The Bowen Formula : Dimension Theory and Thermodynamic Formalism

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Chapter 1

Introduction

The purpose of these notes is to offer glimpses of the relation between ergodic theory and the dimension theory of dynamical systems. Billinglsey (see [8] and references therein) was one of the first who formally established this relation. Ever since, the connection between the two theories has become stronger and deeper.

The mathematical core of the theory of dynamical systems is the study of the global orbit structure of maps and flows. In order to analyse a system, structure on the phase space and restrictions on the map (or the flow) are required. Ergodic Theory is the study of dynamical systems for which the phase space is a measure space and the map (or the flow) preserves a probability measure (see Chapter 3). A large amount of work has been developed over the last years in this area. Not only because the techniques used in the field have proved to be extremely useful to describe the long term behaviour of orbits for a large class of systems, but also, because these techniques have been successfully applied in several other branches of mathematics, most notably in number theory (see for exmple [18]).

An important tool used in ergodic theory is the associated thermodynamic formalism. This is a set of ideas and techniques which derive from statistical mechanics. It can be though of as the study of certain procedures for the choice of invariant measures. Let us stress that a large class of interesting dynamical systems have many invariant measure (see Example 3.4), hence the problem of choosing relevant ones. The main object on the field is the so called topological pressure.

The dimension theory of dynamical systems has remarkably flourished over the last fifteen years. The main goal of the field is to compute the *size* of dynamically relevant subsets of the phase space. For example, sets where the complicated dynamics is concentrated (*repellers* or *attractors*). Usually, the geometry of these sets is rather complicated. That is why there are several notions of *size* that can be used. One could say that a set is *large* if it contains a great deal of disorder on it. Formally, one would say that the dynamical system restricted to that subset has large *entropy* (see Chapter 3). Another way of measuring the size of a set is using geometrical tools. In order to do so, finer notions of dimension are required. In these notes we will discuss one of the first of such notions, which was introduced around 1919, the so called *Hausdorff dimension* (see Chapter 2).

In these notes we will consider a rather simple class of dynamical systems (cookie-cutters) that, nevertheless, exhibits a very complicated orbit structure. The subset of the phase space that is dynamically relevant for these systems is a Cantor set (for the definition and properties see Chapter 2). The Bowen formula (also known as dimension formula) connects the ergodic theory of the cookie cutter with the Hasudorff dimension of the corresponding Cantor set. Indeed, the Hausdorff dimension of the Cantor set corresponds to the unique zero a a certain pressure function (see Chapter 4). This remarkable connection was first established by Rufus Bowen in 1979 [9].

These notes are divided in three Chapters. In the first we introduce the definition, properties and some methods to calculate the Hausdorff dimension. Also the Cantor sets are defined and studied. The second Chapter is devoted to the study of Ergodic Theory. Some of the important definitions and ideas on thermodynamic formalism are discussed. Our main example, that is the cookie cutter, is studied in more detail. Finally, in the third Chapter we establish the connection between the two previous ones by proving the Bowen formula. We also include several applications of the ideas developed in these notes to the study of certain number theoretically defined subsets of the unit interval.

Maybe it would be easier for the reader when first skimming through the text to consider piecewise linear cookie cutters. The statements and the proofs become much easier due to the fact that, in such case, we have an explicit closed form for the topological pressure.

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Chapter 2

Hasudorff Dimension

2.1 Introduction

This section is devoted to give an appropriate notion of *dimension* which generilise the more intuitive notions used in classical geometry. There are certain sets for which the intuitive idea of dimension is fairly clear. For example, a point has dimension zero, a line has dimension one, the plane has dimension two and so on. The classical notion of dimension that assigns these numbers to these geometrical objects is the so called *topological dimension*. Note that the topological dimension is always an integer number. There are sets that naturally appear in mathematics that have a more complicated structure and, in a sense that we would make precise, are "larger" than a point and "smaller" that an interval. The canonical example is the $1/3-$Cantor set (see Section 2.3), which arises naturally as a dynamically defined set. The intuitive idea of topological dimension can be traced back, at least, to Poincaré. But it was only around 1922 that Urysohn [44] and Menger [33] formalised this notion.

In this section we study the Hausdorff dimension. In a sense, this is a finer notion of dimension that allows us to distinguish in between sets having the same topological dimension. For instance, a point has Hausdorff dimension equal to zero, whereas the $1/3-$Cantor set has Hasudorff dimension equal to $\log 2/\log 3$. Both sets have zero topological dimension. Note that the notion of Hausdorff dimension assigns a non negative real number to each set (not necessarily a natural number). This notion of dimension is not invariant under homeomorphism, that is why initially it got less attention than the topological dimension (even though historically the formal definition came before).

A way to think of the Hausdroff dimension is as the number for which there is an "equilibrium" between the number of balls that are needed to cover the set and the diameter of these balls.

This notion of dimension is named after Felix Hausdroff, a Polish mathematician born in 1862. He introduced this definition, generalising results of Carathéodory, in his 1919 paper [21]. Being of jewish origin, in 1935 he was forced out university. The situation became more difficult for him in the years to come. The 26th of January of 1942, he committed suicide together with his wife.

2.2 Definition of Hausdroff Dimension

The definition of Hausdorff dimension is done in several steps.

2.2.1 The Hausdorff Measure

Let $X \subset \mathbb{R}^n$, the *diameter* of the set X is defined by

$$|X| = \sup\left\{|x-y| : x, y \in X\right\},$$

where $|x|$ denotes the norm of the vector $x \in X$.

Definition 2.1. *A countable collection of subsets $U_i \subset \mathbb{R}^n$ is called a $\delta-$*cover *of X, if $X \subset \cup_{i=1}^{\infty} U_i$ and for each $i \in \mathbb{N}$ we have that $|U_i| \leq \delta$.*

Let $s > 0$ and $\delta > 0$, we define

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ a } \delta\text{-cover of } X\right\}. \tag{2.1}$$

That is, we are minimising the sum of the s-th powers of the diameters of the sets belonging to $\delta-$covers of X. Note that as δ decreases the number of $\delta-$covers of X also decreases. Thus the infimum $\mathcal{H}^s_\delta(X)$ increases. Therefore, the following limit exits

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

Note, though, that this limit can be infinite.

Definition 2.2. *The $s-$*Hausdorff measure *of the set X is the number $\mathcal{H}^s(X)$.*

The set function $\mathcal{H}^s(X)$ is actually a measure, it satisifes,

Proposition 2.1. *Let $E, F \subset \mathbb{R}^n$ and $s > 0$ then*

1. *$\mathcal{H}^s(\emptyset) = 0$;*

2. *If $E \subset F$ then $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$.*

3. *If $\{E_i\}$ is a disjoint collection of Borel sets then*

$$\mathcal{H}^s(\cup_{i=0}^{\infty} E_i) = \sum_{i=0}^{\infty} \mathcal{H}^s(E_i).$$

Proposition 2.2. *Let $X \subset \mathbb{R}^n$ and $\lambda > 0$, consider the set*

$$\lambda X = \{\lambda x : x \in X\},$$

then

$$\mathcal{H}^s(\lambda X) = \lambda^s \mathcal{H}^s(X).$$

Proof. Note that if $\{U_i\}$ is a $\delta-$cover of X then $\{\lambda U_i\}$ is a $\lambda\delta-$cover of λX, therefore

$$\mathcal{H}^s_{\lambda\delta}(\lambda X) \leq \sum_{i=0}^{\infty} |\lambda U_i|^s = \lambda^s \sum_{i=0}^{\infty} |U_i|^s. \tag{2.2}$$

This holds for any δ-cover. Hence, letting $\delta \to 0$ we obtain $\mathcal{H}^s(\lambda X) \leq \lambda^s \mathcal{H}^s(X)$. In order to obtain the reverse inequality just replace λ by $1/\lambda$ and X by λX. □

More generally,

Proposition 2.3. *Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a Hölder map, that is, there exists constants $\alpha > 0$ and $C > 0$ such that*

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

then

$$\mathcal{H}^{s/\alpha}(f(X)) = C^{s/\alpha} \mathcal{H}^s(X).$$

Proof. Note that $|f(X \cap U_i)| \leq C|U_i|^\alpha$. Hence, if $\{U_i\}$ is a $\delta-$cover of X then $\{f(X \cap U_i)\}$ is an $\epsilon-$cover of $f(X)$, with $\epsilon = C\delta^\alpha$. Thus

$$\sum_{i=0}^{\infty} |f(X \cap U_i)|^{s/\alpha} \leq C^{s/\alpha} \sum_{i=0}^{\infty} |U_i|^s.$$

Therefore, $\mathcal{H}^{s/\alpha}_\epsilon(f(X)) = C^{s/\alpha} \mathcal{H}^s_\delta(X)$. Since $\delta \to 0$ implies that $\epsilon \to 0$ we obtain the desired result. □

Remark 2.1. *Let $n \in \mathbb{N}$ be an even number then we have the following relation between the n-dimensional Lebesgue measure and the $n-$Hausdorff measure:*

$$\mathcal{H}^n(A) = \frac{2^n(\frac{n}{2})!}{\pi^{\frac{n}{2}}} Leb_n(A),$$

where Leb_n denotes the n-dimensional Lebesgue measure. There is an analogous statement for $n \in \mathbb{N}$ odd.

2.2.2 The Hausdorff Dimension.

Let us start by studying how does the function $\mathcal{H}^s(X)$ changes with the parameter $s > 0$. Note that if $t > s$ and $\{U_i\}$ is a $\delta-$cover of X we have

$$\sum_{i=0}^{\infty} |U_i|^t \leq \delta^{t-s} \sum_{i=0}^{\infty} |U_i|^s.$$

Hence, $\mathcal{H}^t_\delta(X) \leq \delta^{t-s}\mathcal{H}^s_\delta(X)$. Letting $\delta \to 0$ we see that if $\mathcal{H}^s(X) < \infty$ then $\mathcal{H}^t(X) = 0$ for $t > s$. Therefore, there is a critical parameter at which the function $s \to \mathcal{H}^s(X)$ changes its value from infinity to zero.

