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C^1-GENERIC SYMPLECTIC DIFFEOMORPHISMS: PARTIAL HYPERBOLICITY AND ZERO CENTRE LYAPUNOV EXPONENTS

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Abstract We prove that if f is a C^1-generic symplectic diffeomorphism then the Oseledets splitting along almost every orbit is either trivial or partially hyperbolic. In addition, if f is not Anosov then all the exponents in the centre bundle vanish. This establishes in full a result announced by Mañé at the International Congress of Mathematicians in 1983. The main technical novelty is a probabilistic method for the construction of perturbations, using random walks.

Keywords: symplectic diffeomorphisms; partial hyperbolicity; Lyapunov exponents; generic properties

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Contents

1. Introduction

One of the cornerstones of differentiable ergodic theory is the Theorem of Oseledets [**28**]. Given a diffeomorphism $f : M \to M$ of a closed manifold M, a point $x \in M$ is called *regular* if there exists a *Oseledets (or Lyapunov) splitting* $E^1(x) \oplus \cdots \oplus E^{k(x)}(x)$ of the tangent space T_xM, and corresponding *Lyapunov exponents* $\hat{\lambda}_1(x) > \cdots > \hat{\lambda}_{k(x)}(x)$, so that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x) \cdot v\| = \hat{\lambda}_j(x) \quad \text{for all non-zero } v \in E^j(x). \tag{1.1}$$

(Here $\|\cdot\|$ is any Riemannian metric on M.) The Theorem of Oseledets asserts that regular points form a full probability subset R of M (meaning that $\nu(R) = 1$ for any f-invariant probability measure ν). Now, quoting Mañé [**25**]:

> Oseledets' theorem is essentially a measure theoretical result and therefore the information it provides holds only in that category. For instance, the Lyapunov splitting is just a measurable function of the point and the limits defining the Lyapunov exponents are not uniform. It is clear that this is not a deficiency of the theorem but the natural counterweight to its remarkable generality. However, one can pose the problem... of whether these aspects can be substantially improved by working under generic conditions.

These words suggest that a theory of generic dynamical systems must include improved versions of the Oseledets Theorem. Indeed, the paper [**13**] by Viana and the author establishes such a result for the class of volume-preserving C^1-diffeomorphisms.

The present work obtains the C^1-generic improvement of the Oseledets Theorem for the class of *symplectic diffeomorphisms*. Our main result is precisely the strongest one stated and left open by Mañé in 1983 [**25**].

Let $\Lambda \subset M$ be an invariant set for a diffeomorphism $F : M \to M$ of a closed manifold. A Df-invariant splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ into $k \geqslant 2$ non-zero bundles of constant dimensions is called a *dominated splitting* if there is a constant $\tau > 1$ such that, up to a change* of the Riemannian metric on M:

$$\frac{\|Df(x)\cdot v_i\|}{\|v_i\|} > \tau \frac{\|Df(x)\cdot v_j\|}{\|v_j\|} \quad \text{for all } x \in \Lambda, \text{ non-zero } v_i \in E^i(x),\ v_j \in E^j(x) \text{ with } i < j. \tag{1.2}$$

* The usual definition without change of the metric is explained in § 2.1. Here we are using Gourmelon's *adapted metric* [**22**] to simplify the exposition.

Dominated splittings enjoy strong properties: they can be uniquely extended to the closure of Λ, the spaces E^i vary continuously, and the angles between them are uniformly bounded away from zero. Domination is also called *projective hyperbolicity* (see [**11**]).

From now on we assume that the closed manifold M is *symplectic*, that is, it supports a closed non-degenerate 2-form ω. Let $2N$ be the dimension of M. Let $\mathrm{Diff}^1_\omega(M)$ be the space of ω-preserving C^1 diffeomorphisms, endowed with the C^1 topology. Let μ be the measure induced by the volume form $\omega^{\wedge N}$. We assume that ω is normalized so that $\mu(M) = 1$. All the 'almost sure' statements in the sequel refer to this measure.

Here is the generic improvement of the Oseledets Theorem obtained in this paper.

Theorem 1.1. *There exists a residual $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M)$ such that if $f \in \mathcal{R}$ then for almost every point x, the Oseledets splitting $T_xM = E^1(x) \oplus \cdots \oplus E^{k(x)}(x)$ is either trivial or dominated along the orbit of x.*

The first alternative means that $k(x) = 1$, that is, all Lyapunov exponents at x are zero. In the second alternative, we can in fact obtain even sharper information, using the general fact (proven in [**12**]) that *for symplectic maps, dominated splittings are automatically partially hyperbolic.* Let us postpone the precise statement to § 2.1, and explain the consequences for the generic maps from Theorem 1.1.

First, the Lyapunov exponents of any symplectic diffeomorphism are *symmetric*: if λ is an exponent at the point x then so is $-\lambda$, and they have the same multiplicity. (The multiplicity of the Lyapunov exponent $\hat{\lambda}_j(x)$ as in (1.1) is defined as $\dim E^j(x)$.)

From the Oseledets splitting at a regular point x, we form the *zipped Oseledets splitting*:

$$T_xM = E^+(x) \oplus E^0(x) \oplus E^-(x), \tag{1.3}$$

where $E^+(x)$, $E^0(x)$ and $E^-(x)$ are the sums of the spaces $E^j(x)$ corresponding to positive, zero and negative $\hat{\lambda}_j(x)$, respectively. By symplectic symmetry, $\dim E^+(x) = \dim E^-(x)$ and $\dim E^0(x)$ is even.

Assume that the point x is such that the full Oseledets splitting along the orbit of x is dominated. Then so is the zipped splitting $E^+ \oplus E^0 \oplus E^-$. Besides, *the space E^+ is uniformly expanding and the space E^- is uniformly contracting.* In other words, there is a constant $\sigma > 1$ such that, up to a change of the Riemannian metric on M,

$$\left.\begin{aligned} \|Df(y)\cdot v_+\| &\geqslant \sigma\|v_+\| \\ \|Df(y)\cdot v_-\| &\leqslant \sigma^{-1}\|v_-\| \end{aligned}\right\} \quad \text{for all } y = f^n(x),\ n \in \mathbb{Z},\ v_+ \in E^+(y),\ v_- \in E^-(y).$$

We say that the zipped Oseledets splitting is *partially hyperbolic.* It is evident that this is a much stronger conclusion than just the asymptotic expansion/contraction provided by the bare Oseledets Theorem.

In the case that $E^0 = \{0\}$, partial hyperbolicity becomes the usual notion of *uniform hyperbolicity.* Another useful fact (also from [**12**]) is that uniformly hyperbolic sets generically have either zero or full volume. Thus (see § 2.1 for full details) we obtain the following complement of Theorem 1.1:

Corollary 1.2. *A C^1-generic symplectic diffeomorphism f satisfies one and only one of the alternatives below.*

(1) *f is an Anosov diffeomorphism; that is, there exists a uniformly hyperbolic splitting $TM = E^+ \oplus E^-$ that coincides with the zipped Oseledets splitting at almost every point.*

(2) *For almost every point $x \in M$, either all Lyapunov exponents at x are zero, or the zipped Oseledets splitting $T_\Lambda M = E^+ \oplus E^0 \oplus E^-$ over the orbit Λ of x is partially hyperbolic with centre dimension $\dim E^0$ at least 2.*

The statement of Corollary 1.2 is due to Mañé (see [**25**]). Its two-dimensional version, asserting that a generic area-preserving diffeomorphism either is Anosov or has zero metric entropy, was established by the author in [**9**]. Some of the key ideas of the proof in [**9**] came from the outline [**26**] left by Mañé. In [**13**], Viana and the author proved a weaker version of Corollary 1.2 (without the partial hyperbolicity). The paper [**13**] also proves the full version of Theorem 1.1 for volume-preserving diffeomorphisms. (The statement is word-by-word the same, only replacing the symplectic form ω by a volume form.)

There are results of similar nature for volume-preserving and hamiltonian flows (currently only in low dimensions; see [**7**, **8**]) and for linear cocycles (deterministic products of matrices; see [**10**, **13**]).

While this paper is the symplectic counterpart to [**13**], the present proofs required much more than technical adaptations. To achieve our goal, we develop here a new perturbation method that uses random walks. See § 2.3 for an overview. Other examples in the literature where probabilistic arguments are used to find dynamical systems with special properties are [**27**] and [**19**, p. 196].

Let us explore some consequences of the results above. If f is a generic non-Anosov map then the manifold is covered mod 0 by two disjoint invariant sets Z and D such that in Z all exponents vanish, and D can be written as a non-decreasing union $D = \bigcup_{n \in \mathbb{N}} D_n$ of compact invariant sets, each admitting a partially hyperbolic splitting of the tangent bundle, with zero centre exponents. Of course it would be nicer if we could conclude that $\mu(Z) = 1$ or $D_n = M$ for some n. That is the case if one of the following holds:

- if f happens to be ergodic;
- if $\dim M = 2$; then we must have $\mu(Z) = 1$ (so we reobtain the main result from [**9**]);
- if some D_n has non-empty interior; since the generic f is transitive by [**4**], we conclude that $D_n = M$.

There is a fourth situation where we can improve the conclusions of Corollary 1.2: when considering globally partially hyperbolic diffeomorphisms, that is, those that have a partially hyperbolic splitting defined on the whole tangent bundle. (See § 2.1 for the definition.) There is no need to stress their relevance (see, for example, the surveys [**23**] and [**32**]).

Let $\mathrm{PH}^1_\omega(M)$ indicate the (open) subset of $\mathrm{Diff}^1_\omega(M)$ formed by partially hyperbolic maps. Then we have the following theorem.

Theorem 1.3. *For the generic f in $\mathrm{PH}^1_\omega(M)$, there is a partially hyperbolic splitting $TM = E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$ such that all Lyapunov exponents in the centre bundle vanish for almost every point.*

If the partially hyperbolic map f belongs to the residual set given by Corollary 1.2, then to get the conclusion of Theorem 1.3 we have to ensure that $\dim E^0(x)$ is almost everywhere constant. In the lack of ergodicity, the key property we use is *accessibility*, which is known to be C^1 open and dense, by [**21**]. See § 7 for the detailed proof.

Let us now discuss briefly the topic of abundance of ergodicity, and the relevance of Theorem 1.3 in this context.

An important problem in the literature is to determine geometric conditions on a volume preserving dynamics that imply ergodicity of the Lebesgue measure. Partial hyperbolicity seems to be a natural condition to start with. Maybe not much more is needed: Pugh and Shub conjectured in [**29**] that ergodic maps must form a C^2-open and dense set among the partially hyperbolic ones.

Remark 1.4. A more natural (but more difficult) condition to be imposed in the search for ergodicity is the existence of a global dominated splitting. That is so because this condition is satisfied for stably ergodic maps* (see [**3**]) and there exist stably ergodic diffeomorphisms that are not partially hyperbolic (see [**34**]). The situation for *symplectic* maps is simpler, because partial hyperbolicity is the same as dominance. Stably ergodic symplectomorphisms are indeed partially hyperbolic (see [**24**, **33**]).

Improving significantly the results of Pugh and Shub [**29**], Burns and Wilkinson [**17**] gave the following list of conditions that are sufficient for ergodicity: partial hyperbolicity, C^2 smoothness, essential accessibility and centre bunching. The latter condition roughly means that the derivative restricted to the centre bundle is close to conformal.

On the other hand, Theorem 1.3 says that generic maps in $\mathrm{PH}^1_\omega(M)$ have a *non-uniform centre bunching* property (which by semicontinuity is transmitted to nearby C^2 maps). It is natural to ask if this property has interesting consequences. Indeed it does: non-uniform centre bunching is used in [**6**] to prove that *generic diffeomorphisms in* $\mathrm{PH}^1_\omega(M)$ *are ergodic.*

Let us close this introduction with a few comments on the choice of the topology.† For C^r topologies with $r \geqslant 2$, the perturbations we make in this paper definitely do not apply, and indeed the main results do not extend.

The knowledge of C^1-generic dynamics has seen recently very significant progress; see Chapter 10 of [**15**] and the references therein. Despite the fact that some fundamental questions are still open, a broad understanding is perhaps starting to emerge. In contrast, few generic properties are known for topologies C^r with $r > 1$ (with the notable exception of one-dimensional dynamics): even the Closing Lemma is open.

* A (volume-preserving or symplectic) diffeomorphism f is called *stably ergodic* if it is of class C^2 and every C^2 (volume-preserving or symplectic) map sufficiently C^1-close to f is ergodic.

† Here I borrowed some arguments from [**5**].

Sometimes C^1-generic and smoother behaviours are much different. This is especially true for measure-theoretical properties related to distortion. Despite these differences, concrete examples and phenomena that arise from the study of C^1-dynamics often turn out to be important in smoother contexts. Some situations that illustrate this point are the following.

