MULTIFRACTAL ANALYSIS FOR COUNTABLE MARKOV SHIFTS

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Abstract. We study the multifractal analysis of the pointwise dimension for equilibrium measures on countable Markov shifts. The main difficulty is that the space is not compact. In order to overcome this, we use an approximation argument based on the theory of convergence of Fenchel pairs developed by Wijsman. The results of Pesin and Weiss on multifractal analysis for compact spaces are used as well. We also prove a Bowen formula for countable Markov shifts. It turns out that, in this setting, this formula provides the Hausdorff dimension of the set of recurrent points.

1. Introduction

This paper is devoted to the study of the multifractal analysis for countable Markov shifts. In a general setting multifractal analysis can be described as follows. Let μ be a finite measure on a metric space (Σ, d) and $B(x,r) := \{y \in \Sigma : d(x,y) < r\}$. The *pointwise dimension* of μ at the point $x \in \Sigma$ is defined by

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

provided the limit exists. This function describes the power law behaviour of $\mu(B(x,r))$ as $r \to 0$. The pointwise dimension quantifies how concentrated a measure is around a point: the larger it is the less concentrated the measure is around that point. Note that if μ is an atomic measure supported at the point x_0 then $d_\mu(x_0) = 0$ and if $x_1 \neq x_0$ then $d_\mu(x_1) = \infty$. This function induces a decomposition of the space into level sets:

$$J_\alpha = \{x \in \Sigma : d_\mu(x) = \alpha\} \ , \ J' = \{x \in \Sigma : \text{the limit } d_\mu(x) \text{ does not exist}\}.$$

The set J' is called the *irregular set*. The decomposition:

$$\Sigma = \left(\biguplus_\alpha J_\alpha \right) \biguplus J'$$

is called *multifractal decomposition*. The *multifractal spectrum* is defined by

$$f_\mu(\alpha) = \dim_H(J_\alpha),$$

where $\dim_H$ denotes the Hausdorff dimension. Several authors have successfully studied the case when Σ is a finite state Markov shift, see [9] for a general account. Pesin and Weiss [10] developed techniques to describe the multifractal spectrum in very general settings. They define an auxiliary function, usually denoted by $T(q)$. It is possible to describe this function by means of the thermodynamical formalism. It turns out that $T(q)$ and the multifractal spectrum form a Legendre pair, therefore it is possible to characterise the multifractal spectrum based on the properties of $T(q)$. In the finite state case the multifractal spectrum is real analytic, concave

and with bounded domain [10], [20].
The countable alphabet case has been studied by Nakaishi [8], Pollicott and Weiss [11] and Hanus, Mauldin and Urbanski [5], among others. They noticed that the multifractal spectrum does not necessarily have the same properties as in the compact setting. In particular it can have points of non analyticity or can have unbounded domain.
Let Σ denote the symbolic space and μ the *equilibrium measure* corresponding to the potential $\log \phi$ (see Section 2). Consider

$$T(q) := \begin{cases} \inf\{t : P_G(-t\log\psi + q\log\phi) \le 0\} & \text{if } q \ge q^*, \\ \infty & \text{if } q < q^*, \end{cases}$$

where P_G denotes the Gurevich pressure (Definition 2.5) and $\log\psi$ a *metric potential* (see Definition 2.10), for precise statements see Definitions 4.2 and 4.1. In Theorem 4.1 we prove that,

Theorem. The multifractal spectrum f_μ is the Fenchel Transform of T.

Our approach is based on an approximation argument. Subsets K of Σ that are compact and invariant will be considered. In those sets the Pesin and Weiss techniques can be applied. For each such a set K we obtain a Legendre pair $(T_K, f_{\mu,K})$, where $f_{\mu,K}$ is the multifractal spectrum of the measure associated to the potential restricted to K. Using of the infimal convergence introduced by Wijsman in [23] and [22], we prove the convergence of $(T_K, f_{\mu,K})$ to another Legendre pair. We prove that this limit correspond to the function $T(q)$ and to the multifractal spectrum of the measure μ.
This approach allows us to explain the different properties of the multifractal spectrum for countable Markov shifts, in terms of the Fenchel transform relation between $T(q)$ and f_μ. In particular, if $q^* = 0$ then f_μ has unbounded domain (see Proposition 4.5). The existence of points of non analyticity of f_μ can also be understood in terms of the behaviour of $T(q)$ near q^* (see Proposition 4.6, Theorem 4.7). Sufficient conditions will be given in order for the multifractal spectrum to have unbounded domain and points of non analyticity (see Proposition 4.7 , Theorem 5.2). The case in which the function $T(q)$ is linear for $q > q' \ge 1$ is also studied (see Theorem 5.3).
This approach also provides a framework to develop the multifractal analysis on spaces that do not satisfy the BIP property (see Section 5).
In the first part of this paper we prove a Bowen type formula (see [2]) for countable Markov shifts. In this setting this formula provides the Hausdorff dimension of the *Recurrent set* (Definition 3.1), which we denote by $\mathcal{R}$. Assume the potential $\log\psi$ to be a metric potential (see Definition 2.10). In Theorem 3.1 we prove that,

Theorem. $\dim_H(\mathcal{R}) = \inf\{t : P_G(-t\log\psi) \le 0\}$.

This formula was first proved for finite Markov shifts by Bowen [2] and Ruelle [13]. Mauldin and Urbanski (see [7]) generalised it to the case of countable Markov shifts with the BIP property (Definition 2.9). They noticed, in contrast to the finite state case, that the Bowen equation might not have a root and that conformal measures might not exist. We treat the case of arbitrarily topologically mixing countable Markov shifts (Definition 2.2). In this setting the Bowen formula can only account for the Hausdorff dimension of the Recurrent set. Note that it is possible for the Hausdorff dimension of this set to be strictly smaller than the

Hausdorff dimension of the repeller (Example 3.3). The problem of the existence of conservative conformal measures is also treated (see Proposition 3.1 and Proposition 3.2).

2. Preliminaries

The purpose of this section is to define the objects and constructions needed below. Let S be a countable (infinite) set and $T = (t_{ij})_{S\times S}$ a matrix of zeroes and ones. Let

$$\Sigma := \{x \in S^{\mathbb{N}\cup\{0\}} : t_{x_i x_{i+1}} = 1 \text{ for every } i \geq 0\}.$$

A cylinder of length n is defined by

$$C_{i_0 i_1 \dots i_{n-1}} := \{x \in \Sigma : x_j = i_j \text{ for } 0 \leq j \leq n-1\}.$$

An admissible word is an element $\underline{a} \in S^n$ such that $C_{\underline{a}} \neq \emptyset$.

Definition 2.1. *The function $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_i = x_{i+1}$ is called the left shift. The dynamical system (Σ, σ) is called a topological Markov chain and the matrix T is called transition matrix. Note that T can be represented by a directed graph G.*

Definition 2.2. *A topologically Markov shift (Σ, σ) is called topologically mixing if for every $a, b \in S$ there exists N such that for all $n \geq N$ there exists an admissible word $\underline{a}$ of length n such that $a_0 = a$ and $a_{n-1} = b$.*

Unlike the finite state case, this does not imply that some power of the transition matrix is positive. Throughout this paper we will always assume the system to be topologically mixing.

Definition 2.3. *Let $\log\phi : \Sigma \to \mathbb{R}$. The variations of $\log\phi$ are defined by*

$$V_n := \sup\{|\log\phi(x) - \log\phi(y)| : x, y \in \Sigma, x_i = y_i, 0 \leq i \leq n-1\}.$$

Definition 2.4. *Let $\log\phi : \Sigma \to \mathbb{R}$. We say that $\log\phi$ is locally Hölder continuous (with parameter θ) if there exists $B > 0$ and $\theta \in (0,1)$ such that for all $n \geq 1$,*

$$V_n(\log\phi) \leq B\theta^n.$$

This does not imply that $\log\phi$ is bounded. In this setting it is possible to define the pressure. The so called Gurevich pressure, which was defined by Sarig (see [15]) based on the work of Gurevich (see [3]), is the most appropriate to our purposes. Let $\log\phi : \Sigma \to \mathbb{R}$ be a locally Hölder potential and fix a symbol $i_0 \in S$.

Definition 2.5. *The Gurevich Pressure of $\log\phi$ is defined by*

$$P_G(\log\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\Big(\sum_{i=0}^{n-1} \log\phi(\sigma^i x)\Big) 1_{C_{i_0}}(x),$$

where 1_{C_i} is the indicator function of the cylinder C_i. The Gurevich pressure is convex and if $\mathcal{K} := \{K \subset \Sigma : K \text{ compact and } \sigma\text{-invariant}, K \neq \emptyset\}$ then

$$P_G(\log\phi) = \sup\{P(\log\phi|K) : K \in \mathcal{K}\}, \tag{1}$$

where $P(\log\phi|K)$ is the topological pressure of $\log\phi$ restricted to the compact set K (for definition and properties see [21] chapter 9). If the system is topologically mixing then its value does not depend on the cylinder C_{i_0} considered in the indicator function.

Definition 2.6. *The first return time map to C_i is defined by*

$$\varphi_i(x) := 1_{C_i}(x) \inf\{n \geq 1 : \sigma^n x \in C_i\},$$

Let

$$[\varphi_1 = n] := \{x \in \Sigma : \varphi_1(x) = n\}.$$

A locally Hölder continuous potential $\log \phi$ can be classified according to its recurrence properties (see [17]). In order to do so, consider the first return time map φ_1.

