THE DIMENSION THEORY OF NUMBER THEORETICALLY DEFINED SETS

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Abstract. In these notes we study different ways of writing a real number. Namely, continued fractions, backward continued fractions, Lüroth expansion and the $\beta-$expansion. Associated to each expansion there exists a dynamical system. By studying the ergodic properties of each system we are able to compute the Hausdorff dimension (the *size*) of sets defined by number theoretical properties and to determine some number theoretical properties.

1. Introduction

The purpose of these notes is to offer glimpses of the relation between ergodic theory and the dimension theory of dynamical systems. Billinglsey (see [7] and also [6, 10]) was one of the first who formally established this relation. Ever since, the connection between the two theories has become stronger and deeper.

The mathematical core of the theory of dynamical systems is the study of the global orbit structure of maps and flows. In order to analyse a system, structure on the phase space and restrictions on the map (or the flow) are required. Ergodic Theory is the study of dynamical systems for which the phase space is a measure space and the map (or the flow) preserves a probability measure. A large amount of work has been developed over the last years in this area. Not only because the techniques used in the field have proved to be extremely useful to describe the long term behaviour of orbits for a large class of systems, but also, because these techniques have been successfully applied in several other branches of mathematics, most notably in number theory (see for exmple [14]).

An important tool used in ergodic theory is the associated thermodynamic formalism. This is a set of ideas and techniques which derive from statistical mechanics. It can be though of as the study of certain procedures for the choice of invariant measures. Let us stress that a large class of interesting dynamical systems have many invariant measure, hence the problem of choosing relevant ones. The main object on the field is the so called topological pressure.

The dimension theory of dynamical systems has remarkably flourished over the last fifteen years. The main goal of the field is to compute the *size* of dynamically relevant subsets of the phase space. For example, sets where the complicated dynamics is concentrated (*repellers* or *attractors*). Usually, the geometry of these sets is rather complicated. That is why there are

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several notions of *size* that can be used. One could say that a set is *large* if it contains a great deal of disorder on it. Formally, one would say that the dynamical system restricted to that subset has large *entropy.* Another way of measuring the size of a set is using geometrical tools. In order to do so, finer notions of dimension are required. In these notes we will discuss one of the first of such notions, which was introduced around 1919 [17], the so called *Hausdorff dimension.*

In these notes the theory of dynamical systems will be used to determine the *size* of subsets of the real line defined in terms of number theoretical properties. For example, we will study in detail the continued fraction expansion. That is, the unique way of writing a real number $x \in [0,1]$ as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_1 a_2 a_3 \dots],$$

where $a_i \in \mathbb{N}$. One of the main advantages of this way of writing a number is that we can easily obtain the best rational approximations of an irrational number. Associated to this algorithm there exists a dynamical system. By studying the ergodic properties of this system we will be able to compute the size of the set of irrationals which are approximated by rationals at a fixed speed.

We will also study the backward continued fraction. That is, the unique way of writing a real number $x \in [0,1]$ as

$$x = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}}} = [a_1 a_2 a_3 \dots]_B,$$

where the coefficients $\{a_i\}$ are integers such that $a_i > 1$. With techniques coming from the theory of dynamical systems we are able to show that, even though this decomposition has several advantages, it does not provide good rational approximations.

In a different direction we also study the so called Lüroth expansion. Applying methods from ergodic theory we compute the size of sets defined by the frequency at which certain digits appear. It is worth emphasizing that the same methods can be applied to other ways of writing a real number.

We also discuss the base β expansion of a real number. When $\beta = 10$ we obtain the classical decimal expansion. But note that we study the case in which β is an arbitrary real number.

2. Preliminaries from dimension theory and from ergodic theory

We recall in this section all the notions and results form ergodic theory and dimension theory that are needed in these notes.

2.1. **Dimension theory.** We briefly recall some basic definitions and results from dimension theory (see [11, 12, 28] for details). We say that a countable collection of sets $\{U_i\}_{i\in N}$ is a δ-cover of $F \subset \mathbb{R}$ if $F \subset \bigcup_{i\in\mathbb{N}} U_i$, and U_i has diameter $|U_i|$ at most δ for every $i \in \mathbb{N}$. Given $s > 0$, we define

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ a } \delta\text{-cover of } F \right\}.$$

The *Hausdorff dimension* of the set F is defined by

$$\dim_H F = \inf \{s > 0 : \mathcal{H}^s(F) = 0\}.$$

This definition is due to Felix Hausdorff who introduced it in 1919 [17]. Given a finite Borel measure μ in F, the *pointwise dimension* of μ at the point x is defined by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

whenever the limit exists, where $B(x, r)$ is the ball at x of radius r.