- The concept of dominated splitting in dynamical systems originated from the research of Liao and Mañé on the Smale C^1-stability conjecture. It is increasingly important in smooth ergodic theory: see, for example, [**2**] and also Remark 1.4.

- The proof [**20**] that for every compact manifold other than the circle there is a volume-preserving Bernoulli diffeomorphism uses C^1-perturbation techniques from [**9**].

- The blenders introduced in [**14**] to create new examples of C^1-robustly transitive diffeomorphisms now appear as a ingredient for ergodicity in [**31**].

2. Preliminaries and plan of the proof

2.1. Review on dominated and partially hyperbolic splittings

Let $f : M \to M$ be a C^1 diffeomorphism, and let $\Lambda \subset M$ be an f-invariant set.

A splitting $T_\Lambda M = E \oplus F$ is called *m-dominated*, where $m \in \mathbb{N}$, if it is Df-invariant, the dimensions of E and F are constant and positive, and*

$$\frac{\|Df^m|E(x)\|}{\boldsymbol{m}(Df^m|F(x))} \leqslant \frac{1}{2} \quad \text{for all } x \in \Lambda.$$

We call $T_\Lambda M = E \oplus F$ a *dominated splitting* if it is m-dominated for some m. We also say that E *dominates* F. The dimension of E is called the *index* of the splitting.

More generally, a Df-invariant splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ into non-zero bundles of constant dimensions is called *dominated* if $E^1 \oplus \cdots \oplus E^j$ dominates $E^{j+1} \oplus \cdots \oplus E^k$ for each $j < k$. This definition coincides with the one (1.2) given in the introduction, due to a result of Gourmelon [**22**].

A dominated splitting over the invariant set Λ extends continuously to its closure; so Λ can be assumed to be compact when necessary. See, for example, [**15**] for the proof of this and other properties of dominated splittings.

A Df-invariant splitting $T_\Lambda M = E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$ is called *partially hyperbolic* if it is dominated, the bundle E^{u} is uniformly expanding and the bundle E^{s} is uniformly contracting. The latter two conditions mean that there is a uniform $m \in \mathbb{N}$ such that $\boldsymbol{m}(Df^m|E^{\mathrm{u}}) \geqslant 2$ and $\|Df^m|E^{\mathrm{s}}\| \leqslant \frac{1}{2}$ on Λ. As it is customary, we extend the definition of partial hyperbolicity to allow E^{c} to be $\{0\}$, that is, to include uniform hyperbolicity.

Let us mention an equivalent definition of partial hyperbolicity that is also frequent in the literature: there is a Riemannian metric $\|\cdot\|$ on M (called an *adapted metric*) and

* The *co-norm* of a linear map A is $\boldsymbol{m}(A) = \inf_{\|v\|=1} \|Av\|$; it equals $\|A^{-1}\|^{-1}$ if A is invertible.

continuous functions α, β, γ, δ on the compact set Λ such that the following inequalities hold at each point of Λ:

$$\left.\begin{aligned} \alpha > 1 > \delta, \\ \boldsymbol{m}(Df|E^{\mathrm{u}}) \geqslant \alpha > \beta > \|Df|E^{\mathrm{c}}\| \geqslant \boldsymbol{m}(Df|E^{\mathrm{c}}) \geqslant \gamma > \delta \geqslant \|Df|E^{\mathrm{s}}\|. \end{aligned}\right\} \tag{2.1}$$

The equivalence of the two definitions is shown in [**22**].

Remark 2.1. If one asks α, β, γ, δ in (2.1) to be constants, then one has a stronger notion of partial hyperbolicity, called *absolute*. The weaker notion used in this paper is called *relative* (or *pointwise*) partial hyperbolicity. See [**1**] for a detailed discussion.

The precise meaning of the sentence 'dominated splittings are automatically partially hyperbolic in the symplectic case' is given by the following result.

Theorem 2.2 (Theorem 11 in [12]). *Let f be a symplectic diffeomorphism and let $T_\Lambda M = E \oplus F$ be a dominated splitting over a f-invariant set Λ. Assume $\dim E \leqslant \dim F$ and let $E^{\mathrm{u}} = E$. Then F splits invariantly as $E^{\mathrm{c}} \oplus E^{\mathrm{s}}$ with $\dim E^{\mathrm{u}} = \dim E^{\mathrm{s}}$, and the splitting $T_\Lambda M = E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$ is partially hyperbolic.*

Theorem 2.3 (Corollary B.1 in [12]). *A hyperbolic set of a generic symplectic diffeomorphism has either zero or full volume.*

It is now easy how Corollary 1.2 reduces to Theorem 1.1.

Proof of Corollary 1.2. By Theorem 2.3, there is a residual subset $\mathcal{R}_1 \subset \mathrm{Diff}^1_\omega(M)$ formed by maps that either are Anosov or have no hyperbolic sets of positive measure. Let $\mathcal{R}_2$ be residual set given by Theorem 1.1, and let $f \in \mathcal{R}_1 \cap \mathcal{R}_2$. By Theorem 2.2, the zipped Oseledets splitting along the orbit of almost every point x is either uniformly hyperbolic (if $\dim E^0(x) = 0$), or partially hyperbolic with three non-zero bundles (if $2 \leqslant \dim E^0(x) \leqslant 2N-2$), or trivial (if $\dim E^0(x) = 2N$). The first option occurs for a positive measure set if and only if f is Anosov. So f satisfies the stated conclusions. $\square$

2.2. Discontinuity of the Lyapunov exponents

Given $f \in \mathrm{Diff}^1_\omega(M)$ and a regular point $x \in M$, rewrite the list of Lyapunov exponents in non-increasing order and repeating each according to its multiplicity:

$$\lambda_1(f,x) \geqslant \cdots \geqslant \lambda_{2N}(f,x).$$

For $p = 1, \ldots, N$, we consider the *integrated p-exponent* of the diffeomorphism f:

$$\mathrm{LE}_p(f) = \int_M (\lambda_1(f,x) + \cdots + \lambda_p(f,x))\,\mathrm{d}\mu(x).$$

The map $\mathrm{LE}_p : \mathrm{Diff}^1_\omega(M) \to \mathbb{R}$ is upper-semicontinuous, and therefore its points of continuity constitute a residual subset of $\mathrm{Diff}^1_\omega(M)$. On the other hand, continuity of the integrated exponents has strong consequences.

Theorem 2.4. *Let $f \in \mathrm{Diff}^1_\omega(M)$ be such that each map $\mathrm{LE}_1, \dots, \mathrm{LE}_N$ is continuous at f. Then for μ-almost every $x \in M$, the Oseledets splitting of f is either dominated or trivial along the orbit of x.*

The main result we prove is Theorem 2.4, and Theorem 1.1 is itself an immediate corollary. Theorem 2.4 has a more quantitative version, Proposition 6.3, which is used in the proof of Theorem 1.3.

2.3. A preview of the proof

This subsection contains an informal outline of the proof of Theorem 2.4. It is logically independent from the rest of the paper. However, it should help the reader to go through the complete proof.

Assume that the Oseledets splitting of a symplectic diffeomorphism f is non-trivial and not dominated. To prove Theorem 2.4 (and hence Theorem 1.1), we need to show that for some p, the integrated exponent LE_p is discontinuous at f. The proof has two parts.

(1) Assume that the Oseledets splitting $T_{\mathrm{orb(x)}}M = E^1 \oplus \cdots \oplus E^k$ along the orbit of some point x is non-trivial and not dominated: that is, for some i, $E = E^1 \oplus \cdots \oplus E^i$ does not dominate $F = E^{i+1} \oplus \cdots \oplus E^k$. Let $p = \dim E$; for symplectic reasons it suffices to consider the case $p \leqslant N = \frac{1}{2}\dim M$.

Some positive iterate y of x will enter a zone where the non-dominance of the splitting $E \oplus F$ manifests itself. (More on this later.) Then one constructs by hand a C^1-perturbation g of f with the following properties. For some $m \in \mathbb{N}$, $Dg^m(y)$ sends some (non-zero) vector in the space E into the space F. The support of the perturbation is a small neighbourhood $U \sqcup f(U) \sqcup \cdots \sqcup f^{m-1}(U)$ (called a tower) of the orbit segment $\{y, \dots, f^{m-1}y\}$. Furthermore, it is important that some vectors from $E(\tilde{y})$ are sent by $Dg^m(\tilde{y})$ into $F(\tilde{y})$ not only at the point $\tilde{y} = y$, but also for most (in the sense of measure) points $\tilde{y}$ in the base U of the tower.

(2) The global procedure is to cover most of the manifold by many disjoint tall and thin towers. Approximately in the middle of each tower, a perturbation as sketched in part (1) above is performed. The result is the different expansion rates of E and F are blended, and the integrated p-exponent of the new diffeomorphism dropped. So one concludes that LE_p is discontinuous at f, as desired.

This general strategy is the same followed in the papers [**9**] and [**13**]. More detailed (and still informal) descriptions of it can be found in [**11**] and [**12**]. It is clear that the methods would fail for topologies finer than C^1.

To explain the difficulties of the symplectic case, let us return to the first step of the strategy, and look more closely at how the non-dominance of the splitting $E \oplus F$ manifests itself at the point y. There are four possibilities.

(I) Either the angle $\measuredangle(E, F)$ gets very small at y.

(II) Or there is some $m \in \mathbb{N}$ and there are unit vectors $v \in E(y)$, $w \in F(y)$ such that w gets much more expanded than v by $Df^m(y)$.

(III) Or there is some large $m \in \mathbb{N}$ and there are non-zero vectors $v \in E(y)$ and $w \in F(y)$ with $\omega(v, w) \neq 0$ and such that no vector in the plane P spanned by them gets much expanded nor contracted by $Df^j(y)$ for all $j = 1, \ldots, m$. This means that after a bounded change of the Riemannian metric, the restriction of $Df^j(y)$ to P becomes an isometry, for all $j = 1, \ldots, m$. Notice the symplectic form ω restricted to P is non-degenerate (because $\omega(v, w) \neq 0$).

(IV) Or there is some large $m \in \mathbb{N}$ and there are non-zero vectors $v \in E(y)$ and $w \in F(y)$ spanning a plane P that is (up to time m) *uniformly expanding* and *conformal*. That is, there exists $\tau > 1$ such that after a bounded change of the Riemannian metric we have that $Df^j(y)/\|Df^j(y)\|$ is an isometry and $\|Df^j(y)\| \geqslant \tau^j$ for all $j = 1, \ldots, m$. Since the plane P is expanded it must be *null* (meaning that the symplectic form vanishes on $P \times P$).

Let us explain how in each case one sends a vector from E into F by perturbing f. Since we will work on very small neighbourhoods of a segment of orbit, we can assume f is locally linear.

In case (I), one composes f with a small rotation supported around y. Let us be a little more precise. If $\dim M = 2$, pretend $M = \mathbb{R}^2$ and $y = 0$, and let $\alpha = \measuredangle(E(y), F(y))$; then the perturbation will be given by $g(x) = f(R_{\theta(x)}(x))$, where θ vanishes outside a small disk $D = B_r(0)$ and is constant equal to α on a smaller $D_1 = B_{r_1}(0)$. It is very important that the measure of the *buffer* $D \setminus D_1$ is small compared with that of the support D. For $\dim M > 2$, the rotation is made around a codimension 2 axis, and disks are replaced by cylinders.

The second case is similar: we make two rotations, one around y and other around $f^m y$.

Case (III) is more delicate: one has to make small rotations around each of the points $y, fy, \ldots, f^{m-1}y$. The rotations must be *nested*, that is, the buffer of each rotation is mapped by f to the next buffer. (This is necessary to control the measure of the set where the perturbation will be effective.) Since the ambient space M has dimension $2N > 2$, each rotation is around a $(2N-2)$-dimensional axis X, and the actual support is a thin cylinder along X. Moreover, in order to preserve the symplectic form, X needs to be the symplectic complement of the plane P. Thus the fact that ω is non-degenerate on P is also used.

The treatment of the first three cases explained above is the same as in [**13**]. In fact, case (IV) does not occur if $\dim E = \dim F$. That is the precise reason why it does not appear in [**13**]. (Let us remark that in the *volume-preserving* situation dealt with in [**13**] there are only three cases, similar to those explained above. The construction of the nested rotations has some extra subtleties, however.)

The main novelty of the present paper is a perturbation method that permits us to treat the case (IV). Before explaining it, let us see what the difficulties are.

It seems natural to try nested rotations again in case (IV), because Df acts conformally on the plane P. However, a linear map that rotates P and is the identity on a space complementary to P *cannot* preserve the symplectic form. The reason is that P is a null space. To preserve the symplectic form, one also needs to rotate another two-dimensional space Q; then the linear map can be taken as the identity on a certain 'axis' of dimension $(2N-4)$ (that is the symplectic complement of $P \oplus Q$). Thus the situation becomes essentially four dimensional. Indeed, let us from now on assume $\dim M = 4$ (and pretend that $M = \mathbb{R}^4$) to simplify the discussion. Therefore, $\dim E = 1$ and $\dim F = 3$.

