exist $p \in \mathbb{N}$ and $D > 0$ such that given any $u \in B_n(X), v \in B_m(X)$, $n, m \in \mathbb{N}$, there exists $w \in B_k(X), 0 \leq k \leq p$ such that

$$\sup\{f_{n+m+k}(x) : x \in [uwv]\} \geq D \sup\{f_n(x) : x \in [u]\} \sup\{f_m(x) : x \in [v]\}.$$

(C3) There exists a finite set $W \subset \bigcup_{k=0}^{p} B_k(X)$ consisting of elements w for which the property (C2) holds.

(C4) $Z_1(\mathcal{F}) := \sum_{i\in\mathbb{N}} \sup\{f_1(x) : x \in [i]\} < \infty$.

In addition, we consider in Section 4.2 sequences satisfying the following weaker condition.

(D2) There exist $p \in \mathbb{N}$ and a positive sequence $\{D_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ such that given any $u \in B_n(X), v \in B_m(X)$, $n, m \in \mathbb{N}$, there exists $w \in B_k(X), 0 \leq k \leq p$ such that

$$\sup\{f_{n+m+k}(x) : x \in [uwv]\} \geq D_{n,m} \sup\{f_n(x) : x \in [u]\} \sup\{f_m(x) : x \in [v]\},$$

where $\lim_{n\to\infty}(1/n)\log D_{n,m} = \lim_{m\to\infty}(1/m)\log D_{n,m} = 0$. Without loss of generality, we assume that $D_{n,m} \geq D_{n,m+1}$ and $D_{n,m} \geq D_{n+1,m}$.

(D3) There exists a finite set $W \subset \bigcup_{k=0}^{p} B_k(X)$ consisting of elements w for which the property (D2) holds.

If $\mathcal{F}$ on X satisfies (C2) ((D2), respectively) with $w \in B_p(X)$ for all w, then we say that $\mathcal{F}$ on X satisfies (C2) ((D2), respectively) with the strong specification.

It is easy to see that if (X, σ) is a subshift on a countable alphabet and $\mathcal{F}$ is a Bowen sequence on X satisfying (C1) and (C2), then $W = \{\varepsilon\}$ in (C3) if and only if (X, σ) is the full shift on a countable alphabet and $\mathcal{F}$ is almost-additive on the full shift.

Definition 2.7. Let (X, σ) be an irreducible subshift on a countable alphabet and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a sequence of continuous functions on X with tempered variation satisfying (C1). Define $Z_n(\mathcal{F})$ by $Z_n(\mathcal{F}) := \sum_{i_1\ldots i_n\in B_n(X)} \sup\{f_n(x) : x \in [i_1 \ldots i_n]\}$ and the *topological pressure* of $\mathcal{F}$ by

$$P(\mathcal{F}) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}), \tag{2.3}$$

if $\limsup_{n\to\infty}(1/n)\log Z_n(\mathcal{F})$ exists, including possibly ∞ and $-\infty$.

If $Z_1(\mathcal{F}) < \infty$ then sub-additivity of the sequence $\mathcal{F} + C$ implies that $-\infty \leq P(\mathcal{F}) < \infty$. We study the case when $Z_1(\mathcal{F}) = \infty$ in Section 4.

Remark 2.1. Definition 2.7 generalizes the definition by Mauldin and Urbański [34] on the topological pressure on finitely irreducible countable Markov shifts. Assumption (C2) was introduced by Feng [17] and appeared in the study of dimension of non-conformal repellers [18, 49]. Feng [18] studied thermodynamic formalism for the class of the Bowen sequences on subshifts on finite alphabets satisfying (C1) and (C2) (see Theorem 5.2). Käenmäki and Reeve [29] extended the work of Feng [17, 18] by studying thermodynamic formalism for the Bowen sequences on the full shift on a countable alphabet satisfying (C1), (C2) and (C3).

Next we define the Gurevich pressure. Throughout the paper, we identify the set of a countable alphabet with $\mathbb{N}$.

Definition 2.8. Let (X, σ) be an irreducible subshift on a countable alphabet and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a sequence of continuous functions on X with tempered variation satisfying (C1) and (D2). For $a \in \mathbb{N}$, define $Z_n(\mathcal{F}, a) := \sum_{x:\sigma^n x = x} f_n(x)\chi_{[a]}(x)$, where $\chi_{[a]}(x)$ is a characteristic function of the cylinder $[a]$. The *Gurevich pressure* of $\mathcal{F}$ on X, denoted by $P_G(\mathcal{F})$, is defined by

$$P_G(\mathcal{F}) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}, a), \tag{2.4}$$

if $\limsup_{n\to\infty}(1/n)\log Z_n(\mathcal{F}, a)$ is independent of $a \in \mathbb{N}$.

Remark 2.2. The Gurevich entropy was first introduced by Gurevich for countable Markov shifts. This notion was later extended by Sarig [42] where he defines the Gurevich pressure of regular potentials defined on topologically mixing countable Markov shifts. The Gurevich pressure has been studied in [21, 11] for regular potentials defined on more general countable Markov shifts, and in [26] for almost-additive sequences. We stress that the definition given here extends both the class of sequences of potentials and the class of shifts (satisfying the weak specification) previously considered in the literature.

It was shown by Mauldin and Urbański [35] and by Sarig [44] that when restricted to topologically mixing countable Markov shifts satisfying the BIP property for a regular potential, Definitions 2.7 and 2.8 coincide. The next result extends this observation.

Proposition 2.1. *Let (X, σ) be a countable Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a sequence on X with tempered variation satisfying* (C1) *and* (D2). *If $P(\mathcal{F}) < \infty$, then*

$$P(\mathcal{F}) = P_G(\mathcal{F}). \tag{2.5}$$

If $\mathcal{F}$ satisfies (D2) *with the strong specification, then* $\limsup$ *in (2.4) can be replaced by* $\lim$.

Proof. Observe that $P(\mathcal{F}) < \infty$ if and only if $Z_1(\mathcal{F}) < \infty$ (see Proposition 4.1). Let $a \in \mathbb{N}$ be fixed and $c_n \in B_n(X)$. By (D2) there exist allowable words w_1, w_2 with $0 \leq |w_1|, |w_2| \leq p$, such that $aw_1c_nw_2a$ is an allowable word of length $n + 2 + |w_1| + |w_2|$ satisfying

$$\begin{aligned}&\sup\{f_{n+2+|w_1|+|w_2|}(x) : x \in [aw_1c_nw_2a]\}\\ &\geq D_{1,n}D_{1+p+n,1}\sup\{f_n(x) : x \in [c_n]\}(\sup\{f_1(x) : x \in [a]\})^2.\end{aligned}$$

Since $\mathcal{F}$ has tempered variation, for any $x \in [aw_1c_nw_2a]$ we have that

$$\begin{aligned}&\sup\{f_{n+2+|w_1|+|w_2|}(x) : x \in [aw_1c_nw_2a]\}\\ &\le M_{n+2p+2} f_{n+2+|w_1|+|w_2|}(x) \le M_{n+2p+2} f_{n+1+|w_1|+|w_2|}(x) \sup\{f_1(x) : x \in [a]\} e^{C}.\end{aligned}$$

Since $\bar{x} = (aw_1c_nw_2)^\infty = (aw_1c_nw_2aw_1c_nw_2aw_1c_nw_2\dots)$ is a periodic point with period $n + |w_1| + |w_2| + 1$, we obtain

$$f_{n+|w_1|+|w_2|+1}(\bar{x}) \ge \frac{D_{1,n}D_{1+p+n,1}e^{-C}}{M_{n+2p+2}} \sup\{f_n(x) : x \in [c_n]\} \sup\{f_1(x) : x \in [a]\}.$$

Since $\mathcal{F}$ has tempered variation, we have that $\sup\{f_1(x) : x \in [a]\}$ is bounded. Setting $d_n = (D_{1,n}D_{1+p+n,1} \sup\{f_1(x) : x \in [a]\})/(e^C M_{n+2p+2})$ and summing over all allowable words $c_n \in B_n(X)$, we obtain

$$\sum_{i=n+1}^{n+2p+1} Z_i(\mathcal{F}, a) \ge d_n Z_n(\mathcal{F}) > 0.$$

Hence, there exists $n+1 \le i_n \le n+2p+1$ such that $Z_{i_n}(\mathcal{F}, a) \ge (d_n Z_n(\mathcal{F}))/(2p+1)$.

$$\frac{1}{i_n} \log Z_{i_n}(\mathcal{F}, a) \ge \frac{1}{n+2p+2} \left(\log \frac{1}{2p+1} + \log d_n + \log Z_n(\mathcal{F}) \right).$$

Thus $\limsup_{n\to\infty}(1/i_n) \log Z_{i_n}(\mathcal{F}, a) \ge P(\mathcal{F})$. Since $Z_{i_n}(\mathcal{F}, a) \le Z_{i_n}(\mathcal{F})$ for all i_n and a is arbitrary, we obtain (2.5). If $\mathcal{F}$ satisfies (D2) with the strong specification, we obtain $\limsup_{n\to\infty}(1/n) \log Z_n(\mathcal{F}, a) = \liminf_{n\to\infty}(1/n) \log Z_n(\mathcal{F}, a)$ by using similar arguments. □

Remark 2.3. In Section 4, we obtain (2.5) when $Z_1(\mathcal{F}) = \infty$ under certain assumptions on (X, σ) and $\mathcal{F}$. Clearly, $P_G(\mathcal{F}) \le P(\mathcal{F})$. It is well known that for a compact irreducible sofic shift $P_G(0) = P(0)$ (see [33, Theorem 4.3.6]). However, even for topologically mixing countable Markov shifts these two notions can be different, we can have $P_G(0) < P(0)$. The numbers $P(0)$ and $P_G(0)$ are called the *entropy* and the *Gurevich entropy* respectively and we denote them by $h(\sigma)$ and $h_G(\sigma)$ respectively.

2.3. **Factor maps.** The goal of this section is to study certain subshifts which are images of countable Markov shifts under factor maps. Let (X, σ_X) and (Y, σ_Y) be subshifts on countable alphabets. A *one-block code* is a map $\pi : X \to Y$ for which there exists a function, denoted again by π, $\pi : B_1(X) \to B_1(Y)$ such that $(\pi(x))_i = \pi(x_i)$ for all $i \in \mathbb{N}$. For $u = x_1 \dots x_k \in B_k(X)$, $k \in \mathbb{N}$, we denote $\pi(x_1)\dots\pi(x_k) \in B_k(Y)$ by $\pi(u)$. A map $\pi : X \to Y$ is a *factor map* if it is continuous, surjective and satisfies $\pi \circ \sigma_X = \sigma_Y \circ \pi$. For a one-block factor map $\pi : X \to Y$ where X is an irreducible countable Markov shift, let $v \in B_k(Y)$. We denote by $\pi^{-1}(v)$ the set of allowable words u of length k of X such that $\pi(u) = v$ and by $|\pi^{-1}(v)|$ the cardinality of the set. Throughout the paper, we only consider one-block factor maps $\pi : X \to Y$ such that $|\pi^{-1}(i)| < \infty$ for any $i \in \mathbb{N}$. Hence for each $k \in \mathbb{N}$, $v \in B_k(Y)$, we have $|\pi^{-1}(v)| < \infty$.

In general, the image of a shift space on a countable alphabet under a sliding block code is not closed and hence it is not a subshift (see [33]).

Lemma 2.2. *Let (X, σ_X) be a subshift on a countable alphabet, (Σ, σ) the full shift on a countable alphabet and $\pi : X \to \Sigma$ a one-block code such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. Let $Y := \pi(X)$. Then (Y, σ_Y) is a subshift on a countable alphabet.*

Proof. It is enough to show that Y is closed. For $m \in \mathbb{N}$, let $y^{(m)} = \{y_n^{(m)}\}_{n=1}^{\infty} \in Y$. Let $\{y^{(m)}\}_{m=1}^{\infty}$ be a sequence in Y converging to $y = \{y_i\}_{i=1}^{\infty}$. Since Y is the image of X under π, for each $m \in \mathbb{N}$, pick an $x^{(m)} \in X$ such that $\pi(x^{(m)}) = y^{(m)}$ and let $x^{(m)} = \{x_n^{(m)}\}_{n=1}^{\infty}$. Fix $l \in \mathbb{N}$. There exists $M \in \mathbb{N}$ such that $d(y^{(m)}, y) < 1/2^l$ for all $m \geq M$. Then $y_i^{(m)} = y_i$ for all $m \geq M$, $1 \leq i \leq l+1$. Consider $\{x^{(m)}\}_{m=M}^{\infty}$. Then we have $x_i^{(m)} \in \pi^{-1}(y_i)$ for $1 \leq i \leq l+1, m \geq M$. Since there are finitely many symbols in $\pi^{-1}(y_1)$, there exists $x_1^* \in \pi^{-1}(y_1)$ such that x_1^* is the initial symbol of $x^{(m)}$, for infinitely many $m \geq M$. We extract a subsequence $\{x^{1,n}\}_{n=1}^{\infty}$ of sequences with the initial symbol x_1^* from $\{x^{(m)}\}_{m=M}^{\infty}$. Define $\{x^{0,n}\}_{n=1}^{\infty} := \{x^{(m)}\}_{m=M}^{\infty}$. Repeating this process, for each $1 \leq i \leq l+1$, there exist $x_i^* \in \pi^{-1}(y_i)$ and a sequence $\{x^{i,n}\}_{n=1}^{\infty}$ of sequences with the i th symbol x_i^* such that $\{x^{i,n}\}_{n=1}^{\infty}$ is a subsequence of $\{x^{i-1,n}\}_{n=1}^{\infty}$. We define x_i^* for $i = l+i$,$i \geq 2$ similarly. Given $l+1$, there exists $M_1 \in \mathbb{N}$ such that $d(y^{(m)}, y) < 1/2^{l+1}$ for all $m \geq M_1$. Then we have $y_i^{(m)} = y_i$ for all $m \geq M_1$, $1 \leq i \leq l+2$. Let $\{z^{l+1,n}\}_{n=1}^{\infty} := \{x^{l+1,n}\}_{n=1}^{\infty} \cap \{x^m\}_{m=M_1}^{\infty}$. Since there are finitely many symbols in $\pi^{-1}(y_{l+2})$, there exists $x_{l+2}^* \in \pi^{-1}(y_{l+2})$ such that x_{l+2}^* is the $(2+l)$ th symbol of $\{z^{l+1,n}\}_{n=1}^{\infty}$ for infinitely many n. Extract a subsequence $\{x^{l+2,n}\}_{n=1}^{\infty}$ of sequences with the $(2+l)$ th symbol x_{l+2}^* from $\{z^{l+1,n}\}_{n=1}^{\infty}$. Since $|\pi^{-1}(k)| < \infty$ for each $k \in \mathbb{N}$, repeating this process, for each $i \geq 2$ there exist $x_{l+i}^* \in \pi^{-1}(y_{l+i})$ and a sequence $\{x^{l+i,n}\}_{n=1}^{\infty}$ with the i th symbol x_i^*, each of which is a subsequence of $\{x^{l+i-1,n}\}_{n=1}^{\infty}$. Define $x^* = \{x_i^*\}_{i=1}^{\infty}$. By a diagonal argument, $\{x^{l+i,i}\}_{i=1}^{\infty}$ converges to x^*. Since X is closed, we obtain that $x^* \in X$. Then $\pi(x^*) = \{\pi(x_i^*)\}_{i=1}^{\infty} = \{y_i\}_{i=1}^{\infty} = y$. □

Recall that if (X, σ) is a finite state Markov shift, then the image of X under a one-block factor map is a sofic shift [33, 7].

