# FREQUENCY OF DIGITS IN THE LÜROTH EXPANSION

LUIS BARREIRA AND GODOFREDO IOMMI

ABSTRACT. In this note we consider the Lüroth expansion of a real number, and we study the Hausdorff dimension of a class of sets defined in terms of the frequencies of digits in the expansion. We also study the speed at which the approximants obtained from the Lüroth expansion converge. In addition, we describe the multifractal properties of the level sets of the Lyapunov exponent, which measures the exponential speed of approximation obtained from the approximants. Finally, we describe the relation of the Lüroth expansion with the continued fraction expansion and the $\beta$-expansion. We remark that our work is still another application of the theory of dynamical systems to number theory.

## 1. INTRODUCTION

Given constants $a_n \geq 2$ for every $n \in \mathbb{N}$, each real number $x \in (0,1]$ can be written in the form

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)a_2 \cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots .$$

This series expansion, called *Lüroth expansion*, was introduced in 1883 by Lüroth [12]. Each irrational number has a unique infinite expansion of this form and each rational number has either a finite expansion or a periodic one. We denote the Lüroth series expansion of $x \in (0,1)$ by

$$x = [a_1(x)a_2(x)\cdots] = [a_1a_2\cdots].$$

The series is closely related to the dynamics of the *Lüroth map* $T\colon [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} n(n+1)x - n, & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}], \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \in [\frac{1}{n+1}, \frac{1}{n}]$, then $a_1(x) = n$, and

$$a_k(x) = a_1(T^{k-1}(x)), \tag{1}$$

that is, the Lüroth map acts as a shift on the Lüroth series. The Lebesgue measure is $T$-invariant and is ergodic (see [10]). For other properties of the map Lüroth see [7, 6].

In this note we exploit the consequences of identity (1). Namely, by studying the ergodic properties of the map $T$ we will deduce number theoretical properties of the Lüroth expansion.

---

2000 *Mathematics Subject Classification.* Primary: 37C45, 11A67.

*Key words and phrases.* frequency of digits, Lüroth expansion.

L. B. was supported by FCT through CAMGSD, Lisbon. G. I. was supported by Proyecto Fondecyt 11070050 and by Research Network on Low Dimensional Dynamics, CONICYT, Chile.

We first want to study the Hausdorff dimension of the level sets determined by the frequency of digits in the Lüroth series expansion. More precisely, for each $n, k \in \mathbb{N}$ and $x \in (0,1)$ let

$$\tau_k(x,n) = \operatorname{card}\big\{i \in \{1,\dots,n\} : a_i(x) = k\big\}.$$

Whenever the limit

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x,n)}{n} \tag{2}$$

exists, it is called the *frequency* of the number $k$ in the Lüroth expansion of $x$. Since the Lebesgue measure is ergodic, by Birkhoff's ergodic theorem, for Lebesgue-almost every $x \in [0,1]$ we have $\tau_n(x) = 1/[n(n-1)]$. Now let $\alpha = (\alpha_1\alpha_2\cdots)$ be a stochastic vector, i.e., a vector such that $\alpha_i \geq 0$ and $\sum_{i=1}^\infty \alpha_i = 1$. We consider the set

$$F(\alpha) = \big\{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for each } k \in \mathbb{N}\big\}. \tag{3}$$

We have already seen that if $\alpha_n = 1/[n(n-1)]$ for each $n \in \mathbb{N}$, then the level set $F(\alpha)$ has full Lebesgue measure. Of course, any other level set has zero Lebesgue measure.