Definition 2.7. *Assume that $\log \phi$ has finite Gurevich pressure $P_G(\log \phi) = \log A$. We call $\log \phi$*

(1) recurrent *if*

$$\sum_{n \geq 1} A^{-n} \sum_{\sigma^n x = x} \exp\Big(\sum_{i=0}^{n-1} \log \phi(\sigma^i x)\Big) 1_{C_1}(x) = \infty$$

and transient *otherwise.*

(2) positive recurrent *if it is recurrent and*

$$\sum_{n \geq 1} n A^{-n} \sum_{\sigma^n x = x} \exp\Big(\sum_{i=0}^{n-1} \log \phi(\sigma^i x)\Big) 1_{[\varphi_1 = n]}(x) < \infty.$$

(3) null recurrent *if it is recurrent and*

$$\sum_{n \geq 1} n A^{-n} \sum_{\sigma^n x = x} \exp\Big(\sum_{i=0}^{n-1} \log \phi(\sigma^i x)\Big) 1_{[\varphi_1 = n]}(x) = \infty.$$

Remark 2.1. *If we consider the potential $\log \phi = 0$ then we say that a graph G is recurrent (positive or null) or transient according to the previous definition. The number $P_G(0)$ is called Gurevich entropy. See* [3], [4] *for properties and related concepts.*

In [17] Sarig generalises the Ruelle Perron Frobenius Theorem to countable Markov shifts. Let $\log \phi$ be a locally Hölder continuous potential of finite pressure $\log A$, and let

$$L_{\log \phi} g(x) := \sum_{\sigma y = x} \exp(\log \phi(y)) g(y)$$

denotes the Ruelle operator (see [17]). He proves that if $\log \phi$ is:

(1) Positive recurrent then there exists a conservative conformal measure ν and a continuous function g such that $L^*_{\log \phi} \nu = A\nu$, $L_{\log \phi} g = Ag$ and $\int g d\nu < \infty$.
(2) Null recurrent then there exists a conservative conformal measure ν and a continuous function g such that $L^*_{\log \phi} \nu = A\nu$, $L_{\log \phi} g = Ag$ and $\int g d\nu = \infty$.
(3) Transient then there is no conservative conformal measure.

In the positive recurrent case the measure $m = g d\nu$ is an equilibrium measure provided $- \int \log \phi dm < \infty$.

The inducing procedure in the context of topological Markov chains will be summarised. Let (Σ, σ) be a topologically mixing Markov chain with transition matrix $T = (t_{ij})_{S\times S}$. The *induced system* on the symbol 1 will be denoted by $(\Sigma_1, \overline{\sigma})$. It is defined as the full-shift on the new alphabet,

$$\{C_{\overline{a}} : a_i = 1 \text{ iff } i = 0, C_{\overline{a}1} \neq \emptyset\}.$$

Recall that φ_1 is the first return map to C_1. For every locally Hölder potential $\log\phi : \Sigma \to \mathbb{R}$ set

$$\overline{\log\phi} := \Big(\sum_{k=0}^{\varphi_1 - 1} \log\phi \circ \sigma^k\Big) \circ \pi,$$

where $\pi : \Sigma_1 \to C_1$ is defined by

$$\pi(C_{\underline{a}_0}, C_{\underline{a}_1}, \dots) = (\underline{a}_0, \underline{a}_1, \dots).$$

The pair $(\Sigma_1, \overline{\log\phi})$ is called the induced system and $\overline{\log\phi}$ is called the induced potential. Note that if the potential $\log\phi$ is locally Hölder then $\overline{\log\phi}$ is locally Hölder. This follows from the estimates $V_n(\overline{\log\phi}) \leq \sum_{j=n+1}^{\infty} V_j(\log\phi)$.
Next we define the notion of Gibbs measure.

Definition 2.8. *A probability measure μ is called a* Gibbs measure *for the potential $\log\phi$ if there exists two constants M and P, such that for every cylinder $C_{i_0\dots i_{n-1}}$ and every $x \in C_{i_0,\dots i_{n-1}}$*

$$\frac{1}{M} \leq \frac{\mu(C_{i_0\dots i_{n-1}})}{\exp(-nP + \sum_{j=0}^{n-1}\log\phi(\sigma^j x))} \leq M.$$

Let us consider a topologically mixing graph G with transition matrix $T = (t_{ij})$. The following definition appears in [19].

Definition 2.9. *We say that the graph G (or the system Σ) satisfies the big images and preimages property (BIP property) if there exists $\{b_1, b_2 \dots b_n\}$ in the alphabet S such that*

$$\forall a \in S \ \exists i, j \text{ such that } t_{b_i a} t_{a b_j} = 1. \tag{2}$$

Sarig (see [19]) proved that a locally Hölder potential $\log\phi$ of finite Gurevich pressure has an invariant Gibbs measure if and only if the system has the BIP property. Moreover, if Σ satisfies the BIP property and $P_G(\log\phi) < \infty$ then the function $t \to P_G(t\log\phi)$ is real analytic for $t > 1$. Note that the induced system satisfies the BIP property.
Let d be a metric on Σ.

Definition 2.10. *A positive, locally Hölder potential $\log\psi : \Sigma \to \mathbb{R}$ with the property that there exits a constant $C > 0$ such that*

$$\frac{1}{C}\prod_{j=0}^{n-1}\big(\psi(\sigma^j x)\big)^{-1} \leq diam(C_{i_0\dots i_{n-1}}) \leq C\prod_{j=0}^{n-1}\big(\psi(\sigma^j x)\big)^{-1}, \tag{3}$$

for every $x = (i_0, i_1, \dots) \in C_{i_0,\dots i_{n-1}}$ will be called metric potential.

We describe now part of the theory of sequences of Fenchel pairs developed by Wijsman in [23], [22].

Definition 2.11. *Let h be a convex function, we say that (h, g) form a* Fenchel pair *if*

$$g(p) = \sup_x\{px - h(x)\}.$$

Remark 2.2. *If h is a convex C^2 function then the function g is also called Legendre transform. In this case,*

$$g(\alpha) = h(q) + q\alpha,$$

where $\alpha(q) = -h'(q)$.

Let h be a convex function, $\rho > 0$ and

$$_\rho h(x) := \inf\{h(y) : |x - y| < \rho\}.$$

Definition 2.12 (Wijsman [23], [22])**.** *Let $\{h_n\}$ be a sequence of convex functions. We say that $\{h_n\}$ converges* infimally *to the function h if*

$$\lim_{\rho \to 0} \liminf_{n \to \infty} {}_\rho h_n(x) = \lim_{\rho \to 0} \limsup_{n \to \infty} {}_\rho h_n(x) = h(x).$$

We will write in this case $h_n \xrightarrow{\inf} h$.

Remark 2.3 (Wijsman [23])**.** *There are examples where this notion of convergence does not coincide with the pointwise convergence. In general we have that, if $\{h_n\}$ converges infimally to h and $\{h_n\}$ converges pointwise to h_p then $h(x) \leq h_p(x)$.*

Theorem 2.1 (Wijsman [23])**.** *Let (h_n, g_n) and (h, g) be Fenchel pairs. Then*

$$h_n \xrightarrow{\inf} h \text{ iff } g_n \xrightarrow{\inf} g.$$

3. A dimension formula for the Gurevich Pressure.

In this section we prove a Bowen type formula (see [2]) for the Gurevich pressure. In the setting of countable Markov shifts this formula relates the pressure function with the Hausdorff dimension of the recurrent set.

Definition 3.1. *A point $x \in \Sigma$ is called* Recurrent *if there exists a sequence of non equal natural numbers n_i, such that*

$$\lim_{i \to \infty} \sigma^{n_i} x = x.$$

Denote by $\mathcal{R}$ be the set of recurrent points.

Let (Σ, σ) be a topologically mixing Markov chain and $\log \psi$ locally Hölder metric potential (see Definition 2.10).

Theorem 3.1.

$$\dim_H(\mathcal{R}) = \inf\{t : P_G(-t \log \psi) \leq 0\}.$$

In the finite state case Ruelle [13] proved that the Bowen equation

$$P(-t \log \psi) = 0$$

always has a root and that it is equal to the Hausdorff dimension of the repeller. In [7] Mauldin and Urbanski generalised this result to the case of infinite iterated function systems. They first noticed that the Bowen equation might not have a root (see Figure 1) and that conservative conformal measures might not exists. We consider a broader class of systems and prove that the pressure function can only account for the recurrent set. We give an example of a piecewise linear map of the interval such that the Hausdorff dimension of the recurrent set is strictly less than the Hausdorff dimension of the repeller (see Example 3.2).

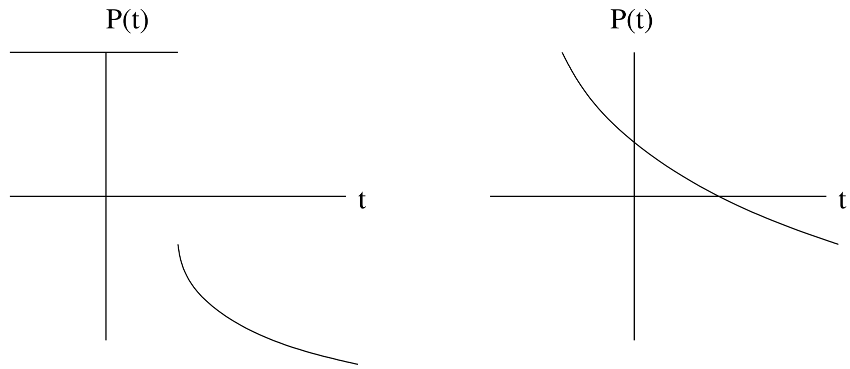

FIGURE 1. Possible behaviour of $t \to P(-t \log \psi)$

Proof Theorem 3.1. Let $d := \inf\{t : P_G(-t \log \psi) \leq 0\}$. We begin by proving that $\dim_H(\mathcal{R}) \geq d$. Let $K \subset \Sigma$ be a topologically mixing finite Markov shift. There exists a unique $t_K > 0$ such that $P_K(-t_K \log \psi) = 0$ and $\dim_H(K \cap \mathcal{R}) = t_K$ (see [9]). By the approximation property of the Gurevich pressure (1) we have that $t_K \leq d$. Moreover,

$$\sup\{t_K : K \in \mathcal{K}\} = d. \tag{4}$$

In fact, assume by way of contradiction that there exists $\epsilon > 0$ such that

$$\sup\{t_K : K \in \mathcal{K}\} = d - \epsilon$$

then

$$\sup\{P_K\big((d - \epsilon/2) \log \psi\big) : K \in \mathcal{K}\} \leq 0 < P_G\big((d - \epsilon/2) \log \psi\big),$$

which contradicts (1). From the Bowen equation for finite shifts and equation (4) we obtain that

$$\dim_H(\mathcal{R}) \geq d.$$

In order to prove the other inequality we will consider the induced system. Denote by

$$\overline{\psi_t}(x) := \prod_{k=0}^{\varphi_1(x)-1} \Big(\psi \circ \sigma^k\Big)^t \circ \pi(x).$$

