PERFECT MATCHINGS IN RANDOM BIPARTITE GRAPHS IN RANDOM ENVIRONMENT

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Abstract. In this note we study random bipartite graphs in random environment, as an extension of the classical Erdös-Rényi random graphs. We show that the expected number of perfect matchings obeys a precise quenched asymptotic.

1. Introduction

In their seminal paper [ER], Erdös and Rényi studied the following random graphs that now bear their names. Consider a bipartite graph with set of vertices given by $W = \{w_1, \dots, w_n\}$ and $M = \{m_1, \dots, m_n\}$. Let $p \in [0,1]$ and consider the independent random variables $X_{(ij)}$ with law

$$X_{(ij)}(x) = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1-p. \end{cases}$$

Denote by $G_n(x)$ the bipartite graph with vertices W and M and edges $E(x)$, where the edge (w_i, m_j) belongs to $E(x)$ if and only if $X_{(ij)}(x) = 1$. Let $\mathrm{pm}(G_n(x))$ be the number of *perfect matchings* of the graph $G_n(x)$ (see Sec. 3 for precise definitions). Erdös and Rényi [ER, p.460] observed that the mean of the number of perfect matchings was given by

$$\mathbb{E}(\mathrm{pm}(G_n(x))) = n! p^n. \tag{1.1}$$

This number has been also studied by Bollobás and McKay [BolMc, Theorem 1] in the context of $k-$regular random graphs and by O'Neil [O, Theorem 1] for random graphs having a fixed (large enough) proportion of edges. We refer to the text by Bollobás [Bol] for further details on the subject of random graphs.

This paper is devoted to study certain sequences of random bipartite graphs $G_{n,\omega}$ in a random environment $\omega \in \Omega$ (definitions are given in Sec. 2). Our main result (see Theorem 3.2 for precise statement) is that there exists a constant $c \in (0,1)$ such that for almost every environment $\omega \in \Omega$ and for large $n \in \mathbb{N}$

$$\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x))) \asymp n! c^n. \tag{1.2}$$

Moreover, we have an explicit formula for the number c. This result implies that $\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x)))$ is a quenched variable.

The result in equation (1.2) should be understood in the sense that the mean number of perfect matchings for random bipartite graphs in a random environment is asymptotically the same as the one of Erdös-Rényi graphs in which $p = c$. The

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number c is the so-called *scaling mean* of a function related to the random graphs. Scaling means were introduced, in more a general setting, in [BIP] and are described in Sec. 3.

2. Random Bipartite Graphs in Random Environment

Consider the following generalization of the Erdös-Rényi graphs. Let $W = \{w_1, \dots, w_n\}$ and $M = \{m_1, \dots, m_n\}$ be two disjoint sets of vertices. For every pair $1 \leqslant i, j \leqslant n$, let $a_{ij} \in [0,1]$ and consider the independent random variables $X_{(ij)}$, with law

$$X_{(ij)}(x) = \begin{cases} 1 & \text{with probability } a_{(ij)}; \\ 0 & \text{with probability } 1 - a_{(ij)}. \end{cases}$$

Denote by $G_n(x)$ the bipartite graph with vertices W, M and edges $E(x)$, where the edge (w_i, m_j) belongs to $E(x)$ if and only if $X_{(ij)}(x) = 1$. In this note we consider random bipartite graphs in random environments, that is, the laws of $X_{(ij)}$ (and hence the numbers $a_{(ij)}$) are randomly chosen following an exterior environment law. This approach to stochastic processes has developed since the ground breaking work by Solomon [Sol] on Random Walks in Random Environment and subsequent work of a large community (see [Bog] for a survey on the subject).

The model we propose is to consider the vertex sets W, M as the environment and to consider that the number $a_{(ij)}$, which is the probability that the edge connecting w_i with m_j occurs in the graph, is a random variable depending on w_i and m_j. We now describe precisely this model.