However, the sets in (3) can have positive Hausdorff dimension. Our first objective is to obtain an explicit formula for the Hausdorff dimension of sets $F(\alpha)$ for an arbitrary stochastic vector $\alpha$. Analogous problems were considered for the base-$m$ representation of a number (with $m \in \mathbb{N}$). The frequency of a digit $k \in \{0,\dots,m-1\}$ in the base-$m$ representation is defined analogously to the one in (2), and we denote it by $\tau_{k,m}(x)$. Let $\alpha = (\alpha_1,\dots,\alpha_m)$ be a stochastic vector. Eggleston [8] proved that the set

$$F_m(\alpha) = \big\{x \in [0,1] : \tau_{k,m}(x) = \alpha_k \text{ for } k \in \{0,\dots,m-1\}\big\}$$

has Hausdorff dimension

$$\dim_H F_m(\alpha) = \frac{\sum_{k=0}^{m-1} \alpha_k \log \alpha_k}{\log m}.$$

This result was recovered and generalized by Barreira, Saussol and Schmeling [2, 3] using a multidimensional version of multifractal analysis. In Section 3 we show that if $\alpha = (\alpha_1\alpha_2\cdots)$ is a stochastic vector satisfying a certain summability condition, then

$$\dim_H F(\alpha) = \frac{-\sum_{n=1}^\infty \alpha_n \log \alpha_n}{\sum_{n=1}^\infty \alpha_n \log(n(n+1))}.$$

Moreover, we show that there exists an ergodic measure, say $\mu_\alpha$, concentrated on $F(\alpha)$ such that $\dim_H F(\alpha) = \dim_H \mu_\alpha$, where $\dim_H \mu_\alpha$ denotes the Hausdorff dimension of the measure (see Section 2.3 for the definition).

The second problem that we want address is to describe the speed of convergence of the Lüroth series expansion. The $n$-th approximant of an irrational number $x \in (0,1)$ is the rational number defined by

$$\begin{aligned}\frac{p_n}{q_n} &= [a_1\cdots a_n] \\ &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)a_2\cdots a_{n-1}(a_{n-1}-1)a_n}.\end{aligned} \tag{4}$$

We study in detail the speed at which $p_n/q_n \to x$ when $n \to \infty$. In particular, we study the real numbers $\lambda(x)$ that give an asymptotic exponential rate

$$\left| x - \frac{p_n}{q_n} \right| \asymp \exp(-n\lambda(x)).$$

We show that the range of possible values for $\lambda(x)$ is the interval $[\log 2, +\infty)$. Moreover, we compute the Hausdorff dimension of the sets

$$J(\gamma) = \{x \in (0,1) : \lambda(x) = \gamma\}.$$

Finally, we compare the Lüroth series expansion with other well known series expansions of a real number, namely the continued fraction expansion and the $\beta$-expansion.

## 2. Preliminaries from ergodic theory and dimension theory

We recall in this section all the notions and results form ergodic theory and dimension theory that are needed in the paper.

**2.1. The symbolic model.** The dynamical system defined by the Lüroth map $T|[0,1)$ can be coded by a full-shift on a countable alphabet. Indeed, let $\mathcal{A} = \{2, 3, 4, \dots\}$ be the alphabet and consider the space of sequences

$$\Sigma = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{A} \text{ for } n \in \mathbb{N}\}.$$

The *shift map* $\sigma \colon \Sigma \to \Sigma$ is defined by $\sigma(x_1x_2x_3\cdots) = (x_2x_3\cdots)$. We note that the map $\psi \colon \Sigma \to [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(0)$ defined by

$$\psi(a_1a_2a_3\cdots) = [a_1a_2a_3\cdots]$$

is a topological conjugacy between the symbolic system $(\Sigma, \sigma)$ and the map $T$ restricted to $[0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(0)$. Let

$$C_{a_1a_2\cdots a_n} = \big\{(x_n)_{n\in\mathbb{N}} \in \Sigma : a_1 = x_1, a_2 = x_2, \dots, a_n = x_n\big\}, \tag{5}$$

be the cylinder of length $n$ determined by $(a_1a_2\cdots a_n)$. Then $\psi(C_{a_1a_2\cdots a_n}) \subset (0,1)$ is an interval of length

$$|\psi(C_{a_1a_2\cdots a_n})| = \frac{1}{a_1(a_1-1)\cdots a_n(a_n-1)}, \tag{6}$$

where $|\cdot|$ denotes the Euclidean length (see [7, p. 40]).

