

① ANOSOV REPRESENTATIONS & DOMINATED SPLITTINGS (j.w. J. Bochi & A. Sambarino)

$\Gamma$  finitely generated group &  $\rho: \Gamma \rightarrow G$  representation with  $G$  Lie group ( $G \subseteq GL(d, \mathbb{R})$ )

GOAL: Understand representations which are: faithful & discrete (ing-  $\rho(\Gamma) \subseteq G$  discrete subset)  
 Particularly interesting is when the embedding  $\rho: \Gamma \rightarrow G$  is quasi-isometric.

Some examples:  $\rightarrow$  (Quasi)-Fuchsian representations  
 $\rightarrow$  Hitchin representations  
 $\rightarrow$  Benoist representations  
 $\rightarrow$  Schottky groups ...  
 $\rightarrow$  Those for which  $G/\rho(\Gamma)$  is compact  
 $\} \rightarrow$  Anosov representations (in the sense of Labourie)

Thm (Weil/Eshermann-Thurston) If  $\rho: \Gamma \rightarrow G$  is faithful & discrete and  $G/\rho(\Gamma)$  is compact  $\Rightarrow \forall \rho'$  small deformation of  $\rho$ ,  $G/\rho'(\Gamma)$  is still compact and  $\rho'$  is F&D (quasi-isom).

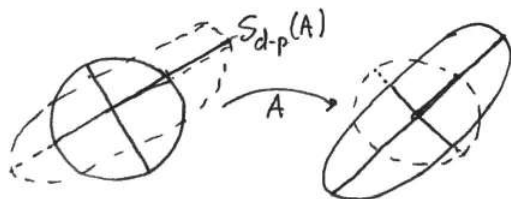
$\rho'$  is a deformation of  $\rho$  if  $\exists \{t_t\}$  continuous path of deformations from  $\rho$  to  $\rho'$ .

Rmk: This is a transversality result. Generalized by Labourie (for  $\pi_1(M)$  with  $M$  negatively curved) and Guichard-Wienhard (for general Gromov hyp. groups).

The generalization involves the geodesic flow of Gromov hyp. groups and equivariant maps from  $\mathcal{D}\Gamma \rightarrow G/P$  with  $P < G$  parabolic.  
 $\leftarrow$  hard to establish its existence.

Makes sense to search for equivalent formulations (GGKW & KLP)

Def  $\rho: \Gamma \rightarrow GL(d, \mathbb{R})$  is p-dominated if  $\exists C, \lambda > 0$  such that if  $\gamma \in \Gamma$  one has  $\frac{\sigma_p(\rho(\gamma))}{\sigma_{p+1}(\rho(\gamma))} > C e^{\lambda|\gamma|}$  where  $\sigma_i(A)$   $i$ th singular value of  $A$   
 $|\gamma|$  word length of  $\gamma \in \Gamma$  resp fixed  $^{(sym)}$  generator  $S$ .  
 sing value: eigenvalue of  $A^t A$



(Rem: No assumption on  $\Gamma$ , just finite gen. set  $S$ .)

(Rem: One can do the same for general Lie groups using Cartan's projection...  
 $p=1$  is the most important case  $\leftarrow$  Reduced via Tits & GW.)

② I will explain the main ideas of our proof (j.w. J. Bochi & A. Sambarino) of the following results (that first appeared in KLP).

Thm 1 The set of  $p$ -dominated representations is open. (And all are quasi-isometric.)  $\square$

Thm 2 If  $\Gamma$  admits a  $p$ -dominated representation  $\rho: \Gamma \rightarrow G \rightarrow \Gamma$  is Gromov hyperbolic and  $\rho$  is  $P$ -Anosov (in the sense of Labourie/Guichard-Wienhard for an adequate parabolic  $P$ )  $\square$   
 $\hookrightarrow$  in part  $\exists$  limit maps from  $\partial\Gamma$  to  $G/P$ .

Both results use DOMINATED SPLITTING (from smooth dynamics) which we will explain.

QUESTION: Does the interior of the set of discrete & faithful representations from  $\Gamma$  (word hyp) to  $GL(d, \mathbb{R})$  contains representations which are not  $p$ -Anosov for no  $p \in \{1, \dots, d-1\}$ .  $\square$

- G. G. K. W. constructs QI repr. from  $\mathbb{F}_2$  which is not stable.
- If  $\text{rank}(G) = 1$ , quasi-isometry  $\Rightarrow$   $p$ -dominated (convex cocompact)
- Avila-Bochi-Yoccoz: In  $SL(2, \mathbb{R})$ , for  $\mathbb{F}_2$ , interior of faithful (and discrete) representations are 1-dominated.
- Sullivan: In  $SL(2, \mathbb{C})$  for  $\mathbb{F}_2$  (and other Kleinian groups), the interior of faithful (and discrete) representations are convex cocompact ( $p$ -dom for some  $p$ ).
- Goldman:  $PSL(2, \mathbb{R})$  for  $\pi_1(\Sigma_g)$  non 1-dominated are either non faithful or non discrete.