Note that the diameter of a cylinder of length n on the induced system is such that

$$diam(C_{i_0 \dots i_{n-1}}) \leq C\overline{\psi_1}(x), \tag{5}$$

where $x \in C_{i_0 \dots i_{n-1}}$. Let $t > d$. Associated to the induced potential $\overline{-t \log \psi}$ there exists a Gibbs measure m (the induced system satisfies the BIP property). Thus, there exists $M > 0$ such that

$$\frac{1}{M} \leq \frac{m(C_{i_0 \dots i_{n-1}})}{\exp(-nP + \log \overline{\psi_t}(x))} \leq M,$$

where $P = P_G(\overline{-t \log \psi})$. Since $P \leq 0$ we have

$$\frac{1}{M} \overline{\psi_t}(x) \leq m(C_{i_0 \dots i_{n-1}}). \tag{6}$$

Combining (5) and (6) we obtain,

$$diam(C_{i_0 \ldots i_{n-1}})^t \leq M_1 m(C_{i_0 \ldots i_{n-1}}).$$

Denote by $C(n)$ the set of cylinders belonging to the induced system of length n. We have

$$\sum_{C \in C(n)} (diam(C))^t \leq \sum_{C \in C(n)} M_1 m(C) \leq M_1.$$

Note that $C(n)$ is a cover and the diameters of the cylinders tend to zero as n tends to infinity. Therefore, the $t-$Hausdorff measure of $\mathcal{R}$ is finite. Since $t > d$ was arbitrary this completes the proof. □

Remark 3.1. *Since the system is topologically mixing the result does not depend on the symbol where we induce.*

Proposition 3.1. *If Σ has finite Gurevich entropy and $\dim_H(\Sigma) < \infty$ then there exists $d > 0$ such that $P_G(-d \log \psi) = 0$. Moreover,*

1. *If the potential $-d \log \psi$ is recurrent then there exists a conservative $d-$conformal measure.*
2. *If the potential $-d \log \psi$ is transient then there is no conservative $d-$conformal measure.*

Proof. Since $P_G(0) < \infty$ the existence of d is a consequence of the continuity of the pressure. The (non) existence of a conservative d-conformal measure follows from the generalisation of the Ruelle Perron Frobenius theorem (see Section 2). □

Proposition 3.2. *Assume Σ to have infinite Gurevich entropy and $\dim_H(\Sigma) < \infty$,*

1. *If there exists $d > 0$ such that $P_G(-d \log \psi) = 0$ then,*
   - (a) *If the potential $-d \log \psi$ is recurrent then there exists a conservative $d-$conformal measure.*
   - (b) *If the potential $-d \log \psi$ is transient then there is no conservative $d-$conformal measure.*
2. *If $P_G(-d \log \psi) < 0$ then there is no conservative $d-$conformal measure.*

Proof. It follows from the Ruelle Perron Frobenius theorem for countable Markov shifts (see Section 2). □

Remark 3.2. *Assume that Σ satisfies the BIP property. If there exists $d > 0$ such that $P_G(-d \log \psi) = 0$ then $-d \log \psi$ is recurrent, therefore there exists a $d - conformal$ measure. Mauldin and Urbanski [7] called such a system* regular. *If there is no such d then there is no conservative $d - conformal$ measure. In the terminology of Mauldin and Urbanski, these systems are called* irregular.

In what follows we give examples that illustrates the theorem. We give them in the context of one dimensional dynamics, they can be translated to the countable Markov shift setting.

Example 3.1 (No root for the Bowen equation)**.** *Let $k' \in \mathbb{N}$ be such that*

$$\sum_{k=k'}^{\infty} \frac{1}{k(\log(2k))^2} < 1.$$

Let $\{I_k : k \geq k'\}$ be a sequence of closed intervals contained in $[0, 1]$ with disjoint interiors. Assume that $|I_k| = \frac{1}{k(\log(2k))^2}$, where $|\cdot|$ denotes the length of the interval.

Define the map $F : \cup_{k=k'}^{\infty} I_k \to [0,1]$ *such that* $F|I_k$ *is linear, onto* $[0,1]$ *and of positive slope for every* $k \geq k'$. *The pressure function is given by,*

$$P_G(-t \log |F'|) = \log \sum_{k=k'}^{\infty} 1/(k^t (\log 2k)^{2t}).$$

If $t < 1$ *then* $P_G(-t\log|F'|) = \infty$ *and if* $t \geq 1$ *then* $P_G(-t\log|F'|) < 0$. *Therefore the Bowen equation has no root and*

$$\dim_H(\mathcal{R}) = 1 = \inf\{t : P_G(-t\log|F'|) \leq 0\}.$$

This potential was first used by Sarig [16] *for another purposes. Mauldin and Urbanski* [7] *produced other examples where the Bowen equation has no root.*

Example 3.2 (Infinite number of roots for the Bowen equation)**.** *The Manneville-Pomeau transformation* $F : [0,1] \to [0,1]$ *is defined by*

$$F(x) := x + x^{1+\alpha},$$

where $\alpha \in (0,1)$. *This map has two branches defined on the intervals,* $[0,a], [a,1]$. *The induced map on* $[a,1]$ *satisfies the assumptions of our theorem. The pressure function* $P_G(-t\log|F'|)$ *is positive for* $t < 1$ *and equal to zero for* $t \geq 1$ *(see* [12]*). Therefore,*

$$\dim_H(\mathcal{R}) = 1 = \inf\{t : P_G(-t\log|F'|) \leq 0\}.$$

Example 3.3 (Small Recurrent set)**.** *We construct a topologically mixing interval map such that the Hausdorff dimension of the recurrent set is strictly less than the Hausdorff dimension of the repeller. Recall that the repeller of a map* F *is defined by,*

$$J := \{x \in [0,1] : F^n(x) \text{ is well defined for all } n \in \mathbb{N}\}.$$

Let $r = \frac{1}{2}$ *and* $I_1 = [0,r]$. *If* $n > 1$ *and* $n \in \mathbb{N} \setminus \{2^{i+1} : i \in \mathbb{N}\}$ *let* $I_n := [\sum_{i=1}^{n-1} r^i, \sum_{i=1}^{n} r^i]$. *For each positive integer* $n > 1$ *there exists a unique interval in* $[0,1] \setminus \biguplus I_i$ *of length* r^{2^n}. *Divide this interval into two intervals,*

$$I_{a_n} = [\sum_{i=1}^{2^n-1} r^i, a_n + \sum_{i=1}^{2^n-1} r^i] \text{ and } I_{2^n} = [a_n + \sum_{i=1}^{2^n-1} r^i, \sum_{i=1}^{2^n} r^i].$$

In this way we obtain a partition of the unit interval $[0,1]$. *Let* F *be a piecewise linear map defined by,*

$$F(I_n) = \begin{cases} I_1 & \text{if } n \in \{a_{i+1} : i \in \mathbb{N}\} \\ I_n \cup I_{a_n} \cup I_{n+1} & \text{if } n \in \{2^{i+1} - 1 : i \in \mathbb{N}\}. \\ I_n \cup I_{n+1} & \text{in any other case.} \end{cases} \tag{7}$$

Let $0 < c < 1$. *Consider a strictly increasing sequence* (s_i) *such that* $s_i \in (0,c)$ *and* $c = \sup\{s_i\}$. *Denote by* F_1 *the restriction of* F *to the set* $I_{a_2} \bigcup \left(\bigcup_{j=1}^{4} I_j\right)$. *Note that if* $a_2 = 0$ *then the Hausdorff dimension of the recurrent set for* F_1 *is zero. If* $a_2 = r^4$ *the the Hausdorff dimension of the recurrent set for* F_1 *is one. Choose* a_2 *such that the Hausdorff dimension of the recurrent set is* s_1, *that is* $P(-s_1 \log|F_1'|) = 0$.

We define a_n *inductively. Suppose that we have already defined* $\{a_2 \ldots a_{n-1}\}$. *Denote by* F_n *the restriction of* F *to* $\left(\bigcup_{j=2}^{n} I_{a_j}\right) \bigcup \left(\bigcup_{j=1}^{2^n} I_j\right)$. *Note that if* $a_n = 0$ *then the Hausdorff dimension of the recurrent set of* F_n *is* s_{n-1}, *and if* $a_n = r^{2^n}$

then the Hausdorff dimension of the recurrent set of F_n *is one. Choose* a_n *such that,*

$$P(-s_n \log |F_n'|) = 0.$$

Note that $c = \inf\{t : P_G(-t \log |F'|) \leq 0\}$*. Therefore* $\dim_H(\mathcal{R}) = c < 1 = \dim_H(J)$ *as required.*

4. Multifractal Analysis, the BIP case.

The purpose of this section is to develop techniques that enables to carry out the multifractal analysis for countable Markov shifts. Recall that Pesin and Weiss [10] studied the case in which the space is compact (finite state case) and μ is an equilibrium measure for a Hölder potential. They proved that the multifractal spectrum is concave and real analytic. Schmeling [20] proved that it has bounded domain. Note that the domain is defined by $\{\alpha \in \mathbb{R} : J_\alpha \neq \emptyset\}$. As mentioned in the introduction, the multifractal analysis for countable Markov shifts has also been studied in [8], [11] and [5]. Our techniques are slightly different. The lower bounds for the Hausdorff dimension of the level sets are obtain by means of an approximation argument that relies on some tools of convex analysis (the infimal convergence). This approach allows us to develop the multifractal analysis in more general symbolic spaces, gives a simple explanation to the differences between the finite and the countable state cases and enables to give sufficient conditions in order for the multifractal spectrum to have unbounded domain and points of non analyticity. The main ingredients that will be used are:

- The infimal convergence of Wijsman.
- The multifractal analysis over compact spaces.
- The approximation property of the Gurevich pressure (1).