Given a stochastic vector $\alpha = (\alpha_1\alpha_2\cdots)$ we denote by $\nu_\alpha$ the $\sigma$-invariant Bernoulli measure in $\Sigma$ such that $\nu_\alpha(C_k) = \alpha_k$ for each $k$. We denote by $\mu_\alpha$ the projection via $\psi$ of this measure onto the interval $[0,1]$. The *entropy* of the measures $\nu_\alpha$ and $\mu_\alpha$ is given by

$$h(\mu_\alpha) = h(\nu_\alpha) = -\sum_{n=1}^{\infty} \alpha_n \log \alpha_n.$$

We note that the entropy may be infinite. The *Lyapunov exponent* of the map $T$ at the point $x \in (0,1)$ is defined by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{n=1}^{\infty} |T'(T^n(x))|,$$

whenever the limit exists. By Birkhoff's ergodic theorem, for $\mu_\alpha$-almost every $x \in (0,1)$ we have

$$\lambda(x) := \lambda(\mu_\alpha) = \int_{[0,1]} \log |T'| \, d\mu_\alpha = \sum_{n=1}^{\infty} \alpha_n \log(n(n+1)). \tag{7}$$

2.2. **Thermodynamic formalism.** Let $\mathcal{M}_T$ be the set of $T$-invariant probability measures in $[0,1]$. Given a continuous function $g\colon [0,1] \to \mathbb{R}$, we define the *topological pressure* of $g$ with respect to $T$ by

$$P(g) = \lim_{n\to\infty} \frac{1}{n} \sum_{T^n(x)=x} \exp \sum_{i=0}^{n-1} g(T^i(x)),$$

This notion was introduced independently by Mauldin and Urbański [13] and Sarig [17]. It satisfies the variational principle

$$P(g) = \sup \left\{ h(\mu) + \int_{[0,1]} g \, d\mu : \mu \in \mathcal{M}_T \text{ such that } -\int_{[0,1]} g \, d\mu < \infty \right\}.$$

A measure $\mu_g \in \mathcal{M}_T$ for which the supremum is attained is called an *equilibrium measure* of $g$.

If the function $g$ is locally constant, i.e., for every $n > 1$ we have $g|\psi(C_n) = \theta_n$ for some $\theta_n > 0$, where $C_n$ is a cylinder set of length 1 (see (5)), then there is an explicit formula for the topological pressure of $\log g$. Indeed,

$$\begin{aligned} P(-t \log g) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n(x)=x} \prod_{i=0}^{n-1} (g(T^i(x)))^{-t} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{j_0, j_2, \ldots, j_{n-1} \in \mathbb{N}} (\theta_{j_0} \cdots \theta_{j_{n-1}})^{-t} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left( \sum_{i=1}^{\infty} (\theta_i^{-t}) \right)^n = \log \sum_{i=1}^{\infty} \theta_i^{-t}. \end{aligned}$$

In particular, for the function $-t \log |T'|$ we obtain

$$p(t) := P(-t \log |T'|) = \log \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right)^t.$$

Hence, if $t \le 1/2$, then $p(t) = \infty$, and if $t > 1/2$, then $p(t)$ is real analytic, strictly decreasing, strictly convex, and has a unique zero at $t = 1$. We note that $\exp p(t)$ is a slight variation of Riemann's zeta function. As a direct application of results in [5, 17] we obtain that for $t > 1/2$ there exists a unique equilibrium measure $\mu_t$ of $-t \log |T'|$.

2.3. **Dimension theory.** We briefly recall some basic definitions and results from dimension theory (see for example [14] for details). We say that a countable collection of sets $\{U_i\}_{i\in N}$ is a $\delta$-cover of $F \subset \mathbb{R}$ if $F \subset \bigcup_{i\in\mathbb{N}} U_i$, and $U_i$ has diameter $|U_i|$ at most $\delta$ for every $i \in \mathbb{N}$. Given $s > 0$, we define

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ a } \delta\text{-cover of } F \right\}.$$

The *Hausdorff dimension* of the set $F$ is defined by

$$\dim_H F = \inf\{s > 0 : \mathcal{H}^s(F) = 0\}.$$

Given a finite Borel measure $\mu$ in $F$, the *pointwise dimension* of $\mu$ at the point $x$ is defined by

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

whenever the limit exists, where $B(x,r)$ is the ball at $x$ of radius $r$.