Not much more results. Related to STABILITY CONJECTURE IN DIFF. DYNAMICS.

§§ DOMINATED SPLITTING: (Matié ~ 1970, appears in works on diff eqns from 50's, 60's other name)  
 $X$  compact metric space &  $\{\phi^t: X \rightarrow X\}_{t \in \mathbb{T}}$  continuous dynamical system where

$\mathbb{T} = \mathbb{Z}$  or  $\mathbb{R}$ .

Let  $E \xrightarrow{\pi} X$  a vector bundle over  $X$ , with fixed Riem. metric, we denote  $E_x = \pi^{-1}(x)$  the fibers.

action flow  $\{\psi^t: E \rightarrow E\}_{t \in \mathbb{T}}$  is called a linear flow over  $\{\phi^t\}$  if  $\pi \circ \psi^t = \phi^t \circ \pi$  (i.e.  $\psi^t: E_x \rightarrow E_{\phi^t(x)}$ )

and  $\psi^t: E_x \rightarrow E_{\phi^t(x)}$  is a linear automorphism.

Examples:  $\rightarrow E = X \times \mathbb{R}^d$  if  $\mathbb{T} = \mathbb{Z} \Rightarrow \psi^t$  is encoded by a choice  $A: X \rightarrow GL(d, \mathbb{R})$

$\Rightarrow$  providing  $\psi^1(v) = A(x)v$  if  $v \in E_x$ . (linear cocycle over  $\mathbb{T} = \phi^1$ )

$\rightarrow$  If  $E = TM$  and  $\phi^1 = f: M \rightarrow M$  diffeo  $\Rightarrow Df: TM \rightarrow M$  is linear flow ...  $\square$

③ We say that  $\{\psi^t: E \rightarrow E\}$  admits a dominated splitting if  $\exists E = E^{cs} \oplus E^{cu}$  continuous  $\psi^t$ -invariant splitting and constants  $C, \lambda > 0$  such that

$$\forall t > 0 \text{ one has: } \frac{\|\psi^t(v^{cs})\|}{\|v^{cs}\|} \leq \frac{1}{2} \frac{\|\psi^t(v^{cu})\|}{\|v^{cu}\|} \quad \text{where } \begin{matrix} v^{cs} \in E_x^{cs} \setminus \{0\} \\ v^{cu} \in E_x^{cu} \setminus \{0\}. \end{matrix}$$

Typo:  $C \lambda^{nt}$   
instead of  $1/2$

(Rem: Independent of the Riemannian metric.)

The following result was shown by Bochi-Gurmelan generalizing a 2-dim result of Avila-Bochi-Yoccoz  
Correction: the 2-dim result is due to Yoccoz.

Thm (BG) The following are equivalent:

1)  $\psi^t$  admits a dominated splitting with  $\dim E^{cu} = p$ .

2)  $\exists C, \lambda > 0$  s.t.  $\frac{\sigma_p(\psi^t/E_x)}{\sigma_{p+1}(\psi^t/E_x)} > Ce^{\lambda t}$  ( $\sigma_i$  has sense since there is a fixed Riem. metric)

3)  $\exists \psi^t$ -invariant cone field of dimension  $p$ .

Sketch: 1)  $\Leftrightarrow$  3) Classical: "Fibred Perron-Frobenius"

1)  $\Rightarrow$  2) Direct.

2)  $\Rightarrow$  1) Condition 2) implies that  $U_p(\psi^t/E_x)$  converges uniformly ( $\epsilon$  exp) to a continuous  $\psi^t$ -inv bundle  $E^{cu}(x)$ .

Also  $S_{d-p}(\psi^{-t}/E_x) \rightarrow E^{cs}(x)$ .

One needs to show that  $E^{cs}(x) \oplus E^{cu}(x) \forall x \in X$ : Use of Oseledets thm.

Rem GGKW proves similar statement without use of Oseledets thm (condition CLI) and works for p-adic groups. ☑

### §§ PROOF OF THEOREM 1

Quasi-isometry is immediate as the distance in  $GL(d, \mathbb{R})$  is comparable to  $\log \|A\| - \log \det(A)$ . An exponential gap in singular values  $\Rightarrow$  translates into quasi-isometry of the embedding.

To show that p-dominated is open we shall assume that  $\Gamma$  is Gromov hyperbolic and use a properties of such groups which is not elementary. (discrete vers. of geod. flow)

④ Given  $\gamma \in \Gamma$  we define its CONE TYPE as  $C^+(\gamma) = \{\eta \in \Gamma : |\eta\gamma| = |\eta| + |\gamma|\}$   
 (we have fixed sym. generating set  $S$ ).