Definition 2.9. A *countable sofic shift* is a subshift on a countable alphabet which is the image of a countable Markov shift under a one-block factor map π such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. In particular, an *irreducible countable sofic shift* is the image of an irreducible countable Markov shift.

It is well known that if (X, σ_X) and (Y, σ_Y) are subshifts on finite alphabets such that there exists a factor map $\pi : X \to Y$, then $h(\sigma_X) \geq h(\sigma_Y)$. In the non-compact case, this is in general not true (see the discussion in [33, Section 13.9]).

Lemma 2.3. *Let (X, σ_X) and (Y, σ_Y) be topologically mixing countable Markov shifts and $\pi : X \to Y$ a one-block factor map such that $|\pi^{-1}(n)| < \infty$ for every $n \in \mathbb{N}$. Then $h(\sigma_X) \geq h(\sigma_Y)$.*

Proof. Recall that the Gurevich entropy satisfies the following approximation property by compact sets [23, 24]

$$\begin{aligned} h_G(\sigma_X) &= \sup\{h(\sigma_X|_K) : K \subset X \text{ compact and invariant}\} \\ &= \sup\{h(\sigma_X|_{\Sigma_K}) : \Sigma_K \subset X \text{ topologically mixing finite Markov shift}\}. \end{aligned}$$

Since for every $n \in \mathbb{N}$ we have that $|\pi^{-1}(n)| < \infty$, for every $\Sigma_K \subset Y$ topologically mixing finite Markov shift we have that $\pi^{-1}(\Sigma_K)$ is a compact subshift of X.

Therefore, by [32, Proposition 4.16] we have that $h_G(\sigma_X|_{\pi^{-1}(\Sigma_K)}) \geq h_G(\sigma_Y|_{\Sigma_K})$. The result now follows. □

3. Examples

3.1. Differences between the (C2) condition and almost-additivity. In this section, we establish relations between almost-additivity, (C2) and (D2) introduced in Section 2.2. The results depend upon the combinatorial structure of the shifts.

Remark 3.1. If (X, σ) is an irreducible Markov shift defined on a finite alphabet (compact), then any almost-additive Bowen sequence on X satisfies (C2).

The next lemma generalizes the result in Remark 3.1.

Lemma 3.1. *Let (X, σ) be a finitely irreducible subshift on a countable alphabet and $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ an almost-additive sequence on X with tempered variation. Then $\mathcal{G}$ satisfies* (C1), *(D2) and* (D3). *If $\mathcal{G}$ is an almost-additive Bowen sequence on X, then it satisfies* (C1), (C2) *and* (C3).

Proof. Since (X, σ) is a finitely irreducible subshift on a countable alphabet, there exist $p \in \mathbb{N}$ and a finite set $W_1 \subset \bigcup_{i=0}^{p} B_i(X)$ such that for any $n, m \in \mathbb{N}$ and $u \in B_n(X), v \in B_m(X)$ there exists $w \in W_1$ such that uwv is an allowable word. Since W_1 is a finite set and $\mathcal{G}$ has tempered variation, there exists $Q_1 > 0$ such that

$$\sup_{w \in W_1, |w| \geq 1} \left\{ g_{|w|}(y) : y \in [w] \right\} > Q_1.$$

For $n \in \mathbb{N}$, let M_n is defined as in Definition 2.6. Let $x \in [uwv]$, where $|w| = k \geq 1$. Then

$$g_{n+m+k}(x) \geq e^{-2C} g_n(x) g_k(\sigma^n x) g_m(\sigma^{k+n} x) \geq \frac{e^{-2C} Q_1}{M_p} g_n(x) g_m(\sigma^{k+n} x). \tag{3.1}$$

Consider a pair $u \in B_n(X), v \in B_m(X)$ such that uv is an allowable word. If $x \in [uv]$, then $g_{n+m}(x) \geq e^{-C} g_n(x) g_m(\sigma^n x)$. Let $Q = \min\{Q_1, 1\}$. Then (D2) holds in particular for p equal to the same p that appears in the specification property and we obtain the result by setting $D_{n,m} = (e^{-2C} Q)/(M_p M_n M_m)$ in (D2) and $W = W_1$ in (D3). If $\mathcal{G}$ is an almost-additive Bowen sequence, then we replace M_p, M_n and M_m by M. □

Lemma 3.2. *Let (X, σ) be a subshift on a countable alphabet, $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ an almost-additive sequence on X with tempered variation, and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a sequence on X satisfying* (C1),(D2) *and* (D3). *Define $\mathcal{H} := \{\log(f_n/g_n)\}_{n=1}^{\infty}$. Then $\mathcal{H}$ satisfies* (C1),(D2) *and* (D3).

Proof. The proof is straightforward and similar to that of Lemma 3.1. □

Example 3.1. A continuous function on a finitely irreducible subshift with tempered variation. Let f be a continuous function defined on a finitely irreducible subshift X. Denote by $A_n := \sup\{|(S_n f)(x) - (S_n f)(y)| : x_i = y_i, 1 \leq i \leq n\}$. Then f has *tempered variation* if $A_n < \infty$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} (1/n) A_n = 0$. We remark that sometimes (see for example [21]) the definition of tempered variation is given without the finiteness assumption $A_n < \infty$. For each $n \in \mathbb{N}$, define $f_n(x) = e^{(S_n f)(x)}$ and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$. Then $\mathcal{F}$ is additive and satisfies (D2) and (D3).

Example 3.2. An almost-additive sequence on a countable Markov shift which does not satisfy (C2). Let (X, σ) a countable Markov shift determined by the transitions given by Figure 1. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lambda_n \in (0,1)$ and $\sum_{j=1}^{\infty} \lambda_j < \infty$. Let $\{\log c_n\}_{n=1}^{\infty}$ be an almost-additive sequence of real numbers. Hence there exists a constant $C \geq 0$ such that $e^{-C} c_n c_m \leq c_{n+m} \leq e^{C} c_n c_m$. For $n \in \mathbb{N}$, define $g_n : X \to \mathbb{R}$ by $g_n(x) = c_n \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}, x \in [i_1 \dots i_n]$. Let $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$. These sequences have been studied in [26, Example 1] when defined on the full shift.

FIGURE 1. The graph defining X in Example 3.2

Lemma 3.3. *The sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ defined on X is an almost-additive Bowen sequence. However, it does not satisfy* (C2).

Proof. Observe that (X, σ) is topologically mixing and that 3 is a strong specification number and, moreover, X is not finitely irreducible.

Claim 3.1. *Let (X, σ) be a subshift on a countable alphabet and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a Bowen sequence on X satisfying* (C1) *and* (C2). *Let $w \in \bigcup_{i=1}^{p} B_i(X)$ be an allowable word from* (C2). *Then there exists $C' > 0$ such that for any w of length k, we have $\sup\{f_k(x) : x \in [w]\} \geq C'$.*

Proof. Since (C2) is satisfied, let $u \in B_n(X), v \in B_m(X)$ and $w \in B_k(X)$ be defined as in (C2). We consider only uwv with the length k of $w \geq 1$. For any $x \in [uwv]$, it is a consequence of (C1) and the Bowen property of $\mathcal{F}$ that, $Me^{2C} f_n(x) f_k(\sigma^n x) f_m(\sigma^{k+n} x) \geq D f_n(x) f_m(\sigma^{k+n} x)$. Hence $\sup\{f_k(x) : x \in [w]\} \geq D/(Me^{2C}) = C'$. □

Assume by way of contradiction that the sequence $\mathcal{G}$ satisfies (C2) for some $p \in \mathbb{N}$. Consider the symbol 3 and $3n$ for some $n \in \mathbb{N}$. To connect 3 and $3n$, the symbol $3n+1$ must be passed through. Let w be a word of length $k \leq p$ such that $3w(3n)$ is allowable and satisfies (C2). Then $3n+1$ must appear in some $w_i, 1 \leq i \leq k$, where $k \geq 1$. Since λ_j is bounded above by some constant $C'' > 1$ for all $j \in \mathbb{N}$, we obtain

$$\sup\{g_k(x) : x \in [w]\} \leq \max_{1 \leq k \leq p} \{c_p\} C''^{p-1} \lambda_{3n+1}.$$

By Claim 3.1, λ_{3n+1} is bounded below by a positive constant for all $n \in \mathbb{N}$. This is a contradiction. □

Example 3.3. A sequence satisfying (C1),(C2) and (C3). In this example, we will make use of the notion of factor map (see Section 2.3). Let (X, σ_X) be a countable Markov shift, (Y, σ_Y) subshift on a countable alphabet, and $\pi : X \to Y$

one-block factor map such that $|\pi^{-1}(i)| < \infty$, for every $i \in \mathbb{N}$. Define $\phi_n : Y \to \mathbb{R}$ by $\phi_n(y) = \log |\pi^{-1}(y_1 \dots y_n)|$ and $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$. Let ε_X and ε_Y be the empty words of X and Y respectively. By convention, let $\pi(\varepsilon_X) = \varepsilon_Y$. In general Φ is not almost-additive (see [49, 50]).

Lemma 3.4. *If X is a finitely irreducible countable Markov shift, then $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is a Bowen sequence on Y satisfying* (C1),(C2) *and* (C3). *If W_1 is a finite set from Definition* 2.3*, then let $\pi(W_1) = \{\pi(w) : w \in W_1\}$. For any $u \in B_n(Y), v \in B_m(Y), n, m \in \mathbb{N}$, there exists $w' \in \pi(W_1)$ such that $|\pi^{-1}(uw'v)| \geq (1/|W_1|)|\pi^{-1}(u)||\pi^{-1}(v)|$.*

Proof. See [18, Lemma 5.7] in which the above result was studied for the case when X is an irreducible subshift on finite alphabets. This implies the result. □

3.2. **More examples of sequences on irreducible countable sofic shifts.** The examples in this section can only occur in non-compact settings and show some of the new phenomena that have to be considered in the countable alphabet situation. Let Φ be the sequence of functions as in Example 3.3.

Example 3.4. A sequence on a finitely irreducible countable Markov shift satisfying (C1), (C2) and (C3). Let (X, σ) be a countable Markov shift determined by the transitions given by Figure 2. Let $\pi : \mathbb{N} \to \mathbb{N}$ be the function defined by $\pi(-i + n(n+1)/2) = n$, $i = 0, \dots, n-1$ for $n \in \mathbb{N}$ and Σ be the full shift on a countable alphabet. Define $\pi : X \to \Sigma$ by $(\pi(x))_i = \pi(x_i)$ for all $i \in \mathbb{N}$ and denote $\pi(X)$ by Y. Then the map $\pi : X \to Y$ is a one-block factor map. Then $|\pi^{-1}(i)| = i$ for $i \in \mathbb{N}$ and we stress that this property cannot occur when X is a finite state Markov shift. X has a strong specification number equal to 2, just by considering $W = \{12, 22\}$. The countable Markov shift Y also has a strong specification number 2. Then Φ is not almost-additive on Y. Let A be the transition matrix for X. It was shown in [49, Example 5.6] that Φ is not almost-additive on $\pi(X_{A|_{\{1,2,3\}\times\{1,2,3\}}})$. Let $k \geq 3$ be fixed. For $y = (y_1, \dots, y_n, \dots) \in Y$, define $\psi_n(y) = \phi_n(y)/(|\pi^{-1}(y_1)| \cdots |\pi^{-1}(y_n)|)^k$ and $\Psi = \{\log \psi_n\}_{n=1}^{\infty}$. Ψ is not almost-additive but it is sub-additive. By Lemma 3.4, (C2) holds with $p = 2$. For $u \in B_n(Y)$ and $v \in B_m(Y)$, there exists a word $w \in \{\pi(12), \pi(22)\}$ of length 2 such that

$$\sup\{\psi_{n+m+2}(y) : y \in [uwv]\} \geq \frac{1}{2^{2k+1}} \cdot \sup\{\psi_n(y) : y \in [u]\} \sup\{\psi_m(y) : y \in [v]\}.$$

FIGURE 2. The graph defining X in Example 3.4

Example 3.5. We study a general case of Example 3.4. Let (X, σ_X) be a finitely irreducible countable Markov shift, (Y, σ_Y) a subshift, and $\pi : X \to Y$ be a one-block factor map such that $|\pi^{-1}(i)| < \infty$ for any $i \in \mathbb{N}$. Thus, (Y, σ_Y) is a finitely irreducible countable sofic shift. Suppose there exist $C_1, C_2 > 0, l \geq 1$ such that for every $i \in \mathbb{N}$ we have $C_1 i^l \leq |\pi^{-1}(i)| \leq C_2 i^l$. Define $\Psi = \{\log \psi_n\}_{n=1}^{\infty}$ as in Example 3.4. Ψ is a sub-additive Bowen sequence on Y satisfying (C1), (C2) and (C3).

Example 3.6. A sequence satisfying (C1) and (C2) but not (C3). Let (X, σ) be the countable Markov shift determined by the transitions given by Figure 3. We partition the alphabet defining X in the following way: $F_1 = \{1\}, F_2 = \{2, 3\}, F_3 = \{4, 5, 6\}, \dots$, in general F_n consists of n symbols, such that the subshift of X restricted to the symbols of F_n is the full shift on n symbols. Let $\pi : \mathbb{N} \to \mathbb{N}$ be the function defined by $\pi(a) = n$ if $a \in F_n, n \in \mathbb{N}$. Let $\pi : X \to \Sigma$ and Y be defined as in Example 3.4. Then X is not finitely irreducible and Y is finitely primitive. Since $|\pi^{-1}(i)| = i$ and $|\pi^{-1}(i1)| = 1$, $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is not almost-additive. Let $u = u_1 \dots u_n \in B_n(Y)$ and $v = v_1 \dots v_m \in B_m(Y)$. Observe that $|\pi^{-1}(uu_n 1 v_1 v)| \geq |\pi^{-1}(u)||\pi^{-1}(v)|$. Thus (C2) is satisfied. To see this, consider a preimage $\bar{u}$ of u and $\bar{v}$ of v. Then $\bar{u}_n \in F_s$ and $\bar{v}_1 \in F_t$ for some $s, t \in \mathbb{N}$. Let $s \neq 1$ and $t \neq 1$. Define $a_s \in F_s$ and $a_t \in F_t$ such that $1a_s1$ and $1a_t1$ are allowable words. $\bar{u}a_s 1 a_t \bar{v}$ is an allowable word of X and $\pi(\bar{u}a_s 1 a_t \bar{v}) = uu_n 1 v_1 v$. Other cases are studied similarly. Φ is a sub-additive Bowen sequence on Y satisfying (C2) with the strong specification. It is easy to show by a proof of contradiction that for any $p \in \mathbb{N}$ (C3) is not satisfied.