**Proposition 2.1.** *Given a finite Borel measure $\mu$, if $d_\mu(x) \le d$ for every $x \in F$, then $\dim_H F \le d$.*

The *Hausdorff dimension* of the measure $\mu$ is defined by

$$\dim_H \mu = \inf\{\dim_H Z : \mu(X \setminus Z) = 0\}.$$

**Proposition 2.2.** *Given a finite Borel measure $\mu$, if $d_\mu(x) = d$ for $\mu$-almost every $x \in F$, then $\dim_H \mu = d$.*

## 3. Frequency of digits on the Lüroth series

3.1. **Main result.** In this section we obtain an explicit formula for the Hausdorff dimension of the set $F(\alpha)$ in (3), where $\alpha = (\alpha_1 \alpha_2 \cdots)$ is a stochastic vector satisfying a certain summability condition corresponding to a finite Lyapunov exponent.

**Theorem 3.1.** *If $\alpha = (\alpha_1 \alpha_2 \cdots)$ is a stochastic vector such that*

$$\lambda(\mu_\alpha) = \sum_{n=1}^{\infty} \alpha_n \log(n(n+1)) < \infty,$$

*then*

$$\dim_H F(\alpha) = \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}$$

*Proof.* By Ruelle's inequality, we have $h(\mu_\alpha) \le \lambda(\mu_\alpha) < \infty$. This will ensure that all the series considered below are convergent. We note that $\mu_\alpha(F(\alpha)) = 1$. Moreover, for $\mu_\alpha$-almost every $x \in F(\alpha)$ we have

$$\begin{aligned} d_{\mu_\alpha}(x) &= \lim_{r \to 0} \frac{\log \mu_\alpha(B(x,r))}{\log r} \\ &= \lim_{n \to \infty} \frac{\log \mu_\alpha(\psi(C_{i_1 \cdots i_n}))}{\log \prod_{i=1}^{n} |T'(T^i(x))|^{-1}} \\ &= \lim_{n \to \infty} \frac{-\log \prod_{j=1}^{n} \alpha_{i_j}}{\log \prod_{j=1}^{n} i_j(i_j+1)} \\ &= -\frac{\int_{[0,1]} \log \phi \, d\mu_\alpha}{\int_{[0,1]} \log |T'| \, d\mu_\alpha} = \frac{h(\mu_\alpha)}{\lambda(\mu_\alpha)} \\ &= \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}, \end{aligned}$$

where $\phi(x) = \alpha_k$ for each $x \in \psi(C_k)$. The second equality follows from standard arguments in dimension theory (see for example [14]), showing that we can replace the interval $B(x,r)$ by the set $\psi(C_{i_1\cdots i_n})$. The fourth equality follows from Birkhoff's ergodic theorem. This implies that

$$\dim_H \mu_\alpha = \frac{-\sum_{n=1}^\infty \alpha_n \log \alpha_n}{\sum_{n=1}^\infty \alpha_n \log(n(n+1))},$$

and since $\mu_\alpha(F(\alpha)) = 1$ it follows from Proposition 2.2 that

$$\dim_H F(\alpha) \geq \dim_H \mu_\alpha = \frac{-\sum_{n=1}^\infty \alpha_n \log \alpha_n}{\sum_{n=1}^\infty \alpha_n \log(n(n+1))}.$$

For each $x \in F(\alpha)$ we have $\tau_k(x) = \alpha_k$ for all $k$, and hence,

$$\begin{aligned} d_{\mu_\alpha}(x) &= \lim_{n\to\infty} \frac{-\log \prod_{j=1}^n \alpha_{i_j}}{\log \prod_{j=1}^n i_j(i_j+1)} \\ &= \lim_{n\to\infty} \frac{-\frac{1}{n}\sum_{k=1}^\infty \tau_k(x,n)\log\alpha_k}{\frac{1}{n}\sum_{k=1}^\infty \tau_k(x,n)\log(k(k+1))} \\ &= \lim_{n\to\infty} \frac{-\sum_{k=1}^\infty \frac{\tau_k(x,n)}{n}\log\alpha_k}{\sum_{k=1}^\infty \frac{\tau_k(x,n)}{n}\log(k(k+1))} \\ &= \frac{-\sum_{n=1}^\infty \alpha_n \log \alpha_n}{\sum_{n=1}^\infty \alpha_n \log(n(n+1))}. \end{aligned}$$