The space of environments is as follows. Fix $\alpha \in \mathbb{N}$ and a stochastic vector $(p_1, p_2, \dots, p_\alpha)$. Endow the set $\{1, \dots, \alpha\}$ with the probability measure P_W defined by $P_W(\{i\}) = p_i$. Denote by Ω_W the product space $\prod_{i=1}^{\infty} \{1, 2, \dots, \alpha\}$ and by μ_W the corresponding product measure. Let (Ω_M, μ_M) be the analogous probability measure space for the set $\{1, 2, \dots, \beta\}$ and the stochastic vector $(q_1, q_2, \dots, q_\beta)$. The *space of environments* is $\Omega = \Omega_W \times \Omega_M$ with the measure $\mu_\Omega = \mu_W \times \mu_M$ and an *environment* is an element $\omega \in \Omega$. Note that every environment defines two sequences

$$W(\omega) = (w_1, w_2, \dots) \in \Omega_W \quad \text{and} \quad M(\omega) = (m_1, m_2, \dots) \in \Omega_M.$$

For each environment $\omega \in \Omega$ we now define the edges distribution $X_{\omega,(ij)}$. Let $F = [f_{sr}]$ be a $\alpha \times \beta$ matrix with entries f_{sr} satisfying $0 \leqslant f_{sr} \leqslant 1$ and let $f : \{1, 2, \dots, \alpha\} \times \{1, 2, \dots, \beta\} \to [0,1]$ be the function defined by $f(w, m) = f_{wm}$. For each $\omega \in \Omega$ let

$$a_{(ij)}(\omega) := f\left(w_i(\omega), m_j(\omega)\right) = f_{w_i(\omega), m_j(\omega)}. \tag{2.1}$$

Given an environment $\omega \in \Omega$ the corresponding *edges distributions* are the random variables $X_{\omega,(ij)}$ with laws

$$X_{\omega,(ij)}(x) = \begin{cases} 1 & \text{with probability } a_{(ij)}(\omega); \\ 0 & \text{with probability } 1 - a_{(ij)}(\omega). \end{cases}$$

Given an environment $\omega \in \Omega$, we construct a sequence of random bipartite graphs $G_{n,\omega}$ considering the sets of vertices

$$W_{n,\omega} = (w_1(\omega), \dots, w_n(\omega)) \quad \text{and} \quad M_{n,\omega} = (m_1(\omega), \dots, m_n(\omega)),$$

and edges distributions $X_{\omega,(ij)}$ given by the values of $a_{(ij)}(\omega)$ as in (2.1). We denote by $\mathbb{P}_{n,\omega}$ the law of the random graph $G_{n,\omega}$.

Example 2.1. Given a choice of an environment $\omega \in \Omega$, the probability that the bipartite graph $G_{n,\omega}(x)$ equals the complete bipartite graph K_n, using independence of the edge variables, is

$$\mathbb{P}_{n,\omega}\left(G_{n,\omega}(x) = K_n\right) = \prod_{1\leqslant i,j\leqslant n} \mathbb{P}_{n,\omega}(X_{\omega,(ij)} = 1) = \prod_{1\leqslant i,j\leqslant n} a_{(ij)}(\omega).$$

3. Counting Perfect Matchings

Recall that a perfect matching of a graph G is a subset of edges containing every vertex exactly once. We denote by $\mathrm{pm}(G)$ the number of perfect matchings of G. When the graph G is bipartite, and the corresponding partition of the vertices has the form $W = \{w_1, w_2, \dots, w_n\}$ and $M = \{m_1, m_2, \dots, m_n\}$, a perfect matching can be identified with a bijection between W and M, and hece with a permutation $\sigma \in S_n$. From this, the total number of perfect matchings can be computed as

$$\mathrm{pm}(G) = \sum_{\sigma\in S_n} x_{1\sigma(1)}x_{2\sigma(2)}\cdots x_{n\sigma(n)}, \tag{3.1}$$

where x_{ij} are the entries of the incidence matrix X_G of G, that is $x_{ij} = 1$ if (w_i, m_j) is an edge of G and $x_{ij} = 0$ otherwise. Of course, the right hand side of (3.1) is the *permanent*, $\mathrm{per}(X_G)$, of the matrix X_G.