Proposition 1. *Given a finite Borel measure μ, if $d_\mu(x) \leq d$ for every $x \in F$, then $\dim_H F \leq d$.*

The *Hausdorff dimension* of the measure μ is defined by

$$\dim_H \mu = \inf \{\dim_H Z : \mu(X \setminus Z) = 0\}.$$

Proposition 2. *Given a finite Borel measure μ, if $d_\mu(x) = d$ for μ-almost every $x \in F$, then $\dim_H \mu = d$.*

2.2. **Thermodynamic formalism.** An important branch within ergodic theory is the so called *thermodynamic formalism.* This is a set of ideas which derive from statistical mechanics. It can be though of as the study of certain procedures for the choice of invariant measures. Let us stress that a large class of interesting dynamical systems (in particular the ones considered in these notes) have *many* invariant measures, hence the problem of choosing relevant ones. The main object in the theory is the *topological pressure* (see Definition 1), which quantifies the disorder of the system. A remarkable result in the field, which ties together topological objets with objects of a measure theoretical nature, is that the topological pressure can be expressed as the supremum of a weighted measure theoretical entropy, where the supremum is taken over the set of all invariant probability measures (see Theorem 3). This result provides a natural way to pick up measures.

Let $T : X \to X$ be a continuous map of the metric space X. Denote by $\mathcal{M}_T$ the set of T-invariant probability measures. Let $\phi : X \to \mathbb{R}$ be a Hölder continuous function, that we will call *potential.*

Definition 1. *The* topological pressure *of the map T at the potential ϕ is defined by*

$$P_T(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x) \right). \quad (1)$$

Let us note that using sub-additivity arguments it is possible to prove that the above limit exists.

In order to understand the definition, let us start considering the null-potential, that is $\phi \equiv 0$. In this case we obtain

$$P_T(0) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} 1.$$

Therefore, $P_T(0)$ quantifies the exponential growth of periodic orbits,

$$\sum_{T^n x = x} 1 \asymp \exp(nP_T(0)).$$

The number $P_T(0)$ is usually called *topological entropy* and it is denoted by $h_{top}(T)$. The topological pressure can be thought of as a weighted topological entropy. Indeed, each point on the periodic orbit $\{x, Tx, T^2x, \dots, T^{n-1}x\}$, is given *weight* $\phi(T^i x)$. The topological pressure quantifies the exponential growth of these weighted periodic orbits,

$$\sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x)\right) \asymp \exp(nP_T(\phi)).$$

The are several properties of the pressure that can easily be deduced from the definition, for instance,

1. If $\phi < \psi$ then $P_T(\phi) \leq P_T(\psi)$.
2. The pressure function $P(\cdot)$ is convex.
3. If $c \in \mathbb{R}$ then $P_T(\phi + c) = P_T(\phi) + c$.
4. $P_T(\phi) = P_T(\phi + \psi \circ T - \psi)$.

One of the most important results in thermodynamic formalism is the so called variational principle. This theorem relates the topological pressure with the measure theoretic entropy. It was first proved by Ruelle [32] and then in full generality by Walters [34].

Theorem 3 (Variational principle)**.** *Let $\phi : X \to \mathbb{R}$ be a Hölder potential, then*

$$P_T(\phi) = \sup\left\{ h(\mu) + \int \phi \, d\mu : \mu \in \mathcal{M}_T \right\}. \tag{2}$$

The number $h(\mu)$ denotes the entropy of the measure μ.

Note that if $\phi \equiv 0$ then we obtain the variational principle for the topological entropy

$$h_{top}(T) = \sup\{h(\mu) : \mu \in \mathcal{M}_T\}.$$

This theorem relates the topological complexity of the system with the measure theoretic complexity of it.

Definition 2. *A measure $\mu \in \mathcal{M}_T$ such that*

$$P_T(\phi) = h(\mu) + \int \phi \, d\mu,$$

is called equilibrium measure *for ϕ.*

Questions about existence, uniqueness and ergodic properties of equilibrium measures are at the core of the theory. In the particular case of hyperbolic dynamical systems and regular potentials (Hölder) the situation it is completely understood, see for example [8, 20, 32, 33],

2.3. **Lyapunov exponent.** Dynamical systems that are sufficiently hyperbolic have the property that almost all orbits move away from each other on time. When the system is defined on a compact space this produces a certain degree of mixing. In such a situation, the orbit structure becomes rather complicated. Therefore, it is of interest to quantify the rate at which the orbits become separated.

In these note we consider (piecewise) differentiable maps $T : I \to I$, where $I \subset \mathbb{R}$ is a bounded union of closed intervals. The *Lyapunov exponent* of the map T at the point $x \in I$ is defined by

$$\lambda_T(x) = \lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|,$$

whenever the limit exists. It measures the exponential rate of divergence of infinitesimally close orbits.