Standard symplectic coordinates p_1, p_2, q_1, q_2 on $\mathbb{R}^4$ can be found with the following properties: the p_1p_2- and q_1q_2-planes are P and Q, respectively, E is the p_1-axis, and F is the space $p_2q_1q_2$. Moreover, the derivatives take the following form:

$$Df(f^i y) : (p_1, p_2, q_1, q_2) \mapsto (\tau_i p_1, \tau_i p_2, \tau_i^{-1} q_1, \tau_i^{-1} q_2), \quad \text{where } \tau_i \geqslant \tau > 1$$

(for $1 \leqslant 1 \leqslant m$). So the splitting $P \oplus Q$ has a uniformly hyperbolic behaviour: P is expanded and Q is contracted.

Now start with a nice domain D (say, a disk in the plane P times a disk of the same size in the plane Q) for the support for the first perturbation. By the uniform hyperbolicity of the splitting $P \oplus Q$, the images $Df^i(y)(D)$ get quickly very deformed. Nesting means that the effective support (that is, the support minus the buffer) of each perturbation is the f-image of the previous one. But the perturbations must also be C^1-small, so it becomes hard to rotate P and Q by a fixed angle. This is the main obstacle for the use of nested rotations in case (IV). (And there is another, more subtle, obstacle: if the support is a box D as above, it is unclear how to rotate by a constant angle while keeping a small buffer. That is because the rotations we want arise from the linear flow generated by the hamiltonian $H = p_2q_1 - p_1q_2$, and since this quadratic form has no definite sign, it cannot be flattened outside of D like in the proof of Lemma 5.5 from [**13**].)

Finally, let us explain the main idea. We abandon nested rotations and buffers.

Start with a small box neighbourhood D of y as above, and consider the field of directions v_0 spanned by the constant vector field $\partial/\partial p_1$. Due to the hyperbolicity of the splitting $P \oplus Q$, there is a strictly invariant cone around the expanding space P. (Of course the cone field will be also invariant under a perturbation g of f.) Given two directions in the cone, we project them on P along Q, and measure the obtained oriented angle; let us call this the p_1p_2-angle between the two directions. Notice f preserves p_1p_2-angles.

Take a symplectic diffeomorphism $h_0 : \mathbb{R}^4 \to \mathbb{R}^4$ that is C^1-close to the identity, is the identity outside of D, and does *not* leave the field v_0 invariant. The perturbation of f in the neighbourhood of y is $g = f \circ h_0$. *Any* h_0 with those properties works, and will be the base for the rest of the construction.

The perturbation around $f(y)$ must be supported on $f(D) = g(D)$. On $g(D)$ we have a field of directions v_1 that is the image of the constant field v_0 by Dg.

Then take many disjoint boxes $D_i \subset g(D)$ covering all of $g(D)$, except for a set of very small measure. The boxes are taken so small so that the variation of the field v_1 on each of them is very small. So let us pretend that the linefield v_1 is constant in each D_i. (See figure 1.)

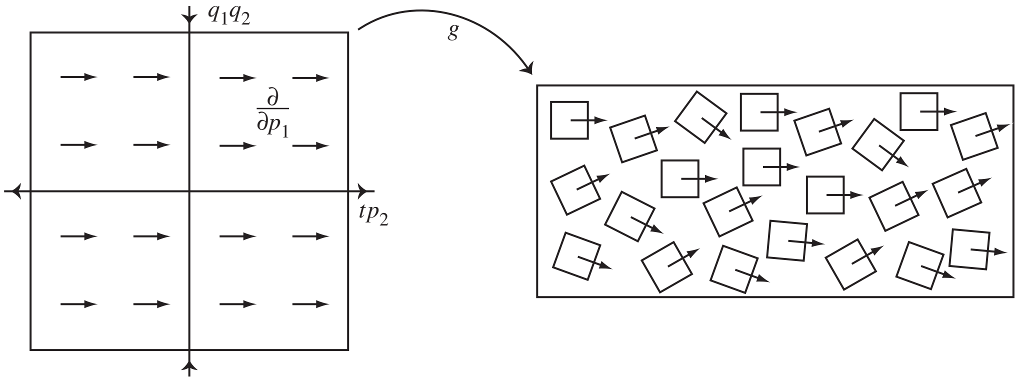

Figure 1. The first step of the perturbation: disjoint boxes D_i cover most of the image of the box D.

Each D_i is a shrunk copy of D: there is an affine map $T_i : D_i \to D$ that takes v_1 to v_0. Let h_1 be a map that equals $T_i^{-1} \circ h_0 \circ T_i$ on each D_i, and the identity outside of $\bigcup D_i$. With the necessary precautions, h_1 becomes symplectic and C^1-close to the identity. Now define the perturbation g on $g(D)$ as equal to $f \circ h_1$.

Let X_0 and X_1 be the p_1p_2-angles turned in the first and second steps, respectively. That is, for $x \in D$, let $X_0(x)$ be the (oriented) p_1p_2-angle between v_0 and $Dh(x) \cdot v_0$, and let $X_1(x)$ be the (oriented) p_1p_2-angle between $v_1(g(x))$ and $Dh_1(g(x)) \cdot v_1(g(x))$. Notice that X_0 is not identically zero by construction. Since the linefield v_0 is Df-invariant, the p_1p_2-angle between v_0 and $Dg(x) \cdot v_0$ equals X_0. Also, the p_1p_2-angle between v_0 and $Dg^2(x) \cdot v_0$ is $X_0 + X_1$.

Let us rescale Lebesgue measure μ so that $\mu(D) = 1$. So X_0 and X_1 can be thought as a *random variables*. The key observation is that they are *independent and identically distributed*.

We continue in an analogous way: in the next step we cover each $g(D_i)$ by still smaller boxes D_{ij}, each of them so that the field of directions $v_2 = Dg \cdot v_1$ is almost constant. In each D_{ij} the perturbation g is modelled on the map h_0 as described above. Continuing in this way, we obtain sequences of maps $g : g^i(D) \to g^{i+1}(D)$ and independent and identically distributed random variables X_i such that Dg^n turns the vector $\partial/\partial p_1$ by an angle $S_n = X_0 + \cdots + X_{n-1}$ in the p_1p_2-plane.

This construction gives a *random walk* S_n on the real line. The probability that a path of the random walk says for all time confined in some compact interval is zero. Moreover, the steps X_n are small. Thus for almost every orbit there is a first time the angle S_n becomes close to $\pm\pi/2$. Then we modify the construction: we perturb one last time to make the angle exactly $\pm\pi/2$, and then perturb no more along that orbit. In other words, the angles behave as a random walk with absorbing barriers around $\pm\pi/2$.

The conclusion is that in some large but finite time, for the majority of orbits of g, the images of the vector $\partial/\partial p_1$ in E eventually have p_1p_2-angle equal to $\pm\pi/2$, and this means the one-dimensional space E has been sent into the three-dimensional space $p_2q_1q_2$, that is, F. So the perturbation g has the desired properties, and case (IV) is settled.

2.4. Organization of the rest of the paper

As explained in § 2.3, the proof of Theorem 2.4 splits into a local and a global part. The local part of the proof fills §§ 3–5.

In § 3 we introduce the ad hoc concept of flexibility, which summarizes the properties our perturbations need to have. (Namely, to make two bundles of a splitting collide for a set of points of large measure.) Flexibility replaces the notion of realizable sequences from [**13**], which is not sufficient for our purposes.

In § 4 we show that lack of dominance can be classified in four types. The proof consists of symplectic linear algebra.

In § 5 we show that each of the four cases has the desired flexibility property. The fourth case is dealt with in § 5.4, where the probabilistic method for the construction of the perturbations is explained in detail.

In § 6 we complete the proof of Theorem 2.4 giving its global part. This part is essentially contained in [**13**], but we will present a simplified proof.

In the final section (§ 7) we prove Theorem 1.3.

3. Flexibility

3.1. Split sequences on $\mathbb{R}^{2N}$ and the flexibility property

Let N be fixed. We consider $\mathbb{R}^{2N} = \{(p_1, \dots, p_N, q_1, \dots, q_N)\}$ endowed with the standard symplectic form $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$, and with Lebesgue measure μ. The euclidian norm on $\mathbb{R}^{2N}$ and also the induced operator norm are indicated by $\|\cdot\|$.

A *split sequence* of length n is composed of the following objects:

- a (finite) sequence of linear ω-preserving maps
$$\mathbb{R}^{2N} \xrightarrow{A_0} \mathbb{R}^{2N} \xrightarrow{A_1} \cdots \xrightarrow{A_{n-1}} \mathbb{R}^{2N};$$
- non-trivial linear splittings $\mathbb{R}^{2N} = E_i^1 \oplus E_i^2$, for $0 \leqslant i \leqslant n$, that are invariant in the sense that $A_i \cdot E_i^* = E_{i+1}^*$, $* = 1, 2$.

The constant $p = \dim E_i$ is called the *index* of the split sequence.

Let $\varepsilon > 0$ and $\kappa > 0$. We say that a split sequence $\{A_i, E_i^{1,2}\}$ of length n is (ε, κ)-*flexible* if for every $\gamma > 0$, there exists a bounded open neighbourhood U of 0 in $\mathbb{R}^{2N}$ and there exist symplectomorphisms $g_0, \dots, g_{n-1} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that:

(1) g_i equals A_i outside $A_{i-1} \circ\circ\circ A_0(U)$ for each $i = 0, \dots, n-1$;*

(2) $\|D(A_i^{-1} \circ g_i) - \mathrm{Id}\,\| < \varepsilon$ uniformly, for each $i = 0, \dots, n-1$;

(3) there is a set $G \subset U$ such that $\mu(G) > (1-\kappa)\mu(U)$ and†
$$\measuredangle(D(g_{n-1} \circ\circ\circ g_0)(x) \cdot E_0^1, E_n^2) < \gamma \quad \forall x \in G.$$

* Note that '$\circ\circ\circ$' is shorthand for '$\circ \cdots \circ$'.

† The angle $\measuredangle(E, F) \in [0, \pi/2]$ between non-zero linear subspaces E, $F \subset \mathbb{R}^{2N}$ is defined as the minimum of the angles $\measuredangle(v, w)$ over non-zero vectors $v \in E$, $w \in F$.

Informally, the linear maps A_i can be (nonlinearly) perturbed so that the space E^1 is sent after time n very close to the space E^2, for most points in the support of the perturbation.

Remark 3.1. Flexibility appears implicitly in [**13**]. The main difference is that in all situations considered there, the map $x \mapsto D(g_{n-1} \circ \circ \circ g_0)(x)$ is approximately (to error γ) constant on G. This will not be always the case here.

Loosely speaking, the next lemma says that flexibility is preserved by changes of coordinates.

Lemma 3.2. *Consider two split sequences of the same length:*

$$\{E_i^1 \oplus E_i^2 \xrightarrow{A_i} E_{i+1}^1 \oplus E_{i+1}^2\}_{0 \leqslant i < n} \quad \textit{and} \quad \{F_i^1 \oplus F_i^2 \xrightarrow{B_i} F_{i+1}^1 \oplus F_{i+1}^2\}_{0 \leqslant i < n}.$$

Assume that there are linear symplectic maps $C_0, \ldots, C_n : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that $C_{i+1} \circ A_i = B_i \circ C_i$ and $C_i(E_i^*) = F_i^*$. Let $K = \max_i \|C_i\|$. If the split sequence $\{A_i, E_i^{1,2}\}$ is (ε, κ)-flexible then $\{B_i, F_i^{1,2}\}$ is $(K^2\varepsilon, \kappa)$-flexible.

Proof. The proof is straightforward, but let us give it anyway. Given $\gamma > 0$, let U, g_i and G be given by the (ε, κ)-flexibility of the sequence $\{A_i, E_i^{1,2}\}$. Define $\hat{U} = C_0(U)$, $\hat{g}_i = C_{i+1} \circ g_i \circ C_i^{-1}$, and $\hat{G} = C_0(G)$. Let us check that these objects satisfy conditions (1), (2) and (3) in the definition of $(K^2\varepsilon, \kappa)$-flexibility. The first one is obvious. Since the linear map C_i is symplectic, $\|C_i\| = \|C_i^{-1}\|$ and so

$$\|D(B_i^{-1} \circ \hat{g}_i) - \mathrm{Id}\,\| \leqslant \|C_i \circ (D(A_i^{-1} \circ g_i) - \mathrm{Id}) \circ C_i^{-1}\| < K^2\varepsilon,$$

which is condition (2). Given $y \in \hat{G}$, let $x = C_0^{-1}(y)$. The spaces $D(\hat{g}_{n-1} \circ \circ \circ \hat{g}_0)(y) \cdot F_0^1$ and F_n^2 are the respective images by C_n of the spaces $D(g_{n-1} \circ \circ \circ g_0)(x) \cdot E_0^1$ and E_n^2. The angle between the latter pair of spaces is less than γ, therefore the angle formed by the earlier pair is at most $K'\gamma$, where $K' = K'(K)$. (In fact, $K' = \frac{1}{2}\pi K^2$ works: see [**13**, Lemma 2.7].) Since $\gamma > 0$ was arbitrarily chosen, condition (3) is verified. $\square$

The following lemma is trivial.

