

PARTIAL QUOTIENTS OF CONTINUED FRACTIONS AND β -EXPANSIONS

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ABSTRACT. For each real number, we obtain an asymptotics for the number of partial quotients in the continued fraction expansion that can be obtained from the first n terms of its β -expansion. A novelty of our approach is the use of methods of the theory of dynamical systems.

1. INTRODUCTION

In this note we establish relations between different forms of writing a real number, namely its continued fraction expansion and its β -expansion for $\beta > 1$ (both expansions are recalled in the main text). More precisely, we consider the following problem:

Given $x \in [0, 1]$ and $n \in \mathbb{N}$, how many partial quotients $k_n(x)$ in the continued fraction expansion of x can be obtained from the first n terms of its β -expansion?

We give an asymptotics for $k_n(x)$ using the theory of dynamical systems.

To formulate a rigorous statement, let $G: (0, 1] \rightarrow (0, 1]$ be the Gauss map. It is well known that G is closely related to the continued fraction expansions (see Section 3 for details). We define the Lyapunov exponent of G at the point $x \in (0, 1)$ by

$$\lambda_G(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |(G^n)'(x)|, \quad (1)$$

whenever the limit exists. We obtain the following asymptotics.

Theorem 1. *For each $x \in (0, 1)$ we have*

$$\lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\lambda_G(x)},$$

whenever the limit exists.

Theorem 1 follows from a much more general statement in Theorem 2 for repellers of conformal expanding maps.

It turns out that the Lyapunov exponent $\lambda_G(x)$ is the exponential speed of approximation of a number by its approximants $p_n(x)/q_n(x)$ (see (4)). By Theorem 1, this implies that if x is well-approximated by rational numbers, then the amount of information about the continued fraction expansion that can be obtained from its β -expansion is small. Moreover, the larger β is (that is, the more symbols we use to code a number x), the more information

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about the continued fraction expansion we obtain. In the particular case when $\beta = 10$, the statement in Theorem 1 was obtained by Faivre [2] for a particular class of numbers, and by Wu [13] in full generality. We emphasize that their methods are different from ours. In particular, they never use methods of the theory of dynamical systems.

An immediate corollary of Theorem 1 (see Section 3 for details) is that

$$\lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta}{\pi^2} \quad (2)$$

for Lebesgue-almost everywhere $x \in (0, 1)$. This statement was established by Lochs [5] in the particular case when $\beta = 10$. Of course, the almost everywhere existence of the limit in (2) does not mean that it always exists, or that the value in the right-hand side is the only one attained by the limit in the left-hand side. As an application of the theory of multifractal analysis, for each α we compute the Hausdorff dimension of the sets of points $x \in (0, 1)$ for which

$$\lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\alpha}.$$

Moreover, the irregular set K of points $x \in (0, 1)$ for which the limit does not exist has Hausdorff dimension equal to 1. In [13], Wu obtained lower estimates for the Hausdorff dimension of K .

The structure of the paper is as follows. In Section 2 we prove a general version of Theorem 1 for conformal expanding maps on smooth manifolds using the Markov structure of the maps. As a corollary we obtain the statement in Theorem 1 for any integer $\beta > 1$. In Section 3 we recall several properties of continued fractions and β -expansions, and we prove Theorem 1 for an arbitrary $\beta > 1$. We emphasize that the main difficulty is that β -transformations need not be Markov. Finally, in Section 4 we apply the theory of multifractal analysis to obtain the results on the Hausdorff dimension of certain special subsets.

2. THE MARKOV CASE

2.1. Basic notions. In this section we prove our main theorem for Markov maps. Let M be a smooth manifold, and let $f, g: M \rightarrow M$ be $C^{1+\alpha}$ maps. We consider a compact set $J \subset M$ such that

$$f(J) = g(J) = J.$$

We assume that f and g are conformal expanding maps, that J is a repeller for both maps, and that f and g are topologically mixing. This means that:

- (1) there exist constants $c > 0$ and $\beta > 1$ such that

$$\|d_x f^n v\| \geq c\beta^n \|v\| \quad \text{and} \quad \|d_x g^n v\| \geq c\beta^n \|v\|$$

for every $n \in \mathbb{N}$, $x \in J$, and $v \in T_x M$;

- (2) $d_x f$ and $d_x g$ are multiples of isometries for every $x \in J$;
(3) given open sets U and V with nonempty intersection with J there exists $n \in \mathbb{N}$ such that for every $m > n$,

$$f^m(U) \cap V \cap J \neq \emptyset \quad \text{and} \quad g^m(U) \cap V \cap J \neq \emptyset.$$

We recall that a finite cover of J by nonempty closed sets C_1, \dots, C_p is a *Markov partition* of J with respect to f if:

- (1) $\overline{\text{int } C_i} = C_i$ for each i ;
- (2) $\text{int } C_i \cap \text{int } C_j = \emptyset$ whenever $i \neq j$;
- (3) $f(C_i) \supset C_j$ whenever $f(\text{int } C_i) \cap \text{int } C_j \neq \emptyset$.