The Birkhoff ergodic theorem implies that if μ is an ergodic $T-$invariant measure, such that $\int \log |T'| \ d\mu$ is finite, then $\lambda_T(x)$ is constant μ-almost everywhere. Nevertheless, it is possible for the Lyapunov exponent to attain a whole interval of values. In this note we address the problem of describing the range of these possible values and computing the size of the level sets determined by the Lyapunov exponent.

There is a deep connection between the pressure function $t \to P(-t \log |T'|)$ and the Lyapunov exponent. Indeed, if μ_t denotes the equilibrium measure corresponding to $-t \log |T'|$ then

$$\frac{d}{dt} P(-t \log |T'|) = \int \log |T'| \ d\mu_t.$$

That is, the derivative of the pressure function can be understood as the Lyapunov exponent of the system T that can be seen with the measure μ_t. This point of view will be used in these notes.

3. Continued fractions

Every real number $x \in (0,1)$ can be written in a unique way as a *continued fraction* of the form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_1 a_2 a_3 \dots],$$

where $a_i \in \mathbb{N}$. This way representing a real number has several advantages with respect to the (more used) decimal system. This is basically due to the fact that this representation is not related to any system of calculation. Therefore it only reflects the properties of the number and not it relationship with a system of calculation. For instance, if the the continued fraction of the number x has only finitely many terms a_n then the number is rational, whereas if it has infinitely many of them then it is irrational. The strong disadvantage of the continued fraction is that there is no simple rule to do arithmetic operations, for instance the sum. By this we mean that there is no simple way of finding the sum of $[a_1a_2a_3\dots]$ with $[b_1b_2b_3\dots]$. For a general account on continued fractions see [16, 22].

Other great advantage of the continued fractions is that it allows us to obtain the best possible rational approximations of an irrational number. To be precise, let us say that a rational number a/b is the best approximation of the real number x if every other rational number with the same or smaller denominator differs from x in a grater amount. In other words, if one defines the complexity of the rational approximation by the size of its denominator, then continued fraction representation allows us to obtain the simpler approximations of a given order. This is the historical reason for the discovery and study of continued fractions. The $n-th$ *approximant* $p_n(x)/q_n(x)$ of the number $x \in [0,1]$ is defined by

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \frac{1}{a_n}}}} \tag{3}$$

A classical result in diophantine approximation states that, if $p_n(x)/q_n(x)$ is the $n-th$ *approximant* of the number $x \in [0,1]$ then

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| < \frac{1}{(q_n(x))^2}.$$

It is possible to associate a dynamical system to the continued fraction system. The Gauss map $G : (0,1] \to (0,1]$, is the interval map defined by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

This map is closely related to the continued fraction expansion. Indeed, for $0 < x < 1$ with $x = [a_1a_2a_3\dots]$ we have that $a_1 = [1/x], a_2 = [1/Gx], \dots, a_n = [1/G^{n-1}x]$. In particular, the Gauss map acts as the shift map on the continued fraction expansion,

$$a_n = \left[1/G^{n-1}x\right].$$

In particular, if $x = [a_1a_2a_3\dots]$ then $Gx = [a_2a_3a_4\dots]$. The Lyapunov exponent of the Gauss map G at the point x , whenever the limit exists, satisfies (see [30])

$$\lambda(x) = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right|, \tag{4}$$

Therefore, the Lyapunov exponent of the Gauss map quantifies the exponential speed of approximation of a number by its approximants (see [30]). That is,

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \asymp \exp\left(-n\lambda(x)\right).$$

There exists an absolutely continuous ergodic $G-$invariant measure, μ_G, called the *Gauss measure*, defined by

$$\mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx,$$

where $A \subset [0,1]$ is a Borel set. From the Birkhoff ergodic theorem we obtain that μ_G-almost everywhere (and hence Lebesgue almost everywhere)

$$\lambda(x) = \frac{\pi^2}{6 \log 2}. \tag{5}$$

Therefore, for Lebesgue almost every point $x \in [0,1]$ we have that

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \asymp \exp\left(\frac{-n\pi^2}{6 \log 2} \right).$$

Note that the range of values of the Lyapunov exponent is

$$\left[2 \log \left(\frac{1+\sqrt{5}}{2} \right), \infty \right)$$

(see [21, 30]). If $x_1 = \frac{-1+\sqrt{5}}{2} = [1,1,1,1,\dots]$ then $\lambda(x_1) = 2\log\left(\frac{1+\sqrt{5}}{2}\right)$. In particular, the set of number for which the frequency of digits equal to 1 in the continued fraction expansion is equal to one, are called *noble numbers*.