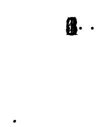

FIGURE 3. The graph defining X in Example 3.6

4. VARIATIONAL PRINCIPLE

The variational principle establishes a relation between the pressure (which is defined by means of the topological structure of the system) and the sum of the metric entropy and the integral with respect to an invariant measure (which is defined by means of the Borel structure of the system).

4.1. **Variational principle for countable Markov shifts with the the weak specification property without the finiteness condition (C3).** In this section we study about the Bowen sequences defined on countable Markov shifts satisfying (C1) and (C2).

Proposition 4.1. *Let (X, σ) be a subshift on a countable alphabet. If $\mathcal{F}$ is a sequence on X with tempered variation satisfying* (C1) *and* (D2)*, then $P(\mathcal{F}) < \infty$ if and only if $Z_1(\mathcal{F}) < \infty$.*

Proof. It is enough to show that $P(\mathcal{F}) < \infty$ if and only if $Z_n(\mathcal{F}) < \infty$ for all $n \in \mathbb{N}$. If $Z_n(\mathcal{F}) < \infty$ for all $n \in \mathbb{N}$, then $P(\mathcal{F}) < \infty$. If $P(\mathcal{F}) < \infty$, then there exists $N \in \mathbb{N}$ such that $Z_n(\mathcal{F}) < \infty$ for all $n \geq N$. Let $u_N \in B_N(X)$ and $v_1 \in B_1(X)$.

Then by (D2), $\sum_{i=0}^{p} Z_{N+i+1}(\mathcal{F}) \geq D_{N,1} Z_N(\mathcal{F}) Z_1(\mathcal{F})$. Since $Z_N(\mathcal{F})$ is bounded below by a constant, we obtain that $Z_1(\mathcal{F}) < \infty$ and hence $Z_n(\mathcal{F}) < \infty$ for all $n \in \mathbb{N}$. □

Remark 4.1. See [35, Proposition 1.6] for a result related to Proposition 4.1.

If X is an irreducible countable Markov shift, by rearranging the set $\mathbb{N}$ of the symbols of X, there exists a transition matrix A for X and an increasing sequence $\{k_n\}_{n=1}^{\infty}$ such that the matrix $A|_{\{1,\dots,k_n\}\times\{1,\dots,k_n\}}$ is irreducible. Define $A_{k_n} := A|_{\{1,\dots,k_n\}\times\{1,\dots,k_n\}}$ (see [35]). We will assume the following property on a sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$.

(P1) There exist an increasing sequence $\{l_n\}_{n=1}^{\infty}$ and constants $D_1, p_1 > 0$ such that for each l_n the matrix A_{l_n} is irreducible and $\mathcal{F}|_{X_{A_{l_n}}}$ satisfies (C2) with constants D_{l_n} and $p_{l_n} \in \mathbb{N}$ such that $D_{l_n} \geq D_1$, and $p_{l_n} \leq p_1$ for every $n \in \mathbb{N}$.

For a finite state Markov shift $Y \subset X$, let $\mathcal{F}|_Y := \{\log f_n|_Y\}_{n=1}^{\infty}$.

Lemma 4.1. *If $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is a Bowen sequence on an irreducible countable Markov shift X satisfying* (P1)*, then $\mathcal{F}$ satisfies* (C2).

Proof. Let $u \in B_n(X)$ and $v \in B_m(X)$ for $n, m \in \mathbb{N}$. Then there exists $N \in \mathbb{N}$ such that u, v are allowable words of $X_{A_{l_N}}$. Call $Y := X_{A_{l_N}}$. Then there exists $w \in B_k(Y)$, $0 \leq k \leq p_{l_N} \leq p_1$ such that

$$\begin{aligned}&\sup\{f_{n+m+k}(x) : x \in [uwv]\} \geq \sup\{f_{n+m+k}|_Y(x) : x \in [uwv]\}\\&\geq D_{l_N} \sup\{f_n|_Y(x) : x \in [u]\} \sup\{f_m|_Y(x) : x \in [v]\}\\&\geq \frac{D_1}{M^2} \sup\{f_n(x) : x \in [u]\} \sup\{f_m(x) : x \in [v]\}.\end{aligned}$$

□

Proposition 4.2. *Let (X, σ) be a countable Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a sequence on X with tempered variation satisfying* (C1) *and* (P1)*. Then $P(\mathcal{F}) > -\infty$. If $Z_1(\mathcal{F}) = \infty$, then $P(\mathcal{F}) = \infty$, in particular (2.5) holds and*

$$P(\mathcal{F}) = \sup_{l_n, n\in\mathbb{N}} \left\{ P(\mathcal{F}|_{X_{A_{l_n}}}) \right\}. \tag{4.1}$$

Proof. Let $f'_n(x) = e^C f_n(x)$ for all $x \in X$ and $\mathcal{F}' = \{\log f'_n\}_{n=1}^{\infty}$. Then $\mathcal{F}'$ is sub-additive and $P(\mathcal{F}) = P(\mathcal{F}')$. Note by Proposition 2.1 that we obtain $P_G(\mathcal{F}'|_{X_{l_n}}, a) = P(\mathcal{F}'|_{X_{l_n}})$ for each l_n. Since $\mathcal{F}$ has tempered variation, if $Z_1(\mathcal{F}) = \infty$, then given $L > 0$, there exists $N \in \mathbb{N}$ such that $Z_1(\mathcal{F}|_{X_{l_N}}) > L$ and thus $Z_1(\mathcal{F}'|_{X_{l_N}}) > Le^C$. Let $Y := X_{l_N}$. Then (P1) implies that for each $n \in \mathbb{N}$ there exists $0 \leq i_n \leq p_1(n-1)$ such that

$$Z_{n+i_n}(\mathcal{F}'|_Y) \geq (\frac{D'_1}{p_1+1})^{n-1} (Z_1(\mathcal{F}'|_Y))^n, \tag{4.2}$$

where $D'_1 = D_1/e^C$. Hence $P(\mathcal{F}') \geq P(\mathcal{F}'|_Y) \geq d + (1/(p_1+1)) \log(Le^C)$ where $d = (1/(p_1+1)) \log(D'_1/(p_1+1))$. Letting $L \to \infty$, we obtain $P(\mathcal{F}) = P(\mathcal{F}') = \infty$. Then clearly (4.1) holds. To see that (2.5) holds, we apply Proposition 2.1. Since $P_G(\mathcal{F}|_Y, a) = P_G(\mathcal{F}'|_Y, a) = P(\mathcal{F}'|_Y)$ and $P_G(\mathcal{F}, a) \geq P_G(\mathcal{F}|_Y, a)$, the result follows by letting $L \to \infty$. Clearly, by (4.2), we obtain that $P(\mathcal{F}) > -\infty$. □

Lemma 4.2. *Let $\mathcal{F}$ be a Bowen sequence on a countable Markov shift satisfying* (C1) *and* (P1)*. If $\mathcal{F}$ fails to have* (C3)*, then $P(\mathcal{F}) = \infty$.*

Proof. Assume $P(\mathcal{F}) < \infty$. Since $\mathcal{F}$ satisfies (C2), $\sum_{i=1}^{p} Z_i(\mathcal{F}) = \infty$ by Claim 3.1. This is a contradiction. □

Lemma 4.3. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a Bowen sequence on an irreducible countable Markov shift X satisfying* (C1)*. Then*

(1) If $\mathcal{F}$ satisfies (C2) *and* (C3)*, then $\mathcal{F}$ satisfies* (P1).

(2) If $\mathcal{F}$ satisfies (P1) *and $Z_1(\mathcal{F}) < \infty$, then $\mathcal{F}$ satisfies* (C2) *and* (C3).

Proof. To see (2), we apply Lemmas 4.2 and 4.1. To show (1), let W be a finite set from (C3). Consider A_{l_q} where l_q is large enough so that $\{1, \dots, l_q\}$ contains all the symbols that appear in $W - \{\varepsilon\}$. Then, for $n \geq q$, $\mathcal{F}|_{X_{A_{l_n}}}$ satisfies (C2) replacing D by D/M. □

We establish the variational principle for the Bowen sequences on irreducible countable Markov shifts satisfying (P1). By Proposition 4.2 and Lemma 4.3, we first study the case when $\mathcal{F}$ satisfies (C1), (P1) and $Z_1(\mathcal{F}) = \infty$. The case when $\mathcal{F}$ satisfies (C1), (P1) and $Z_1(\mathcal{F}) < \infty$ is studied in Section 4.2. Let $M(X, \sigma)$ denote the set of σ-invariant Borel probability measures on X.

Proposition 4.3. *Let (X, σ) be a countable Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a Bowen sequence on X satisfying* (C1) *and* (P1)*. If $Z_1(\mathcal{F}) = \infty$, then*

$$\begin{aligned} P_G(\mathcal{F}) &= P(\mathcal{F}) \\ &= \sup_{\mu \in M(X,\sigma)} \left\{ h_\mu(\sigma) + \limsup_{n\to\infty} \frac{1}{n} \int \log f_n d\mu : \limsup_{n\to\infty} \frac{1}{n} \int \log f_n d\mu > -\infty \right\}. \end{aligned}$$

Proof. We apply the similar arguments as in the Proof of (4.5) in Section 4.2 by using (4.1). □

Example 4.1. Let Φ be defined as in Example 3.6. Let X_n be the subshift of X on the symbols $\{F_1, \dots, F_n\}$. Let $Y_n = \pi(X_n)$. For $n \geq 3$, each $\Phi|_{Y_n}$ satisfies (C2) with $p = 3$ and $D = 1$. Hence (P1) is satisfied. By Lemma 4.2 and Proposition 4.2, $P(\Phi) = P_G(\Phi) = \infty$ and Proposition 4.3 holds.

In section 4.2, we see the proof of (4.1) for the case $Z_1(\mathcal{F}) < \infty$ by using the topological pressure. We can alternatively use the Gurevich pressure to show (4.1). The proof is valid for the Bowen sequences $\mathcal{F}$ satisfying (C1) and (P1), without (C3). Here we sketch some ideas. Define $f'_n(x)$ and $\mathcal{F}'$ as in the proof of Proposition 4.2. Then for $a \in \mathbb{N}$, $P(\mathcal{F}') = \lim_{n\to\infty} B_n$, where $B_n = \sup_{k \geq n} (1/k) \log Z_k(\mathcal{F}', a)$ and $B_n < \infty$ for all $n \in \mathbb{N}$. Then given $\epsilon > 0$, there exist $q, n_1 \in \mathbb{N}$ such that

$$\frac{1}{q} \log Z_q(\mathcal{F}', a) < \frac{1}{q} \log Z_q(\mathcal{F}'|_{X_{A_{l_{n_1}}}}, a) + \epsilon.$$

We apply the following lemma. Let $Y = X_{A_{l_{n_1}}}$ and $p_Y = p_{l_{n_1}}$.

Lemma 4.4. *For $n, m \in \mathbb{N}$, there exists $0 \leq i_{n,m} \leq p_Y$ such that*

$$\frac{(p_Y + 1)M}{D_1} Z_{i_{n,m}+n+m}(\mathcal{F}'|_Y, a) \geq Z_n(\mathcal{F}'|_Y, a) Z_m(\mathcal{F}'|_Y, a).$$

Setting $m = n = q$ in Lemma 4.4, apply the lemma $(k-1)$ times. Approximating $(1/q) \log Z_q(\mathcal{F}'|_Y, a)$ from above and letting $k \to \infty$ imply the results.

4.2. **Variational principle for finitely irreducible countable sofic shifts.** In this section, we prove the variational principle for sequences $\mathcal{F}$ with tempered variation on finitely irreducible countable sofic shifts. Therefore the space X is not a Markov shift and it has the finiteness property. The regularity condition on $\mathcal{F}$ is weaker than what was assumed in Section 4.1.

Let (X, σ) be an irreducible countable sofic shift. Then by Definition 2.9 there exist an irreducible countable Markov shift $(\bar{X}, \sigma_{\bar{X}})$ and a one-block factor map $\pi : \bar{X} \to X$ such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. Rearranging the set $\mathbb{N}$, there is a transition matrix A for $\bar{X}$ and an increasing sequence $\{l_n\}_{n=1}^{\infty}$ such that the matrix $A_{l_n} = A|_{\{1,\dots,l_n\}\times\{1,\dots,l_n\}}$ is irreducible. For each $n \in \mathbb{N}$, let $S_{l_n} = \{\pi(i) : 1 \le i \le l_n\}$. Then $(\pi(\bar{X}_{A_{l_n}}), \sigma_{\pi(\bar{X}_{A_{l_n}})})$ is a sofic shift on the set S_{l_n} of finitely many symbols. Clearly, $\pi(\bar{X}_{A_{l_n}}) \subseteq \pi(\bar{X}_{A_{l_{n+1}}}) \subset X$ and $\mathbb{N} = \cup_{n\in\mathbb{N}} S_{l_n}$. We note that we can extract a subsequence $\{l_{n_j}\}_{j=1}^{\infty}$ such that $\pi(\bar{X}_{A_{l_{n_j}}}) \subset \pi(\bar{X}_{A_{l_{n_{j+1}}}}) \subset X$ for all $n_j, j \in \mathbb{N}$.

We continue to use the notation above throughout this section. The following lemma is important and will be also applied in Section 5.

Lemma 4.5. *Let (X, σ) be an irreducible countable sofic shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a sequence on X with tempered variation satisfying* (D2) *and* (D3)*. Let p be defined as in* (D2) *and W be defined as in* (D3)*. Then there exists $q \in \mathbb{N}$ such that for each $k \ge q$ there exists an irreducible subshift $(X_{l_k}, \sigma_{X_{l_k}})$ on the set S_{l_k} of finitely many symbols such that $\pi(\bar{X}_{A_{l_k}}) \subseteq X_{l_k} \subset X$ and $X_{l_k} \subseteq X_{l_{k+1}}$. Moreover, for any $n, m \in \mathbb{N}, k \ge q, u \in B_n(X_{l_k}), v \in B_m(X_{l_k})$, there exists $w \in W$ such that uwv is an allowable word of X_{l_k} and*

$$\begin{aligned}&\sup\{f_{n+m+|w|}|_{X_{l_k}}(x) : x \in [uwv]\}\\ &\ge \frac{D_{n,m}}{M_{n+m+p}} \sup\{f_n|_{X_{l_k}}(x) : x \in [u]\} \sup\{f_m|_{X_{l_k}}(x) : x \in [v]\},\end{aligned} \tag{4.3}$$

where M_n is defined as in Definition 2.6*.*

Remark 4.2. In the above lemma, we can extract a subsequence $\{l_{n_j}\}_{j=1}^{\infty}$ such that $X_{l_{n_j}} \subset X_{l_{n_{j+1}}}$ for all $n_j, j \in \mathbb{N}$ and $S_{l_{n_j}} \subset S_{l_{n_{j+1}}}$. Let $\mathcal{F}|_{X_{l_k}} := \{\log f_n|_{X_{l_k}}\}_{n=1}^{\infty}$. If $\mathcal{F}$ is a Bowen sequence, (4.3) implies that (C2) holds for $\mathcal{F}|_{X_{l_k}}, k \ge q$, replacing D in (C2) by D/M.