It follows from Proposition 2.1 that

$$\dim_H F(\alpha) \leq \frac{-\sum_{n=1}^\infty \alpha_n \log \alpha_n}{\sum_{n=1}^\infty \alpha_n \log(n(n+1))}$$

This completes the proof of the theorem. □

It follows from the proof of Theorem 3.1 that the ergodic $T$-invariant measure $\mu_\alpha$ satisfies $\mu_\alpha(F(\alpha)) = 1$ and

$$\dim_H F(\alpha) = \dim_H \mu_\alpha = \frac{h(\mu_\alpha)}{\lambda(\mu_\alpha)}.$$

**Example 3.1.** *Let $\alpha_n = 1/n!$ for every $n \geq 2$, and let $\alpha_1 = 1 - \sum_{n=2}^\infty \alpha_n$. We denote by $\mu_\alpha$ the corresponding Bernoulli measure. Then*

$$\begin{aligned} &\dim_H F(\alpha) = \dim_H \mu_\alpha \\ &\quad = \left(-\alpha_1\log\alpha_1 + \sum_{n=2}^\infty \frac{\log n!}{n!}\right) \Bigg/ \left(\alpha_1 \log 2 + \sum_{n=2}^\infty \frac{\log(n(n+1))}{n!}\right). \end{aligned}$$

3.2. **Linear relations.** We also consider the problem of estimating from below the Hausdorff dimension of sets defined by linear relations between the frequencies of numbers in the Lüroth expansion.

Given a sequence $a = (a_n)_{n\in\mathbb{N}}$, we consider the set

$$R_a = \{x \in [0,1] : \tau_{2n-1}(x) = a_n \tau_{2n}(x)\}.$$

We note that the choice of $2n-1$ and $2n$ is arbitrary for the purpose of computing the Hausdorff dimension, and indeed we could have chosen any other bijection from $\mathbb{N}$ to itself. Let us consider the set $A_a$ of stochastic vectors $(\alpha_1\alpha_2\cdots)$ such that $\alpha_{2n-1} = a_n\alpha_{2n}$ for each $n \in \mathbb{N}$. Each vector $\alpha \in A_a$ determines uniquely a Bernoulli measure $\mu_\alpha$ by the identity $\mu_\alpha(\psi(C_n)) = \alpha_n$ for each $n \in \mathbb{N}$. We have $\mu_\alpha(R_a) = 1$.

We note that the Lebesgue measure $m$ satisfies

$$\dim_H m = \sup\{\dim_H \mu : \mu \in \mathcal{M}_T\} \quad \text{and} \quad m\left(\left(\frac{1}{n+1}, \frac{1}{n}\right)\right) = \frac{1}{n(n+1)}.$$

Since

$$\alpha_{2n} + \alpha_{2n-1} = \frac{1}{2n(2n+1)} + \frac{1}{(2n-1)2n} = \frac{2}{4n^2-1},$$

we can consider the unique vector $\alpha(a) \in A_a$ such that

$$\alpha_{2n} + \alpha_{2n-1} = \frac{2}{4n^2-1} \quad \text{and} \quad \alpha_{2n-1} = a_n \alpha_{2n}.$$

This yields

$$\alpha_{2n} = \frac{2}{(4n^2-1)(1+a_n)} \quad \text{and} \quad \alpha_{2n-1} = \frac{2a_n}{(4n^2-1)(1+a_n)}.$$

Let $\mu_{\alpha(a)}$ be the corresponding Bernoulli measure.

We have the following criterion to obtain a lower bound for the Hausdorff dimension of $R_a$.