For $\alpha \in [2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty)$ consider the level set

$$J(\alpha) = \{x \in [0,1] : \lambda(x) = \alpha\} = \left\{ x \in [0,1] : \lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = \alpha \right\}.$$

Consider also the *irregular set*,

$$J' = \{x \in [0,1] : \text{ the limit } \lambda(x) \text{ does not exists}\}.$$

These level sets induced the so called *multifractal decomposition*

$$[0,1] = (\cup_\alpha J(\alpha)) \cup J'.$$

The function that encodes this decomposition is called *multifractal spectrum* and it is defined by

$$\alpha \to L(\alpha) := \dim_H J(\alpha).$$

By means of the thermodynamic formalism it is possible to describe this function and the decomposition. The following result was obtained by Pollicott and Weiss [30] and by Kessembomer and Stratman [21]

Theorem 4. *The following results are valid for the multifractal decomposition for Lyapunov exponents of the Gauss map,*

1. *for every* $\alpha \in \left(2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty\right)$ *the level set* $J(\alpha)$ *is dense in* $[0,1]$ *and has positive Hausdorff diemension.*
2. *The function* $L(\alpha)$ *is real analytic, it has a unique maximum at* $\alpha = \frac{\pi^2}{6\log 2}$, *it has an inflection point and*

$$\lim_{\alpha \to \infty} L(\alpha) = \frac{1}{2}.$$

3. $\dim_H J' = 1$.

Sketch of proof. The first step in the proof of this theorem is to obtain a good description of the topological pressure. This has been done, among others, by Mayer [26]. Indeed he obtained the following result:

Lemma 1 (Mayer). *The pressure function $t \to P(-t\log|G'(x)|)$ satisfies the following properties,*

1. *If $t < \frac{1}{2}$ then $P(-t\log|G'|) = \infty$.*
2. *If $t > \frac{1}{2}$ then $P(-t\log|G'|)$ is finite, strictly decreasing, strictly convex and real analytic.*
3. *The behaviour at the phase transition $t = \frac{1}{2}$ is the following*
$$\lim_{t\to\frac{1}{2}} P(-t\log|G'|) = \infty.$$
4. *For each $t > \frac{1}{2}$ there exists a unique equilibrium measure μ_t. Moreover, for every open set $O \subset [0,1]$ we have that $\mu_t(O) > 0$.*
5. *The Bowen equation $P(-t\log|G'|) = 0$ has a unique root at $t = 1$.*

The second step is the following, given $\alpha \in \left(2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty\right)$ consider the equilibrium measure $\mu_{t(\alpha)}$ such that
$$\frac{d}{dt}P(-t\log|G'|)\Big|_{t=t(\alpha)} := \int \log|G'|d\mu_{t(\alpha)} = \alpha.$$
Note that this measure exists since $P(-t\log|G'|)$ is real analytic and
$$\lim_{t\to\frac{1}{2}} \frac{d}{dt}P(-t\log|G'|) = \infty \text{ and } \lim_{t\to\infty} \frac{d}{dt}P(-t\log|G'|) = 2\log\left(\frac{1+\sqrt{5}}{2}\right).$$
It turns out that this measure satisfies the following properties:

1. $\mu_{t(\alpha)}(J(\alpha)) = 1$,
2. For every $x \in J(\alpha)$ we have that $d_{\mu_{t(\alpha)}}(x) = \frac{h(\mu_{t(\alpha)})}{\alpha}$.

Therefore
$$\dim_H(J(\alpha)) = \frac{h(\mu_{t(\alpha)})}{\alpha}.$$
Note that since the equilibrium measure gives positive measure to every open set, we immediately obtain that each set $J(\alpha)$ is dense in $[0,1]$.

Now, since the measure $\mu_{t(\alpha)}$ is the equilibrium measure corresponding to the potential $-t(\alpha)\log|G'|$, with the property that $\int \log|G'|d\mu_{t(\alpha)} = \alpha$, we obtain from the variational principle the following formula:
$$L(\alpha) == \frac{1}{\alpha}\inf_{t\in\mathbb{R}}(P(-t\log|T'|) + t\alpha).$$
Hence, the regularity of the pressure function is transfered to the multifractal spectrum. The fact that the irregular set has full Hausdorff dimension follows from the work of Barreira and Schmeling [5]. □

It is worth stressing how remarkable this result is. Note that even though the decomposition is extremely complicated, each level set is dense in the whole space, the function that encodes it, $L(\alpha)$, is as regular as it can be! This phenomenon is usually refer at as *multifractal miracle*. Also note that the irregular part of the system is as large as the system itself. Recall that the set J' has zero measure for every invariant measure, hence it is not relevant from the point of view of ergodic theory (but it is large from the point of view of dimension theory).