Lemma 3.3. *Let $\{E_i^1 \oplus E_i^2 \xrightarrow{A_i} E_{i+1}^1 \oplus E_{i+1}^2\}_{0 \leqslant i < n}$ be a split sequence. If there are $0 \leqslant i_0 < i_1 \leqslant n$ such that the shorter split sequence $\{E_i^1 \oplus E_i^2 \xrightarrow{A_i} E_{i+1}^1 \oplus E_{i+1}^2\}_{i_0 \leqslant i < i_1}$ is (ε, κ)-flexible, then so is the whole split sequence of length n.*

The next lemma says that the domain U in the definition of flexibility can be chosen arbitrarily.

Lemma 3.4. *Assume that $\{A_i, E_i^{1,2}\}$ is an $(\varepsilon, \kappa/2)$-flexible split sequence of length n. Then for any non-empty bounded open set $U \subset \mathbb{R}^{2N}$ and any $\gamma > 0$, there exist maps $g_0, \ldots, g_{n-1} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ satisfying the three conditions in the definition of (ε, κ)-flexibility.*

Proof. Given $\gamma > 0$, the $(\varepsilon, \kappa/2)$-flexibility of the splitting sequence $\{A_i, E_i^*\}$ provides a set $\hat{U}$ and symplectomorphisms $\hat{g}_0, \dots, \hat{g}_{n-1}$ with the following properties: (1) each $\hat{g}_i$ equals A_i outside $A_{i-1} \circ\circ\circ A_0(U)$; (2) the derivative of $A_i^{-1} \circ \hat{g}_i$ is ε-close to the identity; and (3) the image of E_0^1 by the derivative of $\hat{g}_{n-1} \circ\circ\circ \hat{g}_0$ is γ-close to E_n^2 for all points in a set $\hat{G}$ with measure at least $(1 - \kappa/2)\mu(\hat{U})$.

Now fix some non-empty bounded open set U. By the Vitali Covering Lemma, we can find a finite family of disjoint sets $\hat{U}_j \subset U$ such that the measure of $U \setminus \bigsqcup_j \hat{U}_j$ is less than $\frac{1}{2}\kappa\mu(U)$, and each $\hat{U}_j$ is equal to $T_j(\hat{U})$, where $T_j : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is a homothety composed with a translation.

For $i = 0, \dots, n-1$, let

$$T_{j,i} = A_{i-1} \circ\circ\circ A_0 \circ T_j \circ (A_{i-1} \circ\circ\circ A_0)^{-1}.$$

Of course, $T_{j,i}$ is a homothety composed with a translation. Define $g_i : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ as equal to A_i outside $\bigsqcup_j A_{i-1} \circ\circ\circ A_0(\hat{U}_j)$ and equal to

$$A_i \circ T_{j,i} \circ A_i^{-1} \circ \hat{g}_i \circ T_{j,i}^{-1}$$

inside each $A_{i-1} \circ\circ\circ A_0(\hat{U}_j)$. Let us see that these maps satisfy the three conditions in the definition of (ε, κ)-flexibility. The first one is obvious. We have $D(A_i^{-1} \circ g_i)(x) = D(A_i^{-1} \circ \hat{g}_i)(T_{j,i}^{-1}(x))$, on $A_{i-1} \circ\circ\circ A_0(\hat{U}_j)$, so the second condition holds (and g_i is symplectic). Finally, let $G = \bigsqcup_j T_j(\hat{G})$ Then $\mu(G) > (1 - \kappa/2)^2\mu(U)$. Moreover, the image of E_0^1 by the derivative of

$$g_{n-1} \circ\circ\circ g_0 = T_{j,n} \circ \hat{g}_n \circ\circ\circ \hat{g}_0 \circ T_{j,0}^{-1}$$

is γ-close to E_n^2 for all points in $T_j(\hat{G}) \subset G$. This proves condition (3). □

3.2. Flexibility on the tangent bundle

Let M be a fixed closed symplectic manifold of dimension $2N$. By Darboux's Theorem, there exists an atlas $\{\phi_i : V_i \to \mathbb{R}^{2N}\}$ formed by charts that take the symplectic form on M to the standard symplectic form on $\mathbb{R}^{2N}$. Let $K_{\mathcal{A}} > 1$ be such that such an atlas can be chosen with $\|D\phi_i\|$, $\|D\phi_i^{-1}\| < K_{\mathcal{A}}$ everywhere. *Fix $K_{\mathcal{A}}$ once and for all, and let $\mathcal{A}$ be the maximal symplectic atlas obeying the bounds above.* That is, $\mathcal{A}$ is the set of all symplectic maps $\phi : V \to \mathbb{R}^{2N}$, where $V \subset M$ is open, such that $\|D\phi(x)\| < K_{\mathcal{A}}$ for all $x \in V$ and $\|D\phi^{-1}(y)\| < K_{\mathcal{A}}$ for all $y \in \phi(V)$.

Choose a finite atlas $\mathcal{A}_0 \subset \mathcal{A}$. For each $z \in M$, choose and fix some chart $\phi_z : V_z \to \mathbb{R}^{2N}$ in $\mathcal{A}_0$ with $V_z \ni z$. For any $x \in V_z$, we define a linear isomorphism

$$i_x^z : T_z M \to T_x M \quad \text{by } i_x^z = [D\phi_z(x)]^{-1} \circ D\phi_z(z). \tag{3.1}$$

Now we extend the notions of split sequences and flexibility to the tangent bundle TM.

Fixing $f \in \mathrm{Diff}^1_\omega(M)$ and a non-periodic point $z \in M$, a *split sequence on TM* is composed of the objects:

- the (finite) sequence of linear maps $Df(f^i z)$, where $0 \leqslant i < n$;
- non-trivial splittings $T_{f^i z}M = E_i^1 \oplus E_i^2$, for $0 \leqslant i \leqslant n$, invariant in the sense that $Df(f^i z) \cdot E_i^* = E_{i+1}^*$.

Using charts, a split sequence on TM induces a split sequence on $\mathbb{R}^{2N}$. More precisely, for each $i = 0, \dots, n$, let ϕ_i be a chart in the atlas $\mathcal{A}$ whose domain contains $f^i z$. Then we consider the split sequence on $\mathbb{R}^{2N}$:

$$\{\hat{E}_i^1 \oplus \hat{E}_i^2 \xrightarrow{A_i} \hat{E}_{i+1}^1 \oplus \hat{E}_{i+1}^2\}_{0 \leqslant i < n},$$
$$\text{where } A_i = D(\phi_{i+1} \circ f \circ \phi_i^{-1})(\phi_i(f^i z)),\ \hat{E}_i^* = D\phi_i(f^i z) \cdot E_i^*.$$

A split sequence on TM is called (ε, κ)-*flexible* if so is a induced split sequence on $\mathbb{R}^{2N}$, for *some* choice of the charts.

Given a split sequence on TM, we can find special perturbations of the diffeomorphism f, as described in the lemma below.

Lemma 3.5. *Given $f \in \mathrm{Diff}^1_\omega(M)$ and a neighbourhood $\mathcal{V}$ of f in $\mathrm{Diff}^1_\omega(M)$, there exists $\varepsilon > 0$ such that the following holds. Let $z \in M$ be a non-periodic point for f. Assume that $Df(f^i z) : E_i^1 \oplus E_i^2 \to E_{i+1}^1 \oplus E_{i+1}^2$ $(0 \leqslant i < n)$ is an (ε, κ)-flexible split sequence.*

Then for every $\gamma > 0$ there exists $r > 0$ with the following properties. First, the closed ball $\bar{B}_r(z)$ is disjoint from its n first iterates. Second, given any non-empty open set $U \subset B_r(z)$, there exists $g \in \mathcal{V}$ with the following properties:

(1) *g equals f outside $\bigsqcup_{i=0}^{n-1} f^i(U)$;*

(2) *there is a set $G \subset U$ with $\mu(G) > (1-\kappa)\mu(U)$ such that*

$$\text{for every } x \in G, \quad \measuredangle(Dg^n(x) i_x^z \cdot E_0^1, i_{g^n x}^{f^n z} \cdot E_n^2) < \gamma.$$

Proof. Let $\varepsilon = \varepsilon(f, \mathcal{V})$ be small (to be specified later).

Let $z \in M$, $n \in \mathbb{N}$, $\kappa > 0$ and $T_{f^i z} M = E_i^1 \oplus E_i^2$ be as in the assumptions of the lemma. That is, there exist charts $\phi_i : V_i \to \mathbb{R}^{2N}$ (for $0 \leqslant i \leqslant n$) in the atlas $\mathcal{A}$ such that $V_i \ni f^i z$ and the split sequence $\{A_i, \hat{E}_i^*\}$ defined by

$$A_i = D(\phi_{i+1} \circ f \circ \phi_i^{-1})(\phi_i(f^i z)), \qquad \hat{E}_i^* = D\phi_i(f^i z) \cdot E_i^*$$

is (ε, κ)-flexible. Without loss of generality, assume that $\phi_i(f^i z) = 0$ and that $V_i = f^i(V_0)$.

We can also assume that *the expression of f in the charts is linear*, that is, $\phi_{i+1} \circ f \circ \phi_i^{-1}$ is the restriction of the linear map A_i to $\phi_i(V_i)$. To see this, let $\psi_i = A_{i-1} \circ \circ \circ A_0 \circ \phi_0 \circ f^{-i}$, for $0 \leqslant i \leqslant n$. Then ψ_i is a symplectomorphism from a neighbourhood of $f^i z$ to a neighbourhood of 0 in $\mathbb{R}^{2N}$. Also, it follows from the definition of the A_is that $D\psi_i(f^i z) = D\phi_i(f^i z)$. Therefore, $\psi_i : W_i \to \mathbb{R}^{2N}$ are charts in the atlas $\mathcal{A}$, provided we choose sufficiently small neighbourhoods W_i of $f^i z$. Moreover, $\psi_{i+1} \circ f \circ \psi_i^{-1}$ equals A_i (where the former is defined). So we just need to replace ϕ_i with ψ_i.

Now the proof becomes straightforward. Let $\gamma > 0$ be given. Choose r with $0 < r < \varepsilon$ such that the closed ball $\bar{B}_r(z)$ is contained in V_0 and is disjoint from its first n iterates.

Given a non-empty open set $U \subset B_r(z)$, let $\hat{U} = \phi_0(U)$. Take $\gamma' \ll \gamma$. The flexibility of the split sequence $\{A_i, \hat{E}_i^{1,2}\}$, together with Lemma 3.4, implies that there exist symplectomorphisms $g_i : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ (for $0 \leqslant i < n$) such that:

- g_i equals A_i outside $A_{i-1} \circ \circ \circ A_0(\hat{U}) = \phi_i(f^i(U))$;
- $\|D(A_i^{-1} \circ g_i) - \mathrm{Id}\,\| < \varepsilon$;
- there is a set $\hat{G} \subset \hat{U}$ such that $\mu(\hat{G}) > (1-\kappa)\mu(\hat{U})$ and

$$\measuredangle(D(g_{n-1} \circ\circ\circ g_0)(\hat{x}) \cdot \hat{E}_0^1, \hat{E}_n^2) < \gamma' \quad \forall \hat{x} \in \hat{G}.$$

Define $g : M \to M$ by

$$g(x) = \begin{cases} \phi_{i+1}^{-1} \circ g_i \circ \phi_i(x) & \text{if } x \in V_i = f^i(V_0),\ 0 \leqslant i < n, \\ f(x) & \text{otherwise.} \end{cases}$$

Then g is a symplectomorphism that equals f outside $\bigsqcup_{i=0}^{n-1} f^i(U)$; moreover if ε is small enough then g is close to f, that is, $g \in \mathcal{V}$. Now, if r is sufficiently small then for every $x \in G = \phi_0^{-1}(\hat{G})$, the space $D\phi_0(x) \circ i_x^z \cdot E_0^1$ is close to $\hat{E}_0^1$, while $D\phi_n(g^n x) \circ i_{g^n x}^{f^n z} \cdot E_n^2$ is close to $\hat{E}_n^2$. Then the second condition in the statement of the lemma follows. □

3.3. A special split sequence

Let us now focus on some specific split sequences that come from the Oseledets splitting.