**Theorem 3.2.** *If*

$$d_{\mu_{\alpha(a)}}(x) = \frac{\sum_{n=1}^{\infty} \alpha_n \log \alpha_n^{-1}}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))} = d < \infty \tag{8}$$

*for $\mu_{\alpha(a)}$-almost every $x \in [0,1]$, then*

$$\dim_H R_a \geq \dim_H \mu_{\alpha(a)} = d.$$

*Proof.* The statement is an immediate consequence of Proposition 2.2. □

We note that if $\lim_{n\to\infty}(n^{\varepsilon}/a_n) < \infty$ for some $\varepsilon > 1$, then $d_{\mu_\alpha}(x) < \infty$ for $\mu_\alpha$-almost every $x$.

**Example 3.2.** *Consider the set*

$$R_a = \left\{x \in [0,1] : \tau_{2n-1}(x) = \frac{1}{2^n}\tau_{2n}(x)\right\}.$$

*The unique vector $\alpha = \alpha(a)$ is given by*

$$\alpha_{2n} = \frac{2^{n+1}}{(2^n+1)(4n^2-1)} \quad \textit{and} \quad \alpha_{2n-1} = \frac{2}{(2^n+1)(4n^2-1)}.$$

*Then* (8) *holds for $\mu_\alpha$-almost every $x$, and* $\dim_H R_a \geq \dim_H \mu_\alpha = d$.

## 4. Multifractal analysis of the Lyapunov exponent: speed of approximation

4.1. **Main result.** Let $x \in (0,1)$ be an irrational number and denote its Lüroth expansion by $x = [a_1 a_2 \cdots]$. The $n$-th approximant $p_n/q_n$ of $x$ is the rational number defined by (4). We study in this section the speed at which $p_n/q_n \to x$ when $n \to \infty$. Let $m$ be the Lebesgue measure. By (6), for every $x \in (0,1)$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| &= \lim_{n\to\infty} \frac{1}{n} \log m(\psi(C_{a_1\cdots a_n})) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} m(\psi(C_{a_i})) \\ &= -\lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = -\lambda(x).\end{aligned}$$

This shows that the Lyapunov exponent measures the exponential speed of approximation of a number by its approximants. Since the measure $m$ is ergodic, the speed of approximation of number by its approximants is given $m$-almost everywhere by

$$\lambda(m) := \sum_{n=1}^{\infty} \frac{\log(n(n+1))}{n(n+1)}.$$

However, some numbers $x \in (0,1)$ may attain other values for the Lyapunov exponent. We consider the sets

$$J(\gamma) = \big\{x \in (0,1) : \lambda(x) = \gamma\big\}$$

for each $\gamma \in \mathbb{R}$, and

$$J' = \big\{x \in (0,1) : \lambda(x) \text{ does not exist}\big\}.$$

We obtain the *multifractal decomposition*

$$(0,1) = \bigcup_{\gamma} J(\gamma) \cup J',$$

and we define the *multifractal spectrum* by $L(\gamma) = \dim_H J(\gamma)$.

**Theorem 4.1.** *The following properties hold:*

(1) *the domain of $L$ is* $[\log 2, +\infty)$, *and*

$$L(\gamma) = \frac{1}{\gamma} \inf_{t\in\mathbb{R}} \big[P(-t \log |T'|) + t\gamma\big] ;$$

(2) *the spectrum $L$ real analytic, has a unique maximum at $\gamma = \lambda(m)$, has an inflection point, and satisfies*

$$\lim_{\gamma\to+\infty} L(\gamma) = \frac{1}{2};$$

(3) $\dim_H J' = 1$.

*Proof.* The proof of the first two statements follows the proofs of statements in [9, 11, 15] with obvious modifications. Indeed, the results in [11, 15] describe completely the multifractal spectrum for the Gauss map (the Lüroth map $T$ is essentially a piecewise linear version of the Gauss map). We note that the results in [9] hold in fact in a more general setting (such as for example for the Rényi map). The last statement follows from results in [4]. □

We can also show that for every $\gamma \in (\log 2, +\infty)$ there exists an ergodic invariant measure $\mu_\gamma$ such that $\mu_\gamma(J(\gamma)) = 1$ and $\dim_H \mu_\gamma = \dim_H J(\gamma)$.