Proof. Since (X, σ) is an irreducible countable sofic shift, there exist an irreducible countable Markov shift $(\bar{X}, \sigma_{\bar{X}})$ and a one-block factor map $\pi : \bar{X} \to X$ such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. Since W is a finite set, only finitely many symbols appear in W. Assume that W contains a nonempty allowable word. Call S_W the set of symbols that appear in $W - \{\varepsilon\}$. Let $\pi^{-1}(S_W)$ be the set of preimages of the symbols of S_W in $\bar{X}$. Then $\pi^{-1}(S_W)$ is a finite set.

Consider a transition matrix A for $\bar{X}$ and an increasing sequence $\{l_k\}_{k=1}^{\infty}$ such that the matrix $A_{l_k}$is irreducible for each l_k. Then there exists $q \in \mathbb{N}$ such that $\pi^{-1}(S_W) \subset \{1, \dots, l_k\}$ for all $k \ge q$. Thus, for $k \ge q$ the subshift $(\pi(\bar{X}_{A_{l_k}}), \sigma_{\pi(\bar{X}_{A_{l_k}})})$ is a sofic shift on the set S_{l_k} of finitely many symbols that contains S_W. For a fixed $k \ge q$, consider the set $\pi^{-1}(S_{l_k})$ of the preimages of the set S_{l_k} and call it P. P contains $\{1, \dots, l_k\}$ and it is a finite set. Let $\bar{Y}_P := X_{A|_{P\times P}} \subset X$ be the finite state Markov shift on the symbols of P and define $Y = \pi(\bar{Y}_P)$. Y is a subshift on the set of S_{l_k} of finitely many symbols which contains S_W. Observe that $\pi(\bar{X}_{A_{l_k}}) \subseteq Y \subset X$.

Then Y is irreducible. Fix $n, m \in \mathbb{N}$. Let $u \in B_n(Y)$ and $v \in B_m(Y)$. Since these are allowable words of X, there exists $w \in W$ of length l, $0 \leq l \leq p$, such that uwv is allowable in X and (D2) holds. Since uwv is allowable in X, there exists $\bar{u}_1 \dots \bar{u}_n \bar{w}_1 \dots \bar{w}_l \bar{v}_1 \dots \bar{v}_m \in B_{n+m+l}(\bar{X})$ such that $\pi(\bar{u}_1 \dots \bar{u}_n \bar{w}_1 \dots \bar{w}_l \bar{v}_1 \dots \bar{v}_m) = uwv$. Since all the symbols that appear in the preimages of u, v, w are in the set P, we obtain that $\bar{u}_1 \dots \bar{u}_n \bar{w}_1 \dots \bar{w}_l \bar{v}_1 \dots \bar{v}_m \in B_{n+m+l}(\bar{Y}_P)$. Therefore, uwv is allowable in Y.

$$\sup\{f_{n+m+|w|}|_Y(y) : y \in [uwv]\} \geq \frac{1}{M_{n+m+p}} \sup\{f_{n+m+|w|}(x) : x \in [uwv]\}$$
$$\geq \frac{D_{n,m}}{M_{n+m+p}} \sup\{f_n(x) : x \in [u]\} \sup\{f_m(x) : x \in [v]\}$$
$$\geq \frac{D_{n,m}}{M_{n+m+p}} \sup\{f_n|_Y(y) : y \in [u]\} \sup\{f_m|_Y(y) : y \in [v]\}.$$

For each $k \geq q$, we can construct a such Y. Setting $Y = X_{l_k}$, we obtain the results. By construction, $X_{l_k} \subseteq X_{l_{k+1}}$. If $W = \{\varepsilon\}$, we make a similar argument. □

Theorem 4.1. *Let (X, σ) be an irreducible countable sofic shift. If $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is a sequence on X with tempered variation satisfying* (C1), (D2) *and* (D3)*, then*

(4.4)
$$P(\mathcal{F}) = \sup_{\substack{l_n \\ n \geq q}}\{P(\mathcal{F}|_{X_{l_n}})\}$$
$$= \sup\{P(\mathcal{F}|_Y) : Y \subset X \text{ is an irreducible sofic shift on a finite alphabet}\},$$

where X_{l_n}, q are defined as in Lemma 4.5, and $P(\mathcal{F}) \neq -\infty$. The following variational principle holds.

(4.5)
$$P(\mathcal{F}) = \sup_{\mu \in M(X,\sigma)} \left\{ h_\mu(\sigma) + \limsup_{n \to \infty} \frac{1}{n} \int \log f_n d\mu : \limsup_{n \to \infty} \frac{1}{n} \int \log f_n d\mu > -\infty \right\}.$$

If $\sup f_1 < \infty$, then $\limsup$ *in (4.5) can be replaced by* $\lim$.

Proof of (4.4). We first consider the case when $Z_1(\mathcal{F}) < \infty$. Then $P(\mathcal{F}) < \infty$. Let $(\bar{X}, \sigma_{\bar{X}})$ be an irreducible countable Markov shift $(\bar{X}, \sigma_{\bar{X}})$ and $\pi : \bar{X} \to X$ a one-block factor map such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. We show first (4.4) using a modification of the proof of [35, Theorem 1.2]. Let $f'(x) = e^C f_n(x)$ and $\mathcal{F}' = \{\log f'_n\}_{n=1}^{\infty}$. Then $\mathcal{F}'$ is sub-additive and $P(\mathcal{F}) = P(\mathcal{F}')$. Let M_n be defined for $\mathcal{F}$ as in Definition 2.6. Let $\epsilon > 0$. Fix $m \in \mathbb{N}$ such that $(1/m) \log M_m < \epsilon$, $(1/(m+p))|\log(D_{m,m}/e^C)| < \epsilon$ and $1 - \epsilon < (m/(m+p))$. Note that $Z_m(\mathcal{F}') < \infty$.

We apply Lemma 4.5 and consider X_{l_k} where $k \geq q$. Then for each $n \in \mathbb{N}$, we have

$$Z_n(\mathcal{F}'|_{X_{l_k}}) = \sum_{w \in B_n(X_{l_k})} \sup\{f'_n|_{X_{l_k}}(x) : x \in [w]\}.$$

Since $w \in B_m(X_{l_k})$ implies that $w \in B_m(X)$, let $S_{l_k}(\mathcal{F}') := \sum_{w \in B_m(X_{l_k})} \sup\{f'_m(x) : x \in [w]\}$. Noting that for each $x_1 \dots x_m \in B_m(X)$ there exists $i \in \mathbb{N}$ such that $x_1 \dots x_m \in B_m(X_{l_i})$, we have that $Z_m(\mathcal{F}') = \lim_{i \to \infty} S_{l_i}(\mathcal{F}')$, where $\{S_{l_i}(\mathcal{F}')\}_{i=1}^{\infty}$is monotone increasing. Hence, for every $\epsilon > 0$, there exists $k_1 > q$ such that

$$\frac{1}{m} \log Z_m(\mathcal{F}') - \frac{1}{m} \log S_{l_{k_1}}(\mathcal{F}') < \epsilon.$$

Since $\mathcal{F}$ has tempered variation, we have that $M_m Z_m(\mathcal{F}'|_{X_{l_k}}) \geq S_{l_k}(\mathcal{F}')$. Since $\mathcal{F}'$ is sub-additive, we obtain

$$\frac{1}{m}\log Z_m(\mathcal{F}'|_{X_{l_{k_1}}}) \geq \frac{1}{m}\log Z_m(\mathcal{F}') - \epsilon - \frac{\log M_m}{m} \geq P(\mathcal{F}') - 2\epsilon. \tag{4.6}$$

Now, for $0 \leq i \leq n$, $n \in \mathbb{N}$, let $u_i \in B_m(X_{l_{k_1}})$. Since $\mathcal{F}$ satisfies (D2) and (D3), letting W be a finite set from (D3), there exist $w_1, \dots, w_{n-1}$ in W such that $u_1 w_1 \dots w_{n-1} u_n$ is an allowable word of length $nm + |w_1| + \dots + |w_{n-1}|$ of X, such that

$$\begin{aligned} &\sup\{f'_{nm+|w_1|+\dots+|w_{n-1}|}(x) : x \in [u_1 w_1 \dots w_{n-1} u_n]\} \\ &\geq \left(\frac{D_{m,m}}{e^C}\right)^{n-1} \prod_{i=1}^{n} \sup\{f'_m(x) : x \in [u_i]\}. \end{aligned} \tag{4.7}$$

By the construction of X_{l_k}, $k \geq q$, in the proof of Lemma 4.5, we note that $u_1 w_1 \dots w_{n-1} u_n$ is an allowable word of $X_{l_{k_1}}$. Therefore, applying (4.7), we obtain

$$\begin{aligned} &M_{nm+p(n-1)} \sup\{f'_{nm+|w_1|+\dots+|w_{n-1}|}|_{X_{l_{k_1}}}(x) : x \in [u_1 w_1 \dots w_{n-1} u_n]\} \\ &\geq \left(\frac{D_{m,m}}{e^C}\right)^{n-1} \prod_{i=1}^{n} \sup\{f'_m(x) : x \in [u_i]\} \geq \left(\frac{D_{m,m}}{e^C}\right)^{n-1} \prod_{i=1}^{n} \sup\{f'_m|_{X_{l_{k_1}}}(x) : x \in [u_i]\}. \end{aligned}$$

Summing over all allowable words $u_i \in B_m(X_{l_{k_1}})$, $0 \leq i \leq n$, we obtain

$$\sum_{0 \leq t \leq p(n-1)} Z_{nm+t}(\mathcal{F}'|_{X_{l_{k_1}}}) \geq \left(\frac{D_{m,m}}{e^C}\right)^{n-1} \cdot \frac{1}{M_{nm+p(n-1)}} (Z_m(\mathcal{F}'|_{X_{l_{k_1}}}))^n.$$

Hence, there exists $0 \leq i_{n,m} \leq p(n-1)$ such that

$$Z_{nm+i_{n,m}}(\mathcal{F}'|_{X_{l_{k_1}}}) \geq \left(\frac{D_{m,m}}{e^C}\right)^{n-1} \cdot \frac{1}{M_{nm+p(n-1)}} \cdot \frac{1}{p(n-1)+1} \cdot \left(Z_m(\mathcal{F}'|_{X_{l_{k_1}}})\right)^n.$$

(4.8)

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{nm+i_{n,m}} \log(Z_{nm+i_{n,m}}(\mathcal{F}'|_{X_{l_{k_1}}})) \\ &\geq \frac{1}{m+p}\log\frac{D_{m,m}}{e^C} + \frac{m}{m+p}\cdot\frac{1}{m}\log Z_m(\mathcal{F}'|_{X_{l_{k_1}}}) \geq -2\epsilon - \epsilon P(\mathcal{F}') + 2\epsilon^2 + P(\mathcal{F}'). \end{aligned}$$

Therefore, we obtain (4.4).

Next assume $Z_1(\mathcal{F}) = \infty$. We first show that $P(\mathcal{F}) = \infty$. Given $L > 0$, there exists $X_{l_s}, s \geq q$ such that $Z_1(\mathcal{F}|_{X_{l_s}}) > L$. Then $Z_1(\mathcal{F}'|_{X_{l_s}}) > Le^C$. Let $Y := X_{l_s}$. Then for each $n \in \mathbb{N}$ there exists $0 \leq i_{n,1} \leq p(n-1)$ such that

(4.9)

$$\begin{aligned} &\frac{1}{n+i_{n,1}}\log(Z_{n+i_{n,1}}(\mathcal{F}'|_Y)) \\ &\geq \frac{1}{n+p(n-1)}\log\left(\left(\frac{D_{1,1}}{e^C}\right)^{n-1}\cdot\frac{1}{M_{n+p(n-1)}}\cdot\frac{1}{p(n-1)+1}\right) + \frac{n}{n+p(n-1)}\log Z_1(\mathcal{F}'|_Y). \end{aligned}$$

A similar argument as in the proof of Proposition 4.2 implies $P(\mathcal{F}) = \infty$. The approximation property (4.4) is obvious from (4.9).

□

Proposition 4.4. *Let (X,σ) be a subshift on a countable alphabet and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on X with tempered variation satisfying* (C1) *and* (D2)*. If $P(\mathcal{F}) < \infty$, then for any $\mu \in M(X,\sigma)$ such that $\lim_{n\to\infty}(1/n)\int \log f_n d\mu > -\infty$ we have*

$$h_\mu(\sigma) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n d\mu \le P(\mathcal{F}). \tag{4.10}$$

Remark 4.3. The assumptions of Proposition 4.4 imply that $\lim_{n\to\infty}(1/n)\int \log f_n d\mu$ exists, and possibly $-\infty$ (see the proof below). Note that (D2) implies that $P(\mathcal{F}) \neq -\infty$.

Proof. We follow the proof of [35, Theorem 1.4] and slightly modify the proof in order to take into account of the sub-additive sequence $\mathcal{F}' := \{\log(e^C f_n)\}_{n=1}^{\infty}$. Since $P(\mathcal{F}) < \infty$, Proposition 4.1 implies $Z_1(\mathcal{F}) < \infty$ and thus $\sup f_1 < \infty$. Hence we obtain that $\int (\log e^C f_1)^+ d\mu < \infty$. Applying Kingman's sub-additive ergodic theorem to $\mathcal{F}'$, we obtain that the limit in (4.10) exists. Note by Proposition 4.1 that $0 < Z_n(\mathcal{F}) < \infty$ for each $n \in \mathbb{N}$. Using the sub-additivity of $\mathcal{F}'$, it follows that for every $n,m \in \mathbb{N}$ $(1/(nm))\int \log f_{nm} d\mu \le (1/n)(\int \log f_n d\mu + C)$. Letting $m \to \infty$, we obtain $-\infty < (1/n)(\int \log f_n d\mu + C)$. For each $n \in \mathbb{N}$

$$\sum_{w\in B_n(X)} \mu([w]) \log\left(\sup\{f_n(x) : x \in [w]\}\right) \ge \int \log f_n d\mu > -\infty.$$

For $n \ge 1$, letting $h(x) = -x\log x$, we have

$$\begin{aligned}
&- \sum_{w_n\in B_n(X)} \mu([w]) \log \mu([w]) + \int \log f_n d\mu \\
&\le \sum_{w\in B_n(X)} \mu([w]) \left(\log(\sup\{f_n(x) : x \in [w]\}) - \log \mu[w]\right) \\
&= Z_n(\mathcal{F}) \sum_{w\in B_n(X)} \frac{\sup\{f_n(x) : x \in [w]\}}{Z_n(\mathcal{F})} h\left(\frac{\mu([w])}{\sup\{f_n(x) : x \in [w]\}}\right) \\
&\le Z_n(\mathcal{F}) h\left(\sum_{w\in B_n(X)} \frac{\mu([w])}{Z_n(\mathcal{F})}\right) \le Z_n(\mathcal{F}) h\left(Z_n(\mathcal{F})^{-1}\right) = \log Z_n(\mathcal{F}),
\end{aligned}$$

where in the third inequality we use the concavity of h. Therefore, for every $n \ge 1$ we have that $-\sum_{w\in B_n(X)} \mu([w]) \log \mu([w]) < \infty$. In particular, if we let $\alpha = \{[u] : u \in B_1(X)\}$, then α is a generator for σ. Hence

$$\begin{aligned}
h_\mu(\sigma) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n d\mu &\le \lim_{n\to\infty}\left(-\frac{1}{n}\sum_{w\in B_n(X)} \mu([w]) \log \mu([w]) + \frac{1}{n}\int \log(e^C f_n) d\mu\right) \\
&\le P(\mathcal{F}).
\end{aligned}$$

□

Lemma 4.6. *Let (X,σ) and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be defined as in Proposition 4.4. If $P(\mathcal{F}) = \infty$, then for any $\mu \in M(X,\sigma)$ such that $\limsup_{n\to\infty}(1/n)\int \log f_n d\mu > -\infty$,*

$$h_\mu(\sigma) + \limsup_{n\to\infty}\frac{1}{n}\int \log f_n d\mu \le P(\mathcal{F}). \tag{4.11}$$

If $\sup f_1 < \infty$*, then* $\limsup$ *can be replaced by* $\lim$.