In this section we will consider a countable Markov shift that satisfies the BIP property and a metric potential $\log \psi$ (see Definition 2.10). Throughout the rest of the paper we will work on the recurrent set of Σ. Let $\log \phi : \Sigma \to \mathbb{R}$ be a locally Hölder potential of zero pressure, not cohomologous to $\log \psi$. Denote by μ the corresponding Gibbs measure.

Remark 4.1. *The metric that we are considering is such that*

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \lim_{n \to \infty} \frac{\log \mu(C_{i_0 \dots i_{n-1}}(x))}{\log \prod_{i=0}^{n-1} \left(\psi(\sigma^i x)\right)^{-1}}.$$

Definition 4.1.

$$q^* := \inf\{q : \textit{there exists } t \in \mathbb{R} \textit{ such that } P_G(-t \log \psi + q \log \phi) \leq 0\}.$$

Lemma 4.1. *If* $\dim_H(\Sigma) < \infty$ *then either* $q^* = 0$ *or* $q^* = -\infty$*.*

Proof. Note that $P_G(-\dim_H(\Sigma) \log \psi) \leq 0$, therefore $q^* \leq 0$.
Assume by way of contradiction that $q^* \in (-\infty, 0)$. Let $0 < \epsilon, \epsilon' < |q^*|/2$. Since $q^* + \epsilon > q^*$ and $q^* + \epsilon' > q^*$ there exists $T(\epsilon)$ and $T(\epsilon')$ such that

$$P_G(-T(\epsilon) \log \psi + (q^* + \epsilon) \log \phi) \leq 0 \text{ and } P_G(-T(\epsilon') \log \psi + (q^* + \epsilon') \log \phi) \leq 0.$$

Since $P_G(f + g) \leq P_G(f) + P_G(g)$ then

$$P_G(-(T(\epsilon) + T(\epsilon')) \log \psi + (2q^* + \epsilon + \epsilon') \log \phi) \leq 0.$$

Therefore $\epsilon + \epsilon' > |q^*|$. This contradiction proves the statement. □

Remark 4.2. *If* $\dim_H(\Sigma) = \infty$ *it is possible for* q^* *to be larger than zero. See Example* 4.3.

Definition 4.2. *The function* $T(q)$ *is implicitly defined by*

$$T(q) := \begin{cases} \inf\{t : P_G(-t\log\psi + q\log\phi) \leq 0\} & \text{if } q \geq q^*, \\ \infty & \text{if } q < q^*. \end{cases} \tag{8}$$

We prove that,

Theorem 4.1. *The multifractal spectrum* f_μ *is the Fenchel Transform of* T.

Equivalent definitions of $T(q)$ will be required to prove the theorem. Let $K \in \mathcal{K}$ and let $P_K(\cdot)$ denote the topological pressure restricted to the set K.

Proposition 4.1.

$$T(q) = \sup\{t : P_K(-t\log\psi + q\log\phi) = 0,\ K \in \mathcal{K}\}.$$

Proof. The result is clear for $q < q^*$. Let $q \geq q^*$. For every compact invariant set K and for every $q \in \mathbb{R}$ there exist $T_K(q)$ such that (see [10]),

$$P_K(-T_K(q)\log\psi + q\log\phi) = 0.$$

Since $K \subset \Sigma$ then by (1) we have $P_K(-t\log\psi + q\log\phi) \leq P_G(-t\log\psi + q\log\phi)$, and therefore $T_K(q) \leq T(q)$. Note that the pressure is monotone, that is, if $K_1 \subset K_2$ then

$$P_{K_1}(-t\log\psi + q\log\phi) \leq P_{K_2}(-t\log\psi + q\log\phi).$$

In particular $T_{K_1}(q) \leq T_{K_2}(q)$. Assume by way of contradiction that

$$A := \sup\{T_K(q) : K \in \mathcal{K}\} < T(q). \tag{9}$$

Let $B \in (A, T(q))$. Since the pressure function is decreasing in t, we have that

$$P_G(-B\log\psi + q\log\phi) > 0.$$

On the other hand, because of the approximation property (1) and the assumption (9),

$$P_G(-B\log\psi + q\log\phi) \leq 0.$$

This contradiction proves the proposition. □

Proposition 4.2.

$$T(q) = \sup\{t : P_K\big(-t\log\psi + q(\log\phi - P_K(\log\phi))\big) = 0, K \in \mathcal{K}\}.$$

Proof. Again the result is clear for $q < q^*$. Let $K \in \mathcal{K}$ and $q \geq q^*$. Define $T_K^*(q)$ implicitly by

$$P_K\big(-T_K^*(q)\log\psi + q(\log\phi - P_K(\log\phi))\big) = 0. \tag{10}$$

Note that

$$|P_K(-t\log\psi + q\log\phi - qP_K(\log\phi)) - P_K(-t\log\psi + q\log\phi)| \leq |qP_K(\log\phi)| \tag{11}$$

Since $P_G(\log\phi) = 0$ and because of (1) we have that

$$\sup\{qP_K(\log\phi) : K \in \mathcal{K}\} = 0. \tag{12}$$

The result follows from (11), (12), (1) and Proposition 4.1. □

Proposition 4.3. *The function* $T(q)$ *is convex and decreasing.*

Proof. Let $K \in \mathcal{K}$. Pesin and Weiss (see [10]) using the fact that the pressure is analytic and considering the formulæ for its derivatives, showed that the function $T_K^*(q)$ is analytic, decreasing and convex. Since

$$T(q) = \sup\{T_K^*(q) : K \subset \Sigma \text{ compact invariant}\}$$

the desired properties follows. □

The next proposition has been proved in different settings, in particular see [10]. It relates the pointwise dimension to the Birkhoff sums.

Proposition 4.4. *If $x \in \Sigma$ then*

$$d_\mu(x) = \lim_{n\to\infty} \frac{\log \mu(C_{i_0\dots i_{n-1}}(x))}{\log \prod_{i=0}^{n-1} \big(\psi(\sigma^i x)\big)^{-1}} = \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \log \phi(\sigma^i(x))}{\sum_{i=0}^{n-1} \log\big(\psi(\sigma^i x)\big)^{-1}},$$

whenever one of the limits exists.

Proof. The proof follows from the Gibbs property of the measure μ and from Remark 4.1. □

Let $q > q^*$. Associated to the potential $\phi_q := \phi_{T(q),q} = -T(q)\log\psi + q\log\phi$ there exists a (normalised) Gibbs measure μ_q (see [19]).

Definition 4.3. *Let $q > q^*$ and consider*

$$\alpha(q) := -\frac{\int \log\phi d\mu_q}{\int \log\psi d\mu_q}.$$

Note that, since the system satisfies the BIP property, for $q > q^*$ the potential ϕ_q is strongly positive recurrent (see [16]). In particular the function $P_G(-t\log\psi + q\log\phi)$ is real analytic in each variable (when finite). Moreover $\log\psi \in L^1_{\mu_q}$ and $\log\phi \in L^1_{\mu_q}$. Using the formula for the derivative of the pressure (see [9]) we have, $-T'(q) = \alpha(q)$.

Remark 4.3. *Note that μ_q-almost every $x \in \Sigma$ is such that*

$$d_\mu(x) = \lim_{n\to\infty} -\frac{\sum_{i=0}^{n-1} \log\phi(\sigma^i(x))}{\sum_{i=0}^{n-1} \log\psi(\sigma^i(x))} = -\frac{\int \log\phi d\mu_q}{\int \log\psi d\mu_q} =: \alpha(q).$$

Let (K_n) be a sequence in $\mathcal{K}$, such that $T_{K_n}^*(q) := T_n^*(q)$ converges pointwise to $T(q)$. We prove that $T_n^*(q)$ converges infimally to $T(q)$.

Lemma 4.2. $T_n^*(q) \xrightarrow{\text{inf}} T(q)$.

Proof. Fix a positive integer n. The function $T_n^*(q)$ is decreasing in q. Thus,

$${}_\rho T_n^*(x) := \inf\{T_n^*(y) : |x-y| < \rho\} = T_n^*(x+\rho).$$

Since $T_n^*(q)$ converges pointwise to $T(q)$, we have

$$\lim_{n\to\infty} {}_\rho T_n^*(x) = T(x+\rho)$$

If $q > q^*$ then $T(q)$ is continuous and convex, so $\lim_{\rho\to 0} T(q+\rho) = T(q)$. Assume $q^* = 0$. If $q < 0$ then for ρ small enough we have $T(q+\rho) = \infty$, so $\lim_{\rho\to 0} T(q+\rho) = T(q)$. If $q = 0$ then the result follows because $T(0) = \lim_{q\to 0^+} T(q)$. Therefore,

$$T_n^*(q) \xrightarrow{\text{inf}} T(q).$$

□

Proof of Theorem 4.1. Denote by L the Fenchel transform of T. We first prove that if $q > q^*$ then $\dim_H(J_{\alpha(q)}) = T(q) + q\alpha(q)$. Recall that $\mu_q(J_{\alpha(q)}) = 1$ and that μ_q is a Gibbs measure. The arguments developed by Pesin and Weiss ([10] Lemma 2) apply and yield:

$$d_{\mu_q}(x) \leq T(q) + q\alpha(q),$$

for every $x \in J_{\alpha(q)}$. Therefore $\dim_H(J_{\alpha(q)}) \leq T(q) + q\alpha(q)$. Note that in order for the argument to work we just need $P_G(\phi_q) \leq 0$. In particular we have proved that $f_\mu \leq L$.