Definition 2.3. *The* Hausdorff dimension *of the set X is defined by*

$$\dim_H X = \inf \{s > 0 : \mathcal{H}^s(X) = 0\}.$$

Note that $\mathcal{H}^s(X) = \infty$ if $s < \dim_H(X)$ and $\mathcal{H}^s(X) = 0$ if $s > \dim_H(X)$.

Proposition 2.4. *The Hasudorff dimension satisfies the following properties*

1. *If $O \subset \mathbb{R}^n$ is an open set then* $\dim_H(O) = n$.
2. *If $M \subset \mathbb{R}^n$ is a smooth $m-$dimensional sub-manifold then* $\dim_H(M) = m$.
3. *If $E \subset F$ then* $\dim_H E \leq \dim_H F$.
4. *If $\{A_i\}$ is a countable sequences of sets then*

$$\dim_H \left(\cup_{i=0}^{\infty} A_i\right) = \sup \{\dim_H A_i : i \in \mathbb{N}_0\}.$$

5. *If the set $A \subset \mathbb{R}^n$ is countable then* $\dim_H(A) = 0$.

Proposition 2.5. *Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ be $\alpha-$Hölder function. then*

$$\dim_H(f(X)) \leq \frac{1}{\alpha} \dim_H(X).$$

Proof. If $\dim_H(X) < s$ then $\mathcal{H}^{s/\alpha}(f(X)) \leq C^{s/\alpha}\mathcal{H}^{s/\alpha}(X) = 0$, therefore

$$\dim_H(f(X)) \leq \frac{s}{\alpha}$$

for every $\dim_H(X) < s$. $\square$

A $\alpha-$Hölder function with constant $\alpha = 1$ is called *Lipschitz* function. Hausdorff dimension is preserved by bi-Lipschitz functions.

Remark 2.2. *Let $f : X \to X$ be a Lipschitz bijection with Lipschitz inverse, then $\dim_H(X) = \dim_H(f(X))$. Let us stress that, since the definition of Hausdorff dimension depends on the metric of the space, the natural class of maps for which it is preserved is the isometries. Fortunately, as we just have seen, it is preserved by Lipschitz maps, which form a larger class.*

2.3 The Cantor set

In this subsection we will study a special subset of the unit interval called the $1/3-$Cantor set. Sets with similar topological properties will appear as dynamically defined sets.

The Cantor set is a closed subset of the unit interval $[0, 1]$ which is obtained in the following way: remove from $[0, 1]$ the interval $(1/3, 2/3)$. After that, remove the middle third interval of each of the remaining intervals $[0, 1/3], [2/3, 1]$. The set that is left is $E_3 := [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Remove now the the middle third interval of each of these intervals. Repeat this process inductively. The set of points K that were not removed is called *Cantor* set. This set has several topological properties (see, for example, [28, Capítulo V]):

1. The set K is compact.
2. It does not contain intervals.
3. All of its points are accumulation points.
4. It is uncountable.

We are interested in its metric properties, in particular in determining its Hausdorff dimension.

Lemma 2.1. *Let K be the middle third Cantor set, then $\dim_H(K) = \frac{\log 2}{\log 3}$.*

Proof. We call the intervals of length 3^{-k} that make up the sets E_k in the construction of K *basic sets*. The covering $\{U_i\}$ of K consisting of the 2^k intervals of E_k of length 3^{-k} gives that

$$\mathcal{H}^s_{3^{-k}}(K) \leq \sum_{i=0}^{\infty} |U_i|^s = 2^k 3^{-ks}.$$

If $s = \frac{\log 2}{\log 3}$ then $\mathcal{H}^s_{3^{-k}}(K) \leq 1$. Therefore,

$$\lim_{k\to\infty} \mathcal{H}^s_{3^{-k}}(K) \leq 1.$$

That is $\mathcal{H}^s(K) \leq 1$. Now we prove the lower bound. Let $\{U_i\}$ be a finite collection of closed intervals that form a cover of K. For each U_i let k be the integer such that

$$3^{-(k+1)} \leq |U_i| \leq 3^{-k}.$$

Then U_i can intersect at most one basic interval of E_k. If $j \geq k$ then, by construction, U_i intersects at most $2^{j-k} = 2^j 3^{-sk} \leq 2^j 3^s |U_i|^s$ basic intervals of E_j. If we choose j large enough so that $3^{-(j+1)} \leq |U_i|$ for all U_i then, since the $\{U_i\}$ intersects all 2^j basic intervals of length 3^{-j}, counting intervals gives $2^j \leq \sum_i 2^j 3^s |U_i|^s$. Therefore

$$\sum_i |U_i|^s \geq \frac{1}{2} = 3^{-s}.$$

That is $\mathcal{H}^s(K) \geq \frac{1}{2}$. □

The proof of the upper bound was simpler than the one of the lower bound. This is due to the fact that we just need to make computations with *one* good cover to obtain upper bounds, whereas to obtain a lower bound we need to consider *all* covers. In the next section we give a different proof of the lower bound using different techniques.

2.4 Mass Distribution Principle

One of the basic tools to estimate the Hausdorff dimension of a set is the so called *mass distribution principle.* The idea is to study a measure supported on the set and try to gain dimension information from it.

Theorem 2.1 (Mass distribution Principle)**.** *Let $X \subset \mathbb{R}^n$ and let μ be finite measure with $\mu(X) > 0$. Assume that there are numbers $s \geq 0$, $c > 0$ and $\delta_0 > 0$ such that*

$$\mu(U) \leq c|U|^s, \tag{2.3}$$

for all sets U with $|U| \leq \delta_0$. Then

$$\mathcal{H}^s(X) \geq \frac{\mu(X)}{c},$$

and $s \leq \dim_H(X)$.

Proof. Let $\{U_i\}$ be any cover of X by sets of diameter smaller that δ_0 then

$$\mu(X) \leq \mu\left(\cup_i U_i\right) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s.$$

Therefore, if $\delta \leq \delta_0$ we have $\mu(X) \leq C\mathcal{H}^s_\delta(X)$. The result follows letting $\delta \to 0$. □

Proposition 2.6. *Let μ be a measure satisfying the assumptions of the mass distribution principle of the set X and let $c > 0$ be a constant.*

1. *If for all $x \in X$ we have*

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} < c$$

   then $\mathcal{H}^s(X) \geq \mu(X)/c$.

2. *If for all $x \in X$ we have*

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} > c$$

   then $\mathcal{H}^s(X) \leq \frac{2^s \mu(X)}{c}$.

Proof. We will only prove the first statement, the proof of the second statement is similar. For each $\delta > 0$ let

$$X_\delta = \left\{x \in X : \mu(B(x,r)) < (c-\epsilon) r^s \text{ for all } 0 < r \leq \delta, \text{ for some } \epsilon > 0\right\}.$$

Let $\{U_i\}$ be a $\delta-$cover of X and thus of X_δ. For each U_i containing a point $x \in X_\delta$, the ball B of center x and radius $|U_i|$ contains the set U_i. By definition of X_δ we have

$$\mu(U_i) \leq \mu(B) < c|U_i|^s,$$

therefore

$$\mu(X_\delta) \leq \sum_i \left\{\mu(U_i) : U_i \text{ intersects } X_\delta\right\} \leq c \sum_i |U_i|^s.$$

Since $\{U_i\}$ is an arbitrary $\delta-$cover of X, it follows that $\mu(X_\delta) \leq c\mathcal{H}^s_\delta(X) \leq c\mathcal{H}^s(X)$. But the sets X_δ increases to X as δ decreases to zero. Therefore $\mu(X) \leq c\mathcal{H}^s(X)$. □

2.4.1 Example: The Cantor set.

Let us consider the $1/3-$Cantor K defined in Section 2.3. We will construct a *natural* measure supported on K and then we will apply the mass distribution principle in order to obtain a lower bound on the Hausdorff dimension of K.

The following method is often used to construct measures supported on special subsets of $\mathbb{R}^n$. It involves repeated subdivision of mass between the basic sets. Recall that we denote by E_k the collection of 2^k intervals of length 3^{-k} that are obtained at each $k-$th step in the construction of the Cantor set. Let m be a probability measure on the unit interval $[0,1]$. In the first step we distribute the mass of m evenly on the two intervals forming E_2. That is

$$1 = m([0,1]) = \frac{1}{2} + \frac{1}{2} = m([0, \frac{1}{3}]) + m([\frac{2}{3}, 1]).$$