It is well known that expanding maps have Markov partitions of diameter as small as desired. Let

$$\mathcal{A} = \{C_1, \dots, C_p\} \quad \text{and} \quad \mathcal{B} = \{D_1, \dots, D_q\}$$

be Markov partitions respectively with respect to f and g (we note that in general $p \neq q$). We denote by $C_m(x)$ and $D_m(x)$ respectively the elements of

$$\bigvee_{k=0}^{m-1} f^{-k} \mathcal{A} \quad \text{and} \quad \bigvee_{k=0}^{m-1} g^{-k} \mathcal{B}$$

that contain x . We call them *cylinder sets* of level m . Except for countably many points, the sets $C_m(x)$ and $D_m(x)$ are well-defined for each $x \in J$.

2.2. Main result. For each $x \in J$ and $n \in \mathbb{N}$ we set

$$k_n(x) := \max\{m \in \mathbb{N} : D_m(x) \subset C_m(x)\}.$$

We can describe $k_n(x)$ as the number of symbols in the symbolic representation of x with respect to f that can be obtained from the first n symbols of the the symbolic representation of x with respect to g . Moreover, for each $x \in J$ we set

$$k(x) = \lim_{n \rightarrow \infty} \frac{k_n(x)}{n},$$

whenever the limit exists.

The following is our main result. For each $x \in J$ we consider the *Lyapunov exponent* of x with respect to f given by

$$\lambda_f(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|d_x f^n\|,$$

whenever the limit exists.

Theorem 2. *For each $x \in J$ we have*

$$k(x) = \frac{\lambda_g(x)}{\lambda_f(x)},$$

whenever there exist simultaneously the limits $k(x)$, $\lambda_f(x)$, and $\lambda_g(x)$.

Proof. Since the map f is conformal, if $c(x) = \|d_x f\|$, then $d_x f = c(x) \text{Isom}_x$ for each $x \in J$, where Isom_x is an isometry. Moreover, since f is a conformal expanding map of class $C^{1+\alpha}$, the following holds (see [8, p. 199] for details).

Lemma 3. *For each $x \in J$ and $m \in \mathbb{N}$, there exist positive numbers $\underline{r}_m(x)$ and $\bar{r}_m(x)$ such that:*

- (1) *the set $C_m(x)$ contains a ball of radius $\underline{r}_m(x)$ and is contained in a ball of radius $\bar{r}_m(x)$;*

(2) *there exist positive constants K_1 and K_2 (independent of x and m) such that for every $y \in C_{i_0 \dots i_m}$ we have*

$$K_1 \prod_{j=0}^m |c(f^j(y))|^{-1} \leq r_m(x) \leq \bar{r}_m(x) \leq K_2 \prod_{j=0}^m |c(f^j(y))|^{-1}.$$

Set

$$\ell_n(x) = \max \left\{ m \in \mathbb{N} : \prod_{j=0}^n |b(g^j(x))|^{-1} \leq \prod_{j=0}^m |c(f^j(x))|^{-1} \right\},$$

where $b(x) = \|d_x g\|$. By Lemma 3, there exists $K > 0$ such that

$$|\ell_n(x) - k_n(x)| < K \tag{3}$$

for every $x \in J$ and $n \in \mathbb{N}$.

Lemma 4. *For each $x \in J$ we have*

$$\frac{\lambda_g(x)}{\lambda_f(x)} \geq k(x),$$

whenever there exist simultaneously the limits $k(x)$, $\lambda_f(x)$, and $\lambda_g(x)$.

Proof. We note that

$$\prod_{j=0}^n |b(g^j(x))|^{-1} \leq \prod_{j=0}^{\ell_n(x)} |c(f^j(x))|^{-1}.$$

Applying logarithms and dividing by n we obtain

$$\frac{1}{n} \log \prod_{j=0}^n |b(g^j(x))|^{-1} \leq \frac{\ell_n(x)}{n} \left(\frac{1}{\ell_n(x)} \log \prod_{j=0}^{\ell_n(x)} |c(f^j(x))|^{-1} \right).$$

Letting $n \rightarrow \infty$ and using (3) we obtain the desired statement. \square

Lemma 5. *For each $x \in J$ we have*

$$k(x) \geq \frac{\lambda_g(x)}{\lambda_f(x)},$$

whenever there exist simultaneously the limits $k(x)$, $\lambda_f(x)$, and $\lambda_g(x)$.