Let $f_{\mu,n}(\alpha)$ be the Fenchel transform of $T_n^*(q)$. We already proved that $T_n^*(q) \xrightarrow{\inf} T(q)$. By Wijsman's Theorem (see Theorem 2.1) we have

$$f_{\mu,n} \xrightarrow{\inf} L.$$

Note that $f_{\mu,n}$ is an increasing sequence and $f_{\mu,n}(\alpha) \leq \alpha$. Thus, there exists a function f_p which corresponds to the pointwise limit of $f_{\mu,n}$. By Wijsman's remark (see Remark 2.3) we have $L \leq f_p$. But by definition we have that $f_p \leq f_\mu$ that is

$$L \leq f_p \leq f_\mu,$$

therefore $L = f_\mu$.

If $q^* = 0$ and $\lim_{q\to 0^+} -T'(q) = \infty$ the result follows from the previous argument. If $q^* = 0$ and $\lim_{q\to 0^+} -T'(q) < \infty$ then for every $\alpha > \lim_{q\to 0^+} -T'(q)$ we have $f_\mu(\alpha) = \dim_H \mathcal{R}$. This concludes the proof. □

Remark 4.4. *It is possible to prove that $f_p = L$ just using the definition of infimal convergence and the fact that the sequence of concave functions $(f_{\mu,n})$ is monotonous.*

In the next two Theorems we prove that the behaviour of the multifractal spectrum for countable Markov shifts can be different from the finite state case (see Figures 2 and 3).

Proposition 4.5. *If $q^* = 0$ then the multifractal spectrum f_μ has unbounded domain.*

Proof. If $q^* = 0$ then $T(q) = \infty$ for every $q < 0$. Recall that

$$\lim_{n\to\infty} T_n(q) = T(q),$$

and that the function $T_n(\cdot)$ is an analytic function for every $n \in \mathbb{N}$.
Recall that, in this setting, when finite the pressure is analytic. Therefore $T(q)$ is analytic on (q^*, ∞). Assume first that the Hausdorff dimension of the system is finite, that is $T(0) < \infty$. Assume as well that the function $T(q)$ has bounded derivative for $q > q^*$, that is

$$-T'(0) = \lim_{q\to 0^+} -T'(q) < \infty.$$

The Mean Value Theorem implies that for every $\alpha > -T'(0)$ there exists $n \in \mathbb{N}$ and $q < 0$ such that $-T_n'(q) = \alpha$. That is $f_{\mu,n}(\alpha) > 0$ and therefore $f_\mu(\alpha) > 0$.
Assume now that,

$$-T'(0) = \lim_{q\to 0^+} -T'(q) = \infty.$$

Then, for every $\alpha > -T'(1)$ there exists $q \in (0,1)$ such that $-T'(q) = \alpha$. Therefore $f_\mu(\alpha) > 0$, which concludes the proof. □

Remark 4.5. *Note that when considering compact spaces the domain is always a bounded interval (see* [20]*).*

Remark 4.6. *Note that Proposition* 4.5 *follows from the Fenchel relation between* $T(q)$ *and* f_μ.

If the multifractal spectrum has unbounded domain then two possible behaviours can occur, as the following Proposition shows (see Figures 2 and 3).

Proposition 4.6. *Assume* $q^* = 0$,

1. *If* $\lim_{q \to 0^+} -T'(q) < \infty$ *then there exists* $\alpha(0)$ *such that for every* $\alpha > \alpha(0)$ *we have* $f_\mu(\alpha) = \dim_H(\mathcal{R})$.
2. *If* $\lim_{q \to 0^+} -T'(q) = \infty$ *then the multifractal spectrum is strictly increasing.*

Proof. First assume $T(0) < \infty$ and that

$$\alpha(0) = -T'(0) = \lim_{q \to 0^+} -T'(q) < \infty.$$

Since $q^* = 0$, for every $q < 0$ we have $T(q) = \infty$. In particular the multifractal spectrum has unbounded domain. Recall that

$$f_\mu(\alpha) = \inf_{-\infty \leq q \leq \infty} \{T(q) + \alpha q\} = \inf_{0 \leq q \leq \infty} \{T(q) + \alpha q\}.$$

Let $\alpha > -T'(0)$. The function $T(q)$ is convex and $-\alpha q + T(0)$ is a support line of $T(q)$. Hence, for every $q > 0$ we have

$$T(q) > -\alpha q + T(0).$$

In particular,

$$T(q) + \alpha q > T(0).$$

Therefore,

$$f_\mu(\alpha) = \inf_{0 \leq q \leq \infty} \{T(q) + \alpha q\} = T(0) = \dim_H(\mathcal{R}).$$

Assume that

$$-T'(0) = \lim_{q \to 0^+} -T'(q) = \infty.$$

Then,

$$f_\mu(\alpha) = \inf_{-\infty \leq q \leq \infty} \{T(q) + \alpha q\} = \inf_{0 \leq q \leq \infty} \{T(q) + \alpha q\} = T(q') - T(q')q',$$

where $q' > 0$ is such that $-T'(q') = \alpha$.

□

Remark 4.7. *Note that in case* (1) *of the previous Proposition the multifractal spectrum has a point of non analyticity. Recall that in the finite state case the multifractal spectrum is analytic (see* [10]*).*

Remark 4.8. *Note that in order for* $\alpha(0)$ *to be infinity we need* $-\int \log \phi d\mu_0 = \infty$. *Recall that* μ_0 *is a Gibbs measure for* $-\dim_H(\mathcal{R}) \log \psi$ *and that it does not depends on* $\log \phi$. *Therefore the integrability of* $\log \phi$ *depends on the relation between* $\psi^{-\dim_H(\mathcal{R})}$ *and* $\log \phi$.

In the next Proposition we provide sufficient conditions in order for the multifractal spectrum to have bounded or unbounded domain.

Proposition 4.7. *Assume* $\dim_H(\Sigma) < \infty$.

(1) *If for every* $x \in \Sigma$
$$(\phi(x))^{-1} \leq \psi(x)$$
then the multifractal spectrum of μ *has bounded domain.*

(2) *If for every* $q < 0$, $t > 0$ *and* $x \in \Sigma$
$$(\phi(x))^{q} > (\psi(x))^{t}$$
then the multifractal spectrum of μ *has unbounded domain. Moreover*

   (a) *If* $\lim_{q \to 0^+} -T'(q) < \infty$ *then there exists a point* $\alpha(0)$ *in the domain of* f_μ *such that for* $\alpha < \alpha(0)$ *the multifractal spectrum is strictly increasing and for* $\alpha > \alpha(0)$, $f_\mu(\alpha) = \dim_H(\mathcal{R})$.

   (b) *If* $\lim_{q \to 0^+} -T'(q) = \infty$ *then the multifractal spectrum is strictly increasing.*

Proof. Note that if $q^* = -\infty$ then f_μ has bounded domain. If $q < 0$ then from the condition given in part (1) of the Theorem we have,
$$-t \log \psi + q \log \phi \leq -(t+q) \log \psi.$$
Therefore,
$$P_G(-t \log \psi + q \log \phi) \leq P_G(-(t+q) \log \psi).$$
If $t > \big(\dim_H(\mathcal{R}) - q\big)$ then
$$P_G(-t \log \psi + q \log \phi) \leq 0.$$
Thus $q^* = -\infty$.
Note that if $q^* = 0$ then f_μ has unbounded domain. Recall that since the system satisfies the BIP property then $P_G(0) = \infty$. The condition given in part (2) of the Theorem ensures that for every $q < 0$ and for every $t \in \mathbb{R}$ we have
$$-t \log \psi + q \log \phi > 0.$$
Therefore $q^* = 0$. If $\lim_{q \to 0^+} -T'(q) < \infty$ and $q^* = 0$ then f_μ is concave, has unbounded domain and $f_\mu(-T'(0)) = \dim_H(\mathcal{R})$. Note that f_μ is bounded above by $\dim_H(\mathcal{R})$. Therefore, if $\alpha > -T'(0)$ then $f_\mu(\alpha) = \dim_H(\mathcal{R})$.
If $\lim_{q \to 0^+} -T'(q) = \infty$ and $q^* = 0$ then f_μ is concave, has unbounded domain and does not have a maximum. □

Example 4.1. *Let* Σ *be the full shift over the positive integers. Let* $\psi\big|_{C_n} = 2^n$ *and let* $\phi\big|_{C_n} = (n!)^{-s}$, *where* $s \in \mathbb{R}^+$ *is such that* $P_G(\log \phi) = 0$. *Let* $q < 0$. *Then for fixed* t *and* n *large enough,*
$$\phi^q\big|_{C_n} = (n!)^{s|q|} > 2^{nt} = \psi\big|_{C_n}.$$
That is, the multifractal spectrum for the equilibrium measure μ *corresponding to the potential* $\log \phi$, *has unbounded domain.*

Example 4.2. *Let* Σ *be the full shift over the positive integers. Let*
$$\psi\big|_{C_n} = 3^n \text{ and } \phi\big|_{C_n} = 2^{-n}.$$
Clearly $\phi^{-1} \leq \psi$ *therefore the multifractal spectrum for the equilibrium measure* μ *corresponding to the potential* $\log \phi$, *has bounded domain.*