4. Backward continued fractions

The map $R : [0,1) \to [0,1)$ is defined by

$$R(x) = \frac{1}{1-x} - \left[\frac{1}{1-x}\right],$$

where $[a]$ denotes the integer part of the number a. It was introduced by Renyi in [31] and we will refer to it as the *Renyi map.* The ergodic properties of this map have been studied, among others, by Adler and Flatto [1] and by Renyi himself [31]. This is a map with infinitely many branches and infinite topological entropy. It has a parabolic fixed point at zero. It is closely related to the backward continued fraction algorithm [15, 13]. Indeed, every irrational number $x \in [0,1)$ has unique infinite backward continued fraction expansion of the form

$$x = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}}} = [a_1 a_2 a_3 \dots]_B,$$

where the coefficients $\{a_i\}$ are integers such that $a_i > 1$. The Renyi map acts as the shift on the backward continued fraction (see [15, 13]). In particular,

$$\text{If } x = [a_1 a_2 a_3 \dots]_B \text{ then } R(x) = [a_2 a_3 \dots]_B.$$

This continued fraction has been used, for example, to obtain results on inhomogenous diophantine approximation (see [29]). Renyi [31] showed that there exists an infinite $\sigma-$finite invariant measure, μ_R, absolutely continuous with respect to the Lebesgue measure. It is defined by

$$\mu_R(A) = \int_A \frac{1}{x}\, dx,$$

where $A \subset [0,1]$ is a Borel set. There is no finite invariant measure absolutely continuous with respect to the Lebesgue measure. As in the case of continued fractions we have can define the $n-th$ *backward approximant* $p_n(x)/q_n(x)$ of the number $x \in [0,1]$ is defined by

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\dots - \frac{1}{a_n}}}} \tag{6}$$

Again, the Lyapunov exponent, $\lambda_R(x)$, of the Renyi map quantifes the exponential speed of approximation of a number by its approximants. That is,

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \asymp \exp\left(-n\lambda_R(x)\right).$$

The range of values that the Lyapunov exponent can attain is $[0, \infty)$. We can decompose the interval in an analogous fashion as in the case of the Gauss map. That is, for $\alpha \in [0, \infty)$ consider

$$J(\alpha) = \{x \in [0, 1] : \lambda_R(x) = \alpha\}$$

Consider also the *irregular set*,

$$J' = \{x \in [0, 1] : \text{ the limit } \lambda_R(x) \text{ does not exists}\}.$$

These level sets induced the so called *multifractal decomposition*

$$[0, 1] = (\cup_\alpha J(\alpha)) \cup J'.$$

The function that encodes this decomposition is called *multifractal spectrum* and it is defined by

$$\alpha \to L_R(\alpha) := \dim_H J(\alpha).$$

By means of a version of the thermodynamic formalism for non-unifomly hyperbolic maps it is possible to describe this multifractal spectrum (see [18]),

Theorem 5. *The following results are valid for the multifractal decomposition of the Renyi map,*

1. *for every $\alpha \in [0, \infty)$ the level set $J(\alpha)$ is dense in $[0, 1]$ and has positive Hausdorff diemension.*
2. *The function $L(\alpha)$ is real analytic, it is stricly decreasing, it has a unique maximum at $\alpha = 0$, it has an inflection point and*
$$\lim_{\alpha \to \infty} L(\alpha) = \frac{1}{2}.$$
3. $\dim_H J' = 1$.

Sketch of proof. The proof of this theorem is, in spirit, the same as the one for the continued fraction. The additional difficulty in this case is that the Renyi map is not uniformly hyperbolic (it has a parabolic fixed point). Therefore the behaviour of the pressure is less regular. We obtain the following [18],

Lemma 2. *The pressure function $t \to P(-t \log |R'(x)|)$ satisfies the following properties,*

1. *If $t < \frac{1}{2}$ then $P(-t \log |R'|) = \infty$.*
2. *If $t > \frac{1}{2}$ then $P(-t \log |R'|)$ is finite, decreasing and convex.*
3. *If $t \in (\frac{1}{2}, 1)$ then $P(-t \log |R'|)$ is positive, strictly decreasing, strictly convex and real analytic.*
4. *For each $t \in (\frac{1}{2}, 1)$ there exists a unique equilibrium measure μ_t. Moreover, for every open set $O \subset [0, 1]$ we have that $\mu_t(O) > 0$.*
5. *If $t > 1$ then $P(-t \log |R'|) = 0$. The only equilibrium measure for $-t \log |R'|$ in this range of values of t is the atomic measure supported at the fixed point $x = 0$.*
6. *The behaviour at the phase transition $t = \frac{1}{2}$ is the following*
$$\lim_{t \to \frac{1}{2}} P(-t \log |R'|) = \infty.$$
The pressure function is differentiable at $t = 1$.