In the framework of Section 2, we estimate the number of perfect matchings for the sequence of random bipartite graphs $G_{n,\omega}$, for a given environment $\omega \in \Omega$. More precisely, we obtain estimates for the growth of the mean of

$$\mathrm{pm}(G_{n,\omega}(x)) = \mathrm{per}(X_{G_{n,\omega}(x)}) = \sum_{\sigma\in S_n} X_{\omega,(1\sigma(1))}\cdots X_{\omega,(n\sigma(n))}. \tag{3.2}$$

Denote by $\mathbb{E}_{n,\omega}$ the expected value with respect to the probability $\mathbb{P}_{n,\omega}$. Since the edges are independent and $\mathbb{E}_{n,\omega}(X_{\omega,(ij)}) = a_{ij}(\omega)$ we have

$$\begin{aligned}\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right) &= \mathbb{E}_{n,\omega}\left(\sum_{\sigma\in S_n} X_{\omega,(1\sigma(1))}\cdots X_{\omega,(n\sigma(n))}\right)\\ &= \sum_{\sigma\in S_n} a_{(1\sigma(1))}(\omega)\cdots a_{(n\sigma(n))}(\omega)\\ &= \mathrm{per}(A_n(\omega)),\end{aligned}$$

where the entries of the matrix are $(A_n(\omega))_{ij} = a_{(ij)}(\omega)$. The main result of this note describes the growth of this expected number for perfect matchings.

The following number is a particular case of a quantity introduced by the authors in a more general setting in [BIP].

Definition 3.1. Let F be an $\alpha \times \beta$ matrix with non-negative entries (f_{rs}). Let $\vec{p} = (p_1, \dots, p_\alpha)$ and $\vec{q} = (q_1, \dots, q_\beta)$ be two stochastic vectors. The *scaling mean* of F with respect to $\vec{p}$ and $\vec{q}$ is defined by

$$\mathrm{sm}_{\vec{p},\vec{q}}(F) := \inf_{(x_r)\in\mathbb{R}_+^\alpha,(y_s)\in\mathbb{R}_+^\beta} \left(\prod_{r=1}^{\alpha} x_r^{-p_r}\right)\left(\prod_{s=1}^{\beta} y_s^{-q_s}\right)\left(\sum_{r=1}^{\alpha}\sum_{s=1}^{\beta} x_r f_{rs} y_s p_r q_s\right).$$

The scaling mean is increasing with respect to the entries of the matrix and lies between the minimum and the maximum of the entries (see [BIP] for details and more properties). Also, the scaling mean can be exponentially approximated using a simple iterative process (the precise value can be computed finding the unique fixed point of a contracting map), and hence, can be implemented with a simple algorithm in short time. It should be stressed that, on the other hand, it has been shown that no such algorithm exists to compute the permanent.

The main result in this note is the following,

Theorem 3.2 (Main Theorem)**.** *Let $(G_{n,\omega})_{n\geqslant 1}$ be a sequence of random bipartite graphs on a random environment $\omega \in \Omega$. If for every pair (r, s) we have $f_{rs} > 0$ then the following pointwise convergence holds*

$$\lim_{n\to\infty} \left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!} \right)^{1/n} = \mathrm{sm}_{\vec{p},\vec{q}}(F), \tag{3.3}$$

for $\mu_W \times \mu_M$-almost every environment $\omega \in \Omega$.