In general, at the $n-$th step the mass is distributed over $E_n = \cup_{i=1}^{2^n} E_n^i$ in the following way

$$\sum_{i=1}^{2^n} m(E_n^i) = 1 \text{ and } m(E_n^i) = \frac{1}{2^n}.$$

If $\mathcal{E}$ denotes the collections of sets E_k together with the sets $\mathbb{R} \setminus E_k$, then the above procedure defines a mass $\mu(\cdot)$ such that $\mu(\mathbb{R} \setminus E_k) = 0$ and $\mu(E_k) = 1$. This mass is defined on $\mathcal{E}$. The definition of μ can be extended to $\mathbb{R}$ so that μ becomes a measure. The value of $\mu(U)$ is uniquely determined for any Borel set U. The support of μ is contained in $\cap_{i=0}^{\infty} \overline{E_i}$. Let $U \subset \mathbb{R}$ such that $|U| < 1$ and let $k \in \mathbb{N}$ be such that

$$3^{-(k+1)} \leq |U| \leq 3^{-k}.$$

Then U can intersect at most one of the intervals of E_k, so

$$\mu(U) \leq 2^{-k} = (3^{-k})^{\frac{\log 2}{\log 3}} \leq (3|U|)^{\frac{\log 2}{\log 3}},$$

hence $\mathcal{H}^{\frac{\log 2}{\log 3}}(K) > 0$ and by the mass distribution principle we obtain

$$\dim_H K \geq \frac{\log 2}{\log 3}.$$

Remark 2.3. *Note that it is possible to construct different Cantor sets. Say, for instance, that instead of removing an interval whose length is a third of the total, we repeat the same procedure but removing intervals whose length are a fifth of the total. In that case the Hasudorff dimension of the Cantor set $K_{1/5}$ obtained in this way is such that:*

$$\dim_H(K_{1/5}) = \frac{\log 2}{\log 5}.$$

Chapter 3

Thermodynamic formalism

3.1 Introduction

Let $T : X \to X$ be a continuous map of the compact metric space (X, d). The theory of dynamical systems is devoted to the study of the long term behaviour of orbits $\{x, Tx, T^2x, \dots\}$, where $T^n x = T \circ T \cdots \circ T(x)$. The techniques used to perform this study depend upon the structure of (X, d). For instance, if (X, d) is a topological space then techniques will be of a topological nature (hence *topological dynamics*). We will mostly be interested in the case that (X, d) is a measure space, (X, μ), and the measure structure is preserved by the map T. A measure μ with this properties is called *invariant measure* (see Definition 3.1). The study of dynamical systems from the perspective of measure theory is called *ergodic theory*. An important branch within ergodic theory is the so called *thermodynamic formalism*. This is a set of ideas which derive from statistical mechanics. It can be though of as the study of certain procedures for the choice of invariant measures. Let us stress that a large class of interesting dynamical systems (in particular the ones considered in these notes) have *many* invariant measures (see Example 3.4), hence the problem of choosing relevant ones. The main object in the theory is the *topological pressure* (see Definition 3.4), which quantifies the disorder of the system. A remarkable result in the field, which ties together topological objets with objects of a measure theoretical nature, is that the topological pressure can be expressed as the supremum of a weighted measure theoretical entropy, where the supremum is taken over the set of all invariant probability measures (see Theorem 3.2). This result provides a natural way to pick up measures. An element realising this supremum is called *equilibrium measure* (see Definition 3.5). In several situations equilibrium measures have strong ergodic properties, they can be physical measures (in the sense of Sinai-Ruelle-Bowen) or can be relevant in dimension theory. Questions about existence, uniqueness and ergodic properties of equilibrium measures are at the core of the theory.

The class of dynamical systems whose ergodic theory is best understood is the class of *uniformly hyperbolic systems*. This is partially due to the fact that these systems have a compact symbolic model whose behaviour is well known. The thermodynamic formalism depends upon tow *forces*, on the one hand the hyperbolicity of the dynamical system plays a very important role, and on the other, the regularity of the potential is essential. In the cases under consideration on these notes the dynamical systems will be uniformly hyperbolic and the potentials will be Hölder continuous. Let us stress that studying thermodynamic formalism for systems satisfying weaker notions of hyperbolicity and/or potentials that are less regular is at the centre of the recent developments in thermodynamic formalism.

Even though we present some of the results in greater generality that what would be used in these notes, the reader might find it easier to have in mind a simple (non-trivial) example. That is the affine cookie cutter, which is a map $T : I_1 \cup I_2 \to [0, 1]$, where I_1, I_2 are two closed disjoint subintervals of $[0, 1]$. The map T being piecewise linear and such that $T(I_i) = [0, 1]$. For these maps the topological pressure has the following simple form:

$$P(-t \log |T'|) = \log(a_1^{-t} + a_2^{-t}),$$

where a_i is the slope of T in the interval I_i. All the results are easier to understand for this map.

There exists several books where the thermodynamic formalism theory is very well exposed. Three classic texts that we can mention are the one by Bowen [10], the one by Ruelle [41] and the one by Walters [45]. The text by Parry and Pollicott [35] has interesting applications of the theory. A more recent view on the theory can be found on the books of Keller [25] and Baladi [2].

3.2 Invariant measures

Let $T : X \to X$ be a continuous map of the compact metric space (X, d). Denote by $\mathcal{B}$ the Borel σ-algebra and by $\mathcal{M}$ the set of probability measures on X.

Definition 3.1. *A measure $\mu \in \mathcal{M}$ is called a $T-$invariant probability measure if for every Borel set $A \in \mathcal{B}$ the following relation holds*

$$\mu(T^{-1}A) = \mu(A). \tag{3.1}$$

Denote by $\mathcal{M}_T$ the set of $T-$invariant probability measures.

Note that if $\mu \in \mathcal{M}_T$ then the map T preserves the measure structure of the probability space $(X, \mathcal{B}, \mu)$. Invariant probability measures are the dynamically relevant measures in $\mathcal{M}$. Indeed, these measures can somehow *see* the dynamics. For instance, the following result shows that invariant measures detect the recurrence of the dynamical system.

Theorem 3.1 (Poincaré Recurrence Theorem)**.** *Let $T : X \to X$ be a continuous map of the compact metric space X and let $\mu \in \mathcal{M}_T$. If $A \in \mathcal{B}$ is such that $\mu(A) > 0$ then $\mu-$almost every point of A returns infinitely often to A under positive iteration of T. That is, there exists a subsequence $\{n_i\}$ such that $T^{n_i}x \cap A \neq \emptyset$.*

The following are examples of invariant measures for certain maps,

Example 3.1 (Rotations on the circle)**.** *Let $T : S^1 \to S^1$ be a rotation of angle $\alpha \in \mathbb{R}$ over the circle S^1. If α is an irrational number then there exists a unique $T-$invariant measure, which is the Haar measure on the circle. If α is rational then each periodic orbit supports a $T-$invariant measure.*

Example 3.2 (The doubling map)**.** *Let $T : [0,1] \to [0,1]$ be defined by $Tx = 2x \mod 1$. Then the Lebesgue measure on the interval is T-invariant. Also, the atomic measure supported at the point zero, δ_0, is T-invariant.*

Example 3.3 (The quadratic map)**.** *Let $T : [0,1] \to [0,1]$ be defined by $Tx = 4x(1-x)$. Then the measure defined by*

$$\mu(A) = \int_A \frac{dx}{\pi\sqrt{x(1-x)}}$$

is $T-$invariant. Note that this is measure is absolutely continuous with respect to the Lebesgue measure. This was proved by Ulam and von Neumannan in 1947.

Example 3.4 (The full-shift on two symbols)**.** *Consider the following set*

$$\Sigma_2 = \{(x_i)_{i\in\mathbb{N}} : x_i \in \{1,2\}\}.$$

That is the space of all one sided sequences of ones and twos. This a compact space when endowed with the product topology. Note that a base for the topology is given by the so called cylinder *sets, that is the collection of sets of the form*

$$C_{i_1 \dots i_n} = \{(x_i)_{i\in\mathbb{N}} : x_1 = i_1; \dots; x_n = i_n\}.$$

Consider the shift map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(x_1x_2x_3\dots) = (x_2x_3\dots)$. The set of invariant measures for this map is extremely complicated. Note that if $x(n) = (x_1x_2\dots x_nx_1\dots x_n\dots)$ is a periodic orbit for σ then the measure

$$\delta_{x(n)} = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i(x(n))}$$

is $\sigma-$invariant. The following properties describe the high complexity of the set $\mathcal{M}_\sigma$,

1. *The set $\mathcal{M}_\sigma$ is compact and convex.*
2. *It is infinite dimensional.*
3. *The extreme points of the convex set $\mathcal{M}_\sigma$ are dense in $\mathcal{M}_\sigma$.*
4. *The measures supported on periodic orbits are dense in $\mathcal{M}_\sigma$.*

This simplex is called Poulsen simplex*, see [17, 29] for more details. We stress that there is nothing particular in the choice of two symbols. Analogous results can be obtained if we consider the full-shift on N symbols, that is, the set*

$$\Sigma_N = \{(x_i)_{i\in\mathbb{N}} : x_i \in \{1, \dots, N\}\}.$$

Together with the shift map $\sigma : \Sigma_N \to \Sigma_N$.

Example 3.5 (Cookie Cutters)**.** *An interval map closely related to the previous example can be constructed in the following way. Let $I_1, I_2 \subset [0, 1]$ be two disjoint closed intervals. Consider a C^2 map, $T : I_1 \cup I_2 \to [0, 1]$ such that $|T'| > 1$ and $T(I_i) = [0, 1]$. This map has an infinite number of periodic points and each of these periodic orbits supports a $T-$invariant measure. Actually, the set $\mathcal{M}_T$ satisfies the same properties as the ones observed in Example 3.4 (it is also the Poulsen simplex).*

Note that the same construction can be made with N disjoint intervals. Maps belonging to these class are called cookie cutters.

It is important to note that the space $\mathcal{M}_T$ is never empty (Krylov-Bogoliubov, 1937 see [45, Chapter 6 p.152]).

3.3 Entropy

The notion of entropy was introduced into ergodic theory in 1958 by Kolmogorov. It is one of the most important invariants in dynamical systems.

The definition of entropy used now is slightly different form Kolmogorov's and it is due to Sinai (1959). In this section we briefly sketch the definition of entropy, for a throughout account see, for example, [36, 45].

Entropy can be thought of as a measure of the disorder of the system. In other words, the entropy of an invariant measure μ quantifies the amount of disorder of the system that can be seen with the measure μ.