The rest of the proof is similar as in the case of the continued fraction. It is possible to prove that,

$$L(\alpha) = \frac{1}{\alpha} \inf_{t \in \mathbb{R}} (P(-t \log |T'|) + t\alpha).$$

Hence, the regularity of the pressure function is transfered to the multifractal spectrum.

□

Note that for this decomposition we have that Lebesgue almost every point is approximated by its $n-$th approximant at sub-exponential speed. It is a *bad* algorithm to find rational approximations.

5. The Lüroth expansion

Given constants $a_n \geq 2$ for every $n \in \mathbb{N}$, each real number $x \in (0, 1]$ can be written in the form

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1 - 1)a_2} + \cdots + \frac{1}{a_1(a_1 - 1)a_2 \cdots a_{n-1}(a_{n-1} - 1)a_n} + \cdots .$$

This series expansion, called *Lüroth expansion*, was introduced in 1883 by Lüroth [24]. Each irrational number has a unique infinite expansion of this form and each rational number has either a finite expansion or a periodic one. We denote the Lüroth series expansion of $x \in (0, 1)$ by

$$x = [a_1(x)a_2(x) \cdots]_L = [a_1a_2 \cdots]_L.$$

The series is closely related to the dynamics of the *Lüroth map* $T\colon [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} n(n + 1)x - n, & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}], \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \in [\frac{1}{n+1}, \frac{1}{n}]$, then $a_1(x) = n$, and

$$a_k(x) = a_1(T^{k-1}(x)), \tag{7}$$

that is, the Lüroth map acts as a shift on the Lüroth series. The Lebesgue measure is T-invariant and is ergodic (see [19]). For other properties of the Lüroth map see [9].

It is possible to exploit the consequences of identity (7). Namely, by studying the ergodic properties of the map T we will deduce number theoretical properties of the Lüroth expansion.

As in the case of the continued fraction expansion and for the backward continued fraction expansion, it is possible to estimate the speed of convergence by the $n - th$ Lüroth approximant. Let $\lambda(m) := \sum_{n=1}^{\infty} \frac{\log(n(n+1))}{n(n+1)}$.

Theorem 6. *The following properties hold:*

1. *the domain of* L *is* $[\log 2, +\infty)$, *and*

$$L(\gamma) = \frac{1}{\gamma} \inf_{t \in \mathbb{R}} \left[P(-t \log |T'|) + t\gamma \right],$$

where $P(\cdot)$ *denotes the topological pressure with respect to* T;

2. *the spectrum L real analytic, has a unique maximum at $\gamma = \lambda(m)$, has an inflection point, and satisfies*

$$\lim_{\gamma \to +\infty} L(\gamma) = \frac{1}{2};$$

3. $\dim_H J' = 1$.

We also study the Hausdorff dimension of the level sets determined by the frequency of digits in the Lüroth series expansion. More precisely, for each $n, k \in \mathbb{N}$ and $x \in (0,1)$ let

$$\tau_k(x,n) = \text{card}\big\{i \in \{1,\dots,n\} : a_i(x) = k\big\}.$$

Whenever the limit

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x,n)}{n} \tag{8}$$

exists, it is called the *frequency* of the number k in the Lüroth expansion of x. Since the Lebesgue measure is ergodic, by Birkhoff's ergodic theorem, for Lebesgue-almost every $x \in [0,1]$ we have $\tau_n(x) = 1/[n(n-1)]$. Now let $\alpha = (\alpha_1\alpha_2\cdots)$ be a stochastic vector, i.e., a vector such that $\alpha_i \geq 0$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. We consider the set

$$F(\alpha) = \big\{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for each } k \in \mathbb{N}\big\}. \tag{9}$$

We have already seen that if $\alpha_n = 1/[n(n-1)]$ for each $n \in \mathbb{N}$, then the level set $F(\alpha)$ has full Lebesgue measure. Of course, any other level set has zero Lebesgue measure.

However, the sets in (9) can have positive Hausdorff dimension. Our objective is to obtain an explicit formula for the Hausdorff dimension of sets $F(\alpha)$ for an arbitrary stochastic vector α. The following result was obtained by Barreira and Iommi [3] and it can be proved showing that there exists an ergodic measure, say μ_α, concentrated on $F(\alpha)$ such that $\dim_H F(\alpha) = \dim_H \mu_\alpha$.