Proof. We note that

$$\prod_{j=0}^{\ell_n(x)-1} |c(f^j(x))|^{-1} \leq \prod_{j=0}^n |b(g^j(x))|^{-1}.$$

Applying logarithms and dividing by n we obtain

$$\frac{\ell_n(x) - 1}{n(\ell_n(x) - 1)} \log \prod_{j=0}^{\ell_n(x)-1} |c(f^j(x))|^{-1} \leq \frac{1}{n} \log \prod_{j=0}^n |b(g^j(x))|^{-1}.$$

Letting $n \rightarrow \infty$ and using (3) we obtain the desired statement. \square

Theorem 2 follows readily from Lemmas 4 and 5. \square

It follows immediately from Theorem 2 that the statement in Theorem 1 holds for any integer $\beta > 1$. This corresponds to consider base- m representations (see Section 3.2).

3. CONTINUED FRACTIONS AND β -EXPANSIONS

This section is devoted to prove Theorem 1 for continued fractions and β -expansions. We emphasize that Theorem 2 cannot be applied directly to this setting since in general the β -transformations are not Markov maps (and thus Markov partitions may not exist).

3.1. Continued fractions. Every irrational number $x \in (0, 1)$ has a continued fraction of the form

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_1 a_2 a_3 \dots],$$

where $a_i \in \mathbb{N}$ for each i . We also consider the n th approximant $p_n(x)/q_n(x)$ of x given by

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}}$$

For a detailed account of continued fractions see [3].

Now we consider the Gauss map $G: (0, 1] \rightarrow (0, 1]$, defined by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x} \right],$$

where $[a]$ denotes the integer part of a . There is a close relation between the continued fraction expansion and the dynamics of the Gauss map. Indeed, for each $x \in (0, 1)$ with $x = [a_1 a_2 a_3 \dots]$ we have $a_n = [1/G^{n-1}(x)]$ for each n . In particular, the Gauss map acts as a shift map on the continued fraction expansion, that is,

$$G^n(x) = [a_{n+1}(x) a_{n+2}(x) \dots].$$

This causes that the study of the dynamics of the Gauss map can be used in particular to understand the distribution of digits in the continued fraction expansion. It is easy to check that the Lyapunov exponent $\lambda_G(x)$ in (1) is given by

$$\lambda_G(x) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right|. \quad (4)$$

This shows that the Lyapunov exponent gives the exponential speed of approximation of a number by its approximants. For each $x \in (0, 1)$, the Lévy constant of x is defined by

$$L(x) := \lim_{n \rightarrow \infty} \frac{\log q_n(x)}{n},$$

whenever the limit exists. We have (see for example [3])

$$\frac{1}{2q_{n+1}^2} \leq \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n^2},$$

and hence $\lambda_G(x) = 2L(x)$.

Moreover, the *Gauss measure* defined by

$$\mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx$$

is an absolutely continuous ergodic G -invariant measure. By Birkhoff's ergodic theorem, for μ_G -almost every x (and hence for Lebesgue-almost every x) we have

$$\lambda_G(x) = \frac{\pi^2}{6 \log 2} \quad \text{and} \quad L(x) = \frac{\pi^2}{12 \log 2}. \quad (5)$$

3.2. Base- m representations. Given $m \in \mathbb{N}$, the base- m expansion of a point $x \in [0, 1]$ is

$$x = \frac{\epsilon_1(x)}{m} + \frac{\epsilon_2(x)}{m^2} + \frac{\epsilon_3(x)}{m^3} + \dots,$$

where $\epsilon_i(x) \in \{0, \dots, m-1\}$ for each i . There is a dynamics associated to this representation, which as in the case of the Gauss map, acts as a shift map: consider the map $T_m: [0, 1] \rightarrow [0, 1]$ defined by $T_m x = mx \pmod{1}$. Clearly, $\lambda_{T_m}(x) = \log m$ for every $x \in [0, 1]$. The following is an immediate consequence of Theorem 2.

Corollary 6. *For each $x \in (0, 1)$ we have*

$$k(x) = \lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{\log m}{\lambda_G(x)} = \frac{\log m}{2L(x)},$$

whenever there exist simultaneously the limits $k(x)$ and $\lambda_G(x)$.