The definition of entropy is done in several steps. Let $T : X \to X$ be a continuous map of the compact space X and let μ be a $T-$invariant probability measure.

3.3.1 Entropy of a Partition.

A *partition* of X is a disjoint collection of elements in the Borel $\sigma-$algebra $\mathcal{B}$, whose union is equal to X. If $P_1 = \{A_1, \dots A_n\}$ and $P_2 = \{C_1, \dots C_m\}$ are two finite partitions of X then their *join* is the partition

$$P_1 \vee P_2 := \{A_i \cap C_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

Definition 3.2. *Let $P_1 = \{A_1, \dots A_n\}$ be a partition of X. The* entropy *of P_1 with respect to μ is defined by*

$$H(P_1) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i).$$

3.3.2 Entropy of a measure preserving transformation.

Let $P_1 = \{A_1, \dots A_n\}$ be a partition of X. This partition can be refined using the dynamics. Indeed, consider the partition

$$\bigvee_{i=0}^{n-1} T^{-i} P_1 = \left\{ \cap_{i=0}^{n-1} T^{-i} A_{i_j} : i_j \in \{1, \dots n\} \right\}$$

The *entropy of* (T, μ) *with respect to* P_1 is defined by

$$h(T, P_1) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i} P_1 \right).$$

Finally,

Definition 3.3. *Let $T : X \to X$ be a continuous map of the compact space X and let $\mu \in \mathcal{M}_T$, then the* entropy of T with respect to μ *is defined by*

$$h(\mu) = \sup\{h(T, P) : P \text{ finite partition of } X\}.$$

Remark 3.1. *Entropy is an invariant under topological conjugacies.*

It is not easy to compute the entropy straight from the definition. Although, there are results that enable us to make explicit computations in certain cases [45, Chapter 4].

Example 3.6. *Let (Σ_2, σ) be the full-shift on two symbols. Denote by μ the $(1/2, 1/2)-$Bernoulli measure then $h(\sigma, \mu) = \log 2$. Denote by $\delta_{\overline{0}}$ the atomic measure supported on the fixed point $\overline{0} = (0, 0, 0, 0, \dots)$ then $h(\sigma, \delta_{\overline{0}}) = 0$. Therefore, the dynamical system (Σ_2, σ) when looked at through the measure $\delta_{\overline{0}}$ is very simple (indeed, it is just a fixed point), instead, when looked at through the measure μ it is more complicated.*

We stress that this definition of entropy only depends on the Borel structure of X.

3.4 Topological Pressure

There are a number of different ways of defining the topological pressure. The most general one (at least when considering continuous maps defined over compact spaces) is using $(n, \epsilon)-$generating sets (see [45, Chapter 9]). Here we will consider a different definition, which coincides with the classical one for dynamical systems that are sufficiently hyperbolic. Even though the following definition holds in greater generality, we will restrict ourselves to symbolic systems and to piecewise expanding interval maps with full branches (*cookie cutters*).

Let $T : X \to X$ be either a full-shift on $N-$symbols (see Example 3.4) or a cookie cutter (see Example 3.5). These will be dynamical systems under consideration. Let $\phi : X \to \mathbb{R}$ be a Hölder continuous function, that we will call *potential.*

Definition 3.4. *The* topological pressure *of the map T at the potential ϕ is defined by*

$$P_T(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{T^n x = x} \exp \left(\sum_{i=0}^{n-1} \phi(T^i x) \right). \tag{3.2}$$

Let us note that using sub-additivity arguments it is possible to prove that the above limit exists.

In order to understand the definition, let us start considering the null-potential, that is $\phi \equiv 0$. In this case we obtain

$$P_T(0) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{T^n x = x} 1.$$

Therefore, $P_T(0)$ quantifies the exponential growth of periodic orbits,

$$\sum_{T^n x = x} 1 \asymp \exp(nP_T(0)).$$

The number $P_T(0)$ is usually called *topological entropy* and it is denoted by $h_{top}(T)$. The topological pressure can be thought of as a weighted topological entropy. Indeed, each point on the periodic orbit $\{x, Tx.T^2x, \dots, T^{n-1}x\}$, is given *weight* $\phi(T^i x)$. The topological pressure quantifies the exponential growth of these weighted periodic orbits,

$$\sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x)\right) \asymp \exp(nP_T(\phi)).$$

The are several properties of the pressure that can easily be deduced from the definition, for instance,

1. If $\phi < \psi$ then $P_T(\phi) \leq P_T(\psi)$.
2. The pressure function $P(\cdot)$ is convex (with respect to the potential)
3. If $c \in \mathbb{R}$ then $P_T(\phi + c) = P_T(\phi) + c$.
4. $P_T(\phi) = P_T(\phi + \psi \circ T - \psi)$.

One of the most important results in thermodynamic formalism is the so called variational principle. This theorem relates the topological pressure with the measure theoretic entropy. It was first proved by Ruelle [42] and then in full generality by Walters [46].

Theorem 3.2 (Variational principle)**.** *Let $\phi : X \to \mathbb{R}$ be a Hölder potential, then*

$$P_T(\phi) = \sup\left\{h(\mu) + \int \phi \, d\mu : \mu \in \mathcal{M}_T\right\}. \tag{3.3}$$

Note that if $\phi \equiv 0$ then we obtain the variational principle for the topological entropy

$$h_{top}(T) = \sup\{h(\mu) : \mu \in \mathcal{M}_T\}.$$

This theorem relates the topological complexity of the system with the measure theoretic complexity of it.

Definition 3.5. *A measure $\mu \in \mathcal{M}_T$ such that*

$$P_T(\phi) = h(\mu) + \int \phi \, d\mu,$$

is called equilibrium measure *for ϕ.*

Questions about existence, uniqueness and ergodic properties of equilibrium measures are at the core of the theory. In the particular case of hyperbolic dynamical systems (full-shifts and cookie cutters) and regular potentials (Hölder) the situation it is completely understood, see for example [10, 25, 35, 41, 45],

Theorem 3.3 (Bowen, Parry, Ruelle, Sinai, Walters)**.** *Let $T : X \to X$ be either a full-shift or a cookie cutter and let $\phi : X \to \mathbb{R}$ be a Hölder potential, then*

1. *The pressure function, $P(t) : \mathbb{R} \to \mathbb{R}$, defined by $P(t) = P(t\phi)$ is real analytic.*
2. *There exists a unique equilibrium measure μ_ϕ corresponding to ϕ.*

3.5 The cookie cutter

In this section we focus our attention in the main example considered in these notes. Let $T : I_1 \cup I_2 \to [0, 1]$ be a cookie cutter as in Example 3.5. We stress that $T|I_i$ need not to be linear. The *repeller* corresponding to T is the set

$$\Lambda = \bigcap_{i=0}^{\infty} T^{-i}([0, 1]). \tag{3.4}$$

Note that

$$\Lambda = \{x \in I_1 \cup I_2 : T^n x \text{ is well defined for every } n \in \mathbb{N}\}.$$

Since $\{T^{-i}([0, 1])\}$ is a decreasing sequence of compact sets we have that Λ is a non-empty compact set. Moreover, it is $T-$invariant. It is called a repeller because points not in Λ are eventually mapped out of $I_1 \cup I_2$ under iteration of T.

Lemma 3.1. *If $\overline{I_1} \cup \overline{I_2} \neq [0, 1]$ then Λ is a Cantor set.*

When Λ is a Cantor set the dynamics of $T : \Lambda \to \Lambda$ is topologically conjugated (the same from the topological point of view) to the full-shift on two symbols. Indeed, for each sequence $\omega = (\omega_1, \omega_2, \dots)$, where $\omega_i \in \{1, 2\}$ we can define the map $h : \Sigma_2 \to \Lambda$ by

$$h(\omega) = \bigcap_{i=1}^{\infty} T^{-i}(I_{\omega_i}).$$

It turns out that the map h is an homemorphism and satisfies the following:

$$(T \circ h)(\omega) = (h \circ \sigma)(\omega).$$

A *cylinder* is the set

$$C_{i_1 \dots i_n} = \{\omega \in \Sigma_2 : \omega_1 = i_1, \dots \omega_n = i_m\}.$$

With a slight abuse of language we will also call a cylinder the set

$$h(C_{i_1 \dots i_n}) = I(i_1 \dots i_n).$$

In this section we will estimate the length of $I(i_1 \dots i_n)$ comparing it with $|(T^n)'|$.

Example 3.7. *Let $I_1 = [0, 1/3]$ and $I_2 = [2/3, 1]$ consider the map $T : I_1 \cup I_2 \to [0, 1]$ defined by $T(x) = 3x \mod 1$. This map is a cookie cutter and Λ is the middle third Cantor set.*

Example 3.8. *Let $I_1 = [0, 1/5]$ and $I_2 = [2/3, 1]$ consider the map $T : I_1 \cup I_2 \to [0, 1]$ defined by $T(x) = 5x$ if $x \in [0, 1/5]$ and $T(x) = 3x - 2$ if $x \in [2/3, 1]$. This map is a cookie cutter and Λ is a Cantor set.*

In terms of the thermodynamic formalism will be interested in a particular family of potentials, that is $\phi_t = -t \log |T'|$, with $t \in \mathbb{R}$. It turns out that the pressure function is very well behaved. Before stating a general result we consider the following example

Example 3.9. *Let $T : I_1 \cup I_2 \to [0, 1]$ be an affine cookie cutter, that is T restricted to each interval I_i is piecewise linear, $T|_{I_i} = a_i x + c_i$. In such a situation we have the following explicit formula for the topological pressure,*

$$P(-t \log |T'|) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \exp\Big(\sum_{i=0}^{n-1} -t \log |(T'(T^i x))| \Big) =$$

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \prod_{i=0}^{n-1} |T'(T^i x)| = \lim_{n\to\infty} \frac{1}{n} \log \sum_{j \in \{1,2\}^n} (a_{i_1} \dots a_{i_n})^{-t}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log(a_1^{-t} + a_2^{-t})^n = \log(a_1^{-t} + a_2^{-t}).$$

That is $P(-t \log |T'|) = \log(a_1^{-t} + a_2^{-t})$.