We emphasize that the results in Theorem 4.1 should be considered surprising, particularly since each level set $J(\gamma)$ is dense in $(0,1)$. Indeed, even though the multifractal decomposition is extremely complicated, the multifractal spectrum is quite regular. Moreover, with the exception of $J(\log 2)$, all levels set have positive Hausdorff dimension.

4.2. **Finer decomposition of level sets.** In this section we will analyze further each level set $J(\gamma)$. Let $\gamma > \log 2$, and consider the set $A_\gamma$ of stochastic vectors $\alpha = (\alpha_1 \alpha_2 \cdots)$ such that

$$\sum_{n=1}^{\infty} \alpha_n \log(n(n+1)) = \gamma.$$

It follows from (7) that the set of points for which the frequency of digits in their Lüroth expansion is equal to $\alpha$ is contained in the level set $J(\gamma)$, that is, $F(\alpha) \subset J(\gamma)$. In fact we have the following statement.

**Proposition 4.1.** *We have*

$$\bigcup_{\alpha \in A_\gamma} F(\alpha) = J(\gamma).$$

*Proof.* Clearly, by the definition of the set $A_\gamma$ we have

$$\bigcup_{\alpha \in A_\gamma} F(\alpha) \subset J(\gamma).$$

Now we establish the reverse inclusion. For each $x \in J(\gamma)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = \gamma.$$

Consider the sequence of measures

$$\eta_{n,x} = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)},$$

where $\delta_y$ is the Dirac measure supported in $y$. Let $\eta_x \in \mathcal{M}_T$ be a weak* limit of the sequence $\{\eta_{n,x}\}_n$. If $\chi_k$ is the characteristic function of $\psi(C_k)$, then

$$\tau_k(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_k(T^i(x)) = \int_{[0,1]} \chi_k \, d\eta_x,$$

passing eventually to a subsequence. This shows that $x \in F(\alpha_x)$, where

$$\alpha_x = \left( \int_{[0,1]} \chi_1 \, d\eta_x, \int_{[0,1]} \chi_2 \, d\eta_x, \dots \right),$$

and we obtain the desired statement. □

We recall that there exists an ergodic measure $\mu_\gamma$ such that

$$\dim_H J(\gamma) = \dim_H \mu_\gamma = \frac{h(\mu_\gamma)}{\lambda(\mu_\gamma)} \geq \sup \left\{ \frac{h(\mu_\alpha)}{\lambda(\mu_\alpha)} : \alpha \in A_\gamma \right\}.$$

## 5. Relation with continued fraction expansions and $\beta$-expansions

We discuss in this section the relation of the Lüroth expansion of a real number with other expansions of the same number.

Namely, given $x \in (0,1)$, let $x = [a_1 a_2 \cdots]$ be its Lüroth expansion and let $x = [b_1 b_2 \cdots]_O$ be some other expansion of the same number (we give specific examples below). We are interested in finding asymptotically how many partial quotients $k_n(x)$ of $[b_1 b_2 \cdots]_O$ can be obtained from the first $n$-terms of the Lüroth expansion. We consider the limit

$$k_O(x) = \lim_{n\to\infty} \frac{k_n(x)}{n},$$

whenever it exists.

5.1. **The $\beta$-expansion.** Let $\beta \in \mathbb{R}$ with $\beta > 1$. The *beta transformation* $T_\beta\colon [0,1) \to [0,1)$ is defined by

$$T_\beta(x) = \beta x \pmod 1.$$

It was shown by Rényi in [16] that each $x \in [0,1)$ has a $\beta$-expansion

$$x = \frac{b_1(x)}{\beta} + \frac{b_2(x)}{\beta^2} + \frac{b_3(x)}{\beta^3} + \cdots = [b_1 b_2 \cdots]_\beta,$$

where $b_n(x) = [\beta T_\beta^{n-1}(x)]$ for each $n$, being $[a]$ the integer part of $a$. We note that $b_n(x) \in \{0, 1, \ldots, [\beta]\}$, and that in general there may exist several ways of writing $x = \sum_{i=1}^\infty a_i/\beta^i$ with $a_i \in \{0, 1, \ldots, [\beta]\}$. When $\beta$ is an integer we recover the classical base-$\beta$ representation.

The following is an immediate consequence of Theorem 2 in [1].