Given $f \in \mathrm{Diff}^1_\omega(M)$ and $p \in \{1, \ldots, N\}$, we define the invariant set

$$\varSigma_p(f) = \{z \in M;\ z \text{ is non-periodic, Oseledets regular and } \lambda_p(f,z) > \lambda_{p+1}(f,z)\}.$$

We consider the splitting

$$T_{\varSigma_p(f)}M = E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}} \tag{3.2}$$

such that at each point E^{u}, E^{c} and E^{s} are the sum of the Oseledets spaces corresponding respectively to the sets of Lyapunov exponents

$$\{\lambda_1, \ldots, \lambda_p\}, \quad \{\lambda_{p+1}, \ldots, \lambda_{2N-p} = -\lambda_{p+1}\} \quad \text{and} \quad \{\lambda_{2N-p+1} = -\lambda_p, \ldots, \lambda_{2N} = -\lambda_1\}.$$

We also define bundles E^{uc}, E^{us}, E^{cs} respectively as $E^{\mathrm{u}} \oplus E^{\mathrm{c}}$, etc.

Two obvious remarks. First, when we speak of E^{u}, E^{c}, E^{s}, the number p is implicitly fixed. Second, despite the notation, the splitting (3.2) has no reason to be partially hyperbolic.

The splitting (3.2) has the following properties:

$$Df\text{-invariance} : Df(z) \cdot E^*(z) = E^*(f(z)), \quad * = \mathrm{u}, \mathrm{c}, \mathrm{s}, \tag{3.3}$$

$$\dim E^{\mathrm{u}} = \dim E^{\mathrm{s}} = p, \qquad \dim E^{\mathrm{c}} = 2(N-p), \tag{3.4}$$

$$\omega(E^u, E^{\mathrm{uc}}) \equiv 0, \qquad \omega(E^c, E^{\mathrm{us}}) \equiv 0, \qquad \omega(E^s, E^{\mathrm{cs}}) \equiv 0. \tag{3.5}$$

The first two are completely obvious, while (3.5) follows from the fact that if $v_i, v_j \in T_xM$ are vectors with respective Lyapunov exponents λ_i, λ_j such that $\lambda_i + \lambda_j \neq 0$ then $\omega(v_i, v_j) = 0$.

The split sequences on TM that we will be interested in are those that come from the splitting $E^{\mathrm{u}} \oplus E^{\mathrm{cs}}$, that is, those of the form

$$\{E^{\mathrm{u}}(f^i z) \oplus E^{\mathrm{cs}}(f^i z) \xrightarrow{Df(f^i z)} E^{\mathrm{u}}(f^{i+1} z) \oplus E^{\mathrm{cs}}(f^{i+1} z)\}_{0 \leqslant i < m},$$

where $z \in \Sigma_p(f)$. To avoid such a cumbersome notation, we write the sequence as $Df(f^i z) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow (0 \leqslant i < m)$.

3.4. The Main Lemma: lack of dominance implies flexibility

If the splitting $E^{\mathrm{u}} \oplus E^{\mathrm{cs}}$ is dominated over the orbit of a point z, then, due to the existence of a strictly invariant cone field, no split sequence $Df(f^i z) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow (0 \leqslant i < m)$ can be (ε, κ)-flexible, provided $\varepsilon > 0$ is small enough. A major part of this paper is devoted to proving the following converse to this fact.

Main Lemma. *Given $f \in \mathrm{Diff}^1_\omega(M)$, $\varepsilon > 0$, $\kappa > 0$, and $p \in \{1, \dots, N\}$, there exist $m_1 \in \mathbb{N}$ with the following properties.*

If $z \in \Sigma_p(f)$ and $m \in \mathbb{N}$ are such that $m \geqslant m_1$ and

$$\frac{\|Df^m(z)|E^{\mathrm{cs}}(z)\|}{\boldsymbol{m}(Df^m(z)|E^{\mathrm{u}}(z))} \geqslant \frac{1}{2}, \tag{3.6}$$

then the split sequence $Df(f^i z) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow (0 \leqslant i < m)$ is (ε, κ)-flexible.

That is, lack of dominance expressed by (3.6) implies flexibility.

Remark 3.6. In addition to (3.6), the only properties about the splitting $E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$ that we are going to use in the proof of the Main Lemma are (3.3), (3.4) and (3.5).

The proof of the Main Lemma will occupy §§ 4 and 5.

4. The four types of non-dominance

The aim of this section is to prove Lemma 4.1 below. That proposition classifies the split sequences considered in the Main Lemma in four types. Each of these four types of sequences will be shown to be flexible in § 5, and this will prove the Main Lemma.

For the rest of this section, let $f \in \mathrm{Diff}_\omega(M)$ and $p \in \{1, \dots, N\}$ be fixed. Recall from § 3.3 the definition of the set $\Sigma_p(f)$ and the splitting $T_{\Sigma_p(f)}M = E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$.

4.1. The classification

A set of the form $\{f^i z;\ 0 \leqslant i < n\}$, where $z \in \Sigma_p(f)$ and $n \in \mathbb{N}$, will be called a *segment of length n*.

A segment $\{z, \dots, f^{n-1} z\}$ is called *of type II* (*with constant* $K_{\mathrm{II}} > 1$) if

$$\frac{\|Df^n|E^{\mathrm{cs}}(z)\|}{\boldsymbol{m}(Df^n|E^{\mathrm{u}}(z))} > K_{\mathrm{II}}.$$

A segment $\{z, \dots, f^{n-1}z\}$ is called *of type III* (*with constant* $K_{\mathrm{III}} > 1$) if for $0 \leqslant i \leqslant n$ there exist symplectic linear maps $\mathcal{L}_i : T_{f^i z}M \to \mathbb{R}^{2N}$ (that is, that send ω to the standard symplectic form $\sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$ on $\mathbb{R}^{2N}$) such that:

- $\|\mathcal{L}_i^{\pm 1}\| \leqslant K_{\mathrm{III}}$;
- the images by $\mathcal{L}_i^{-1}$ of the vectors $\partial/\partial p_1$ and $\partial/\partial q_1$ are contained respectively in the spaces $E^{\mathrm{u}}(f^i z)$ and $E^{\mathrm{s}}(f^i z)$;
- the (symplectic linear) map $A_i = \mathcal{L}_{i+1} \circ Df(f^i z) \circ \mathcal{L}_i^{-1} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is the identity on the 2-plane $p_1 q_1$.

A segment $\{z, \dots, f^{n-1}z\}$ is called *of type IV* (*with constants* $K_{\mathrm{IV}} > 1$, $\tau > 1$) if there exist symplectic linear maps $\mathcal{L}_i : T_{z_i}M \to \mathbb{R}^{2N}$, $0 \leqslant i \leqslant n-1$, such that:

- $\|\mathcal{L}_i^{\pm 1}\| \leqslant K_{\mathrm{IV}}$;
- the images by $\mathcal{L}_i^{-1}$ of the vectors $\partial/\partial p_1$, $\partial/\partial p_2$, $\partial/\partial q_1$ and $\partial/\partial q_2$ are contained respectively in the spaces E^{u}, E^{c}, E^{c} and E^{s};
- the (symplectic linear) map $A_i = \mathcal{L}_{i+1} \circ Df(z_i) \circ \mathcal{L}_i^{-1} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ preserves the 4-plane $p_1 p_2 q_1 q_2$, where it is given by

$$A_i : (p_1, p_2, q_1, q_2) \mapsto (c_i p_1, c_i p_2, c_i^{-1} q_1, c_i^{-1} q_2), \quad \text{where } c_i > \tau.$$

Notice that segments of type IV do not exist if $p = N$, because in that case $E^{\mathrm{c}} = \{0\}$. (That is why type IV does not appear in [**13**].)

Recall that the *symplectic complement* of a vector space E is the space E^ω formed by vectors w such that $\omega(v, w) = 0$ for all $v \in E$. If L is a symplectic linear map then $(L(E))^\omega = L(E^\omega)$. It follows that if A_i is the linear map as in the definition of type III (respectively type IV) then A_i preserves the $(2N-2)$-plane $p_2 \cdots p_{2N} q_2 \cdots q_{2N}$ (respectively the $(2N-4)$-plane $p_3 \cdots p_{2N} q_3 \cdots q_{2N}$).

Lemma 4.1. *Let $\alpha > 0$, $K_{\mathrm{II}} > 1$, $m_0 \in \mathbb{N}$. Then there exist numbers K_{III}, $K_{\mathrm{IV}} > 1$, $\tau > 1$, where K_{III} does not depend on m_0, with the following properties. Assume that $z \in \Sigma_p(f)$ and $m \geqslant m_0$ are such that the non-dominance condition (3.6) is satisfied. Then one of the following holds.*

(I) *There exists i with $0 \leqslant i \leqslant m$, such that $\measuredangle(E^{\mathrm{u}}(f^i z), E^{\mathrm{cs}}(f^i z)) < \alpha$.*

(II) *There exist i and j with $0 \leqslant i < j \leqslant m$ such that the segment $\{f^i z, \dots, f^j z\}$ is of type II with constant K_{II}.*

(III) *There is some i with $0 \leqslant i \leqslant m - m_0$ such that the segment $\{f^i z, \dots, f^{i+m_0} z\}$ is of type III with constant K_{III}.*

(IV) *The segment $\{z, \dots, f^m z\}$ is of type IV with constants K_{IV}, τ.*

4.2. Proof

We start with some generalities about symplectic and Riemannian structures on the manifold.

For each $x \in M$, let $\mathcal{J}_x : T_xM \to T_xM$ be the isomorphism defined by $\omega(v, w) = \langle \mathcal{J}_x v, w\rangle$ for all $v, w \in T_xM$. Observe that the symplectic complement of a subspace $E \subset T_xM$ is $E^\omega = (\mathcal{J}_x(E))^\perp$.

Denote

$$K_\omega = \sup_{x\in M} \|\mathcal{J}_x^{\pm 1}\|.$$

In particular, we have

$$|\omega(v, w)| \leqslant K_\omega \|v\| \, \|w\| \quad \text{for all } v, w \in T_xM. \tag{4.1}$$

Lemma 4.2. *There are functions $\beta_1(B) > 0$ and $B_1(\beta) > 1$ with the following properties.*

Let $x \in M$, and let $E, F \subset T_xM$ be vector spaces with the same dimension, and such that $E^\omega \cap F = \{0\}$.

If $\measuredangle(E^\omega, F) > \beta > 0$ then setting $B = B_1(\beta)$ we have that

$$\exists \text{ isomorphism } J : E \to F \text{ such that } \begin{cases} \|J^{\pm 1}\| \leqslant B, \\ |\omega(v, J(v))| \geqslant B^{-1}\|v\|^2. \end{cases} \tag{4.2}$$

Conversely, if (4.2) holds for some $B > 1$ then $\measuredangle(E^\omega, F) > \beta_1(B)$.

Proof. Assume that $\measuredangle(E^\omega, F) > \beta$. Let $p : T_xM \to F$ be the projection parallel to E^ω; then $\|p\| < 1/\sin\beta$. Let J be the restriction of $p \circ \mathcal{J}_x$ to E. If $v \in E$ then $|\omega(v, J(v))| = |\omega(v, \mathcal{J}_x(v))| = \|\mathcal{J}_x(v)\|^2 \geqslant K_\omega^{-2}\|v\|^2$. Since $E^\omega = (\mathcal{J}_x(E))^\perp$, we have $\|J(v)\| \geqslant \|\mathcal{J}_x(v)\| \geqslant K_\omega^{-1}\|v\|$. Therefore (4.2) holds for some appropriate $B = B_1(\beta)$.

On the other hand, if (4.2) holds then for any unit vectors $v \in E^\omega$, $w \in F$ we have

$$|\omega(w - v, J^{-1}(w))| = |\omega(w, J^{-1}(w))| \geqslant B^{-1}\|J^{-1}(w)\|^2 \geqslant B^{-3}.$$

Using (4.1) we find a lower bound for $\|w - v\|$. This shows that $\measuredangle(E^\omega, F)$ is bigger than some $\beta_1(B) > 0$. □

It follows from the lemma that there is a function $\beta_2(\beta) > 0$ such that

$$\measuredangle(E^\omega, F) > \beta \;\Rightarrow\; \measuredangle(E, F^\omega) > \beta_2(\beta) \tag{4.3}$$

(where E, $F \subset T_xM$ have the same dimension).

An (ordered) set $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_\nu, \boldsymbol{f}_1, \dots, \boldsymbol{f}_\nu\} \subset T_xM$ will be called *orthosymplectic* if

$$\omega(\boldsymbol{e}_i, \boldsymbol{e}_j) = \omega(\boldsymbol{f}_i, \boldsymbol{f}_j) = 0, \quad \omega(\boldsymbol{e}_i, \boldsymbol{f}_j) = \delta_{ij} \quad \text{for all } i, j.$$

If $\nu = N$ then the set is called a *symplectic basis* of T_xM.