Proof. The equation (4.11) is obvious because $P(\mathcal{F}) = \infty$. Kingman's sub-additive ergodic theorem implies the second statement as it is shown in the proof of Proposition 4.4.

□

To show the variational principle, we need the following variational principle for sequences on subshifts on finite alphabets (see [8]).

Theorem 4.2. [8] *Let* (X, σ) *be a subshift on a finite alphabet. If* $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ *is a sequence on* X *with tempered variation satisfying* (C1)*, then*

$$P(\mathcal{F}) = \sup_{\mu \in M(X,\sigma)} \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu \right\},$$

where $P(\mathcal{F})$ *is defined as in Definition* 2.7*. Then* $P(\mathcal{F}) = -\infty$ *if and only if* $\lim_{n\to\infty}(1/n)\int \log f_n d\mu = -\infty$ *for all* $\mu \in M(X, \sigma)$*.*

In Theorem 4.2 an equilibrium measure for $\mathcal{F}$ (see Definition 5.1) always exists.

*Proof of (*4.5*) in Theorem* 4.1. First assume that $Z_1(\mathcal{F}) < \infty$. Then $P(\mathcal{F}) < \infty$. Let $\epsilon > 0$. Applying (4.4) there exists a finite state Markov shift Y such that $P(\mathcal{F}) - P(\mathcal{F}|_Y) < \epsilon$. Let m be an equilibrium measure for $\mathcal{F}|_Y$. Since $m \in M(X, \sigma)$ and $\lim_{n\to\infty}(1/n)\int \log f_n dm > -\infty$, we obtain by Proposition 4.4

$$\begin{aligned} &P(\mathcal{F}|_Y) = h_m(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n dm \\ &\le \sup_{\mu \in M(X,\sigma)} \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu : \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu > -\infty \right\} \le P(\mathcal{F}). \end{aligned}$$

Equation (4.5) also holds when $Z_1(\mathcal{F}) = \infty$ by similar arguments using Lemma 4.6. □

In the following, we study a condition for which $P(\mathcal{F}) = P_G(\mathcal{F})$, when $\mathcal{F}$ is defined on a countable sofic shift.

Proposition 4.5. *Let* (X, σ) *be a finitely irreducible countable sofic shift. If* $\mathcal{G}$ *is an almost-additive sequence on* X *with tempered variation, then* $P(\mathcal{G}) = P_G(\mathcal{G})$*. If* X *is a factor of a finitely primitive countable Markov shift and* $P(\mathcal{G}) < \infty$*, then* $\limsup$ *in (*2.4*) can be replaced by* $\lim$*.*

Proof. First assume $Z_1(\mathcal{G}) < \infty$. Thus $P(\mathcal{G}) < \infty$. Since X is a finitely irreducible countable sofic shift, let $\bar{X}$ and $\pi : \bar{X} \to X$ be as in the proof of Lemma 4.5. Let $p \in \mathbb{N}$ and a finite set W_1 be defined for X as in Definition 2.3. We consider the case when $W_1 \neq \{\varepsilon\}$. Let $x_1 \dots x_n \in B_n(X)$ and $a \in \mathbb{N}$ be a symbol in X. Then there exist allowable words w_1, w_2 in W_1 of length $0 \le k_1, k_2 \le p$ respectively such that $aw_1x_1 \dots x_n w_2 a \in B_{n+2+k_1+k_2}(X)$. Therefore, there exist $\bar{x}_1 \dots \bar{x}_n \in \pi^{-1}(x_1 \dots x_n)$, $a_1, a_2 \in \pi^{-1}(a)$, $\bar{w}_1 \in \pi^{-1}(w_1)$ and $\bar{w}_2 \in \pi^{-1}(w_2)$ such that $a_1\bar{w}_1\bar{x}_1 \dots \bar{x}_n\bar{w}_2a_2 \in B_{n+k_1+k_2+2}(\bar{X})$ and $\pi(a_1\bar{w}_1\bar{x}_1 \dots \bar{x}_n\bar{w}_2a_2) = aw_1x_1 \dots x_nw_2a$. Since $|\pi^{-1}(a)| < \infty$, we have $\pi^{-1}(a) = \{a_1, \dots, a_t\}$ for some $t \in \mathbb{N}$. For each pair $a_i, a_j, 1 \le i, j \le t$, define $k_{i,j} = \min\{|w| : a_iwa_j \in B_{2+|w|}(\bar{X}), |w| \ge 1\}$. Then for each i, j, there exist a word at which the minimum is attained and we call it $\bar{w}_{i,j} \in B_{k_{i,j}}(\bar{X})$. Let $\pi(\bar{w}_{i,j}) = w_{i,j}$. Let $\bar{x} = (a_1\bar{w}_1\bar{x}_1 \dots \bar{x}_n\bar{w}_2a_2\bar{w}_{2,1})^\infty \in \bar{X}$ and $x = \pi(\bar{x})$. Then x has a period $(n + 2 + k_1 + k_2 + k_{2,1})$ in X. We first consider the case when k_1, k_2 are both nonzero. Since $\mathcal{G}$ is almost-additive and has tempered variation, letting $N_a = \sup\{f_1(x) : x \in [a]\}$, we obtain

(4.12)

$$\begin{aligned} &g_{n+k_1+k_2+k_{2,1}+2}(x) \\ &\geq \frac{e^{-5C}}{M_n(M_p)^2(M_1)^2M_k} \sup\{g_n(x) : x \in [x_1 \dots x_n]\}(N_a)^2 \sup\{g_{k_1}(x) : x \in [w_1]\} \\ &\cdot \sup\{g_{k_2}(x) : x \in [w_2]\} \sup\{g_{k_{2,1}}(x) : x \in [w_{2,1}]\}. \end{aligned}$$

Since g has tempered variation, for each $1 \leq i, j \leq t$, there exists constant $C_{w_{i,j}} > 0$ such that $\sup\{g_{k_{i,j}}(x) : x \in [w_{i,j}]\} > C_{w_{i,j}}$. Since we have finitely many i, j, let $B = \min_{i,j} C_{w_{i,j}}$. and $K = \max_{i,j} k_{i,j}$

Now we consider the case when at least one of k_1, k_2 is 0. Observe that if k_1 is 0, then we replace $\sup\{g_{k_1}(x) : x \in [w_1]\}$ in (4.12) by 1. This applies also to k_2. Clearly there exists $\bar{D} > 0$ such that $\min_{w \in W_1, |w| \geq 1} \sup\{g_{|w|}(x) : x \in [w]\} > \bar{D}$. Let$\bar{D}' = \min\{1, \bar{D}\}$. Then, (4.12) implies that

$$\sum_{0 \leq i \leq 2p+K} Z_{n+i+2}(\mathcal{G}, a) \geq \frac{e^{-5C}}{M_n(M_p)^2(M_1)^2M_K} Z_n(\mathcal{G})(N_a)^2 B\bar{D}'^2. \tag{4.13}$$

Thus similar arguments as in the proof of Proposition 2.1 yield

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{G}, a) \geq \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{G}).$$

Since a is arbitrary, we obtain the result.

Next assume that $Z_1(\mathcal{G}) = \infty$. Then $P(\mathcal{G}) = \infty$. Let $\mathcal{G}' = C + \mathcal{G}$. Given $L > 0$, there exists $X_{l_s}, s \geq q$ such that $Z_1(\mathcal{G}|_{X_{l_s}}) > L$. Let $Y := X_{l_s}$. Then (4.9) holds if we replace $\mathcal{F}'$ by $\mathcal{G}'$. Since $P(\mathcal{G}'|_Y) = P_G(\mathcal{G}'|_Y)$, similar arguments as in the proof of Proposition 4.2 imply $P_G(\mathcal{G}) = \infty$ To show the second statement, we use the similar arguments as in the proof of Proposition 2.1. □

Note that Theorem 4.1 generalizes the thermodynamic formalism on non-compact shifts, including now irreducible countable sofic shifts. Indeed,

Corollary 4.1. *Let (X, σ) be a finitely irreducible countable sofic shift. If $\mathcal{F}$ is an almost-additive sequence on X with tempered variation, then Theorem* 4.1 *holds for $\mathcal{F}$ and $P(\mathcal{F}) = P_G(\mathcal{F})$. In particular, Theorem* 4.1 *holds for a continuous function f on X with tempered variation by setting $f_n(x) = e^{(S_nf)(x)}$ for all $x \in X$.*

Remark 4.4. The variational principle is proved in [35, Theorem 1.5] for acceptable functions (uniformly continuous functions with an additional property) on finitely irreducible countable Markov shifts and in [21, Theorem 2.4] for continuous functions with tempered variation on irreducible countable Markov shifts. Applying [21, Proposition 6.2], acceptable functions belong to the class of continuous functions with tempered variation. Corollary 4.1 also generalizes the variational principle [26, Theorem 3.1].

Example 4.2. The sequence $\mathcal{G}$ in Theorem 6.2 is a Bowen sequence defined on a finitely irreducible countable sofic shift satisfying (C1), (D2) and (D3). $\mathcal{G}$ does not satisfy (C2). Theorem 4.1 is applied in Theorem 6.2.

Example 4.3. Let $\Psi = \{\log \psi_n\}_{n=1}^{\infty}$ be defined as in Example 3.5. Since $Z_1(\Psi) < \infty$, (4.5) holds and lim sup in (4.5) can be replaced by lim. The same results hold for Ψ in Example 3.4.

5. Invariant Gibbs measures and uniqueness of Gibbs equilibrium measures

The variational principle provides a criteria to choose relevant invariant measures for the (very large) set $M(X,\sigma)$ of invariant Borel probability measures. Indeed, measures that maximize the supremum have interesting ergodic properties. Major difficulties to prove the existence of these measures are the fact that the space $M(X,\sigma)$ is not compact (when endowed with the weak* topology) and that the entropy map $\mu \mapsto h_\mu(\sigma)$ is not necessarily upper-semi continuous.

5.1. **Invariant Gibbs measures and uniqueness of Gibbs equilibrium measures.**

Definition 5.1. Let (X,σ) be an irreducible countable sofic shift and $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ a sequence on X satisfying (C1), (D2) and (D3). A measure $\mu \in M(X,\sigma)$ is an *equilibrium measure* for $\mathcal{F}$ if the supremum in (4.5) is attained at μ, i.e., $P(\mathcal{F}) = h_\mu(\sigma) + \lim_{n\to\infty}(1/n)\int \log f_n \, d\mu$, where $\lim_{n\to\infty}(1/n)\int \log f_n \, d\mu > -\infty$.

Definition 5.2. Let (X,σ) and $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ be defined as in Definition 5.1. A measure $\mu \in M(X,\sigma)$ is *Gibbs* for $\mathcal{F}$ if there exist constants $C_0>0$ and $P\in\mathbb{R}$ such that for every $n\in\mathbb{N}$ and every $x\in[i_1\dots i_n]$ we have

$$\frac{1}{C_0} \le \frac{\mu([i_1\dots i_n])}{\exp(-nP)f_n(x)} \le C_0.$$

There is an example of a Gibbs measure μ for a continuous function f satisfying $h_\mu(\sigma)=\infty$ and $\int f \, d\mu = -\infty$ which is not an equilibrium measure for f (see [44]). The existence of Gibbs measures was studied in [26, 27] for almost-additive sequences on topologically mixing countable Markov shifts with BIP property and in [29, Theorem 3.7] for a class of sub-additive Bowen sequences on the full shift on a countable alphabet satisfying (C2), (C3) and (C4). In the next theorem we will generalize these results by considering finitely irreducible countable sofic shifts.

Theorem 5.1. *Let (X,σ) be a finitely irreducible countable sofic shift. If $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ is a Bowen sequence on X satisfying* (C1), (C2), (C3) *and* (C4), *then there is a unique invariant ergodic Gibbs measure μ for $\mathcal{F}$. Moreover, if in addition*

$$\sum_{i\in\mathbb{N}} \sup\{\log f_1(x) : x\in[i]\} \sup\{f_1(x) : x\in[i]\} > -\infty,$$

then μ is the unique equilibrium measure for $\mathcal{F}$ on X.

Corollary 5.1. *Let (X,σ) be a finitely irreducible countable Markov shift and $\mathcal{G}=\{\log g_n\}_{n=1}^{\infty}$ an almost-additive Bowen sequence on X. If $\mathcal{G}$ satisfies* (C4), *Theorem* 5.1 *holds for $\mathcal{G}$.*

In Theorem 5.1, we study the case when $W\neq\{\varepsilon\}$. Hence, throughout the rest of the section, without loss of generality we assume

(A1) $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ satisfies (C1), (C2) with some $p\in\mathbb{N}$ and (C3) with a finite set W containing a nonempty word w^* of length p,

and

(A2) In Lemma 4.5, for all $k\ge q$, $w^*\in W$ appears in (4.3) for a pair of allowable words u,v of X_{l_k}.

The idea of the proof of Theorem 5.1 is similar to that of [26, Theorem 4.1], which in turn was proved using techniques of [35, Lemma 2.8] and [3, Lemmas 1, 2 and Theorem 5]. We modify the proof by considering Condition (C2) instead of (2.2) and by considering finitely irreducible sofic shifts instead of finitely primitive Markov shifts. This makes our proof more technical. We use the notation from Lemma 4.5.

Theorem 5.2. [18] *Let (X, σ) be an irreducible subshift on a finite alphabet. If $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is a Bowen sequence on X satisfying* (C1) *and* (C2)*, then there exists a unique invariant Gibbs measure for $\mathcal{F}$. Moreover, it is the unique equilibrium measure for $\mathcal{F}$.*

Proposition 5.1. *For $n \geq q$, there is a unique equilibrium measure for $\mathcal{F}|_{X_{l_n}}$ and it is Gibbs for $\mathcal{F}|_{X_{l_n}}$. The Gibbs constant C_0 (see Definition* 5.2*) can be chosen independently of X_{l_n}.*

Proof. The first part of Proposition 5.1 follows from Theorem 5.2. Indeed, note that since $\mathcal{F}$ is a Bowen sequence satisfying (C1), (C2), (C3) and (C4) and X_{l_n} contains all allowable words in W for $n \geq q$, we have that $\mathcal{F}|_{X_{l_n}}$ is a sequence on $(X_{l_n}, \sigma_{X_{l_n}})$ satisfying (4.3) replacing $D_{n,m}/M_{n+m+p}$ by D/M.