If $\dim_H(\mathcal{R}) = \infty$ the situation is different as the following example shows,

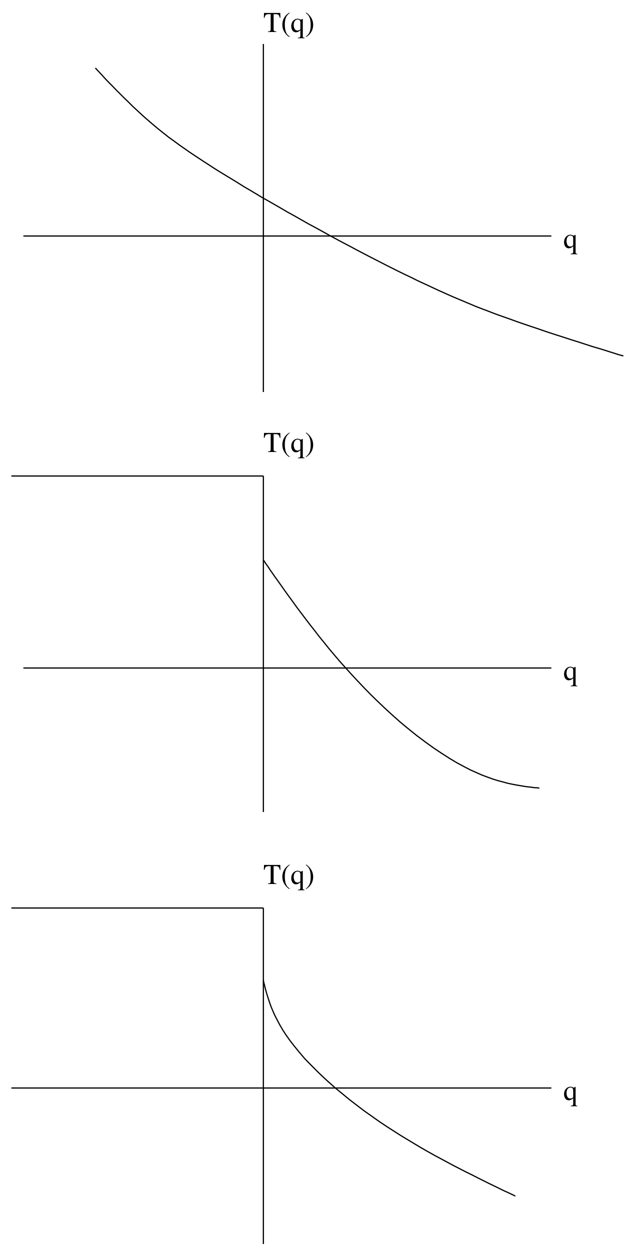

FIGURE 2. Possible behaviour of the function $T(q)$.

Example 4.3. *Assume Σ to satisfy the BIP property and let $\log\psi = \log\lambda$, where $\lambda > 1$. Since the system has infinite Gurevich entropy then $\dim_H(\mathcal{R}) = \infty$. If $\log\phi$ is a potential of zero pressure and μ is the corresponding Gibbs measure then the multifractal spectrum of μ*

(1) *Has unbounded domain*
(2) *Has unbounded image*

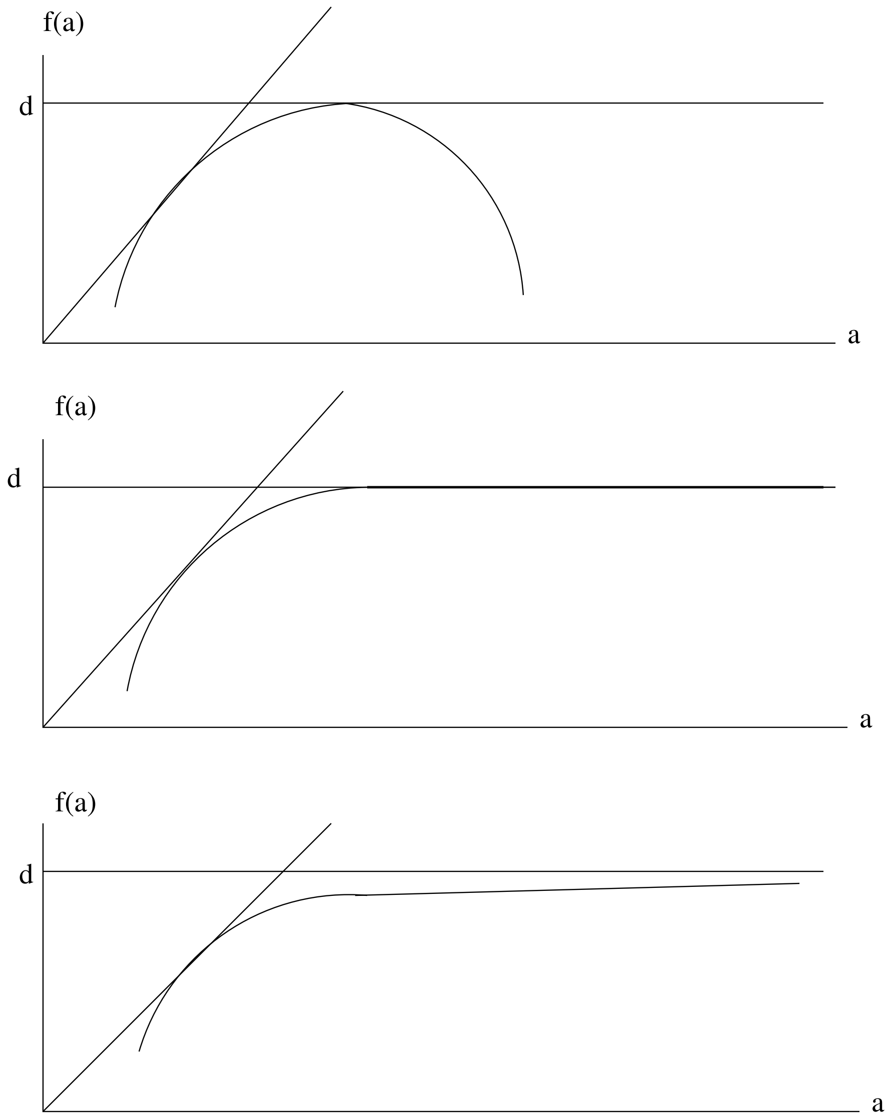

FIGURE 3. Possible behaviour of the multifractal spectrum.

(3) *Is an increasing concave function.*
(4) *If* $P_G(q^* \log \phi) < \infty$ *then there exists an associated equilibrium measure* μ_{q^*}*. We have*
   (a) *If* $\log \phi \in L^1_{\mu_{q^*}}$ *then there exists a point* $-T'(q^*)$ *such that for* $\alpha < -T'(q^*)$ *the multifractal spectrum is strictly concave. And for* $\alpha > -T'(q^*)$ *is linear with slope* q^**.*
   (b) *If* $\log \phi \notin L^1_{\mu_{q^*}}$ *then the multifractal spectrum is strictly concave and one of the asymptotes has slope* q^**.*
(5) *If* $P_G(q^* \log \phi) = \infty$ *then the multifractal spectrum is strictly concave and one of the asymptotes has slope* q^**.*

Let (Σ, σ) be the full-shift on a countable infinite alphabet. Consider the following potentials that illustrates the possibilities in this example. For the locally constant potential defined by $\log \phi_1\big|_{C_n} = -\log(n(n+1))$ we have $q^* = 1/2$.
For the potential $\log \phi_2(x) = -\log(x_0(\log 2x_0)^2)^r$, where $r > 1$ is such that $P_G(\log \phi_2) = 0$. The multifractal spectrum of the corresponding measure is strictly concave for $\alpha < -T'(1)$ and for $\alpha > -T'(1)$ is linear with slope one.
The potential $\log \phi_3\big|_{C_n} = \log n^{-nr}$, where r is a positive number such that $P_G(\log \phi_3) = 0$, is such that the multifractal spectrum of the corresponding measure is strictly concave.

Remark 4.9. *Note that $\dim_H(\mathcal{R}) = \sup\{\dim_H(K) : K \in \mathcal{K}\}$. Let $J'_K := \{x \in K :$ the limit $d_\mu(x)$ does not exists$\}$ be the irregular set of K. Barreira and Schmeling proved in [1] that the set J'_K has full Hausdorff dimension. Therefore,*

$$\dim_H(\mathcal{R}) = \sup\{\dim_H(J'_K) : K \in \mathcal{K}\}.$$

That is, the irregular set J' has the same Hausdorff dimension as the recurrent set.

5. Multifractal Analysis, the non BIP case

This section will be devoted to the case in which the graph G does not satisfy the BIP property. Therefore locally Hölder potentials do not have corresponding Gibbs measures. As in the previous section we will assume the potential $\log \phi$ to be locally Hölder continuous of zero pressure not cohomologous to the metric potential $\log \psi$. We also assume that it is positive recurrent. Our approach is based on the study of the induced system Σ_1 (see section 2). Note that the induced system satisfies the BIP property. Therefore, the results of the previous sections (with some modifications) can be translated to this setting.

Definition 5.1. *Let $T_i(q)$ be defined implicitly by.*

$$P_G(-T_i(q)\overline{\log \psi} + q\overline{\log \phi}) = 0, \tag{13}$$

provided such number exists.

Denote by $\overline{\phi_q} := -T_i(q)\overline{\log \psi} + q\overline{\log \phi}$ and by μ_q the associated Gibbs measure (recall that this potential is defined on the induced system).