This generalizes a result of Wu [13], who established it for $m = 10$. It also follows from (5) that for Lebesgue-almost every x we have

$$\lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log m}{\pi^2}.$$

This was obtained by Lochs [5] when $m = 10$.

3.3. The β -transformations. Now let $\beta \in \mathbb{R}$ with $\beta > 1$. The *beta transformation* $T_\beta: [0, 1) \rightarrow [0, 1)$ is defined by

$$T_\beta(x) = \beta x \pmod{1}.$$

We emphasize that in general T_β is not a Markov map. It was shown by Rényi [10] that each $x \in [0, 1)$ has a β -expansion

$$x = \frac{\epsilon_1(x)}{\beta} + \frac{\epsilon_2(x)}{\beta^2} + \frac{\epsilon_3(x)}{\beta^3} + \dots,$$

where $\epsilon_n(x) = [\beta T_\beta^{n-1}(x)]$ for each n , being $[a]$ the integer part of a . Note that the *digits* in the β -expansion may take values in $\{0, 1, \dots, [\beta]\}$. In general there might exist several ways of writing a number x in the form $x = \sum_{i=1}^{\infty} a_i / \beta^i$, with $a_i \in \{0, 1, \dots, [\beta]\}$.

We obtain a version of Theorem 2 in this setting.

Theorem 7. *For each $x \in (0, 1)$ we have*

$$k(x) = \lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\lambda_G(x)} = \frac{\log \beta}{2L(x)},$$

whenever there exist simultaneously the limits $k(x)$ and $\lambda_G(x)$.

Proof. Let $\beta_1 := \beta - [\beta]$ and $\beta_n := \epsilon_{n-1}(\beta - [\beta])$ for each n . We have

$$1 = \frac{\beta_1}{\beta} + \frac{\beta_2}{\beta^2} + \cdots.$$

Let ξ be the partition of $[0, 1)$ into the intervals

$$\left[0, \frac{1}{\beta}\right), \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \left[\frac{2}{\beta}, \frac{3}{\beta}\right), \dots, \left[\frac{[\beta]}{\beta}, 1\right).$$

The elements of the partition $\xi_n = \bigvee_{i=0}^{n-1} T_\beta^{-i}\xi$ are the sets $\bigcap_{i=0}^{n-1} T_\beta^{-i}A_i$, where $A_i \in \xi$ for each i . These are intervals of the form

$$\left[\frac{i_1}{\beta} + \frac{i_2}{\beta^2} + \frac{i_3}{\beta^3} + \cdots + \frac{i_n}{\beta^n}, \frac{i_1}{\beta} + \frac{i_2}{\beta^2} + \frac{i_3}{\beta^3} + \cdots + \frac{i_n + 1}{\beta^n}\right),$$

where

$$(i_1, i_2, \dots, i_n) < (\beta_1, \beta_2, \dots, \beta_n) \quad \text{and} \quad (i_j, \dots, i_n) < (\beta_1, \beta_2, \dots, \beta_{n-j+1})$$

for $2 \leq j \leq n$, together with

$$\left[\frac{\beta_1}{\beta} + \frac{\beta_2}{\beta^2} + \cdots + \frac{\beta_n}{\beta^n}, 1\right).$$

The order considered is the lexicographic order (see [7, 11]). The elements of ξ_n are called *intervals of rank n* , and those which are mapped by T_β^n onto $[0, 1)$ are called *full intervals of rank n* . We denote by $I_n(x)$ the interval of rank n containing x .

The difficulty in this setting is to control the length of the intervals of the partition. To overcome this we make use of the following lemma of Walters in [11, Lemma 1(i)].

Lemma 8. *For each $N > 0$, the interval $[0, 1)$ is covered by the full intervals of rank at least N .*

We also need the following statement. We denote by $\partial\xi$ the set of endpoints of the intervals in $\bigcup_n \xi_n$.

Lemma 9. *For each $x \in (0, 1) \setminus \partial\xi$ there exists a strictly increasing sequence of positive integers $(n_i)_i$ such that*

$$|I_{n_i}(x)| = \beta^{-n_i}. \tag{6}$$

Proof. Let $N_1 > 0$ be a positive integer and denote by ζ_1 the cover of $[0, 1)$ by the full intervals of rank at least N_1 . Let $I_{n_1}(x)$ be the element of ζ_1 containing x with minimum rank $n_1 \geq N_1$. We assume that $I_{n_j}(x)$ is the element of ζ_{N_j} containing x with minimum rank. Let $N_{j+1} > n_j$ and denote by $\zeta_{N_{j+1}}$ the cover of $[0, 1)$ by the full intervals of rank at least N_{j+1} . Let $I_{n_{j+1}}(x)$ be the element of $\zeta_{N_{j+1}}$ containing x with minimum rank. With this inductive procedure we obtain a sequence of full intervals $I_{n_i}(x)$. Note that $T_\beta^{n_i}$ restricted to $I_{n_i}(x)$ expands distances by a factor β^{n_i} and that $T_\beta^{n_i}(I_{n_i}(x)) = (0, 1)$. This establishes (6). \square