Theorem 7. *If $\alpha = (\alpha_1\alpha_2\cdots)$ is a stochastic vector such that*

$$\lambda(\mu_\alpha) = \sum_{n=1}^{\infty} \alpha_n \log(n(n+1)) < \infty,$$

then

$$\dim_H F(\alpha) = \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}$$

Proof. By Ruelle's inequality, we have $h(\mu_\alpha) \leq \lambda(\mu_\alpha) < \infty$. This will ensure that all the series considered below are convergent. We note that $\mu_\alpha(F(\alpha)) = 1$. Denote by $\psi \colon \Sigma \to [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(0)$ defin ed by

$$\psi(a_1a_2a_3\cdots) = [a_1a_2a_3\cdots]$$

is a topological conjugacy between the full-shift on a countable alphabet (Σ, σ) and the map T restricted to $[0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(0)$. Moreover, for μ_α-almost every $x \in F(\alpha)$ we have

$$\begin{aligned} d_{\mu_\alpha}(x) &= \lim_{r\to 0} \frac{\log \mu_\alpha(B(x,r))}{\log r} \\ &= \lim_{n\to\infty} \frac{\log \mu_\alpha(\psi(C_{i_1\cdots i_n}))}{\log \prod_{i=1}^{n} |T'(T^i(x))|^{-1}} \\ &= \lim_{n\to\infty} \frac{-\log \prod_{j=1}^{n} \alpha_{i_j}}{\log \prod_{j=1}^{n} i_j(i_j+1)} \\ &= -\frac{\int_{[0,1]} \log \phi \, d\mu_\alpha}{\int_{[0,1]} \log |T'| \, d\mu_\alpha} = \frac{h(\mu_\alpha)}{\lambda(\mu_\alpha)} \\ &= \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}, \end{aligned}$$

where $\phi(x) = \alpha_k$ for each $x \in \psi(C_k)$, where C_k is a cylinder of length 1. The second equality follows from standard arguments in dimension theory (see for example [28]), showing that we can replace the interval $B(x,r)$ by the set $\psi(C_{i_1\cdots i_n})$. The fourth equality follows from Birkhoff's ergodic theorem. This implies that

$$\dim_H \mu_\alpha = \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))},$$

and since $\mu_\alpha(F(\alpha)) = 1$ it follows from Proposition 2 that

$$\dim_H F(\alpha) \geq \dim_H \mu_\alpha = \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}.$$

For each $x \in F(\alpha)$ we have $\tau_k(x) = \alpha_k$ for all k, and hence,

$$\begin{aligned} d_{\mu_\alpha}(x) &= \lim_{n\to\infty} \frac{-\log \prod_{j=1}^{n} \alpha_{i_j}}{\log \prod_{j=1}^{n} i_j(i_j+1)} \\ &= \lim_{n\to\infty} \frac{-\frac{1}{n}\sum_{k=1}^{\infty} \tau_k(x,n) \log \alpha_k}{\frac{1}{n}\sum_{k=1}^{\infty} \tau_k(x,n) \log(k(k+1))} \\ &= \lim_{n\to\infty} \frac{-\sum_{k=1}^{\infty} \frac{\tau_k(x,n)}{n} \log \alpha_k}{\sum_{k=1}^{\infty} \frac{\tau_k(x,n)}{n} \log(k(k+1))} \\ &= \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}. \end{aligned}$$

It follows from Proposition 1 that

$$\dim_H F(\alpha) \leq \frac{-\sum_{n=1}^{\infty} \alpha_n \log \alpha_n}{\sum_{n=1}^{\infty} \alpha_n \log(n(n+1))}$$

This completes the proof of the theorem. □

6. Base m representation of a number

Given $m \in \mathbb{N}$, the base-m expansion of a point $x \in [0,1]$ is given by

$$x = \frac{\epsilon_1(x)}{m} + \frac{\epsilon_2(x)}{m^2} + \frac{\epsilon_3(x)}{m^3} + \cdots ,$$

where $\epsilon_i(x) \in \{0, \dots, m-1\}$ for each i. There is a dynamical system associated to this representation, which as in the case of the Gauss map, acts as a shift map: consider the map $T_m \colon [0,1] \to [0,1]$ defined by $T_m x = mx \pmod 1$. Clearly, $\lambda_{T_m}(x) = \log m$ for every $x \in [0,1]$. As in the case of the Lüroth map we consider for each $n \in \mathbb{N}$, $k \in \{0, \dots m-1\}$ and $x \in (0,1)$ the number

$$\tau_k(x,n) = \text{card}\big\{ i \in \{1, \dots, n\} : a_i(x) = k \big\}.$$

Whenever the limit

$$\tau_k(x) = \lim_{n \to \infty} \frac{\tau_k(x,n)}{n} \tag{10}$$

exists, it is called the *frequency* of the number k in the base $m-$expansion of x. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a stochastic vector. Eggleston [10] proved that the set

$$F_m(\alpha) = \big\{ x \in [0,1] : \tau_{k,m}(x) = \alpha_k \text{ for } k \in \{0, \dots, m-1\} \big\}$$

has Hausdorff dimension

$$\dim_H F_m(\alpha) = \frac{\sum_{k=0}^{m-1} \alpha_k \log \alpha_k}{\log m}.$$

This result was recovered and generalized by Barreira, Saussol and Schmeling [4] using a multidimensional version of multifractal analysis.