By the assumptions, any allowable word in W is an allowable word of X_{l_n} for all $n \geq q$. Fix X_{l_n}, $n \geq q$, and call it Z. Define $\alpha_n^Z = \sum_{i_1 \dots i_n \in B_n(Z)} \sup\{f_n|_Z(z) : z \in [i_1 \dots i_n]\}$. For $l \in \mathbb{N}$, let ν_l be the Borel probability measure on Z defined by

$$\nu_l([i \dots i_l]) = \frac{\sup\{f_l|_Z(z) : z \in [i_1 \dots i_l]\}}{\alpha_l^Z}.$$

By the sub-additive property of $\{\log e^C f_n\}_{n=1}^{\infty}$, we have for $l, n \in \mathbb{N}$ that $\alpha_{n+l}^Z \leq e^C \alpha_n^Z \alpha_l^Z$. Hence $\{\log(e^C \alpha_n^Z)\}_{n=1}^{\infty}$ is sub-additive. We show that for some $C_1 > 0$ the sequence $\{\log(C_1 \alpha_n^Z)\}_{n=1}^{\infty}$ is super-additive. Observe that Lemma 4.5 implies that $\sum_{i=0}^{p} \alpha_{n+l+i}^Z \geq (D/M)\alpha_n^Z \alpha_l^Z$. Let $D/M := D_1$. Then for each $n, l \in \mathbb{N}$, there exists $0 \leq i_{n,l} \leq p$ such that $\alpha_{n+l+i_{n,l}}^Z \geq (D_1 \alpha_n^Z \alpha_l^Z)/(p+1)$. By sub-additivity of $\{\log(e^C \alpha_n^Z)\}_{n=1}^{\infty}$, we obtain

$$\alpha_{n+l+i_{n,l}}^Z \leq e^C \alpha_{n+l}^Z \alpha_{i_{n,l}}^Z \leq e^{Cp} \alpha_{n+l}^Z (\alpha_1^Z)^{i_{n,l}}.$$

Letting $K = \max_{0 \leq i \leq p} Z(\mathcal{F})^i$, for any $n, l \in \mathbb{N}$ we have

$$\alpha_{n+l}^Z \geq D_1 \alpha_n^Z \alpha_l^Z / (e^{Cp} K(p+1)). \tag{5.1}$$

Let $C_1 = D_1/(e^{Cp}K(p+1))$. Since $P(\mathcal{F}|_Z) = \lim_{n\to\infty}(1/n)(\log \alpha_n^Z)$, using the sub-additivity of $\{\log(e^C \alpha_n^Z)\}_{n=1}^{\infty}$, the super-additivity of $\{\log(C_1 \alpha_n^Z)\}_{n=1}^{\infty}$ and $Z_1(\mathcal{F}) < \infty$, we obtain that

$$C_1 \alpha_n^Z \leq e^{P(\mathcal{F}|_Z)n} \leq e^C \alpha_n^Z. \tag{5.2}$$

We now construct a Gibbs measure. For fixed $u \in B_n(Z)$, $m \in \mathbb{N}$, we define $\alpha_{n+m}^{Z,u} = \sum_{ua_1 \dots a_m \in B_{n+m}(Z)} \sup\{f_{n+m}|_Z(z) : z \in [ua_1 \dots a_m]\}$.

Lemma 5.1. *There exists $C_2 > 0$ such that for each fixed $u \in B_n(Z)$, for $l > n+2p$, we have*

$$\alpha_l^{Z,u} \geq C_2 \alpha_{l-n-2p}^Z \sup\{f_n|_Z(z) : z \in [u]\}.$$

Note that C_2 is independent of Z.

Proof. For the proof, see Section 5.2. □

By the definition of the measure ν_l and (C1), for a fixed $u = u_1 \dots u_n \in B_n(Z)$, $n < l$, we have that,

$$\nu_l([u]) \le \frac{e^C \sup\{f_n|_Z(z) : z \in [u]\}\alpha^Z_{l-n}}{\alpha^Z_l}.$$

Therefore, using (5.2), we obtain that for each $z \in [u]$

$$\frac{\nu_l([u])}{e^{-nP(\mathcal{F}|_Z)} f_n|_Z(z)} \le \frac{M\nu_l([u])}{e^{-nP(\mathcal{F}|_Z)} \sup\{f_n|_Z(z) : z \in [u]\}} \le \frac{Me^{2C}\alpha^Z_{l-n}\alpha^Z_n}{\alpha^Z_l} \le \frac{Me^{3C}}{C_1^2}.$$

On the other hand, by Lemma 5.1 and (5.2), for each $z \in [u]$, for $l > n + 2p$,

$$\frac{\nu_l([u])}{e^{-nP(\mathcal{F}|_Z)} f_n|_Z(z)} \ge \frac{\alpha^{Z,u}_l}{\alpha^Z_l e^{-nP(\mathcal{F}|_Z)} \sup\{f_n|_Z(z) : z \in [u]\}} \ge C_1C_2e^{-2pP(\mathcal{F}|_Z)-C}.$$

Noting that $e^{-2pP(\mathcal{F}|_Z)} \ge e^{-2pP(\mathcal{F})}$ if $P(\mathcal{F}) \ge 0$ and $e^{-2pP(\mathcal{F}|_Z)} > 1$ if $P(\mathcal{F}) < 0$, there exist $C_3 > 0, C_4 > 0$, both independent of Z, such that for all $l > n + 2p$,

$$C_3 \le \frac{\nu_l([u])}{e^{-nP(\mathcal{F}|_Z)} f_n|_Z(z)} \le C_4 \text{ for all } z \in [u] \text{ in } Z. \tag{5.3}$$

Since the set Z is compact, there exists a subsequence $\{\nu_{n_k}\}_{k=1}^{\infty}$ of $\{\nu_n\}_{n=1}^{\infty}$ that converges to a measure ν and for all $z \in [u]$ in Z

$$C_3 \le \frac{\nu([u])}{e^{-nP(\mathcal{F}|_Z)} f_n|_Z(z)} \le C_4. \tag{5.4}$$

Now let $\mu_n = (1/n)\sum_{i=1}^n \sigma_Z^i \nu$. We claim that any weak limit point μ of $\{\mu_n\}_{n=1}^{\infty}$ is a σ_Z-invariant Gibbs measure on Z.

For each fixed $u \in B_n(Z)$, define $\alpha^Z_{l+n}(u) = \sum_{a_1\dots a_l u \in B_{l+n}(Z)} \sup\{f_{l+n}|_Z(z) : z \in [a_1 \dots a_l u]\}$. Then setting $l = m + i$, for $m \in \mathbb{N}, 0 \le i \le p$, we obtain that $\sum_{0\le i\le p} \alpha^Z_{n+m+i}(u) \ge D_1\alpha^Z_m \sup\{f_n|_Z(z) : z \in [u]\}$. Therefore, there exists $0 \le i_{n,m,u} \le p$ such that

$$\alpha^Z_{n+m+i_{n,m,u}}(u) \ge (D_1/(p+1))\alpha^Z_m \sup\{f_n|_Z(z) : z \in [u]\}.$$

Note that $i_{n,m,u}$ depends on n, m and u.

Lemma 5.2. *There exists $C_5 > 0$ such that for any $0 \le i \le p$, any $n, m \in \mathbb{N}$ and $u \in B_n(Z)$ we have*

$$\alpha^Z_{n+m+i}(u) \ge C_5\alpha^Z_m \sup\{f_n|_Z(z) : z \in [u]\}.$$

Note that C_5 is independent of Z.

Proof. The proof can be found in Section 5.2 □

Let $u \in B_n(Z)$ be fixed and set $M_2 = \max\{0, P(\mathcal{F})\}$. Letting $l = m + i$ for $m \in \mathbb{N}$ and $0 \le i \le p$,

$$\begin{aligned}\nu(\sigma_Z^{-l}[u]) &\ge \sum_{vu\in B_{l+n}(Z)} \frac{C_3}{M} e^{-(l+n)P(\mathcal{F}|_Z)} \sup\{f_{n+l}|_Z(z) : z \in [vu]\} \\ &\ge \frac{C_3C_5}{M} e^{-(m+i+n)P(\mathcal{F}|_Z)}\alpha^Z_m \sup\{f_n|_Z(z) : z \in [u]\} \ge \frac{C_3C_5}{MC_4e^C} e^{-pM_2}\nu([u]),\end{aligned}$$

where in the last inequality we use (5.2). Using (C1), similarly, we obtain

$$\nu(\sigma_Z^{-l}[u]) \leq \frac{C_4 e^C M}{C_1 C_3} \nu([u]).$$

Therefore, using the similar arguments as in the proof of [3, Theorem 5], there exist $\bar{C}_3, \bar{C}_4 > 0$ such that for $u \in B_n(Z)$ and $x \in [u]$ in Z we have

$$\bar{C}_3 \leq \frac{\mu([u])}{e^{-nP(\mathcal{F}|_Z)} f_n|_Z(x)} \leq \bar{C}_4. \tag{5.5}$$

By Theorem 5.2, μ is the unique invariant ergodic Gibbs measure and the unique equilibrium measure for $\mathcal{F}|_Z$. For $n \geq q$, if we let μ_{l_n} be the $\sigma|_{Z_{l_n}}$- invariant Gibbs measure on Z_{l_n}, then it satisfies for each $k \in \mathbb{N}$, $u \in B_k(Z_{l_n})$ and every $z \in [u]$ in Z_{l_n},

$$\bar{C}_3 \leq \frac{\mu_{l_n}([u])}{e^{-kP(\mathcal{F}|_{Z_{l_n}})} f_k|_{Z_{l_n}}(z)} \leq \bar{C}_4. \tag{5.6}$$

Clearly $\bar{C}_3$ and $\bar{C}_4$ are independent of Z_{l_n}. □

In the following proof, we continue to use the notation of the $\sigma|_{Z_{l_n}}$-invariant Gibbs measure μ_{l_n} on Z_{l_n} satisfying (5.6).

Proof of Theorem 5.1. We show that the sequence $\{\mu_{l_n}\}_{n=q}^{\infty}$ of σ-invariant Borel probability measures on X is tight. For this purpose, we apply Prohorov's theorem to the sequence $\{\mu_{l_n}\}_{n=q}^{\infty}$. We note that the same proof of [26, Theorem 4.1] holds (see also the proof of [35, Lemma 2.7]). Here we only state how we modify using the notation of [26, Theorem 4.1]. We first note that in the proof the Gibbs property of μ_{l_n} and the property (C1) of $\mathcal{F}|_{Z_{l_n}}$ are applied. Secondly the fact that, for an irreducible Markov shift X, $X \cap \prod_{k\geq 1}[1, n_k]$ is a compact subset of X is used (see proof of [26, Theorem 4.1] for details). Since we consider a finitely irreducible countable sofic shift X, there exist an irreducible countable Markov shift $\bar{X}$ and one-block factor map $\pi : \bar{X} \to X$ such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. For a fixed k, we first consider the set of preimages of the set of symbols $\{1, \dots, n_k\}$ and call it P_{n_k}. Note that P_{n_k} is a finite set. Then $\bar{X} \cap \prod_{k\geq 1} P_{n_k}$ is a compact subset of $\bar{X}$. Thus $X \cap \prod_{k\geq 1}[1, n_k]$ is a compact subset of X.

Hence there exists a convergent subsequence $\{\mu_{l_{n_k}}\}_{k=1}^{\infty}$ of $\{\mu_{l_n}\}_{n=q}^{\infty}$. We denote by μ a limit point of this subsequence. Then μ is σ-invariant on X. By (5.6), letting $l_{n_k} \to \infty$, we obtain for $n \in \mathbb{N}$, $u \in B_n(X)$ and each $x \in [u]$ that,

$$\bar{C}_3 \leq \frac{\mu([u])}{e^{-nP(\mathcal{F})} f_n(x)} \leq \bar{C}_4. \tag{5.7}$$

Therefore, μ is a Gibbs measure for $\mathcal{F}$ on X. Now we show that any invariant Gibbs measure for $\mathcal{F}$ is ergodic and in particular μ is ergodic.

Lemma 5.3. *For fixed allowable words $u \in B_n(X), v \in B_l(X)$ and $t \in \mathbb{N}$,*

$$\begin{aligned} &\sum_{ua_1\dots a_{i+t}v \in B_{n+l+t+i}(X), 0\leq i\leq 2p} \sup\{f_{n+l+t+i}(x) : x \in [ua_1 \dots a_{t+i}v]\} \\ &\geq D^2 \sup\{f_n(x) : x \in [u]\} \sup\{f_l(x) : x \in [v]\} Z_t(\mathcal{F}). \end{aligned}$$

Proof. The proof can be found in Section 5.2. □

Define $\alpha_n = \sum_{u\in B_n(X)} \sup\{f_n(x) : x \in [u]\}$. Let $M_2 = \max\{0, P(\mathcal{F})\}$. By Lemma 5.3,

$$\begin{aligned}
&\sum_{i=0}^{2p} \mu([u]\cap \sigma^{-(n+t+i)}([v])) = \sum_{i=0}^{2p} \sum_{ua_1\dots a_{t+i}v\in B_{n+l+t+i}(X)} \mu([ua_1\dots a_{t+i}v])\\
&\geq \frac{\bar{C}_3 e^{-(n+l+t)P(\mathcal{F})-2pM_2}}{M} \sum_{i=0}^{2p} \sum_{ua_1\dots a_{t+i}v\in B_{n+l+t+i}(X)} \sup\{f_{n+l+t+i}(x) : x\in [ua_1\dots a_{t+i}v]\}\\
&\geq \frac{\bar{C}_3 D^2 e^{-(n+l+t)P(\mathcal{F})-2pM_2}}{M} \alpha_t \sup\{f_n(x) : x\in [u]\}\sup\{f_l(x) : x \in [v]\}\\
&\geq \frac{\bar{C}_3 D^2 e^{-2pM_2}}{M\bar{C}_4{}^2 e^C} \mu([u])\mu([v]),
\end{aligned}$$

where in the third inequality we use Lemma 5.3 and in the last inequality we use (5.7). Now letting $C_6 = (\bar{C}_3 e^{-2pM_2} D^2)/(M\bar{C}_4{}^2 e^C)$, there exists $0 \leq i_{u,v,t} \leq 2p$ such that $\mu([u]\cap\sigma^{-(n+t+i_{u,v,t})}([v])) \geq (C_6/(2p+1))\mu([u])\mu([v])$. Thus μ is ergodic. Note that the same proof holds for any invariant Gibbs measure for $\mathcal{F}$. The Gibbs property with ergodicity implies that μ is the unique invariant ergodic measure on X that satisfies the Gibbs property for $\mathcal{F}$. Finally we show that, if in addition,

$$\sum_{i\in\mathbb{N}} \sup\{\log f_1(x) : x\in [i]\}\sup\{f_1(x) : x\in[i]\} > -\infty,$$

then μ is the unique equilibrium measure for $\mathcal{F}$. By (5.7) it is easy to see that $-\sum_{i\in\mathbb{N}} \mu([i])\log\mu([i]) < \infty$ if and only if $\sum_{i\in\mathbb{N}} \sup\{\log f_1(x) : x \in [i]\}\sup\{f_1(x) : x \in [i]\} > -\infty$. Hence $h_\mu(\sigma) < \infty$. Using (5.7), we obtain $P(\mathcal{F}) = h_\mu(\sigma) + \lim_{n\to\infty}(1/n)\int \log f_n d\mu$. Thus $\lim_{n\to\infty}(1/n)\int\log f_n d\mu > -\infty$. Hence μ is an equilibrium measure.