Remark 5.1. *Three conditions will be required on the potentials. Assume that there exists an interval I of values of q for which:*

1. *The function $T_i(q)$ is well defined (equation (13) has a root.)*
2. *The potential $-T_i(q)\log \psi + q \log \phi$ on the original system is positive recurrent.*
3. *The potentials $\overline{\log \psi}$, $\overline{\log \phi}$ are integrable with respect to Gibbs measure associated to $-T_i(q)\overline{\log \psi} + q\overline{\log \phi}$.*

Proposition 4.1 is still valid. So the function $T_i(q)$ is convex and decreasing. Denote by $f_{\mu,i}$ the Fenchel transform of $T_i(q)$. As proved in the previous section, this is the multifractal spectrum of the Gibbs measure μ_i associated to $\overline{\log \phi}$.

Remark 5.2. *The potential $\overline{\phi_q}$ has zero pressure and is positive recurrent. This implies that the potential on the original system, $-T_i(q)\log \psi + q \log \phi$ has zero pressure and is recurrent (see [16]). Therefore, if $q \in I$ then the functions $T_i(q)$ defined on the induced system coincides with the function $T(q)$ defined in the original system.*

Denote by f_μ the Fenchel transform of $T(q)$ and by $J_{\alpha,i}$ the level sets of the induced system.

Theorem 5.1. *If $q \in I$ then* $\dim_H(J_{\alpha(q)}) = T(q) + q(-T'(q))$.

Proof. The lower bound $\dim_H(J_{\alpha(q)}) \geq T(q) + q(-T'(q))$ can be obtained using the same arguments (infimal convergence) as in the proof of Theorem 4.1.
In order to obtain the upper bound, note that $J_{\alpha(q)} \subset \pi(J_{\alpha(q),i})$. Since we have that $\dim_H(J_{\alpha(q),i}) = T(q) + q(-T'(q))$ and the projection π is a Lipschitz function we have,

$$\dim_H(J_{\alpha(q)}) \leq \dim_H(\pi(J_{\alpha(q),i})) \leq T(q) + q(-T'(q)). \tag{14}$$

□

Remark 5.3. *As in the previous section, the domain of the multifractal spectrum f_μ can be unbounded. Moreover, assume $\dim_H(\mathcal{R}) = d < \infty$ and that f_μ has unbounded domain. Since (f_μ, T) form a Fenchel pair, we have two possible behaviours. Either f_μ is strictly increasing and concave, or there exists a point α^* such that for $\alpha < \alpha^*$ the multifractal spectrum is strictly increasing and strictly concave and for $\alpha > \alpha^*$ we have $f_\mu(\alpha) = d$.*

In the next theorem we give sufficient conditions in order for the multifractal spectrum to have unbounded domain. Before stating it we need the following definitions. Let $\mathcal{A}_n := \{C_{1i_1\dots i_n} : i_j \neq 1, j \in \{1, \dots, n\}, t_{i_n,1} = 1\}$, where $1i_1 \dots i_n$ are admissible words. Denote by $\Sigma_{\mathcal{A}_n}$ and $\Sigma_{\cup_{i=1}^n \mathcal{A}_i}$ the full-shift over the alphabet $\mathcal{A}_n$ and $\cup_{i=1}^n \mathcal{A}_i$ respectively.

Theorem 5.2. *Let (b_n) be an increasing sequence of real numbers with $b_n > 1$. Assume that for every $x \in [\varphi_1 = n]$ we have $\overline{\log \psi(x)} < \log b_n$. If for every $q < 0$*

$$\lim_{n \to \infty} \frac{P_{\Sigma_{\cup_{i=1}^n \mathcal{A}_i}}(q\overline{\log \phi})}{\log b_n} = \infty, \tag{15}$$

then the multifractal spectrum of μ has unbounded domain.

Proof. Note that,

$$0 = P_{\Sigma_{\cup_{i=1}^n \mathcal{A}_i}}(-T_n(q)\overline{\log \psi} + q\overline{\log \phi}) \geq -T_n(q) \log b_n + P_{\Sigma_{\cup_{i=1}^n \mathcal{A}_i}}(q\overline{\log \phi}).$$

Therefore,

$$T_n(q) \geq \frac{P_{\Sigma_{\cup_{i=1}^n \mathcal{A}_i}}(q\overline{\log \phi})}{\log b_n}.$$

From equation (15) we deduce that for negative values of q

$$\lim_{n \to \infty} T_n(q) = T(q) = \infty.$$

Recall that for every positive integer n the function $T_n(q)$ is analytic. Applying the Mean Value Theorem, we have that for every $\alpha > \alpha(0)$ there exists n and $q_n < 0$ such that

$$-T_n'(q_n) = \alpha,$$

therefore $\dim_H(J_{(\alpha,i)}) > 0$. This implies that for every $\alpha > \alpha(0)$ the multifractal spectrum $f_{\mu_i}(\alpha) > 0$. Hence it has unbounded domain.

□

Remark 5.4. *Note that if the system has finite entropy then the sequence (b_n) is always well defined.*

Remark 5.5. *Let (a_n) be a sequence of positive numbers satisfying*

$$\lim_{n\to\infty} \frac{\log a_n}{\log b_n} = -\infty.$$

If there exists $n^* \in \mathbb{N}$ such that for every $n > n^*$ there exits a periodic point $x_n \in [\varphi_1 = n]$ such that

$$\prod_{i=0}^{n-1} \phi(\sigma^i(x_n)) \leq a_n \tag{16}$$

then, from the variational principle, for every $q < 0$

$$\frac{P_{\Sigma_{\mathcal{A}_n}}(q\overline{\log \phi})}{\log b_n} \geq \frac{q \log a_n}{\log b_n}.$$

From equation (5.5) we obtain that $T(q) = \infty$. Hence the multifractal spectrum of μ has unbounded domain.

Remark 5.6. *Assume the system to have finite entropy $\log h$. Recall that the induced measure μ_i on C_1 is Gibbs and let $C_{1i_1\dots i_n} \in \mathcal{A}_n$. If for every $x \in [\varphi_1 = n]$ we have*

$$\frac{P_{\Sigma_{\mathcal{A}_n}}(q\overline{\log \phi})}{\log b_n} \geq \frac{q \log a_n}{\log b_n},$$

where $(a_n), (b_n)$ are as in Remark 5.5. Then there exists a constant C that does not depends on n or on the cylinder, such that

$$\mu_i(C_{1i_1\dots i_n}) \leq C a_n.$$

Since the system has finite entropy $\log h$ the number of cylinders of length n is bounded above by h^n (see [6] p.213). Therefore

$$\mu_i([\varphi_1(x) = n]) \leq C h^n a_n.$$

This allows us to estimate the tail

$$\mu([\varphi_1 > n]) \leq \mu(C_1) C \sum_{i=n+1}^{\infty} h^i a_i.$$

Example 5.1. *The renewal graph G is defined over the positive integers by the transition matrix $T = (t_{ij})$ whose entries t_{11}, t_{1i} and $t_{i,i-1}$ for $i = 2, 3, \dots$ are one, and the rest of the entries are equal to zero. This is a topologically mixing graph with finite entropy (actually $\log 2$) and is positive recurrent.*
The combinatorial structure of the induced system on the cylinder C_1 is simple. For every $n \geq 1$ we have that $\mathcal{A}_n = \{C_{1n(n-1)\dots 32}\}$. A generalisation of this graph arises naturally when studying tower extensions. Let (d_i) be an increasing sequence of positive integers. The transition matrix $T = (t_{ij})$ associated to this kind of graphs has entries $t_{11}, t_{i,i-1}, t_{1,d_i}$ equal to one. The rest of the entries are equal to zero. Depending on d_n the graph is transient or recurrent. If it is recurrent, that is

$$\sum_{n\geq 1} h^{-n} \sum_{\sigma^n x = x} 1_{[1]}(x) = \infty,$$

where $P_G(0) = \log h$ is the entropy of the system, then the graph is positive recurrent. In fact, note that

$$\sum_{n \geq 1} \frac{n}{h^n} \sum_{\sigma^n x = x} 1_{[\varphi_1 = n]} \leq \sum_{n \geq 1} \frac{n}{h^n}$$

which converges because $h > 1$ (the system has positive entropy). Hence there exists a probability measure of maximal entropy.
Consider the case in which $d_n = 2n + 1$. Let $\log \psi = \log \lambda$, where $\lambda > 1$. The potential is defined by

$$\log \phi(x) = \begin{cases} \log n^{-r} & \text{if } x \in C_n \text{ and } n > 1, \\ 1 & \text{if } x \in C_1. \end{cases} \tag{17}$$

The constant r is such that $P_G(\log \phi) = 0$. In this case the multifractal spectrum of the equilibrium measure μ corresponding to $\log \phi$ has unbounded domain. And since

$$\sum_{n=1}^{\infty} \frac{r \log(2n+1)!}{\lambda^{nT(0)}} < \infty$$

there exists a point $\alpha(0)$ such that for all $\alpha > \alpha(0)$ the multifractal spectrum is constant.

Example 5.2 (Nakaishi)**.** *In [8] Nakaishi studied the multifractal spectrum of the measure of maximal entropy of maps of the interval with a parabolic fixed point. He proved that it has unbounded domain. His example can be understood in our setting as follows. Let G be the renewal graph and $\log \phi(x) = \log 2^{-n}$ for $x \in C_n$. The metric considered is such that $\log \psi\big|_{C_n} = \log n$. Since,*

$$\lim_{n \to \infty} \frac{-n \log 2}{\log n} = -\infty \tag{18}$$

the multifractal spectrum has unbounded domain.

The following example can be understood as a model for a piecewise linear geometrical construction.