Fact: If  $C$  is a cone type and  $a \in S \cap C \Rightarrow aC := C^+(a\gamma)$  is well defined.  
 (i.e.  $C = C^+(\gamma)$  for some  $\gamma$ )

For Gromov hyperbolic groups, as a consequence of the classical Morse-lemma it follows that there exists only finitely many cone types (Cannon).

We can associate to  $(\Gamma, S)$  a graph  $\mathcal{G}$  with labels so that:

Vertices: Cone types.

(labeled) edges If  $\exists a \in S \cap C_1$  s.t.  $aC_1 = C_2$  we have  $C_1 \xrightarrow{a} C_2$ .

One gets a finite automaton and if

$\Lambda = \{ \{a_i\}_{i \in \mathbb{Z}} \}$  admissible sequence of labeled edges one has that

$\Lambda$  is a shift invariant subset of  $S^{\mathbb{Z}}$  and contains ALL bi-infinite geodesics passing ~~to~~ through id. This set is "d-dense" in the group  $\Gamma$ .  
(encodes)

One considers the linear cocycle over  $T: \Lambda \rightarrow \Lambda$  shift determined by  $A: \Lambda \rightarrow GL(d, \mathbb{R})$   
(linear flow)  $\{a_i\} \mapsto C(a_0)$

Translating  $p$ -domination one gets that condition (2) of BG-Theorem is satisfied

Since condition (3) is clearly an open condition one obtains that closely representations also give cocycles with dominated splitting. As  $\Lambda$  is "dense" in  $\Gamma$  one recovers the  $p$ -domination for closely representations.

Remark: The cone fields also allow to locate the subspaces, as an application we show that boundary maps vary analytically (result of BCLS), Hölder exponents...

§§ PROOF OF THEOREM 2 (just that  $\Gamma$  is Gromov hyperbolic) We take the following as a definition.

(Bowditch)  $\Gamma$  is a non-elementary Gromov hyp. group  $\iff \exists X$  compact, perfect metric space s.t.  $\Gamma$  acts on  $X$  and the diagonal action of  $\Gamma \curvearrowright X^{(3)}$  (distinct triples) is properly discontinuous and cocompact.

Remark: In this case  $X \cong \partial\Gamma$  equivariantly.

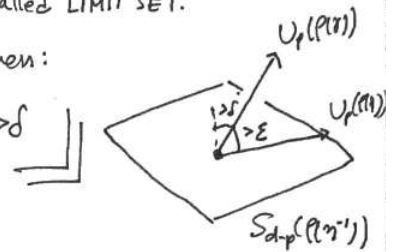
(5) The proof is elementary but quite involved, I'll just present some key ingredients.

Define  $X = \bigcap_{n>0} \overline{\{U_p(\rho(r)) : |r| > n\}} = \{\text{all limits of } U_p(\rho(r_n)) \mid r_n \rightarrow \infty\}$

↳ Benoist for Zariski-dense / GGKW in general called LIMIT SET.

Lemma:  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $r, \eta \in \Gamma$  are large enough, then:

$$d(U_p(\rho(r)), U_p(\rho(\eta))) > \epsilon \implies \angle(U_p(\rho(r)), S_{d-p}(\rho(\eta))) > \delta$$



This allows to push triples together, etc, etc...

To prove this lemma we use BG again,  $\int$  consider

$$D_{C,K,\mu}^P = \left\{ \{A_i\}_{i \in \mathbb{Z}} : \|A_i\| \leq K, \frac{\sigma_p(A_{n+m} \dots A_n)}{\sigma_{p+1}(A_{n+m} \dots A_n)} \geq C e^{\lambda n} \right\}$$

(CLI sequences in GGKW)  
compact & shift invariant subset

The cocycle  $A_0 : D_{C,K,\mu}^P \rightarrow GL(d, \mathbb{R})$  has dominated splitting thanks to BG-Thm and this gives good angles for good sequences.  
 $\{A_i\}_{i \in \mathbb{Z}} \mapsto A_0$

The other ingredient is a linear algebra estimate (that works FOR EVERY group  $\Gamma$ ) which is something like:

$$d(r, \eta) \geq \nu(|r| + |\eta|) - c_0 - c_1 |\log d(U_p(\rho(r)), U_p(\rho(\eta)))|$$

For  $p$ -dominated representations, this is enough to conclude.

Remark: Finer angle estimates also allow us to use these results to obtain a Morse-lemma in the symmetric space (recovering other results of KLP).

FINAL REMARK: All this works in general and does NOT use an important characteristic of (Anosov) representations: ONLY FINITELY MANY MATRICES ARE INVOLVED. In GGKW some results make use of this fact (AMS) Also, in ABY some results use a similar fact and contrast with results false in more general setting. Problem: Understand this better.