Lemma 4.3. *For every $K_1 > 0$ there exist $K_2, K_3 > 0$ with the following properties. Every orthosymplectic set $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_\nu, \boldsymbol{f}_1, \dots, \boldsymbol{f}_\nu\} \subset T_xM$ such that*

$$\|\boldsymbol{e}_i\|, \|\boldsymbol{f}_i\| \leqslant K_1 \quad \text{for } 1 \leqslant i \leqslant \nu$$

can be extended to a symplectic basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_N, \boldsymbol{f}_1, \dots, \boldsymbol{f}_N\}$ such that

$$\|\boldsymbol{e}_i\|, \|\boldsymbol{f}_i\| \leqslant K_2 \quad \text{for } \nu < i \leqslant N. \tag{4.4}$$

Furthermore, if $\mathcal{L} : T_xM \to \mathbb{R}^{2N}$ is the linear map that takes this basis to the canonical symplectic basis $\{\partial/\partial p_1, \dots, \partial/\partial p_N, \partial/\partial q_1, \dots, \partial/\partial q_N\}$ of $\mathbb{R}^{2N}$ then $\|\mathcal{L}^{\pm 1}\| \leqslant K_3$.

Proof. Fix an orthosymplectic set $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_\nu, \boldsymbol{f}_1, \dots, \boldsymbol{f}_\nu\} \subset T_xM$ composed of vectors of norm at most K_1. Let Y be the spanned space; it is a symplectic space (that is, $Y \cap Y^\omega = \{0\}$) of dimension 2ν. Let $P : T_xM \to Y$ be the projection onto Y parallel to Y^ω. It is given by the formula:

$$P(v) = \sum_{i=1}^{\nu} [\omega(v, \boldsymbol{f}_i)\boldsymbol{e}_i - \omega(v, \boldsymbol{e}_i)\boldsymbol{f}_i].$$

By (4.1), $\|P\| \leqslant K_\omega K_1^2$. Now assume $\nu < N$ and let us see how to extend the orthosymplectic set. Take a unit vector $\hat{\boldsymbol{e}}$ orthogonal to Y, and let $\boldsymbol{e}_{\nu+1} = \hat{\boldsymbol{e}} - P(\hat{\boldsymbol{e}})$. Then $\boldsymbol{e}_{\nu+1}$ belongs to Y^ω, and by the Pythagorean Theorem, its norm is at least 1. Consider the vector $\hat{\boldsymbol{f}} = \mathcal{J}_x(\boldsymbol{e}_{\nu+1})/\|\mathcal{J}_x(\boldsymbol{e}_{\nu+1})\|^2$; its norm is at most K_ω, and $\omega(\boldsymbol{e}_{\nu+1}, \hat{\boldsymbol{f}}) = 1$. Let $\boldsymbol{f}_{\nu+1} = \hat{\boldsymbol{f}} - P(\hat{\boldsymbol{f}})$. Then $\boldsymbol{f}_{\nu+1}$ belongs to Y^ω and $\omega(\boldsymbol{e}_{\nu+1}, \boldsymbol{f}_{\nu+1}) = 1$, so the enlarged set $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{\nu+1}, \boldsymbol{f}_1, \dots, \boldsymbol{f}_{\nu+1}\}$ is orthosymplectic. Also, we can bound $\|\boldsymbol{e}_{\nu+1}\|$ and $\|\boldsymbol{f}_{\nu+1}\|$ by functions of K_1. Continuing by induction, we find the desired symplectic basis.

Now let $\mathcal{L}$ be as in the statement of the lemma. Obviously, an upper bound for $\|\mathcal{L}^{-1}\|$ can be found using (4.4). On the other hand, if $\mathcal{L}(v) = (p_1, \dots, p_N, q_1, \dots, q_N)$ then $p_i = \omega(v, \boldsymbol{f}_i)$ and $q_i = -\omega(v, \boldsymbol{e}_i)$. So we can bound $\|\mathcal{L}\|$ as well. □

Let us adopt the following notation. If A and B are positive quantities then

$$A \lesssim B \pmod{a, b, \dots}$$

means that B/A is bigger than some positive quantity depending only on $a, b, \dots$ (and maybe on M, f and p, which are fixed). Then $A \approx B$ and $A \gtrsim B \pmod{a, b, \dots}$ are defined in the obvious ways.

Now we are ready to give the proof of Lemma 4.1.

Proof of Lemma 4.1. Let α, $K = K_{\mathrm{II}}$, m_0 be given. Let z belong to $\Sigma_p(f)$, and let $z_i = f^i z$. Assume that for some $m \geqslant m_0$, the segment $\{z_0, \dots, z_m\}$ is non-dominated, meaning that (3.6) holds.

From now on, assume that

$$\measuredangle(E_i^{\mathrm{u}}, E_i^{\mathrm{cs}}) \geqslant \alpha, \quad \text{for every } i \text{ with } 0 \leqslant i \leqslant m, \tag{4.5}$$

and

$$\frac{\|Df^n|E_i^{\mathrm{cs}}\|}{\boldsymbol{m}(Df^n|E_i^{\mathrm{u}})} \leqslant K, \quad \text{for every } i,\ n \text{ with } 0 \leqslant i < i+n \leqslant m, \tag{4.6}$$

because otherwise we fall in one of the first two cases and there is nothing to prove.

We claim that

$$\measuredangle(E_i^{\mathrm{u}}, E_i^{\mathrm{cs}}), \measuredangle(E_i^{\mathrm{c}}, E_i^{\mathrm{us}}), \measuredangle(E_i^{\mathrm{s}}, E_i^{\mathrm{uc}}) \gtrsim 1 \pmod{\alpha}, \quad \text{for every } i. \tag{4.7}$$

From (3.5) we see that $E^{\mathrm{cs}} = (E^{\mathrm{s}})^\omega$ and $(E^{\mathrm{u}})^\omega = E^{\mathrm{uc}}$. So using (4.5) and (4.3) we get that $\measuredangle(E_i^{\mathrm{s}}, E_i^{\mathrm{uc}}) > \beta_2(\alpha)$. So we got two bounds in (4.7), and the third follows (use for instance Lemma 2.6 from [**13**]).

Sublemma 4.4. *Let* $(\prime, \prime\prime)$ *be either* (u, s), (c, c) *or* (s, u). *Let* $i \in \{0, \dots, m\}$.

(1) *For every unit vector v in E_i', there exists a unit vector $v^\star$ in E_i'' such that*

$$|\omega(v, v^\star)| \gtrsim 1 \pmod{\alpha}.$$

Moreover, if $n \in \mathbb{Z}$ is such that $i + n \in \{0, \dots, m\}$ then we have the following.

(2) *If $v \in E_i'$ is a unit vector then* $\|Df^n(v)\|\,\|Df^n(v^\star)\| \gtrsim 1 \pmod{\alpha}$.

(3) $\boldsymbol{m}(Df^n|E_i')\|Df^n|E_i''\| \approx 1 \pmod{\alpha}$.

(4) *If v is a unit vector in E_i' such that $\|Df^n v\| = \boldsymbol{m}(Df^n|E_i')$ then*

$$\|Df^n(v^\star)\| \approx \|Df^n|E_i''\| \pmod{\alpha}.$$

(That is, if v is the unit vector that is most contracted by $Df^n|E_i'$, then $v^\star$ is a unit vector that is almost-the-most expanded by $Df^n|E_i''$.)

Proof. Let $\prime$, $\prime\prime$, i, n be as in the statement. By (4.7), $\measuredangle((E_i')^\omega, E_i'') \gtrsim 1 \pmod{\alpha}$. Let $J_i' : E_i' \to E_i''$ be given by Lemma 4.2. If $v \in E_i'$ is a unit vector, let $v^\star = J_i'(v)/\|J_i'v\|$. Then $v^\star$ has the properties as in item (1). Item (2) is evident:

$$K_\omega \|Df^n(v)\|\,\|Df^n(v^\star)\| \geqslant |\omega(Df^n(v), Df^n(v^\star))| = |\omega(v, v^\star)| \gtrsim 1 \pmod{\alpha}.$$

Now let v be a unit vector in E_i' such that $\|Df^n v\| = \boldsymbol{m}(Df^n|E_i')$. By item (2),

$$\boldsymbol{m}(Df^n|E_i')\|Df^n|E_i''\| \geqslant \|Df^n(v)\|\,\|Df^n(v^\star)\| \gtrsim 1 \pmod{\alpha},$$

proving one inequality in item (3). The other inequality follows from the first, replacing (i, n) by $(i + n, -n)$. Item (4) follows from items (2) and (3):

$$\|Df^n|E_i''\| \geqslant \|Df^n(v^\star)\| \gtrsim \frac{1}{\|Df^n(v)\|} = \frac{1}{\boldsymbol{m}(Df^n|E_i')} \approx \|Df^n|E_i''\| \pmod{\alpha}.$$

$\square$

Now we extract consequences from (4.6).

Sublemma 4.5. *For any i, n with $0 \leqslant i < i+n \leqslant m$, we have*

$$\|Df^n|E^{\mathrm{s}}_i\| \;\lesssim\; \boldsymbol{m}(Df^n|E^{\mathrm{c}}_i) \;\lesssim\; 1 \;\lesssim\; \|Df^n|E^{\mathrm{c}}_i\| \;\lesssim\; \boldsymbol{m}(Df^n|E^{\mathrm{u}}_i) \quad (\mathrm{mod}\ \alpha, K)$$

Moreover, the matched pairs have product ≈ 1 $(\mathrm{mod}\ \ \alpha, K)$.

Proof. By (4.6),

$$\boldsymbol{m}(Df^n|E^{\mathrm{u}}_i) \gtrsim \|Df^n|E^{\mathrm{cs}}_i\| \geqslant \|Df^n|E^{\mathrm{c}}_i\| \quad (\mathrm{mod}\ K).$$

Then the other assertions follow easily from Sublemma 4.4 (item (3)). □

Sublemma 4.6. *If*

$$\frac{\|Df^{m_0}|E^{\mathrm{s}}_k\|}{\boldsymbol{m}(Df^{m_0}|E^{\mathrm{u}}_k)} \geqslant \frac{1}{2}, \quad \textit{for some } k \textit{ with } 0 \leqslant k \leqslant m - m_0, \tag{4.8}$$

then the segment $\{z_k, \dots, z_{k+m_0}\}$ is of type III (with some constant K_{III} that depends only on α and $K = K_{\mathrm{II}}$).

The interpretation of (4.8) is that the segment $\{z_k, \dots, z_{k+m_0}\}$ is non-dominated in a stronger way: E^{u} does not dominate E^{s}.

Proof. Together with Sublemma 4.5, the assumption (4.8) gives

$$\|Df^{m_0}|E^{\mathrm{s}}_k\| \approx 1 \approx \boldsymbol{m}(Df^{m_0}|E^{\mathrm{u}}_k) \quad (\mathrm{mod}\ \alpha, K).$$

Let v be a unit vector in E^u_k that is least expanded by Df^{m_0}, that is $\|Df^{m_0}v\| = \boldsymbol{m}(Df^{m_0}|E^u_k)$. By Sublemma 4.4, the unit vector $v^\star \in E^s_k$ satisfies $\|Df^{m_0}(v^\star)\| \approx \|Df^{m_0}|E^s_k\|$ (mod α). Using (4.6) we get, for each $i = 0, \dots, m_0$,

$$K \geqslant \frac{\|Df^i(v^\star)\|}{\|Df^i(v)\|} \geqslant \frac{\|Df^{m_0}(v^\star)\|/\|Df^{m_0-i}|E^{\mathrm{s}}_{k+i}\|}{\|Df^{m_0}(v)\|/\boldsymbol{m}(Df^{m_0-i}|E^{\mathrm{u}}_{k+i})} \gtrsim 1 \quad (\mathrm{mod}\ \alpha, K).$$

That is, $\|Df^i(v^\star)\| \approx \|Df^i(v)\|$. In addition, both norms are ≈ 1, by Sublemma 4.5. For each $i = 0, \dots, m_0$, let

$$\boldsymbol{e}_{1,i} = Df^i(v), \qquad \boldsymbol{f}_{1,i} = \frac{Df^i(v^\star)}{\omega(v, v^\star)}.$$

Then $\{\boldsymbol{e}_{1,i}, \boldsymbol{f}_{1,i}\}$ is a orthosymplectic subset of $T_{z_{k+i}}M$. By Lemma 4.3, we can extend it to a symplectic basis $\{\boldsymbol{e}_{1,i}, \boldsymbol{f}_{1,i}, \dots, \boldsymbol{e}_{N,i}, \boldsymbol{f}_{N,i}\}$, and furthermore if $\mathcal{L}_i$ is the linear map that takes this basis to the canonical symplectic basis of $\mathbb{R}^{2N}$ then $\|\mathcal{L}_i^{\pm 1}\| \lesssim 1$ (mod α, K). The map $\mathcal{L}_{i+1} \circ Df(z_{k+i}) \circ \mathcal{L}_i^{-1}$ is the identity on the plane $p_1 q_1$. This shows that the segment being considered is of type III. □

Sublemma 4.6 says that if (4.8) holds then we are done. Assume from now on that (4.8) does not hold, that is,

$$\frac{\|Df^{m_0}|E^{\mathrm{s}}_k\|}{\boldsymbol{m}(Df^{m_0}|E^{\mathrm{u}}_k)} < \frac{1}{2}, \quad \text{for all } k \text{ with } 0 \leqslant k \leqslant m - m_0. \tag{4.9}$$

From now on, all relations $\gtrsim$, $\lesssim$, $\approx$ *will be meant* mod α, K, m_0.