We proceed with the proof of Theorem 7. Let $x \in (0, 1)$ be such that $k(x)$ and $\lambda_G(x)$ exist and let $(n_i)_i$ be a sequence of positive integers as in Lemma 8. We have

$$k(x) = \lim_{i \rightarrow \infty} \frac{k_{n_i}(x)}{n_i}.$$

By Lemma 9 we have

$$k_{n_i}(x) = \max \left\{ m : \beta^{-n_i} \leq \prod_{j=0}^m |G^j(x)|^{-1} \right\}.$$

Note that for each $x \in [0, 1]$ has Lyapunov exponent $\lambda_{T_\beta}(x) = \log \beta$. Therefore, we can apply Lemmas 4 and 5 to obtain

$$k(x) = \lim_{i \rightarrow \infty} \frac{k_{n_i}(x)}{n_i} = \frac{\log \beta}{\lambda_G(x)} = \frac{\log \beta}{2L(x)}.$$

This completes the proof of the theorem. \square

Theorem 7 combined with (5) yields the following.

Theorem 10. *For Lebesgue-almost every $x \in [0, 1]$ we have*

$$k(x) = \lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta}{\pi^2}. \quad (7)$$

4. HAUSDORFF DIMENSION OF LEVEL SETS

We established in the former section that $k(x)$ is constant Lebesgue-almost everywhere. Nevertheless, there exist points $x \in [0, 1]$ for which $k(x)$ is different from the value in (7). Moreover, there also exist points for which $k(x)$ is not defined. More precisely, for each $\alpha \in \mathbb{R}$ set

$$J(\alpha) = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\alpha} \right\},$$

and

$$K = \left\{ x \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{k_n(x)}{n} < \limsup_{n \rightarrow \infty} \frac{k_n(x)}{n} \right\}.$$

We obtain the decomposition

$$[0, 1] = K \cup \bigcup_{\alpha} J(\alpha).$$

Applying the theory of multifractal analysis (see [8, 9, 4]) we compute the Hausdorff dimension of these sets.

The main tool in the theory of multifractal analysis is the thermodynamic formalism. Let \mathcal{M}_G be the set of all G -invariant probability measures. The topological pressure of the function $-t \log |G'|$ (for $t \in \mathbb{R}$) is defined by

$$P_G(-t \log |G'|) = P(t) := \sup \left\{ h_\mu(G) - t \int_{[0,1]} \log |G'| d\mu : \mu \in \mathcal{M}_G \right\},$$

where $h_\mu(G)$ is the Kolmogorov–Sinai entropy of the Gauss map with respect to the measure μ (see for example [12, Chapter 4] for the definition). The map $t \mapsto P(t)$ is infinite for $t < 1/2$, and finite, decreasing, real analytic, and strictly convex for $t > 1/2$ (see [6]). Moreover, for each $t > 1/2$ there exists a unique probability measure $\mu_t \in \mathcal{M}_G$ such that

$$P(t) = h_{\mu_t}(G) - t \int_{[0,1]} \log |G'| d\mu_t.$$

The measure μ_t is called *equilibrium measure* for $-t \log |G'|$.

Pollicott and Weiss [9] initiated the study of the multifractal analysis of the Lyapunov exponents of the Gauss map, which was later completed by

Kesseböhmer and Stratmann [4]. The following statement is a consequence of their results in our setting. Let μ_α be the equilibrium measure of the potential $-t_\alpha \log |G'|$ such that $\alpha = \int_{[0,1]} \log |G'| d\mu_\alpha$.

Theorem 11. *For every $\alpha > (1 + \sqrt{5})/2$ we have*

$$\dim_H J(\alpha) = h_{\mu_\alpha}(G)/\alpha,$$

where \dim_H denotes the Hausdorff dimension.

This extends and improves the results obtained by Wu in [13]. Note that

$$\inf\{\lambda_G(x) : x \in (0, 1)\} = \frac{1 + \sqrt{5}}{2}.$$

The following is a consequence of results of Barreira and Schmeling in [1].

Theorem 12. *The set K has Hausdorff dimension equal to one.*

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