Proposition 3.1. *Let $T : I_1 \cup I_2 \to [0, 1]$ be a cookie cutter, then*

1. *The function* $t \to P(-t \log |T'|)$ *is real analytic, convex and strictly decreasing.*

2. *There exists a unique value* t^* *such that* $P(-t^* \log |T'|) = 0$.

Ideas of the proof. In order to prove that the pressure is real analytic machinery from functional analysis is required. Indeed, the pressure can be represented as the maximal eigenvalue of a certain operator (the so called *transfer operator*). Using perturbation arguments it is possible to prove that if the potential is regular enough and the system has enough hyperbolicity (both conditions which are satisfied in our case) then the eigenvalue depends analytically on the perturbation (see [41, 35]).

The pressure function is decreasing since it is monotonous and the potential $\phi = \log |T'| > 0$. Convexity follows from the definition and Hölder's inequality.

In order to prove that the pressure function has a unique zero we prove that it is positive at $T = 0$ and negative for t sufficiently large. Therefore, by continuity and convexity we obtain the desired result. Note that $P(0) = h_{top}(T)$. Since (T, Σ) is topologically conjugated to the full-shift on two symbols, both systems have the same entropy, hence $P(0) = \log 2$. On the other hand, note that from the variational principle we obtain that

$$\begin{aligned} P(-t \log |T'|) &= \sup \left\{ h(\mu) - t \int \log |T'| \, d\mu : \mu \in \mathcal{M}_T \right\} \\ &\leq \log 2 - t \sup\{\log |T'(x)| : x \in \Lambda\}. \end{aligned}$$

Therefore, if

$$t > \frac{\log 2}{\sup\{\log |T'(x)| : x \in \Lambda\}}$$

then $P(-t \log |T'|) < 0$. Hence, there exists a unique root to the equation $P(-t \log |T'|) = 0$. □

Theorem 3.4. *Let* $T : I_1 \cup I_2 \to [0, 1]$ *be a cookie cutter then there exists a unique equilibrium measure* μ_{t^*} *corresponding to* $-t^* \log |T'|$. *Moreover, this measures is a* Gibbs *measure, that is, there exists a constant* $C > 0$ *such that*

$$\frac{1}{C} \leq \frac{\mu_{t^*}(I(i_0, \dots i_n))}{|(T^n)' x|^{-t^*}} \leq C, \tag{3.5}$$

for every $x \in I(i_0, \dots i_n)$.

In order to prove this theorem we need certain estimates relating the length of a cylinder and the derivative of T. Recall that $\phi = -t^* \log |T'|$ (the results actually holds for any Hölder potential and that is why we keep the notation ϕ for the potential). Denote by $S_n \phi(x) = \sum_{i=0}^{n-1} \phi(T^i x)$.

Lemma 3.2. *There exists a constant $b > 0$ such that*

$$|S_n\phi(x) - S_n\phi(y)| \leq b,$$

for any cylinder $I(i_1, \dots i_n)$ and $x, y \in I(i_1, \dots i_n)$.

Proof. Denote by $T_1^{-1} = T^{-1}|I_1$ and $T_2^{-1} = T^{-1}|I_2$. Since the function T' is continuous and defined over a compact set there exists constants $C_1, C_2 > 0$ such that $C_1 \leq |(T_1^{-1})'|, |(T_2^{-1})'| \leq C_2$. By the Mean Value Theorem we obtain that for every $x, y \in \Lambda$,

$$C_1|x - y| \leq | T_i^{-1}(x) - T_i^{-1}(y)| \leq C_2|x - y|. \tag{3.6}$$

Applying equation (3.6) several times we obtain that

$$|I(i_1 \dots i_n)| = |T_{i_1}^{-1} \circ T_{i_2}^{-1} \circ \dots \circ T_{i_n}^{-1}(I)| \leq C_2^n|I|.$$

Recall that we are assuming the map T' to be Hölder continuous, that is, there exists a constant $a > 0$ and $\alpha > 0$ such that

$$|\phi(x) - \phi(y)| \leq a|x - y|^\alpha.$$

Therefore, if $x, y \in I(i_1, \dots I_n)$ we have

$$|\phi(T^jx) - \phi(T^jy)| \leq a|T^jx - T^jy|^\alpha \leq a|I(i_{j+1} \dots i_n)|^\alpha \leq aC_2^{\alpha(n-j)}|I|^\alpha.$$

Therefore,

$$|S_n\phi(x) - S_n\phi(y)| = \big|\sum_{i=0}^{n-1} \phi(T^ix) - \sum_{i=0}^{n-1} \phi(T^iy)\big| \leq$$

$$\leq \sum_{i=0}^{n-1} \big|\phi(T^ix) - \phi(T^iy)\big| \leq \sum_{i=0}^{n-1} aC_2^{\alpha(n-1-i)}|I|^\alpha \leq a|I|\frac{(C_2)^\alpha}{1 - (C_2)^\alpha}.$$

if we denote by $b = a|I|\frac{C_2}{1-C_2}$ the result is proven. □

Remark 3.2. *The conclusion of the above Lemma can be equivalently written in the following way*

$$\exp(-b) \leq \frac{\exp(S_n(\phi(x))}{\exp(S_n(\phi(y))} \leq \exp(b).$$

Remark 3.3. *Note that as a consequence of equation* (3.6) *we obtain that*

$$C_1|I(i_1 \dots i_n)| \leq |I(i_1 \dots i_n i_{n+1})| \leq C_2|I(i_1 \dots i_n)|.$$

Remark 3.4. *Note that since the map T satisfies the bounded distortion property (this is due to the regularity assumption) we have that we can estimate the length of the interval $I(i_1 \dots i_n)$ in the following way: There exists a constant $b > 0$ such that for any $x \in I(i_1 \dots i_n)$ we have*

$$\frac{1}{b} \leq \frac{|I(i_1 \dots i_n)|}{|(T^n(x))'|} \leq b \tag{3.7}$$

The remarkable fact is that the constant $b > 0$ does not depends on the value of n. This estimate can be used to describe the topological pressure, indeed

$$P(-t \log |T'|) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} |(T^n)'x|^{-t} = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1 \dots i_n)} |I(i_1 \dots i_n)|^{-t}.$$

Another useful consequence of this bounded distortion principle is that the sets $I(i_1 \dots i_n 1)$, $I(i_1 \dots i_n 2)$, are well separated in $I(i_1 \dots i_n)$. Moreover, the sets $I(i_1 \dots i_n)$ are comparable with balls in a uniform way. Indeed, denote by $d = dist(I_1, I_2)$, then

1. *for all $(i_1 \dots i_n)$ we have that*

$$d\frac{1}{C}|I(i_1 \dots i_n)| \leq dist(I(i_1 \dots i_n 1), I(i_1 \dots i_n 2)) \leq |I(i_1 \dots i_n)|.$$

2. *There exists $\lambda > 0$ such that for every $(i_1 \dots i_n)$, if $x \in I(i_1 \dots i_n) \cap \Lambda$ and $|I(i_1 \dots i_n)| \leq r \leq |I(i_1 \dots i_n)| C_1^{-1}$ then*

$$B(x, \lambda r) \cap \Lambda \subset I(i_1 \dots i_n) \cap \Lambda \subset B(x, r). \tag{3.8}$$

Proof of Theorem 3.4. We will construct a measure satisfying the desired property. Let μ_n be the atomic measure defined by

$$\mu_n(A) = \frac{1}{s_n} \sum_{T^n x = x} \exp(S_n \phi(x)) \chi_A(x)$$

where $\chi_A(x)$ is the characteristic function of the Borel set A and $s_n = \sum_{T^n x = x} \exp(S_n \phi(x))$ is the normalisation constant. Recall that the space of probability measures in compact. Therefore, there exists an accumulation point μ of the sequence $\{\mu_n\}$. Note that by construction $\mu(\Lambda) = 1$. Let $k < n$ then

$$\mu_n(I(i_1 \dots i_k)) = \frac{1}{s_n} \sum_{T^n x = x} \exp(S_n \phi(x)) \chi_{I(i_1 \dots i_k)}(x) =$$

$$\frac{1}{s_n} \sum_{T^n x = x} \exp(S_k \phi(x)) \exp(S_{n-k} \phi(T^k x)) \chi_{I(i_1 \dots i_k)}(x)$$

Thus, if $y \in I(i_1 \dots i_n)$ we have by Remark 3.2 that

$$\exp(S_n(\phi(x)) \leq \exp(b)\exp(S_n(\phi(y)),$$

therefore

$$\begin{aligned} &\exp(-b)\mu_n(I(i_1 \dots i_k)) \leq \\ \frac{1}{s_n}\exp(S_k(\phi(y)) \sum_{T^n x = x} &\exp(S_{n-k}\phi(T^k x))\chi_{I(i_1 \dots i_k)}(x) \leq \\ &\exp(b)\mu_n(I(i_1 \dots i_k)). \end{aligned}$$

Since the system is a full-shift and it satisfies the bounded distortion estimate, from the above inequalities we obtain that

$$\begin{aligned} &\exp(-2b)\mu_n(I(i_1 \dots i_k)) \leq \\ \frac{1}{s_n}\exp(S_k(\phi(y)) \sum_{T^{n-k} x = x} &\exp(S_{n-k}\phi(x))\chi_{I(i_1 \dots i_k)}(x) \leq \\ &\exp(2b)\mu_n(I(i_1 \dots i_k)). \end{aligned}$$