Note that the Lebesgue measure is not only in T_m-invariant but also is the measure of maximal entropy. In particular, if $\alpha = (1/m, 1/m, \dots, 1/m)$ then

$$F(\alpha) = 1.$$

7. The β-transformations

Now let $\beta \in \mathbb{R}$ with $\beta > 1$. The *beta transformation* $T_\beta \colon [0,1) \to [0,1)$ is defined by

$$T_\beta(x) = \beta x \pmod 1.$$

We emphasize that in general T_β is not a Markov map. It was shown by Rényi [31] that each $x \in [0,1)$ has a β-expansion

$$x = \frac{\epsilon_1(x)}{\beta} + \frac{\epsilon_2(x)}{\beta^2} + \frac{\epsilon_3(x)}{\beta^3} + \cdots ,$$

where $\epsilon_n(x) = [\beta T_\beta^{n-1}(x)]$ for each n, being $[a]$ he integer part of a. Note that the *digits* in the β-expansion may take values in $\{0, 1, \dots, [\beta]\}$. In general there might exist several ways of writing a number x in the form $x = \sum_{i=1}^{\infty} a_i / \beta^i$, with $a_i \in \{0, 1, \dots, [\beta]\}$.

There exists a unique T_β-invariant measure absolutely continuous with respect to the Lebesgue measure. Moreover, this measure has maximal entropy (see [27]).

8. Comparison of different expansions

In this section we establish relations between different forms of writing a real number, for instance its continued fraction expansion and its β-expansion for $\beta > 1$. More precisely, we consider the following problem:

> Given $x \in [0,1]$ and $n \in \mathbb{N}$, how many partial quotients $k_n(x)$ in the continued fraction expansion of x can be obtained from the first n terms of its β-expansion?

We give an asymptotics for $k_n(x)$ using the theory of dynamical systems. Denote by $\lambda_G(x)$ the Lyapunov exponent of the Gauss map a the point x. In [2] Barreira and Iommi obtained the following asymptotics.

Theorem 8. *For each $x \in (0,1)$ we have*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\lambda_G(x)},$$

whenever the limit exists.

It turns out that the Lyapunov exponent $\lambda_G(x)$ is the exponential speed of approximation of a number by its approximants $p_n(x)/q_n(x)$. By Theorem 8, this implies that if x is well-approximated by rational numbers, then the amount of information about the continued fraction expansion that can be obtained from its β-expansion is small. Moreover, the larger β is (that is, the more symbols we use to code a number x), the more information about the continued fraction expansion we obtain. In the particular case when $\beta = 10$, the statement in Theorem 8 was obtained by Faivre [12] for a particular class of numbers, and by Wu [35] in full generality. Their methods are different from ours. In particular, they never use methods of the theory of dynamical systems.

An immediate corollary of Theorem 8 is that

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta}{\pi^2} \tag{11}$$

for Lebesgue-almost everywhere $x \in (0,1)$. This statement was established by Lochs [23] in the particular case when $\beta = 10$. Of course, the almost everywhere existence of the limit in (11) does not mean that it always exists, or that the value in the right-hand side is the only one attained by the limit in the left-hand side. As an application of the theory of multifractal analysis, for each α we compute the Hausdorff dimension of the sets of points $x \in (0,1)$ for which

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\alpha}.$$

Indeed , if

$$J(\alpha) = \Big\{ x \in [0,1] : \lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log \beta}{\alpha} \Big\},$$

and

$$K = \Big\{ x \in [0,1] : \liminf_{n\to\infty} \frac{k_n(x)}{n} < \limsup_{n\to\infty} \frac{k_n(x)}{n} \Big\}.$$

then

Theorem 9. *For every $\alpha > (1+\sqrt{5})/2$ we have*

$$\dim_H J(\alpha) = h_{\mu_\alpha}(G)/\alpha,$$

where $\dim_H$ *denotes the Hausdorff dimension.*

This extends and improves the results obtained by Wu in [35]. Note that

$$\inf\{\lambda_G(x) : x \in (0,1)\} = \frac{1+\sqrt{5}}{2}.$$

The following is a consequence of results of Barreira and Schmeling in [5].

Theorem 10. *The set K has Hausdorff dimension equal to one.*

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