Example 5.3. *Let (λ_i) be an increasing sequence with $\lambda_i > 1$. Assume $\log \psi\big|_{C_i} = \lambda_i$ and that $diam(C_{i_0 \dots i_n}) = \prod_{j=0}^{n} \lambda_{i_j}^{-1}$. Let G be a positive recurrent graph of finite Gurevich entropy and let $\log \phi$ be a positive recurrent, locally Hölder continuous potential of zero pressure. Denote by μ the corresponding equilibrium measure. Let*

$$M_n := \min\{diam(C_{i_0 \dots i_n}) : C_{i_0 \dots i_n} \in \mathcal{A}_n\}.$$

Let (a_n) be a sequence of positive numbers such that

$$\lim_{n \to \infty} \frac{\log a_n}{\log M_n} = -\infty.$$

If there exists n^* such that for every $n > n^*$ and for every $x \in [\varphi_1 = n]$

$$\prod_{i=0}^{n-1} \phi(\sigma^i(x)) \leq a_n$$

then the multifractal spectrum of μ is continuous, concave and has unbounded domain.

(1) *If the potential $-T(0)\log\psi$ is positive recurrent we denote by μ_m the corresponding equilibrium measure. One of the following alternatives occurs:*
   (a) *If $\log\phi \in L^1_{\mu_m}$ then there exists a point $\alpha(0)$ in the domain of f_μ such that for $\alpha < \alpha(0)$ the multifractal spectrum is increasing and for $\alpha > \alpha(0)$ is constant.*
   (b) *If $\log\phi \notin L^1_{\mu_m}$ then the multifractal spectrum is strictly increasing.*

(2) *If the potential $-T(0)\log\psi$ is null recurrent then the multifractal spectrum is strictly increasing.*

The next Theorem describes the multifractal spectrum for small values of α, when there exists $q' \geq 1$ such that the function $T(q)$ is linear for $q > q'$. Recall that those are the level sets where the measure is highly concentrated (see Figure 4)

Definition 5.2.

$$\alpha(q') := \lim_{q \to (q')^-} -T'(q).$$

Theorem 5.3. *Let $q' \geq 1$. Assume the function $T(q)$ to be strictly convex for $q \in (0, q')$ and linear with slope $-A < 0$ for $q > q'$. Then the multifractal spectrum of μ is linear on the interval $(A, \alpha(q'))$ with slope q'.*

Proof. Let $\alpha \in (A, \alpha(q'))$ we claim that the Fenchel transform of $T(q)$ is given by,

$$\inf_{-\infty<q<\infty} \{T(q) + \alpha q\} = T(q') + q'\alpha.$$

Note that if $q \in (0, q')$ then

$$\frac{T(q') - T(q)}{q' - q} < -\alpha. \tag{19}$$

In fact, assume by way of contradiction that

$$\frac{T(q') - T(q)}{q' - q} > -\alpha.$$

The Mean Value Theorem implies that there exists $q_1 \in (q, q')$ such that $T'(q_1) > -\alpha$. This contradiction leads to the claim. In fact, from equation (19) we obtain,

$$T(q') + \alpha q' < T(q) + \alpha q.$$

Assume now that $q > q'$. Let $C > 0$ be such that

$$T(q) = -Aq + C.$$

Note that $\alpha - A > 0$. Therefore, since $q' < q$ we have,

$$q'(\alpha - A) + C \leq q(\alpha - A) + C$$

that is

$$T(q') + \alpha q' \leq T(q) + \alpha q.$$

This implies,

$$f_\mu(\alpha) = \inf_{-\infty<q<\infty} \{T(q) + \alpha q\} = T(q') + q'\alpha,$$

as required.

□

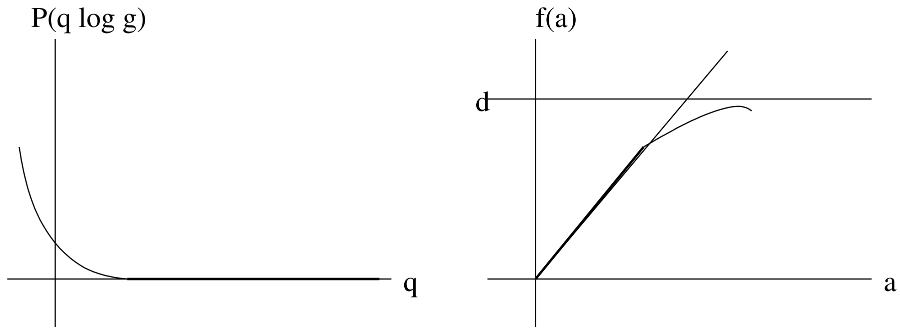

FIGURE 4. Behaviour of the multifractal spectrum of concentrated measures.

Proposition 5.1. *If the potential* $\log\phi$ *is such that* $P_G(q\log\phi) = 0$ *for every* $q \geq 1$ *then the multifractal spectrum is given by*

$$f_\mu(\alpha) = \alpha,$$

for $\alpha \in (0, \alpha(1))$

Proof. Note that if $q \geq 1$ then $P_G(q\log\phi) = 0$, therefore $T(q) \leq 0$. We prove that it is equal to zero. Assume by way of contradiction that $T(q) < 0$. Then the function,

$$t \to P_G(-t\log\psi + q\log\phi)$$

is equal to zero for $t \in (T(q), 0)$ and negative for $t > \dim_H(\mathcal{R})$. This contradicts the convexity of the function. Therefore, if $q \geq 1$ then $T(q) = 0$. The result follows from Theorem 5.3. □

Example 5.4. *Let G be the renewal graph. Sarig proved in* [15] *that if* $\log\phi$ *is a locally Hölder potential then there exists a critical value* q_c *(it might be infinity) such that,*

1. *The potential* $q\log\phi$ *is strongly positive recurrent for* $q \in (0, q_c)$ *and transient for* $q > q_c$.
2. *The pressure function* $P_G(q\log\phi)$ *is real analytic in* $(0, q_c)$ *and linear in* (q_c, ∞). *It is continuous but not analytic at* q_c.
3. *The potential* $q_c\log\phi$ *can be positive recurrent, null recurrent or transient.*

This framework provides a family of examples that illustrate Proposition 5.1.

We finally give an example where the multifractal spectrum has bounded domain.

Example 5.5 (Gauss map and the Renewal Graph)**.** *The Gauss map* $G : [0,1] \to [0,1]$ *is defined by*

$$G(x) := \frac{1}{x} - \left[\frac{1}{x}\right],$$

where $[x]$ *is the integer part of* x. *Denote by F the restriction of the Gauss map G to the renewal graph, that is,*

$$F(x) := \begin{cases} G(x) & \text{If } x \in I_{ij} \text{ and } t_{ij} = 1, \\ \text{Not defined} & \text{Any other case.} \end{cases} \tag{20}$$

Where $I_{ij} := ((i+1)^{-1}, i^{-1}) \cap G^{-1}((j+1)^{-1}, j^{-1})$. *Denote by* f *the induced system on the set* $(\frac{1}{2}, 1)$. *The combinatorics of the induced graph is simple, there is just one cylinder of length* n, *which is* $C_{1n(n-1)\ldots 2}$. *Note that if* $x \in (\frac{1}{n+1}, \frac{1}{n})$ *then,*

$$\Big(\frac{n-1}{n^2-n+1}\Big)^2 \leq |F'(x)|^{-1} \leq \Big(\frac{n}{n^2+1}\Big)^2.$$

If $x \in (\frac{1}{2}, 1)$ *and* $F(x) \in (\frac{1}{n+1}, \frac{1}{n})$ *then* $x \in (\frac{n^2+1}{n^2+n+1}, \frac{n^2-n+1}{n^2})$. *Using the combinatorics of the induced system and the above estimates we can bound the derivatives of the induced system* f. *That enable us to bound the pressure function,*

$$\log\Big((1/4)^t + \sum_{n=2}^{\infty}\Big(\Big(\frac{n^2+1}{n^2+n+1}\Big)^2 \prod_{i=2}^{n}\Big(\frac{i-1}{i^2-i+1}\Big)^2\Big)^t\Big) \leq P_G(-t\log|f'|) \leq \log\Big((4/9)^t + \sum_{n=2}^{\infty}\Big(\Big(\frac{n^2-n+1}{n^2}\Big)^2 \prod_{i=2}^{n}\Big(\frac{i}{i^2+1}\Big)^2\Big)^t\Big).$$

Computer estimates of the root d *of the equation* $P_G(-t\log|f'|) = 0$ *and Theorem* 3.1 *yield to,*

$$\dim_H(\mathcal{R}) = d \in (0.4, 0.6).$$

Note that $\mathcal{R}$ *is the set of recurrent points such that their continued fraction expansion is determined by the renewal graph.*
The potential $-d\log|F'|$ *has zero pressure and is recurrent (see Remark* 5.2*). Using the bounds on the derivative of* F *we obtain,*

$$\sum_{n\geq 1} n \sum_{F^n x = x} \exp(\sum_{i=0}^{n-1}(-d\log|F'(F^i x)|)1_{[\varphi_1 = n]}(x) \leq C + \sum_{n\geq 1}\Big(\frac{1}{n!}\Big)^{2d},$$

where C *is a constant. This series converges, therefore the potential* $-d\log|F'|$ *is positive recurrent and there exists a (finite) equilibrium measure* m. *Since the Gurevich entropy is* $\log 2$ *and* $0.4 \leq d$, *using the variational principle (see* [21],[15]*) we can estimate the Lyapunov exponent,*

$$\lambda_m := \int \log|F'| dm \leq \frac{5}{2}\log 2 \sim 1.732.$$

Note that λ_m *measures the exponential speed of approximation of a number by its approximants. Recall that Lebesgue almost everywhere the Lyapunov exponent is* $\frac{\pi^2}{6\log 2} \sim 2.373$.
Let $\log\phi = -\log 2$ *be a constant potential. It has pressure zero, is positive recurrent and the corresponding equilibrium measure is the* measure of maximal entropy. *Note that* $q^* = -\infty$. *Therefore the multifractal spectrum* f_μ, *is concave, has a unique maximum at* $-T'(0)$ *and has bounded domain.*

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