Sublemma 4.7. *E^{u} is uniformly expanding and E^{s} is uniformly contracting. That is, there exists $\lambda > 1$ and $C > 1$ (depending on α, K, m_0) such that*

$$\left.\begin{aligned} \boldsymbol{m}(Df^n|E^{\mathrm{u}}_i) > C^{-1}\lambda^n \\ \|Df^n|E^{\mathrm{s}}_i\| < C\lambda^{-n} \end{aligned}\right\} \quad \forall i, n \text{ with } 0 \leqslant i < i+n \leqslant m. \tag{4.10}$$

Proof. It follows from (4.9) that

$$\frac{\|Df^n|E^{\mathrm{s}}_i\|}{\boldsymbol{m}(Df^n|E^{\mathrm{u}}_i)} \leqslant \left(\sup_{x\in M} \frac{\|Df(x)\|}{\boldsymbol{m}(Df(x))}\right)^{m_0-1} \left(\frac{1}{2}\right)^{\lfloor n/m_0 \rfloor}.$$

The right-hand side is exponentially small with n. Since $\|Df^n|E^{\mathrm{s}}_i\| \approx 1/\boldsymbol{m}(DF^n|E^{\mathrm{u}}_i)$, the lemma follows. $\square$

For the first time, let us use the hypothesis of non-domination of the segment $\{z_0, \ldots, z_m\}$:

$$\frac{\|Df^m|E^{\mathrm{cs}}_0\|}{\boldsymbol{m}(Df^m|E^{\mathrm{u}}_0)} \geqslant \frac{1}{2}. \tag{4.11}$$

We claim that:

$$\|Df^m|E^{\mathrm{s}}_0\| \approx \boldsymbol{m}(Df^m|E^{\mathrm{c}}_0) \quad \text{and} \quad \|Df^m|E^{\mathrm{c}}_0\| \approx \boldsymbol{m}(Df^m|E^{\mathrm{u}}_0). \tag{4.12}$$

Since $\|Df^m|E^{\mathrm{s}}_0\| \lesssim 1 \lesssim \|Df^m|E^{\mathrm{c}}_0\|$ and $\measuredangle(E^{\mathrm{s}}_0, E^{\mathrm{c}}_0) \approx 1$, we have $\|Df^m|E^{\mathrm{cs}}_0\| \approx \|Df^m|E^{\mathrm{c}}_0\|$. So (4.11), together with Sublemma 4.5, gives the second relation in (4.12). The first relation follows from the second.

Let $v^{\mathrm{u}} \in E^{\mathrm{u}}_0$ and $v^{\mathrm{cs}} \in E^{\mathrm{c}}_0$ be unit vectors such that

$$\|Df^m v^{\mathrm{u}}\| = \boldsymbol{m}(Df^m|E^{\mathrm{u}}_0) \quad \text{and} \quad \|Df^m v^{\mathrm{cs}}\| = \boldsymbol{m}(Df^m|E^{\mathrm{c}}_0).$$

Let $v^{\mathrm{s}} = (v^{\mathrm{u}})^\star \in E^{\mathrm{s}}_0$ and $v^{\mathrm{cu}} = (v^{\mathrm{cs}})^\star \in E^{\mathrm{c}}_0$. Then, by Sublemma 4.4,

$$\|Df^m v^{\mathrm{s}}\| \approx \|Df^m|E^{u}_0\| \quad \text{and} \quad \|Df^m v^{\mathrm{cu}}\| \approx \|Df^m|E^{\mathrm{c}}_0\|.$$

Sublemma 4.8. *If $0 \leqslant i \leqslant m$ then*

$$\|Df^i v^{\mathrm{u}}\| \approx \|Df^i v^{\mathrm{cu}}\| \approx \frac{1}{\|Df^i v^{\mathrm{cs}}\|} \approx \frac{1}{\|Df^i v^{\mathrm{s}}\|}. \tag{4.13}$$

If $n > 0$ and $i + n \leqslant m$ then

$$\boldsymbol{m}(Df^n|E^{\mathrm{u}}_i) \approx \frac{\|Df^{n+i} v^{\mathrm{u}}\|}{\|Df^i v^{\mathrm{u}}\|}. \tag{4.14}$$

Proof. From (4.12), $\|Df^m v^{\mathrm{u}}\| \approx \|Df^m v^{\mathrm{cu}}\|$. Therefore,

$$K \geqslant \frac{\|Df^i v^{\mathrm{cu}}\|}{\|Df^i v^{\mathrm{u}}\|} \geqslant \frac{\|Df^m v^{\mathrm{cu}}\| / \|Df^{m-i}|E_i^{\mathrm{c}}\|}{\|Df^m v^{\mathrm{u}}\| / \boldsymbol{m}(Df^{m-i}|E_0^{\mathrm{u}})} \gtrsim 1,$$

that is, $\|Df^i v^{\mathrm{u}}\| \approx \|Df^i v^{\mathrm{cu}}\|$. Analogously, $\|Df^i v^{\mathrm{s}}\| \approx \|Df^i v^{\mathrm{cs}}\|$. Now, from Sublemma 4.5,

$$\boldsymbol{m}(Df^n|E_i^{\mathrm{u}}) \leqslant \frac{\|Df^{n+i} v^{\mathrm{u}}\|}{\|Df^i v^{\mathrm{u}}\|} \approx \frac{\|Df^{n+i} v^{\mathrm{cu}}\|}{\|Df^i v^{\mathrm{cu}}\|} \leqslant \|Df^n|E_i^{\mathrm{c}}\| \lesssim \boldsymbol{m}(Df^n|E_i^{\mathrm{u}}),$$

proving (4.14). In particular, $\|Df^n v^{\mathrm{u}}\| \approx \boldsymbol{m}(Df^n|E_0^{\mathrm{u}})$. Analogously, $\|Df^n v^{\mathrm{s}}\| \approx \|Df^n|E_0^{\mathrm{s}}\|$. Therefore, $\|Df^n v^{\mathrm{u}}\| \approx 1/\|Df^n v^{\mathrm{s}}\|$, completing the proof of (4.13). $\square$

For $i = 0, \dots, m$, let

$$\begin{aligned} \boldsymbol{e}_{1,i} &= \frac{Df^i v^{\mathrm{u}}}{\|Df^i v^{\mathrm{u}}\|}, & \boldsymbol{f}_{1,i} &= \frac{\|Df^i v^{\mathrm{u}}\| Df^i v^{\mathrm{s}}}{\omega(v^{\mathrm{u}}, v^{\mathrm{s}})}, \\ \boldsymbol{e}_{2,i} &= \frac{Df^i v^{\mathrm{cu}}}{\|Df^i v^{\mathrm{u}}\|}, & \boldsymbol{e}_{2,i} &= \frac{\|Df^i v^{\mathrm{u}}\| Df^i v^{\mathrm{cs}}}{\omega(v^{\mathrm{cu}}, v^{\mathrm{cs}})}. \end{aligned}$$

Then $\{\boldsymbol{e}_{1,i}, \boldsymbol{f}_{1,i}, \boldsymbol{e}_{2,i}, \boldsymbol{f}_{2,i}\}$ is a orthosymplectic subset of $T_{z_i}M$. By Lemma 4.3, we can extend it to a symplectic basis $\{\boldsymbol{e}_{1,i}, \boldsymbol{f}_{1,i}, \dots, \boldsymbol{e}_{N,i}, \boldsymbol{f}_{N,i}\}$, and furthermore if $\mathcal{L}_i$ is the linear map that takes this basis to the canonical symplectic basis of $\mathbb{R}^{2N}$ then $\|\mathcal{L}_i^{\pm 1}\| \lesssim 1$. The restriction of the map $A_i = \mathcal{L}_{i+1} \circ Df(z_i) \circ \mathcal{L}_i^{-1}$ to the 4-plane $p_1 p_2 q_1 q_2$ is given by

$$A_i : (p_1, p_2, q_1, q_2) \mapsto (c_i p_1, c_i p_2, c_i^{-1} q_1, c_i^{-1} q_2), \quad \text{where } c_i = \frac{\|Df^{i+1} v^{\mathrm{u}}\|}{\|Df^i v^{\mathrm{u}}\|}.$$

Unfortunately, c_i is not necessarily always bigger than 1 as required in the definition of type IV. To remedy that, we present the following lemma.

Sublemma 4.9. *Given $C_1 > 0$, $\delta_1 > 0$ and $\ell \in \mathbb{N}$ there exist $C_2 > 0$, $\delta_2 > 0$ with the following properties. Given a sequence $\{a_i\}_{i=0}^{m-1}$ with $|a_i| \leqslant C_1$ for each i and $\sum_{j=i}^{i+\ell-1} a_j > \delta_1$ for $0 \leqslant i \leqslant m-\ell$, there exists a sequence $\{b_i\}_{i=0}^{m}$ such that $|b_i| \leqslant C_2$ and $b_{i+1} + a_i - b_i > \delta_2$ for each i.*

Proof. Let $a_i = C_1$ for $i \geqslant m$. Let

$$b_i = \frac{1}{\ell} \sum_{j=0}^{\ell-1} (\ell - 1 - j) a_{i+j}.$$

Then

$$b_{i+1} + a_i - b_i = \frac{1}{\ell} \sum_{j=0}^{\ell-1} a_{i+j} > \frac{\delta_1}{\ell}.$$

$\square$

Let $a_i = \log c_i$ and let b_i be given by the sublemma. Let $D_i : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be the symplectic linear map defined by $D_i(\partial/\partial p_j) = \mathrm{e}^{b_i}(\partial/\partial p_j)$, $D_i(\partial/\partial q_j) = \mathrm{e}^{-b_i}(\partial/\partial q_j)$. Consider the new map $\hat{\mathcal{L}}_i = D_i \circ \mathcal{L}_i$; then the action of $\hat{\mathcal{L}}_{i+1} \circ Df(z_i) \circ \hat{\mathcal{L}}_i^{-1}$ on the 4-plane $p_1p_2q_1q_2$ is given by

$$(p_1, p_2, q_1, q_2) \mapsto (\hat{c}_i p_1, \hat{c}_i p_2, \hat{c}_i^{-1} q_1, \hat{c}_i^{-1} q_2), \quad \text{where } \hat{c}_i = \mathrm{e}^{b_{i+1}-b_i} c_i.$$

We have $\hat{c}_i > \tau > 1$ where τ depends only on α, K and m_0. This proves that the segment $\{z_0, \dots, z_m\}$ is of type IV, completing the proof of Lemma 4.1. □

5. Proof of flexibility

The goal of this section is to prove the Main Lemma. Thus we will show that each of the cases (I)–(IV) from Lemma 4.1 implies flexibility.

Let the diffeomorphism f, $p \in \{1, \dots, N\}$, $\varepsilon > 0$ and $\kappa > 0$ be fixed throughout this section. For concision, we will say that a segment $\{z, \dots, f^{n-1}z\}$ (with $z \in \Sigma_p(f)$) is *flexible* if the split sequence $Df(f^i z) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow$ $(0 \leqslant i < n)$ is (ε, κ)-flexible.

We now state four lemmas.

Lemma 5.1. *There is $\alpha > 0$ such that if $z \in \Sigma_p(f)$ satisfies $\measuredangle(E^{\mathrm{u}}(z), E^{\mathrm{cs}}(z)) < \alpha$ then the segment (of length 1) $\{z\}$ is flexible.*

Lemma 5.2. *There is $K_{\mathrm{II}} > 1$ such that if a segment $\{z, \dots, f^{n-1}z\}$, with $z \in \Sigma_p(f)$ is of type II with constant K_{II} then it is flexible.*

Lemma 5.3. *Given $K_{\mathrm{III}} > 1$, there exists m_0 such that if a segment $\{z, \dots, f^{m_0-1}z\}$ is of type III with constant K_{III} then it is flexible.*

Lemma 5.4. *Given $K_{\mathrm{IV}} > 1$ and $\tau > 1$ there exists m_1 such that if a segment of length $m \geqslant m_1$ is of type IV with constants K_{IV}, τ then it is flexible.*

Assuming Lemmas 5.1–5.4, we can give the following proof.

Proof of the Main Lemma. Let α and K_{II} be given by Lemmas 5.1 and 5.2, respectively. Let $K_{\mathrm{III}} = K_{\mathrm{III}}(\alpha, K_{\mathrm{II}})$ be given by Lemma 4.1. Let $m_0 = m_0(K_{\mathrm{III}})$ be given by Lemma 5.3. Let $K_{\mathrm{IV}} = K_{\mathrm{IV}}(\alpha, K_{\mathrm{II}}, m_0)$ and $\tau = \tau(\alpha, K_{\mathrm{II}}, m_0)$ be given by Lemma 4.1. Finally, let $m_1 = m_1(K_{\mathrm{IV}}, \tau)$ be given by Lemma 5.4. We can assume $m_1 \geqslant m_0$.