Therefore

$$\exp(-2b)\mu_n(I(i_1 \dots i_k)) \leq \frac{s_{n-k}}{s_n}\exp(S_k(\phi(y)) \leq \exp(2b)\mu_n(I(i_1 \dots i_k)).$$

That is

$$\exp(-3b)\mu_n(I(i_1 \dots i_k)) \leq \frac{1}{s_k}\exp(S_k(\phi(y)) \leq \exp(3b)\mu_n(I(i_1 \dots i_k)).$$

This holds for any $n > k$, in particular it holds for the limit measure μ. Therefore

$$\frac{\exp(-3b)}{s_k} \leq \frac{\mu(I(i_1 \dots i_k))}{\exp(S_k(\phi(y))} \leq \frac{\exp(3b)}{s_k}.$$

But

$$\exp(-b)\exp(kP(\phi)) \leq s_k \leq \exp(b)\exp(kP(\phi))$$

The result now follows (it is worth to emphasize that Gibbs measures can be chosen to be invariant and ergodic [15, p.84]). $\square$

Chapter 4

The Bowen formula

In this section we prove the main result of these notes, that is, the dimension formula that relates the topological pressure and the Hausdorff dimension. This results was first proved by Rufus Bowen [9] in a 1979 paper that appeared after his premature dead (he died in 1978 aged 31 years old from brain hemorrhage). His result was later generalised by Ruelle in [40] and, still now, more general forms of this formula are being proved.

We also include applications of the ideas developed here to the interplay of number theory and dynamical systems.

4.1 The Bowen formula

Let $T : I_1 \cup I_2 \to [0, 1]$ be a cookie cutter map and $\Lambda \subset [0, 1]$ be the repeller of T.

Theorem 4.1. *Let T be a cookie cutter then $t = \dim_H(\Lambda)$ is the unique root of the equation*

$$P(-t \log |T'|) = 0.$$

Remark 4.1. *Note that if T is an affine cookie cutter, using the formulas obtained in Example 3.9, we obtain that $s = \dim_H(\Lambda)$ is the only real number such that*

$$a_1^{-s} + a_2^{-s} = 1.$$

This number is also called similarity dimension*. This formula was first obtained by Morán in 1946.*

Before giving a rigorous proof we present an heuristic argument that can give an idea of why this result should be true. We will consider covers of

Λ by cylinders $\{I(i_1 \dots i_n)\}$ such that all those intervals have length smaller than $\delta > 0$. From the definition of Hausdorff measure we have that

$$\mathcal{H}^t_\delta(\Lambda) \leq \sum_{(i_1 \dots i_n) \in \{1,2\}^n} |I(i_1 \dots i_n)|^t.$$

Letting δ tend to zero we obtain that

$$\mathcal{H}^t(\Lambda) \leq \liminf_{n \to \infty} \sum_{(i_1 \dots i_n) \in \{1,2\}^n} |I(i_1 \dots i_n)|^t.$$

From Remark 3.4 we have the following

$$\sum_{(i_1 \dots i_n) \in \{1,2\}^n} |I(i_1 \dots i_n)|^t \asymp \exp(nP(-t \log |T'|).$$

Therefore, if $t \in \mathbb{R}$ is such that $P(-t \log |T'|) < 0$, we have that

$$\mathcal{H}^t(\Lambda) \leq \liminf_{n \to \infty} \sum_{(i_1 \dots i_n) \in \{1,2\}^n} |I(i_1 \dots i_n)|^t = \lim_{n \to \infty} \exp(nP(-t \log |T'|) = 0.$$

That implies that for each $t \in \mathbb{R}$ such that $P(-t \log |T'|) > 0$ we have that $\mathcal{H}^t(\Lambda) = \infty$. In the same way, we obtain that if $t \in \mathbb{R}$ is such that $P(-t \log |T'|) < 0$ we obtain that $\mathcal{H}^t(\Lambda) = 0$. Therefore

$$P(-\dim_H(\Lambda) \log |T'|) = 0.$$

Remark 4.2. *We will first prove the piecewise linear case. That is, we will assume T to be as in Example 4.1. The upper bound is obtained as much in the similar fashion as in the heuristic argument. Indeed, let $\epsilon > 0$ then $P(-(t^* + \epsilon) \log |T'|) < 0$. Therefore*

$$\begin{aligned}\mathcal{H}^{t^*+\epsilon}(\Lambda) &\leq \liminf_{n \to \infty} \sum_{(i_1 \dots i_n) \in \{1,2\}^n} |I(i_1 \dots i_n)|^{t^*+\epsilon} \\ &= \lim_{n \to \infty} \exp(nP(-(t^* + \epsilon) \log |T'|) = 0.\end{aligned}$$

This implies that $\dim_H(\Lambda) \leq t^*$. In order to prove the lower bound, we will make use of the Mass distribution principle. Indeed the equilibrium measure μ corresponding to $-t^* \log |T'|$ is such that

$$\mu(I(i_1 \dots i_n) = |I(i_1 \dots i_n)|^{t^*}.$$

From this it follows that $\dim_H(\Lambda) \geq t^*$. Let us show that in fact the measure μ has this property.

Lemma 4.1. *Let* (Σ_2, σ) *the full-shift in two symbols and let* $\phi : \Sigma \to \mathbb{R}$ *a locally constant potential such that* $\phi(x_i) = a_{x_i}$*. Denote by* $\mathcal{E}$ *the partition of cylinders of length one. Denote by* μ *the equilibrium measure corresponding to* ϕ*. Assume that* $\mu(I_i) = p_i$*. Then*

$$h(\mu) = h(\mu, \mathcal{E}) \leq -p_1 \log p_1 - p_2 \log p_2.$$

Therefore,

$$\begin{aligned} h(\mu) + a_1 p_1 + a_2 p_2 &\leq -p_1 \log p_1 - p_2 \log p_2 + a_1 p_1 + a_2 p_2 = \\ p_1(a_1 - \log p_1) &+ p_2(a_2 - \log p_2) \leq \log(\exp(a_1) + \exp(a_2)) \end{aligned}$$

Where the equality is only achieved only if

$$p_i = \frac{\exp(a_i)}{\exp(a_1) + \exp(a_2)}.$$

Proof of the Bowen formula. Note that there exists a unique root, $t^* > 0$, to the equation $P(-t \log |T'|) = 0$ (see Proposition 3.1). Moreover, there exists a unique equilibrium measure μ corresponding to the potential $-t^* \log |T'|$. This measure has the following (Gibbs) property (see Theorem 3.4)

$$\frac{1}{C} \leq \frac{\mu_{t^*}(I(i_1, \ldots i_n))}{|(T^n)'x|^{-t^*}} \leq C, \tag{4.1}$$

for every $x \in I(i_1, \ldots i_n)$. In particular we obtain that

$$\frac{1}{C} \leq \frac{\mu_{t^*}(I(i_1, \ldots i_n))}{|I(i_1, \ldots i_n)|^{-t^*}} \leq C, \tag{4.2}$$

The main feature of the proof is to relate the measure μ with the t^*-Hausdorff measure on Λ and then use the mass distribution principle.

Let $x \in \Lambda$ and $r > 0$ small enough. From Remark 3.3 it is possible to find $(i_1 \ldots i_n)$ and $x \in I(i_1 \ldots i_n)$ such that

$$|I(i_1 \ldots i_n)| \leq r < \frac{1}{C}|I(i_1 \ldots i_n)|. \tag{4.3}$$

From equation (3.8) we obtain that

$$\mu(B(x, \lambda r)) \leq \mu(I(i_1 \ldots i_n)) \leq \mu(B(x, r)), \tag{4.4}$$

where $\lambda > 0$ is independent of x and r. From equation 4.2 we obtain that

$$\frac{1}{C}\mu(B(x, \lambda r)) \leq |I(i_1 \ldots i_n)|^{t^*} \leq C\mu(B(x, r)).$$

Therefore
$$\frac{1}{C} r^{t^*} \leq \mu(B(x,r)) \leq C r^{t^*}$$
for every $x \in \Lambda$ and sufficiently small values of r. Therefore form Proposition 2.6 we obtain that
$$\frac{1}{C} \leq \mathcal{H}^{t^*}(\Lambda) \leq C.$$
Thus
$$\dim_H(\Lambda) = t^*.$$

□

4.2 Generalisations

4.2.1 Non-conformal systems.

There are several ways of generalising the above result. One of them is to consider dynamical systems defined in higher dimensions, say $\mathbb{R}^2$. The major difficulty that one has to deal with when considering higher dimensions is the loss of *conformality*. This basically means that the cylinders $I(i_1 \dots i_n)$ are not comparable with balls $B(x,r)$. They might be comparable to ellipses if you have to different rates of contraction. In terms of Hausdorf dimension balls are a special kind of sets. In particular, Hausdorff dimension can be computed using covers by balls (this can not be done using covers by ellipses for example). It is a very difficult task to develop a dimension theory for non-conformal dynamical systems. Only very partial results are known.

Now, if the system is conformal (essentially one dimensional), there is only one rate of contraction, then all the above results apply directly.

4.2.2 Countable number of branches

The results stated here (Theorem 4.1) hold for systems with a finite number of branches.The proof is exactly the same. A natural question is if this result holds for a system with countably many branches (for example the Gauss map). There are several problems that need to be addressed when considering this question. First, the properties of the thermodynamic formalism need to be studied. This has been done by Mauldin and Urbański [32] and by Sarig [43]. A version for the dimension formula holds.