Remark 3.3. As discussed in the introduction Theorem 3.2 shows that there exists a constant $c \in (0, 1)$, such that for almost every environment $\omega \in \Omega$ and for $n \in \mathbb{N}$ sufficiently large

$$\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x))) \asymp n!c^n.$$

Namely $c = \mathrm{sm}_{\vec{p},\vec{q}}(F)$. This result should be compared with the corresponding one obtained by Erdös and Rényi for their class of random graphs, that is

$$\mathbb{E}(\mathrm{pm}(G_n(x))) = n!p^n.$$

Thus, we have shown that for large values of n the growth of the number of perfect matchings for random graphs in a random environment behaves like the simpler model studied by Erdös and Rényi with $p = \mathrm{sm}_{\vec{p},\vec{q}}(F)$.

Remark 3.4. Theorem 3.2 shows that the expected number of perfect matchings is a quenched variable.

Remark 3.5. Using the Stirling formula, the limit in (3.3) can be stated as

$$\lim_{n\to\infty} \left(\frac{1}{n} \log\left(\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)\right) - \log n \right) = \log \mathrm{sm}_{\vec{p},\vec{q}}(F) - 1,$$

which gives a quenched result for the growth of the *perfect matching entropy* for the sequence of graphs $G_{\omega,n}$ (see [ACFG]).

Remark 3.6. Note that we assume a *uniform ellipticity* condition on the values of the probabilities $a_{(ij)}$ as in (2.1). A similar assumption appears in the setting of Random Walks in Random Environment (see [Bog, p.355]).

We now present some concrete examples.

Example 3.7. Let $\alpha = \beta = 2$ and $p_1 = p_2 = q_1 = q_1 = 1/2$. Therefore, the space of environments is the direct product of two copies of the full shift on two symbols endowed with the $(1/2, 1/2)-$Bernoulli measure. The edge distribution matrix F is a 2×2 matrix with entries belonging to $(0, 1)$. In [BIP, Example 2.11], it was shown that

$$\mathrm{sm}_{\vec{p},\vec{q}} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \frac{\sqrt{f_{11}f_{22}} + \sqrt{f_{12}f_{21}}}{2}.$$

Therefore, Theorem 3.2 implies that

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!}\right)^{1/n}=\frac{\sqrt{f_{11}f_{22}}+\sqrt{f_{12}f_{21}}}{2},$$

for almost every environment $\omega\in\Omega$.

Example 3.8. More generally let $\alpha\in\mathbb{N}$ with $\alpha\geqslant 2$ and $\beta=2$. Consider the two stochastic vectors $\vec{p}=(p_1,p_2,\dots,p_\alpha)$ and $\vec{q}=(q_1,q_2)$. The space of environments is the direct product of a full shift on α symbols endowed with the $\vec{p}$-Bernoulli measure with a full shift on two symbols endowed with the $\vec{q}$-Bernoulli measure. The edge distribution matrix F is a $\alpha\times 2$ matrix with entries $f_{r1},f_{r2}\in(0,\infty)$, where $r\in\{1,\dots,\alpha\}$. Denote by $\chi\in\mathbb{R}^+$ the unique positive solution of the equation

$$\sum_{r=1}^{\alpha}\frac{p_r f_{r1}}{f_{r1}+f_{r2}\chi}=q_1.$$

Then

$$\mathrm{sm}_{\vec{p},\vec{q}}(F)=\mathrm{sm}_{\vec{p},\vec{q}}\begin{pmatrix}f_{11}&f_{12}\\ \vdots&\vdots\\ f_{\alpha 1}&f_{\alpha 2}\end{pmatrix}=q_1^{q_1}\left(\frac{q_2}{\chi}\right)^{q_2}\prod_{r=1}^{\alpha}\left(f_{r1}+f_{r2}\chi\right)^{p_r}.$$

Therefore, Theorem 3.2 implies that

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!}\right)^{1/n}=q_1^{q_1}\left(\frac{q_2}{\chi}\right)^{q_2}\prod_{r=1}^{\alpha}\left(f_{r1}+f_{r2}\chi\right)^{p_r},$$

for almost every environment $\omega\in\Omega$. The quantity in the right hand side first appeared in work by Halász and Székely in 1976 [HS], in their study of symmetric means. In [BIP, Theorem 5.1] using a completely different approach we recover their result.