**Proposition 5.1.** *For each $x \in (0,1)$ we have $k_\beta(x) = \log\beta/\lambda(x)$, whenever $k_\beta(x)$ is well-defined.*

This implies that for Lebesgue-almost every $x$ we have

$$k_\beta(x) = \log\beta \Big/ \sum_{n=1}^\infty \frac{\log(n(n+1))}{n(n+1)}.$$

5.2. **The continued fraction expansion.** Every irrational number $x \in (0,1)$ can be expressed as a continued fraction

$$x = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}} = [b_1 b_2 \cdots]_C,$$

where $b_i \in \mathbb{N}$ for each $i$. We consider the Gauss map $G\colon (0,1] \to (0,1]$ defined by $G(x) = 1/x - [1/x]$. We have $b_n = [1/G^{n-1}(x)]$ for each $n$. In particular, the Gauss map acts as a shift on the continued fraction expansion, that is,

$$G^n(x) = [b_{n+1}(x) b_{n+2}(x) \cdots]_C.$$

We consider the limit

$$\lambda_G(x) = \lim_{n\to\infty} \frac{1}{n} \log |(G^n)'(x)|,$$

whenever it exists. The following is again an immediate consequence of Theorem 2 in [1].

**Proposition 5.2.** *For each $x \in (0,1)$ we have $k_C(x) = \lambda_G(x)/\lambda(x)$, whenever $k_C(x)$ is well-defined.*

This implies that for Lebesgue-almost every $x$ we have

$$k_C(x) = \frac{\pi^2}{6\log 2} \Big/ \sum_{n=1}^{\infty} \frac{\log(n(n+1))}{n(n+1)}.$$

## References

[1] L. Barreira and G. Iommi, *Partial quotients of continued fractions and β-expansions*, preprint.
[2] L. Barreira and B. Saussol, *Variational principles and mixed multifractal spectra*, Trans. Amer. Math. Soc. **353** (2001), 3919–3944.
[3] L. Barreira, B. Saussol and J. Schmeling, *Higher-dimensional multifractal analysis*, J. Math. Pures Appl. **81** (2002), 67–91.
[4] L. Barreira and J. Schmeling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, Israel J. Math. **116** (2000), 29–70.
[5] J. Buzzi and O. Sarig, *Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps*, Ergodic Theory Dynam. Systems **23** (2003), 1383–1400.
[6] K. Dajani and C. Kraaikamp, *On approximation by Lüroth series*, J. Théor. Nombres Bordeaux **8** (1996), 331–346.
[7] K. Dajani and C. Kraaikamp, *Ergodic Theory of Numbers*, Carus Mathematical Monographs 29, Mathematical Association of America, 2002.
[8] H. Eggleston, *The fractional dimension of a set defined by decimal properties*, Quart. J. Math. Oxford Ser. **20** (1949), 31–36.
[9] G. Iommi, *Multifractal analysis of Lyapunov exponent for the backward continued fraction map*, preprint.
[10] H. Jager and C. de Vroedt, *Lüroth series and their ergodic properties*, Indag. Math. **31** (1969), 31–42.
[11] M. Kesseböhmer and B. Stratmann, *A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates*, J. Reine Angew. Math. **605** (2007), 133–163.
[12] J. Lüroth, *Ueber eine eindeutige Entwickelung von Zahlen in eine unendliche Reihe*, Math. Ann. **21** (1883), 411–423.
[13] R. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. (3) **73** (1996), 105–154.
[14] Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, 1997.
[15] M. Pollicott and H. Weiss, *Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation*, Comm. Math. Phys. **207** (1999), 145–171.
[16] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar **8** (1957), 477–493.
[17] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Theory Dynam. Systems **19** (1999), 1565–1593.

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
*E-mail address*: barreira@math.ist.utl.pt
*URL*: http://www.math.ist.utl.pt/~barreira/

Facultad de Matemáticas, Pontificia Universidad Católica de Chile (PUC), Avenida Vicuña Mackenna 4860, Santiago, Chile
*E-mail address*: giommi@mat.puc.cl
*URL*: http://www.mat.puc.cl/~giommi/