To show that μ is the unique equilibrium measure, we use the same arguments as in [29] and only mention modified parts for our setting. As in [29, Lemma 3.9], we first claim that if $\nu \neq \mu$ is an equilibrium measure for $\mathcal{F}$ then ν is absolutely continuous with respect μ. Observe that given a sequence $\{C_n\}_{n=1}^{\infty}$, where each C_n is a union of cylinder sets of length n of X, by using the concavity of $h(x) = -x\log x$ and the Gibbs property of μ, we obtain

$$\begin{aligned}
0 =& n(h_\nu(\sigma) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n d\nu - P(\mathcal{F}))\\
&\leq \int \log(f_n e^C)d\nu - nP(\mathcal{F}) - \sum_{w\in B_n(X)} \nu([w])\log\nu([w])\\
&\leq \log 2 + \nu([C_n])\log(\frac{\mu([C_n]}{e^C\bar{C}_3}) + \nu([X\setminus C_n])\log(\frac{\mu([X\setminus C_n])}{e^C\bar{C}_3}).
\end{aligned}$$

Applying the proof of [29, Lemma 3.9] by using the above inequalities, we obtain the claim. Then we follow the same proof found in [29] to show the uniqueness.

□

5.2. **Proofs of Lemmas 5.1, 5.2, and 5.3.**

Proof of Lemma 5.1. Fix $n \in \mathbb{N}$. It is direct from Lemma 4.5 that for any $m \in \mathbb{N}$, $u \in B_n(Z)$,

$$\sum_{0 \le i \le p} \alpha_{n+m+i}^{Z,u} \ge D_1 \alpha_m^Z \sup\{f_n|_Z(z) : z \in [u]\},$$

where $D_1 := D/M$. Thus, there exists $0 \le i_{n,m,u} \le p$ such that $\alpha_{n+m+i_{n,m,u}}^{Z,u} \ge (D_1/(p+1))\alpha_m^Z \sup\{f_n|_Z(z) : z \in [u]\}$. Fix $l > n+2p$ and set $m = l-n-2p$. Then there exists $i_{n,m,u}$ such that

$$\alpha_{l-2p+i_{n,m,u}}^{Z,u} \ge \frac{D_1}{p+1} \alpha_{l-2p-n}^Z \sup\{f_n|_Z(z) : z \in [u]\}. \tag{5.8}$$

Now take $w^* \in W$ such that $|w^*| = p$. Take $ua_1 \dots a_{l-n-2p+i_{n,m,u}} \in B_{l-2p+i_{n,m,u}}(Z)$ and call it v. Then by Lemma 4.5 there exists $w \in W$ such that vww^* is an allowable word of Z and

$$\begin{aligned} &\sup\{f_{l-2p+i_{n,m,u}+|w|+p}|_Z(z) : z \in [vww^*]\} \\ &\ge D_1 \sup\{f_{l-2p+i_{n,m,u}}|_Z(z) : z \in [v]\} \sup\{f_p|_Z(z) : z \in [w^*]\}. \end{aligned}$$

In the similar manner, we can take $\bar{w} \in W$ such that

$$\begin{aligned} &\sup\{f_{l+i_{n,m,u}+|w|+|\bar{w}|}|_Z(z) : z \in [vww^*\bar{w}w^*]\} \\ &\ge {D_1}^2 \sup\{f_{l-2p+i_{n,m,u}}|_Z(z) : z \in [v]\}(\sup\{f_p|_Z(x) : x \in [w^*]\})^2. \end{aligned}$$

Let $|w| = q_1$, $|\bar{w}| = q_2$ and write $ww^*\bar{w}w^* = w_1 \dots w_{2p+q_1+q_2}$. Then using (C1),

$$\begin{aligned} &\sup\{f_{l+i_{n,m,u}+q_1+q_2}|_Z(z) : z \in [vww^*\bar{w}w^*]\} \\ &\le e^{3pC} \sup\{f_l|_Z(z) : z \in [vw_1 \dots w_{2p-i_{n,m,u}}]\} \max_{0 \le i \le 3p} Z_1(\mathcal{F})^i, \end{aligned}$$

if $i_{n,m,u} + q_1 + q_2 \ge 1$. If $i_{n,m,u} = q_1 = q_2 = 0$, then the second line in the above inequalities is simplified. If we let $M' = \max_{0 \le i \le 3p} Z_1(\mathcal{F})^i$, then

$$\begin{aligned} &\sup\{f_l|_Z(z) : z \in [vw_1 \dots w_{2p-i_{n,m,u}}]\} \\ &\ge \frac{{D_1}^2}{e^{3pC}M'} \sup\{f_{l-2p+i_{n,m,u}}|_Z(z) : z \in [v]\}(\sup\{f_p|_Z(z) : z \in [w^*]\})^2 \\ &\ge \frac{{D_1}^2}{e^{3pC}M'M^2} \sup\{f_{l-2p+i_{n,m,u}}|_Z(z) : z \in [v]\}(\sup\{f_p(y) : y \in [w^*]\})^2. \end{aligned}$$

Let $\bar{m} = \min_{w \in W}(\sup\{f_p(y) : y \in [w]\})^2$. Then summing over all allowable words $a_1 \dots a_{l-n-2p+i_{n,m,u}}$ such that $ua_1 \dots a_{l-n-2p+i_{n,m,u}} \in B_{l-2p+i_{n,m,u}}(Z)$, we obtain that

$$\alpha_l^{Z,u} \ge \frac{(\sup\{f_p(y) : y \in [w^*]\})^2 {D_1}^2}{e^{3pC}M'M^2(p+1)} \alpha_{l-2p+i_{n,m,u}}^{Z,u} \ge \frac{\bar{m}{D_1}^2}{e^{3pC}M'M^2(p+1)} \alpha_{l-2p+i_{n,m,u}}^{Z,u},$$

and combining with (5.8) the result follows. □

Proof of Lemma 5.2. Fix $n, m \in \mathbb{N}$ and $u \in B_n(Z)$. There exists $0 \le i_{n,m,u} \le p$ such that $\alpha_{n+m+i_{n,m,u}}^Z(u) \ge (D_1/(p+1))\alpha_m^Z \sup\{f_n|_Z(z) : z \in [u]\}$. We first consider the case when $p \ge 2$. Let $i_{n,m,u} = i_0$ and assume $i_0 \ge 1$. Let $a_1 \dots a_{m+i_0}u \in B_{n+m+i_0}(Z)$ and call it v. Let $w^* = w_1^* \dots w_p^* \in W$ such that $|w^*| = p$. Take

$\bar{C} = \max_{0\le i\le 2p} Z_1(\mathcal{F})^i$ Also, take $D_W = (1/M)\min_{w\in W}\sup\{f_{|w|}(x) : x \in [w]\}$. Then by Lemma 4.5 there exists $w \in W$ such that

$$\sup\{f_{n+m+i_0+p+|w|}|_Z(z) : z \in [w^*wv]\} \tag{5.9}$$

$$\ge \frac{D}{M}\sup\{f_p|_Z(z) : z \in [w^*]\}\sup\{f_{n+m+i_0}|_Z(z) : z \in [v]\}. \tag{5.10}$$

First we show that there exists $C_1 > 0$ such that for any $j \in \mathbb{N}$ such that $i_0 + j \le p$,

$$\alpha^Z_{n+m+i_0+j}(u) \ge C_1\alpha^Z_{n+m+i_0}(u). \tag{5.11}$$

Fix j and we consider two cases depending on $|w|$, $|w| > j$ and $|w| \le j$. Let $w = w_1 \dots w_k$ and suppose $k > j$. Since

$$\begin{aligned}&\sup\{f_{n+m+i_0+p+k}|_Z(z) : z \in [w^*wv]\}\\ &\le e^C \sup\{f_{p+k-j}|_Z(z) : z \in [w^*w_1 \dots w_{k-j}]\}\sup\{f_{n+m+i_0+j}|_Z(z) : z \in [w_{k-j+1}\dots w_kv]\}\\ &\le e^{2pC}\bar{C}\sup\{f_{n+m+i_0+j}|_Z(z) : z \in [w_{k-j+1}\dots w_kv]\},\end{aligned}$$

applying (5.9), we obtain

$$\sup\{f_{n+m+i_0+j}|_Z(z) : z \in [w_{k-j+1}\dots w_kv]\} \tag{5.12}$$

$$\ge \frac{D}{e^{2pC}\bar{C}M}\sup\{f_p|_Z(z) : x \in [w^*]\}\sup\{f_{n+m+i_0}|_Z(z) : z \in [v]\}. \tag{5.13}$$

Next suppose $k \le j \le p - i_0$. Then

$$\sup\{f_{n+m+i_0+p+k}|_Z(z) : z \in [w^*wv]\} \tag{5.14}$$

$$\le e^C \sup\{f_{p-(j-k)}|_Z(z) : z \in [w^*_1 \dots w^*_{p-(j-k)}]\}\sup\{f_{n+m+i_0+j}|_Z(z) : z \in [w^*_{p-(j-k)+1}\dots w^*_pwv]\}. \tag{5.15}$$

Hence

$$\sup\{f_{n+m+i_0+j}|_Z(z) : z \in [w^*_{p-(j-k)+1}\dots w^*_pwv]\} \tag{5.16}$$

$$\ge \frac{D}{e^{pC}\bar{C}M}\sup\{f_p|_Z(z) : x \in [w^*]\}\sup\{f_{n+m+i_0}|_Z(z) : z \in [v]\}. \tag{5.17}$$

For each $a_1 \dots a_{m+i_0}u \in B_{n+m+i_0}(Z)$, finding w satisfying (5.9) and applying (5.13) or (5.17), we obtain

$$\alpha^Z_{n+m+i_0+j}(u) \ge \frac{DD_W}{e^{2pC}\bar{C}}\alpha^Z_{n+m+i_0}(u). \tag{5.18}$$

Next we show that there exists $C_1' > 0$ such that for each $j \in \mathbb{N}$, $0 \le j \le i_0 \le p$, we have $\alpha^Z_{n+m+i_0-j}(u) \ge C_1'\alpha^Z_{n+m+i_0}(u)$. Fix j. For each $v = a_1 \dots a_{m+i_0}u \in B_{n+m+i_0}(Z)$,

$$\begin{aligned}&\sup\{f_j|_Z(z) : z \in [a_1 \dots a_j]\}\sup\{f_{n+m+i_0-j}|_Z(z) : z \in [a_{j+1}\dots a_{m+i_0}u]\}\\ &\ge e^{-C}\sup\{f_{n+m+i_0}|_Z(z) z \in [v]\}.\end{aligned}$$

Noting that $\sup\{f_j|_Z(z) : z \in [a_1 \dots a_j]\} \le e^{(p-1)C}\bar{C}$, we obtain

$$\alpha^Z_{n+m+i_0-j}(u) \ge \frac{1}{\bar{C}e^{pC}}\alpha^Z_{n+m+i_0}(u). \tag{5.19}$$

For the case when $i_0 = 0$, we make similar arguments. For the case when $p = 1$, we consider the case when $i_0 = 0, 1$ in a similar manner. Hence we obtain the results. □

Proof of Lemma 5.3. For a fixed $t \in \mathbb{N}$, fix $c \in B_t(X)$. Then given v and c, there exists $w_1 \in B_{|w_1|}(X)$, $0 \leq |w_1| \leq p$ such that

$$\sup\{f_{t+|w_1|+l}(x) : x \in [cw_1v]\} \geq D \sup\{f_t(x) : x \in [c]\} \sup\{f_l(x) : x \in [v]\}. \tag{5.20}$$

Therefore, there exists $w_2 \in B_{|w_2|}(X)$, $0 \leq |w_2| \leq p$ such that

$$\sup\{f_{n+|w_2|+t+|w_1|+l}(y) : y \in [uw_2cw_1v]\} \tag{5.21}$$
$$\geq D \sup\{f_n(x) : x \in [u]\} \sup\{f_{t+|w_1|+l}(x) : x \in [cw_1v]\} \tag{5.22}$$
$$\geq D^2 \sup\{f_n(x) : x \in [u]\} \sup\{f_t(y) : x \in [c]\} \sup\{f_l(x) : x \in [v]\}. \tag{5.23}$$

This implies the result. □

6. Application to Hidden Gibbs measures on shift spaces over countable alphabets

In this section, we apply the results in the previous sections to problems on factors of invariant Gibbs measures. Let $\pi : X \to Y$ be a one-block factor map between countable sofic shifts such that $|\pi^{-1}(i)| < \infty$ for each $i \in \mathbb{N}$. For every measure $\mu \in M(X, \sigma)$ the map π induces a measure $\nu \in M(Y, \sigma)$ defined by $\nu(B) = \pi\mu(B) := \mu(\pi^{-1}B)$, where $B \subset Y$ is any Borel set. If μ is a Gibbs measure then the measure ν, which is a factor of a Gibbs measure, is sometimes called *hidden Gibbs measure*. In statistical mechanics, the images of Gibbs measures under Renormalization Group transformations are not always Gibbs measures and generalizations of Gibbs measures have been studied (see for example [12, 13]). The study of the factors of Gibbs measures also has attracted a great deal of attention in dynamical systems (see [7]).