4. Proof of the Theorem

The *shift map* $\sigma_W:\Omega_W\to\Omega_W$ is defined by

$$\sigma_W(w_1,w_2,w_3,\dots)=(w_2,w_3,\dots).$$

The shift map σ_W is a μ_W-preserving, that is, $\mu_W(\Lambda)=\mu_W(\sigma_W^{-1}(\Lambda))$ for every measurable set $\Lambda\subset\Omega_W$, and it is ergodic, that is, if $\Lambda=\sigma_W^{-1}(\Lambda)$ then $\mu_W(\Lambda)$ equals 1 or 0. Analogously for σ_M and μ_M. We define a function $\Phi:\Omega_W\times\Omega_M\to\mathbb{R}$ by

$$\Phi(\vec{w},\vec{m})=f_{w_1m_1}.$$

Thus

$$\Phi(\sigma_W^{i-1}(\vec{w}),\sigma_M^{j-1}(\vec{m}))=f_{w_im_j}=a_{(ij)}(\omega).$$

That is, the matrix $A_n(\omega)$ has entries $a_{(ij)}(\omega)=\Phi(\sigma_W^{i-1}(\vec{w}),\sigma_M^{j-1}(\vec{m}))$. We are in the exact setting in order to apply the Law of Large Permanents see [BIP, Theorem 4.1].

Theorem (Law of Large Permanents). Let (X,μ), (Y,ν) be Lebesgue probability spaces, let $T\colon X\to X$ and $S\colon Y\to Y$ be ergodic measure preserving transformations, and let $g\colon X\times Y\to\mathbb{R}$ be a positive measurable function essentially bounded away from zero and infinity. Then for $\mu \times \nu$-almost every $(x, y) \in X \times Y$, the $n \times n$ matrix

$$M_n(x,y) = \begin{pmatrix} g(x,y) & g(Tx,y) & \cdots & g(T^{n-1}x,y) \\ g(x,Sy) & g(Tx,Sy) & \cdots & g(T^{n-1}x,Sy) \\ \vdots & \vdots & & \vdots \\ g(x,S^{n-1}y) & g(Tx,S^{n-1}y) & \cdots & g(T^{n-1}x,S^{n-1}y) \end{pmatrix}$$

verifies

$$\lim_{n\to\infty} \left(\frac{\mathrm{per}\,(M_n(x,y))}{n!} \right)^{1/n} = \mathrm{sm}_{\mu,\nu}(g)$$

pointwise, where $\mathrm{sm}_{\mu,\nu}(g)$ is the scaling mean of g defined as

$$\mathrm{sm}_{\mu,\nu}(g) = \inf_{\varphi,\psi} \frac{\iint_{X\times Y} \varphi(x) g(x,y) \psi(y) d\mu d\nu}{\exp\left(\int_X \log\varphi(x) d\mu\right) \exp\left(\int_Y \log\psi(y) d\nu\right)},$$

where the functions φ and ψ are assumed to be measurable, positive and such that their logarithms are integrable.

We apply this Law of Large Permanents setting $X = \Omega_W, Y = \Omega_M, T = \sigma_W, S = \sigma_M$, $g = \Phi$ and recalling that $f_{rs} > 0$. We have

$$\mathrm{sm}_{\mu_W,\mu_M}(\Phi) = \mathrm{sm}_{\vec{p},\vec{q}}(F)$$

as a consequence of an alternative characterization of the scaling mean given in (see [BIP, Proposition 3.5]). This concludes the proof of the Main Theorem. ∎

Remark 4.1. We have chosen to present our result in the simplest possible setting. That is, the environment space being products of full-shifts endowed with Bernoulli measures. Using the general form of the Law of Large Permanent above our results can be extended for random graphs in more general random environments.

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