Theorem 4.2 (Mauldin and Urbański)**.** *Let $\{I_i\}$ be a countable family of disjoint, closed intervals contained in $[0,1]$. Consider a map $T : \cup_{i=1}^{\infty} I_i \to [0,1]$ be a piecewise C^2 map such that*

1. *There exists* $A > 1$ *such that for every* $x \in \cup_{i=1}^{\infty} I_i$ *we have that* $|T'(x)| > A$.

2. *The map* T *satisfies the bounded distortion assumptions.*

3. *For every* $i \in \mathbb{N}$ *we have that* $T(I_i) = [0, 1]$.

Then

$$\dim_H(\Lambda) = \inf\{t : P(-t \log |T'|) \leq 0\}.$$

Note that in this setting it is possible that if $t < \dim_H(\Lambda)$ then

$$P(-t \log |T'|) = \infty$$

and if $t \geq \dim_H(\Lambda)$ then $P(-t \log |T'|) < 0$.

A more general version of this theorem (for more complicated sub-shifts on countable alphabets) can be found in [22].

4.3 Results related to number theory

In this section we describe some results on the dimension theory of some number theoretically defined sets. The methods used to prove these results are in spirit similar to the ones developed in these notes.

4.3.1 Continued fractions

Every real number $x \in (0, 1)$ can be written in a unique way as a *continued fraction* of the form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_1 a_2 a_3 \dots],$$

where $a_i \in \mathbb{N}$. This way of representing a real number has several advantages with respect to the (more used) decimal system. This is basically due to the fact that this representation is not related to any system of calculation. Therefore it only reflects the properties of the number and not it relationship with a system of calculation. For instance, if the the continued fraction of the number x has only finitely many terms a_n then the number is rational, whereas if it has infinitely many of them then it is irrational. The strong disadvantage of the continued fraction is that there is no simple rule to do arithmetic operations, for instance the sum. By thus we mean that there is no simple way of finding the sum of $[a_1a_2a_3\dots]$ with $[b_1b_2b_3\dots]$. For a general account on continued fractions see [20, 27].

Other great advantage of the continued fractions is that it allows us to obtain the best possible rational approximations of an irrational number. To be precise, let us say that a rational number a/b is the best approximation of the real number x if every other rational number with the same or smaller denominator differs from x in a grater amount. In other words, if one defines the complexity of the rational approximation by the size of its denominator, then continued fraction representation allows us to obtain the simpler approximations of a given order. This is the historical reason for the discovery and study of continued fractions. The $n-th$ *approximant* $p_n(x)/q_n(x)$ of the number $x \in [0,1]$ is defined by

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \frac{1}{a_n}}}} \tag{4.5}$$

A classical result in diophantine approximation states that, if $p_n(x)/q_n(x)$ is the $n-th$ *approximant* of the number $x \in [0,1]$ then

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| < \frac{1}{(q_n(x))^2}.$$

It is possible to associate a dynamical system to the continued fraction system. The Gauss map $G : (0,1] \to (0,1]$, is the interval map defined by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right],$$

This map is closely related to the continued fraction expansion. Indeed, for $0 < x < 1$ with $x = [a_1a_2a_3\dots]$ we have that $a_1 = [1/x], a_2 = [1/Gx], \dots, a_n = [1/G^{n-1}x]$. In particular, the Gauss map acts as the shift map on the continued fraction expansion,

$$a_n = \left[1/G^{n-1}x\right].$$

In particular, if $x = [a_1a_2a_3\dots]$ then $Gx = [a_2a_3a_4\dots]$. The Lyapunov exponent of the Gauss map G at the point x , whenever the limit exists, satisfies (see [38])

$$\lambda(x) = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right|, \tag{4.6}$$

Therefore, the Lyapunov exponent of the Gauss map quantifies the exponential speed of approximation of a number by its approximants (see [38]). That is,

$$\left|x - \frac{p_n(x)}{q_n(x)}\right| \asymp \exp\left(-n\lambda(x)\right).$$

There exists an absolutely continuous ergodic $G-$invariant measure, μ_G, called the *Gauss measure*, defined by

$$\mu_G(A) = \frac{1}{\log 2}\int_A \frac{1}{1+x}dx,$$

where $A \subset [0,1]$ is a Borel set. From the Birkhoff ergodic theorem we obtain that μ_G-almost everywhere (and hence Lebesgue almost everywhere)

$$\lambda(x) = \frac{\pi^2}{6\log 2}. \tag{4.7}$$

Therefore, for Lebesgue almost every point $x \in [0,1]$ we have that

$$\left|x - \frac{p_n(x)}{q_n(x)}\right| \asymp \exp\left(\frac{-n\pi^2}{6\log 2}\right).$$

Note that the range of values of the Lyapunov exponent is

$$\left[2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty\right)$$

(see [26, 38]). If $x_1 = \frac{-1+\sqrt{5}}{2} = [1,1,1,1,\dots]$ then $\lambda(x_1) = 2\log\left(\frac{1+\sqrt{5}}{2}\right)$. In particular, the set of number for which the frequency of digits equal to 1 in the continued fraction expansion is equal to one, are called *noble numbers*.

For $\alpha \in [2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty)$ consider the level set

$$J(\alpha) = \{x \in [0,1] : \lambda(x) = \alpha\} = \left\{x \in [0,1] : \lim_{n\to\infty}\frac{1}{n}\log\left|x - \frac{p_n(x)}{q_n(x)}\right| = \alpha\right\}.$$

Consider also the *irregular set*,

$$J' = \{x \in [0,1] : \text{ the limit } \lambda(x) \text{ does not exists}\}.$$

These level sets induced the so called *multifractal decomposition*

$$[0,1] = (\cup_\alpha J(\alpha)) \cup J'.$$

The function that encodes this decomposition is called *multifractal spectrum* and it is defined by

$$\alpha \to L(\alpha) := \dim_H J(\alpha).$$

By means of the thermodynamic formalism it is possible to describe this function and the decomposition. The following result was obtained by Pollicott and Weiss [38] and by Kessembomer and Stratman [26]

Theorem 4.3. *The following results are valid for the multifractal decomposition for Lyapunov exponents of the Gauss map,*

1. *for every* $\alpha \in \left(2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty\right)$ *the level set* $J(\alpha)$ *is dense in* $[0,1]$ *and has positive Hausdorff diemension.*
2. *The function* $L(\alpha)$ *is real analytic, it has a unique maximum at* $\alpha = \frac{\pi^2}{6\log 2}$*, it has an inflection point and*

$$\lim_{\alpha\to\infty} L(\alpha) = \frac{1}{2}.$$

3. $\dim_H J' = 1$.

4.3.2 Backward continued fractions

The map $R : [0,1) \to [0,1)$ is defined by

$$R(x) = \frac{1}{1-x} - \left[\frac{1}{1-x}\right],$$

where $[a]$ denotes the integer part of the number a. It was introduced by Renyi in [39] and we will refer to it as the *Renyi map.* The ergodic properties of this map have been studied, among others, by Adler and Flatto [1] and by Renyi himself [39]. This is a map with infinitely many branches and infinite topological entropy. It has a parabolic fixed point at zero. It is closely related to the backward continued fraction algorithm [19, 16]. Indeed, every irrational number $x \in [0,1)$ has unique infinite backward continued fraction expansion of the form

$$x = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}}} = [a_1 a_2 a_3 \dots]_B,$$

where the coefficients $\{a_i\}$ are integers such that $a_i > 1$. The Renyi map acts as the shift on the backward continued fraction (see [19, 16]). In particular,

$$\text{If } x = [a_1 a_2 a_3 \dots]_B \text{ then } R(x) = [a_2 a_3 \dots]_B.$$

This continued fraction has been used, for example, to obtain results on inhomogenous diophantine approximation (see [37]). Renyi [39] showed that there exists an infinite $\sigma-$finite invariant measure, μ_R, absolutely continuous with respect to the Lebesgue measure. It is defined by

$$\mu_R(A) = \int_A \frac{1}{x}\, dx,$$

where $A \subset [0,1]$ is a Borel set. There is no finite invariant measure absolutely continuous with respect to the Lebesgue measure. As in the case of continued fractions we have can define the $n-th$ *backward approximant* $p_n(x)/q_n(x)$ of the number $x \in [0,1]$ is defined by

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\dots - \frac{1}{a_n}}}} \tag{4.8}$$

Again, the Lyapunov exponent, $\lambda_R(x)$, of the Renyi map quantifes the exponential speed of approximation of a number by its approximants. That is,

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \asymp \exp\left(-n\lambda_R(x)\right).$$

The range of values that the Lyapunov exponent can attain is $[0,\infty)$. We can decompose the interval in an analogous fashion as in the case of the Gauss map. That is, for $\alpha \in [0,\infty)$ consider

$$J(\alpha) = \{x \in [0,1] : \lambda_R(x) = \alpha\}$$

Consider also the *irregular set*,

$$J' = \{x \in [0,1] : \text{ the limit } \lambda_R(x) \text{ does not exists}\}.$$

These level sets induced the so called *multifractal decomposition*

$$[0,1] = (\cup_\alpha J(\alpha)) \cup J'.$$

Th function that encodes this decomposition is called *multifractal spectrum* and it is defined by

$$\alpha \to L_R(\alpha) := \dim_H J(\alpha).$$

By means of a version of the thermodynamic formalism for non-unifomly hyperbolic maps it is possible to describe this multifractal spectrum (see [?]),

Theorem 4.4. *1. for every $\alpha \in [0, \infty)$ the level set $J(\alpha)$ is dense in $[0, 1]$ and has positive Hausdorff diemension.*

2. The function $L(\alpha)$ is real analytic, it is stricly decreasing, it has a unique maximum at $\alpha = 0$, it has an inflection point and

$$\lim_{\alpha\to\infty} L(\alpha) = \frac{1}{2}.$$

3. $\dim_H J' = 1$.