We study factors of Gibbs measures by applying the results in Section 5 and an approach found in [50]. Let (X, σ_X) and (Y, σ_Y) be finitely irreducible countable sofic shifts. For a one-block factor map $\pi : X \to Y$, $n \in \mathbb{N}, y = (y_1, \dots y_n, \dots) \in Y$, let $E_n(y)$ be a set consisting of exactly one point from each cylinder $[x_1 \dots x_n]$ such that $\pi(x_1 \dots x_n) = y_1 \dots y_n$. Given a sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on X, define

$$g_n(y) = \sup_{E_n(y)} \left\{ \sum_{x \in E_n(y)} f_n(x) \right\}.$$

Theorem 6.1. *Let (X, σ_X) be a finitely irreducible countable sofic shifts, (Y, σ_Y) a subshift on a countable alphabet and $\pi : X \to Y$ a one-block factor map such that for each $i \in \mathbb{N}$, $|\pi^{-1}(i)| < \infty$. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on X. If $Z_1(\mathcal{F}) < \infty$, then there exists a unique invariant ergodic Gibbs measure μ for $\mathcal{F}$ and the projection $\pi\mu$ of the measure μ is the unique invariant ergodic Gibbs measure for $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$. Moreover,*

$$\begin{aligned} &P_G(\mathcal{F}) = P(\mathcal{F}) \\ &= \sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu : \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu > -\infty \right\} \\ &= \sup_{\nu \in M(Y,\sigma_Y)} \left\{ h_\nu(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \log g_n d\nu : \lim_{n\to\infty} \frac{1}{n} \int \log g_n d\nu > -\infty \right\} \\ &= P(\mathcal{G}) < \infty. \end{aligned} \tag{6.1}$$

In addition, if $\sum_{i\in\mathbb{N}}\sup\{\log f_1(x) : x \in [i]\}\sup\{f_1(x) : x \in [i]\} > -\infty$*, then* μ *is the unique equilibrium measure for* $\mathcal{F}$ *and* $\pi\mu$ *is the unique equilibrium measure for* $\mathcal{G}$*. In particular, if* (X, σ_X) *is a factor of a finitely primitive countable Markov shift, then* $\limsup$ *in the definition (2.4) of* $P_G(\mathcal{F})$ *can be replaced by* $\lim$*.*

Remark 6.1. In [50, Theorem 3.1], almost-additive Bowen sequences on finitely primitive subshifts are considered. Another approach to show [50, Theorem 3.1] is to apply [18, Proposition 3.7] concerning a relative variational principle. However, in [18, Proposition 3.7], shift spaces are assumed to be compact (subshifts on finite alphabets) and so we cannot apply the proposition to show Theorem 6.1.

Proof of Theorem 6.1*.* We first note that Y is an irreducible countable sofic shift because X is an irreducible countable sofic shift. Since X is finitely irreducible, there exist $p \in \mathbb{N}$ and a finite set W_1 defined in Definition 2.3. By Lemma 3.1 the sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ satisfies (C1), (C2) with p, (C3) with W_1 and (C4). By Theorem 5.1, there exists a unique invariant ergodic Gibbs measure μ for $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$. Clearly $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ is a Bowen sequence. We show that $\mathcal{G}$ satisfies (C1), (C2), (C3) and (C4). By [50, Lemma 3.4], $\mathcal{G}$ satisfies (C1). To verify that Condition (C4) is fulfilled, note that $Z_1(\mathcal{G}) \leq \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N},\pi(j)=i}\sup\{f_1(x) : x \in [j]\} = Z_1(\mathcal{F}) < \infty$. Next we show that $\mathcal{G}$ satisfies (C2). For $y = (y_1, \dots, y_n, \dots) \in Y$, by the Bowen property,

$$\begin{aligned}\frac{1}{M}&\sum_{u\in B_n(X),\pi(u)=y_1\dots y_n}\sup\{f_n(x) : x \in [u]\} \leq g_n(y)\\ &\leq \sum_{u\in B_n(X),\pi(u)=y_1\dots y_n}\sup\{f_n(x) : x \in [u]\}.\end{aligned}$$

We note that if X is an irreducible subshift on a finite alphabet (compact case), then [18, Lemma 5.7] and the above inequality imply that $\mathcal{G}$ satisfies (C1) and (C2). Making similar arguments, we obtain that given $u \in B_n(Y)$ and $v \in B_m(Y)$, $n, m \in \mathbb{N}$, there exists $w_1 \in \pi(W_1)$ such that uw_1v is allowable in Y and

$$\begin{aligned}&\sup\{g_{n+|w_1|+m}(y) : y \in [uw_1v]\}\\ &\geq \frac{D}{M|\pi(W_1)|}\sup\{g_n(y) : y \in [u]\}\sup\{g_m(y) : y \in [v]\}.\end{aligned} \tag{6.2}$$

$\mathcal{G}$ satisfies (C2) with the same value of p that appears in the weak specification and (C3) with $W = \pi(W_1)$. By Theorem 5.1 $\mathcal{G}$ has a unique invariant Gibbs measure ν. The second and fourth equalities in Theorem 6.1 hold because of the variational principle.

Let μ be the equilibrium measure for $\mathcal{F}$. To show that that $\pi\mu = \nu$, observe that the proof of [50, Theorem 3.7] holds in our setting. By the definition of topological pressure, we obtain $P(\mathcal{F}) = P(\mathcal{G})$. Finally, we show that $\sum_{i\in\mathbb{N}}\sup\{\log g_1(y) : y \in [i]\}\sup\{g_1(y) : y \in [i]\} > -\infty$. Since $\sum_{x_1\in\mathbb{N}}\sup\{f_1(x) : x \in [x_1]\}\sup\{\log f_1(x) :$

$x \in [x_1]\} > -\infty$, a simple calculation shows that

$$\begin{aligned}&\sup\{g_1(y) : y \in [y_1]\} \sup\{\log g_1(y) : y \in [y_1]\}\\ &\geq \frac{1}{M} \cdot \left(\log \frac{1}{M}\right) \sum_{\substack{x_1\in\mathbb{N}\\ \pi(x_1)=y_1}} \sup\{f_1(x) : x \in [x_1]\}\\ &+ \frac{1}{M} \sum_{\substack{x_1\in\mathbb{N}\\ \pi(x_1)=y_1}} \sup\{f_1(x) : x \in [x_1]\} \sup\{\log f_1(x) : x \in [x_1]\}.\end{aligned}$$

Summing over all allowable $y_1 \in \mathbb{N}$, we obtain the result. Applying Theorem 5.1 we have that ν is the unique equilibrium measure for $\mathcal{G}$. For the last statement, we apply Proposition 4.5. □

For a more general case, we have the following theorem.

Theorem 6.2. *Let (X, σ_X), (Y, σ_Y) and $\pi : X \to Y$ be defined as in Theorem 6.1. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive sequence on X with tempered variation. Then*

$$\begin{aligned}&P_G(\mathcal{F}) = P(\mathcal{F})\\ &= \sup_{\mu\in M(X,\sigma_X)} \left\{h_\mu(\sigma_X) + \limsup_{n\to\infty} \frac{1}{n} \int \log f_n d\mu : \limsup_{n\to\infty} \frac{1}{n} \int \log f_n d\mu > -\infty\right\}\\ &= \sup_{\nu\in M(Y,\sigma_Y)} \left\{h_\nu(\sigma_Y) + \limsup_{n\to\infty} \frac{1}{n} \int \log g_n d\nu : \limsup_{n\to\infty} \frac{1}{n} \int \log g_n d\nu > -\infty\right\}\\ &= P(\mathcal{G}).\end{aligned}$$

If $\sup f_1 < \infty$*, then* $\limsup$ *in the above equations can be replaced by* $\lim$*.*

Proof. If $\mathcal{F}$ has tempered variation, (6.2) is replaced by

$$\begin{aligned}&\sup\{g_{n+|w_1|+m}(y) : y \in [uwv]\}\\ &\geq \frac{e^{-C}Q}{M_{n+m+p}M_nM_mM_p|\pi(W_1)|} \sup\{g_n(y) : y \in [u]\} \sup\{g_m(y) : y \in [v]\},\end{aligned}$$

where Q is defined for $\mathcal{F}$ as in Lemma 3.1. Hence $\mathcal{G}$ satisfies (D2) and (D3) and we apply Corollary 4.1 and Theorem 4.1. It is easy to obtain $P(\mathcal{F}) = P(\mathcal{G})$. □

The existence of equilibrium measures for $\mathcal{F}$ and $\mathcal{G}$ in Theorem 6.2 is not known. The following corollary is a generalization of [50, Corollary 3.2]. Let (X, σ_X) be a subshift on a countable alphabet. Recall from Section 2 that a function $f \in C(X)$ belongs to the Bowen class if $\{S_n f\}_{n=1}^{\infty}$ is a Bowen sequence.

Corollary 6.1. *Let (X, σ_X), (Y, σ_Y) and $\pi : X \to Y$ be defined as in Theorem 6.1. Let $f \in C(X)$ be in the Bowen class and suppose $Z_1(f) < \infty$. Then there exists a unique invariant ergodic Gibbs measure μ for f. Setting $f_n = e^{S_n f}$ in $\mathcal{G}$, the projection $\pi\mu$ of the measure μ is the unique invariant ergodic Gibbs measure for $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$. Then*

$$\begin{aligned}P_G(f) = P(f) &= \sup_{\mu\in M(X,\sigma_X)} \left\{h_\mu(\sigma_X) + \int f d\mu : \int f d\mu > -\infty\right\}\\ &= \sup_{\nu\in M(Y,\sigma_Y)} \left\{h_\nu(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \log g_n d\nu : \lim_{n\to\infty} \frac{1}{n} \int \log g_n d\nu > -\infty\right\}\\ &= P(\mathcal{G}) < \infty.\end{aligned}$$

In addition, if $\sum_{i\in\mathbb{N}} \sup\{\log f(x) : x \in [i]\} \sup\{f(x) : x \in [i]\} > -\infty$*, then* μ *is the unique equilibrium measure for* f *and* $\pi\mu$ *is the unique equilibrium measure for* $\mathcal{G}$*.*

7. Other applications

7.1. **Product of matrices and maximal Lyapunov exponents.** A natural and interesting application of the non-additive version of thermodynamic formalism is the study of the norm of products of matrices. Indeed, let $M_d(\mathbb{R})$ be the set of real valued $d \times d$ matrices and $\|\cdot\|$ be a sub-multiplicative norm. Let $\{A_1, A_2, \dots\}$ be a countable set in $M_d(\mathbb{R})$. Let (X, σ) be a finitely irreducible countable sofic shift. If $w = (i_1, i_2, \dots) \in X$, define the sequence of functions $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ by $\phi_n(w) = \|A_{i_n} \cdots A_{i_2} A_{i_1}\|$. Since $\|AB\| \leq \|A\|\|B\|$, the sequence Φ is sub-additive. It is a direct consequence of the sub-additive ergodic theorem [31] that if $\mu \in M(X, \sigma)$ is an ergodic measure, then for μ-almost every $w \in X$

$$\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu = \lim_{n\to\infty} \frac{1}{n} \log \phi_n(w).$$

The number $\lambda(w) := \lim_{n\to\infty}(1/n) \log \phi_n(w)$, is called *Maximal Lyapunov exponent of* w, whenever the limit exists. This number was originally studied in the context in which X is the full shift on a finite alphabet with a finite collection matrices with strictly positive entries (see the work by Furstenberg and Kesten from 1960 [22]). Ever since, the assumptions on the space and on the matrices has been generalized in wide ranges. The techniques developed in this article allow for another generalization that can be thought of as a non-compact version of the results obtained by Feng in [17].

Proposition 7.1. *Let* (X, σ) *be a finitely irreducible countable sofic shift. Let* $\{A_1, A_2, \dots\}$ *be a countable set of matrices in* $M_d(\mathbb{R})$ *having non-negative entries. Let* $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ *be a the sequence of functions such that* $\phi_n : X \to \mathbb{R}$ *is defined by* $\phi_n(w) = \|A_{i_n} \cdots A_{i_2} A_{i_1}\|$*. If* Φ *satisfies* (C2), (C3) *and* (C4)*, then there exists a unique invariant ergodic Gibbs measure* μ *for* Φ*. Moreover, if in addition*

$$\sum_{i=1}^{\infty} \|A_i\| \log \|A_i\| > -\infty$$

then μ *is the unique equilibrium measure for* Φ *on* X*, that is*

$$P(\Phi) = h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n d\mu.$$

Note that ϕ_n is constant in cylinders of length n, therefore the Bowen condition is satisfied. Proposition 7.1 is an extension of [26, Proposition 7.1] in which the same conclusion was obtained under the assumption that X is a countable Markov shift satisfying the BIP condition and Φ is almost-additive.

7.2. **The singular value function.** Thermodynamic formalism has been used, at least since the mid 1970s, to study the (Hausdorff) dimension of certain dynamically defined sets. This approach has been rather successful when the dynamical system is conformal. However, in dimension two (or higher) where a typical dynamical system is non-conformal the results obtained are fairly weak. With the purpose of obtaining better estimates on the dimension of non-conformal repellers, Falconer [14] introduced the singular value function. The singular values $s_1(A), s_2(A)$ of a 2×2 matrix A are the eigenvalues, counted with multiplicities, of the matrix $(A^*A)^{1/2}$, where A^* denotes the transpose of A. The singular values can be interpreted as the length of the semi-axes of the ellipse which is the image of the unit ball under A.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a C^1 map and let $\Lambda \subset \mathbb{R}^2$ be a repeller of f. That is, the set Λ is a (not necessarily compact), f-invariant, and the map f is expanding on Λ, i.e., there exist $c > 0$ and $\beta > 1$ such that $\|d_x f^n(v)\| \geq c\beta^n \|v\|$, for every $x \in \Lambda$, $n \in \mathbb{N}$ and $v \in T_x\mathbb{R}^2$. We will also assume that there exists an open set $U \subset \mathbb{R}^2$ such that $\Lambda \subset U$ and $\Lambda = \cap_{n\in\mathbb{N}} f^n(U)$ and that f restricted to Λ can be coded by an irreducible countable sofic shift. For each $x \in \mathbb{R}^2$ and $v \in T_x R^2$, we define the *Lyapunov exponent* of (x, v) by

$$\lambda(x, v) := \limsup_{n\to\infty} \frac{1}{n} \log \|d_x f^n v\|.$$

For each $x \in \mathbb{R}^2$, there exists a positive integer $s(x) \leq 2$, numbers $\lambda_1(x) \geq \lambda_2(x)$, and linear subspaces

$$\{0\} = E_{s(x)+1}(x) \subset E_{s(x)}(x) \subset E_1(x) = T_x R^2,$$

such that

$$E_i(x) = \left\{v \in T_x\mathbb{R}^2 : \lambda(x, v) = \lambda_i(x)\right\}$$

and $\lambda(x, v) = \lambda_i(x)$ if $v \in E_i(x) \setminus E_{i+1}(x)$. The functions, $\phi_{i,n} : \Lambda \to \mathbb{R}$ are defined by

$$\phi_{i,n}(x) = \log s_i(d_x f^n)$$

and called *singular value functions*. It follows from Oseledets' multiplicative ergodic theorem that for each finite $f-$invariant measure μ there exists a set $X \subset \mathbb{R}^2$ of full μ measure such that

$$\lim_{n\to\infty} \frac{\phi_{i,n}(x)}{n} = \lim_{n\to\infty} \frac{1}{n} \log s_i(d_x f^n) = \lambda_i(x). \tag{7.1}$$

It was proved by Barreira and Gelfert [5, Proposition 4] that if the dynamical system f has dominated splitting (see [4, p.234] for a precise definition) and Λ is compact then the sequences $\{\phi_{i,n}\}_{n=1}^{\infty}$ are almost-additive. The methods developed in this article allow us to study the singular value function in a broader context. In particular, it is a consequence of the variational principle that

Proposition 7.2. *Let (f, Λ) be a non-conformal repeller that can be coded by an irreducible countable sofic shift. If the singular value functions Φ satisfy* (C2), (C3) *and* (C4), *then there exists a unique invariant ergodic Gibbs measure μ for Φ.*

We stress that Gibbs measures are of particular importance in the dimension theory of dynamical systems.

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