4.3.3 The Lüroth expansion

Given constants $a_n \geq 2$ for every $n \in \mathbb{N}$, each real number $x \in (0, 1]$ can be written in the form

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)a_2 \cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots .$$

This series expansion, called *Lüroth expansion*, was introduced in 1883 by Lüroth [31]. Each irrational number has a unique infinite expansion of this form and each rational number has either a finite expansion or a periodic one. We denote the Lüroth series expansion of $x \in (0, 1)$ by

$$x = [a_1(x)a_2(x)\cdots]_L = [a_1a_2\cdots]_L.$$

The series is closely related to the dynamics of the *Lüroth map* $T\colon [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} n(n+1)x - n, & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}], \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \in [\frac{1}{n+1}, \frac{1}{n}]$, then $a_1(x) = n$, and

$$a_k(x) = a_1(T^{k-1}(x)), \tag{4.9}$$

that is, the Lüroth map acts as a shift on the Lüroth series. The Lebesgue measure is T-invariant and is ergodic (see [24]). For other properties of the map Lüroth see [12, 11].

It is possible to exploit the consequences of identity (4.9). Namely, by studying the ergodic properties of the map T we will deduce number theoretical properties of the Lüroth expansion.

As in the case of the continued fraction expansion and for the backward continued fraction expansion, it is possible to estimate the speed of convergence by the $n-th$ Lüroth approximant. Let $\lambda(m) := \sum_{n=1}^{\infty} \frac{\log(n(n+1))}{n(n+1)}$.

Theorem 4.5. *The following properties hold:*

1. *the domain of L is $[\log 2, +\infty)$, and*

$$L(\gamma) = \frac{1}{\gamma} \inf_{t\in\mathbb{R}} \left[P(-t\log|T'|) + t\gamma\right],$$

   where $P(\cdot)$ denotes the topological pressure with respect to T;

2. *the spectrum L real analytic, has a unique maximum at $\gamma = \lambda(m)$, has an inflection point, and satisfies*

$$\lim_{\gamma\to+\infty} L(\gamma) = \frac{1}{2};$$

3. $\dim_H J' = 1$.

We also study the Hausdorff dimension of the level sets determined by the frequency of digits in the Lüroth series expansion. More precisely, for each $n, k \in \mathbb{N}$ and $x \in (0, 1)$ let

$$\tau_k(x, n) = \text{card}\big\{i \in \{1, \ldots, n\} : a_i(x) = k\big\}.$$

Whenever the limit

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x, n)}{n} \tag{4.10}$$

exists, it is called the *frequency* of the number k in the Lüroth expansion of x. Since the Lebesgue measure is ergodic, by Birkhoff's ergodic theorem, for Lebesgue-almost every $x \in [0, 1]$ we have $\tau_n(x) = 1/[n(n-1)]$. Now let $\alpha = (\alpha_1\alpha_2\cdots)$ be a stochastic vector, i.e., a vector such that $\alpha_i \geq 0$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. We consider the set

$$F(\alpha) = \big\{x \in [0, 1] : \tau_k(x) = \alpha_k \text{ for each } k \in \mathbb{N}\big\}. \tag{4.11}$$

We have already seen that if $\alpha_n = 1/[n(n-1)]$ for each $n \in \mathbb{N}$, then the level set $F(\alpha)$ has full Lebesgue measure. Of course, any other level set has zero Lebesgue measure.

However, the sets in (4.11) can have positive Hausdorff dimension. Our objective is to obtain an explicit formula for the Hausdorff dimension of sets $F(\alpha)$ for an arbitrary stochastic vector α. The following result was obtained by Barreira and Iommi [4] and it can be proved showing that there exists an ergodic measure, say μ_α, concentrated on $F(\alpha)$ such that $\dim_H F(\alpha) = \dim_H \mu_\alpha$.

Theorem 4.6. *If $\alpha = (\alpha_1 \alpha_2 \cdots)$ is a stochastic vector such that*

$$\lambda(\mu_\alpha) = \sum_{n=1}^{\infty} \alpha_n \log(n(n+1)) < \infty,$$

then

$$\dim_H F(\alpha) = \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}$$

4.3.4 Base m representation of a number

Given $m \in \mathbb{N}$, the base-m expansion of a point $x \in [0,1]$ is given by

$$x = \frac{\epsilon_1(x)}{m} + \frac{\epsilon_2(x)}{m^2} + \frac{\epsilon_3(x)}{m^3} + \cdots,$$

where $\epsilon_i(x) \in \{0, \ldots, m-1\}$ for each i. There is a dynamical system associated to this representation, which as in the case of the Gauss map, acts as a shift map: consider the map $T_m \colon [0,1] \to [0,1]$ defined by $T_m x = mx \pmod 1$. Clearly, $\lambda_{T_m}(x) = \log m$ for every $x \in [0,1]$. As in the case of the Lüroth map we consider for each $n \in \mathbb{N}$, $k \in \{0, \ldots m-1\}$ and $x \in (0,1)$ the number

$$\tau_k(x,n) = \operatorname{card}\big\{i \in \{1, \ldots, n\} : a_i(x) = k\big\}.$$

Whenever the limit

$$\tau_k(x) = \lim_{n \to \infty} \frac{\tau_k(x,n)}{n} \tag{4.12}$$

exists, it is called the *frequency* of the number k in the base $m-$expansion of x. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a stochastic vector. Eggleston [13] proved that the set

$$F_m(\alpha) = \big\{x \in [0,1] : \tau_{k,m}(x) = \alpha_k \text{ for } k \in \{0, \ldots, m-1\}\big\}$$

has Hausdorff dimension

$$\dim_H F_m(\alpha) = \frac{\sum_{k=0}^{m-1} \alpha_k \log \alpha_k}{\log m}.$$

This result was recovered and generalized by Barreira, Saussol and Schmeling [5, 6] using a multidimensional version of multifractal analysis.

4.3.5 The β-transformations

Now let $\beta \in \mathbb{R}$ with $\beta > 1$. The *beta transformation* $T_\beta \colon [0,1) \to [0,1)$ is defined by

$$T_\beta(x) = \beta x \pmod 1.$$

We emphasize that in general T_β is not a Markov map. It was shown by Rényi [39] that each $x \in [0,1)$ has a β-expansion

$$x = \frac{\epsilon_1(x)}{\beta} + \frac{\epsilon_2(x)}{\beta^2} + \frac{\epsilon_3(x)}{\beta^3} + \cdots ,$$

where $\epsilon_n(x) = [\beta T_\beta^{n-1}(x)]$ for each n, being $[a]$ he integer part of a. Note that the *digits* in the β-expansion may take values in $\{0, 1, \ldots, [\beta]\}$. In general there might exist several ways of writing a number x in the form $x = \sum_{i=1}^{\infty} a_i/\beta^i$, with $a_i \in \{0, 1, \ldots, [\beta]\}$.

Comparison of different expansions.

In this subsection we establish relations between different forms of writing a real number, for instance its continued fraction expansion and its β-expansion for $\beta > 1$. More precisely, we consider the following problem:

> Given $x \in [0,1]$ and $n \in \mathbb{N}$, how many partial quotients $k_n(x)$ in the continued fraction expansion of x can be obtained from the first n terms of its β-expansion?

We give an asymptotics for $k_n(x)$ using the theory of dynamical systems. Denote by $\lambda_G(x)$ the Lyapunov exponent of the Gauss map a the point x. In [3] Barreira and Iommi obtained the following asymptotics.

Theorem 4.7. *For each $x \in (0,1)$ we have*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\lambda_G(x)},$$

whenever the limit exists.

It turns out that the Lyapunov exponent $\lambda_G(x)$ is the exponential speed of approximation of a number by its approximants $p_n(x)/q_n(x)$. By Theorem 4.7, this implies that if x is well-approximated by rational numbers, then the amount of information about the continued fraction expansion that can be obtained from its β-expansion is small. Moreover, the larger β is (that is, the more symbols we use to code a number x), the more information about the continued fraction expansion we obtain. In the particular case when $\beta = 10$, the statement in Theorem 4.7 was obtained by Faivre [15] for a particular class of numbers, and by Wu [47] in full generality. Their methods are different from ours. In particular, they never use methods of the theory of dynamical systems.

An immediate corollary of Theorem 4.7 is that

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{6\log 2\log\beta}{\pi^2} \tag{4.13}$$

for Lebesgue-almost everywhere $x \in (0,1)$. This statement was established by Lochs [30] in the particular case when $\beta = 10$. Of course, the almost everywhere existence of the limit in (4.13) does not mean that it always exists, or that the value in the right-hand side is the only one attained by the limit in the left-hand side. As an application of the theory of multifractal analysis, for each α we compute the Hausdorff dimension of the sets of points $x \in (0,1)$ for which

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log\beta}{\alpha}.$$

Indeed , if

$$J(\alpha) = \Big\{x \in [0,1] : \lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log\beta}{\alpha}\Big\},$$

and

$$K = \Big\{x \in [0,1] : \liminf_{n\to\infty} \frac{k_n(x)}{n} < \limsup_{n\to\infty} \frac{k_n(x)}{n}\Big\}.$$

then

Theorem 4.8. *For every $\alpha > (1+\sqrt{5})/2$ we have*

$$\dim_H J(\alpha) = h_{\mu_\alpha}(G)/\alpha,$$

where $\dim_H$ denotes the Hausdorff dimension.

This extends and improves the results obtained by Wu in [47]. Note that

$$\inf\{\lambda_G(x) : x \in (0,1)\} = \frac{1+\sqrt{5}}{2}.$$

The following is a consequence of results of Barreira and Schmeling in [7].

Theorem 4.9. *The set K has Hausdorff dimension equal to one.*

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