Now, if $m \geqslant m_1$ and the segment $\{z, \dots, f^m z\}$ is non-dominated (meaning that (3.6) is satisfied) then one of the four alternatives in Lemma 4.1 hold. Lemmas 5.1–5.4 imply that in each case the segment contains a flexible subsegment. So, by Lemma 3.3, the whole segment is flexible. □

5.1. Dealing with cases (I) and (II)

Lemma 5.5. *Given $\varepsilon > 0$ and $\kappa > 0$, there exists $\alpha > 0$ with the following properties. If v, w are unit vectors in $\mathbb{R}^{2N}$ with $\measuredangle(v, w) < \alpha$, and $U \subset \mathbb{R}^{2N}$ is a non-empty open set, then there exists $h \in \mathrm{Diff}^1_\omega(\mathbb{R}^{2N})$ that equals the identity outside of U, $\|Dh - \mathrm{Id}\| < \varepsilon$ uniformly, and such that the set G of points $x \in U$ such that $Dh(x) \cdot v = w$ has measure $\mu(G) > (1 - \kappa)\mu(U)$.*

Proof. This follows from Lemmas 5.7 and 5.12 from [**13**]. □

Proof of Lemma 5.1. It follows easily from Lemma 5.5. □

Proof of Lemma 5.2. It follows from Lemma 5.5 applied twice. More precisely, one takes the unit vector in $E^{\mathrm{u}}(z)$ that is least expanded by Df^n, and rotates it (using Lemma 5.5) towards the direction in $E^{\mathrm{cs}}(z)$, which is most expanded by Df^n. The image of the rotated vector by Df^n then gets close to $E^{\mathrm{cs}}(f^n z)$, so with another rotation we are done. The reader can either fill the details for himself, or else see [**13**, p. 1449]. □

5.2. Hamiltonians and dimension reduction

Let us see a procedure that will permit us to essentially reduce the proofs of Lemmas 5.3 and 5.4 to dimensions 2 and 4, respectively.

For $\nu < N$, let

$$\mathbb{R}^{2\nu} = \{(p_1, \dots, p_N, q_1, \dots, q_N) \in \mathbb{R}^{2N};\ p_i = q_i = 0 \text{ for } i > \nu\}.$$

Notice the standard symplectic form on $\mathbb{R}^{2N}$ restricted to $\mathbb{R}^{2\nu}$ coincides with the standard symplectic form on $\mathbb{R}^{2\nu}$. Also, $(\mathbb{R}^{2\nu})^\omega = \{p_i = q_i = 0 \text{ for } i \leqslant \nu\}$, so $\mathbb{R}^{2N} = \mathbb{R}^{2\nu} \oplus (\mathbb{R}^{2\nu})^\omega$. In what follows, we write

$$\mathbb{R}^{2N} = \{(x, y);\ x \in \mathbb{R}^{2\nu},\ y \in (\mathbb{R}^{2\nu})^\omega\}.$$

If a symplectic map $A : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ preserves $\mathbb{R}^{2\nu}$ then it also preserves the symplectic complement $(\mathbb{R}^{2\nu})^\omega$, so A can be written as $A(x, y) = (B(x), C(y))$, where B and C are symplectic maps on $\mathbb{R}^{2\nu}$ and $(\mathbb{R}^{2\nu})^\omega$, respectively.

If H is a smooth (ie, C^∞) function on $\mathbb{R}^{2N}$, then we let φ^t_H denote the Hamiltonian flow generated by H.

Lemma 5.6. *Let $H : \mathbb{R}^{2N} \to \mathbb{R}$ be a smooth function that is constant outside a compact set. Then the associated Hamiltonian flow $\varphi^t_H : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is defined for every time $t \in \mathbb{R}$, and*

$$\|\varphi^t_H(\xi) - \xi\| \leqslant |t| \sup \|DH\|, \qquad \|D(\varphi^t_H)(\xi) - \mathrm{Id}\,\| \leqslant \exp(|t| \sup \|D^2 H\|) - 1$$

for every $\xi \in \mathbb{R}^{2N}$ and $t \in \mathbb{R}$.

Proof. The last assertion follows from a Gronwall inequality applied to the Lipschitz function $u(t) = 1 + \sup \|D\varphi^t_H - \mathrm{Id}\,\|$. □

Lemma 5.7. *Given $\nu \in \{1, \dots, N-1\}$, $\delta > 0$, $\kappa > 0$, and also*

- *symplectic linear maps $A_0, \dots, A_{m-1} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ preserving $\mathbb{R}^{2\nu}$, so that we can write $A_i(x, y) = (B_i(x), C_i(y))$, for $x \in \mathbb{R}^{2\nu}$, $y \in (\mathbb{R}^{2\nu})^\omega$;*
- *for each $i = 0, \dots, m-1$, a smooth function $H_i : \mathbb{R}^{2\nu} \to \mathbb{R}$ such that $\|D^2 H_i\| < \delta$ uniformly and H_i is constant outside of $B_{i-1} \circ \cdots \circ B_0(U)$, where U is the open unit ball in $\mathbb{R}^{2\nu}$.*

Then there exist:

- *a cylinder* $\hat{U} = \{(x, y) \in \mathbb{R}^{2N};\ \|x\| < 1,\ \|y\| < a\}$, *where* $a > 0$;
- *smooth functions* $\hat{H}_i : \mathbb{R}^{2N} \to \mathbb{R}$ *such that* $\|D^2 \hat{H}_i\| < 2\delta$ *uniformly and* $\hat{H}_i$ *is constant outside of* $A_{i-1} \circ\circ\circ A_0(\hat{U})$;
- *a set* $\hat{G} \subset \hat{U}$ *with* $\mu(\hat{G}) > (1-\kappa)\mu(\hat{U})$ *such that if* $(x, y) \in \hat{G}$ *then*

$$\begin{aligned} A_{m-1} \circ \varphi^t_{\hat{H}_{m-1}} \circ\circ\circ A_0 \circ \varphi^t_{\hat{H}_0}(x, y) \\ = (B_{m-1} \circ \varphi^t_{H_{m-1}} \circ\circ\circ B_0 \circ \varphi^t_{H_0}(x), C_{m-1} \circ\circ\circ C_0(y)). \end{aligned} \tag{5.1}$$

Proof. Let $\mathcal{B}_0$, $\mathcal{C}_0$ be the open unit balls in $\mathbb{R}^{2\nu}$, $(\mathbb{R}^{2\nu})^\omega$, respectively. Let $\mathcal{B}_i = B_{i-1} \circ\circ\circ B_0(\mathcal{B}_0)$, $\mathcal{C}_i = C_{i-1} \circ\circ\circ C_0(\mathcal{C}_0)$. Let $0 < \sigma < 1$ be such that $\sigma^{2(N-\nu)} > 1 - \kappa$. Let $\zeta : \mathbb{R} \to [0, 1]$ be a smooth function such that:

$$\zeta(t) = 1 \quad \text{for } t \leqslant \sigma, \qquad \zeta(t) = 0 \quad \text{for } t \geqslant 1, \qquad |\zeta'(t)| \leqslant \frac{10}{1-\sigma}, \qquad |\zeta''(t)| \leqslant \frac{10}{(1-\sigma)^2}.$$

Let $a \gg 1$ (to be specified later). Define

$$\psi_i : (\mathbb{R}^{2\nu})^\omega \to \mathbb{R} \quad \text{by } \psi_i(y) = \zeta(a^{-1}\|C_0^{-1} \cdots C_{i-1}^{-1}(y)\|).$$

Then

$$\psi_i(y) = 1 \quad \text{for } y \in \sigma a \mathcal{C}_i \qquad \text{and} \qquad \psi_i(y) = 0 \quad \text{for } y \notin a\mathcal{C}_i.$$

Letting $c = c(\sigma)$ be an upper bound for the norms of the first and second derivatives of the function $y \in (\mathbb{R}^{2\nu})^\omega \mapsto \zeta(\|y\|)$, we can write

$$\|D\psi_i\| \leqslant ca^{-1}\|C_0^{-1} \cdots C_{i-1}^{-1}\| \quad \text{and} \quad \|D^2\psi_i\| \leqslant ca^{-2}\|C_0^{-1} \cdots C_{i-1}^{-1}\|^2.$$

So if a is large enough, $\|D\psi_i\|$ and $\|D^2\psi_i\|$ are both uniformly small, for every i.

There is no loss in generality if we assume that each H_i is zero outside $\mathcal{B}_i$. Define $\hat{H}_i(x, y) = H_i(x)\psi_i(y)$. Writing $v = (v_x, v_y) \in \mathbb{R}^{2\nu} \oplus (\mathbb{R}^{2\nu})^\omega$ and analogously for w, we compute

$$\begin{aligned} D^2\hat{H}_i(x, y)(v, w) = H_i(x) \cdot D^2\psi_i(y)(v_y, w_y) + DH_i(x)(w_x) \cdot D\psi_i(y)(v_y) \\ + DH_i(x)(v_x) \cdot D\psi_i(y)(w_y) + D^2H_i(x)(v_x, w_x) \cdot \psi_i(y). \end{aligned}$$

Therefore, $\|D^2\hat{H}_i\| < 2\delta$ for every i, provided a is chosen sufficiently large.

Define the subsets of $\mathbb{R}^{2N}$:

$$\hat{U} = \mathcal{B}_0 \oplus (a\mathcal{C}_0) \quad \text{and} \quad \hat{G} = \mathcal{B}_0 \oplus (\sigma a \mathcal{C}_0).$$

The choice of σ implies that $\mu(\hat{G}) > (1-\kappa)\mu(\hat{U})$. We have $\hat{H}_i(x, y) = 0$ if $x \notin \mathcal{B}_i$ or $y \notin a\mathcal{C}_i$, that is, if $(x, y) \notin A_{i-1} \circ\circ\circ A_0(\hat{U})$. Moreover, if $(x, y) \in \mathcal{B}_i \oplus (\sigma a \mathcal{C}_i)$ then $\varphi^t_{\hat{H}_i}(x, y) = (\varphi^t_{H_i}(x), y)$. So (5.1) follows. □

In § 5.4 we will use the following lemma about change of coordinates in hamiltonians. The easy proof is left to the reader.

Lemma 5.8. *Let H be a hamiltonian on $\mathbb{R}^{2N}$, $a > 0$, and $M : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be a symplectic linear map. Define hamiltonians $H_1(x) = a^{-2}H(ax)$ and $H_2(x) = H(M(x))$. Then*

$$D^2H_1(x) \cdot (v,w) = D^2H(ax) \cdot (v,w), \qquad \varphi^t_{H_1}(x) = a^{-1}\varphi^t_H(ax),$$
$$D^2H_2(x) \cdot (v,w) = D^2H(M(x)) \cdot (Mv, Mw), \qquad \varphi^t_{H_2}(x) = M^{-1} \circ \varphi^t_H \circ M(x).$$

5.3. Dealing with case (III)

Proof of Lemma 5.3. We will assume $2N > 2$. (Readers can adapt the arguments for the simpler two-dimensional case if they want to reobtain the results of [**9**].)

Let K_{III} (and also ε, κ) be given. Let $\varepsilon' = (K_{\mathcal{A}}K_{\mathrm{III}})^{-2}\varepsilon$. (Recall the definition of $K_{\mathcal{A}}$ from § 3.2.) Let δ be such that $\mathrm{e}^{2\delta} - 1 = \varepsilon'$. Let $\sigma = 1 - \kappa/2$. Take a smooth function $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\rho(t) = t \quad \text{for } 0 \leqslant t \leqslant \sigma, \qquad \rho(t) = 1 \quad \text{for } t \geqslant 1, \qquad 0 \leqslant \rho'(t) \leqslant 1, \qquad |\rho''(t)| \leqslant \frac{10}{1-\sigma}.$$

Let $\alpha > 0$ and define $H(p_1, q_1) = \frac{1}{2}\alpha\rho(p_1^2 + q_1^2)$. Notice that φ^t_H restricted to the disk $p_1^2 + q_1^2 \leqslant \sigma$ is a rotation of angle $t\alpha$. Choose m big enough so that setting $\alpha = \pi/2m$ we have $\|D^2H\| < \delta$ uniformly. Let us see that $m_0 = m$ has the desired properties.

Take a segment $\{z, \dots, f^{m-1}z\}$ of type III with constant K_{III}. Let $\mathcal{L}_i : T_{f^iz}M \to \mathbb{R}^{2N}$ and $A_i = \mathcal{L}_{i+1} \circ Df(f^iz) \circ \mathcal{L}_i$ be as in the definition of type III. Our aim is to show that the split sequence $Df(f^iz) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow (0 \leqslant i < m)$ is (ε, κ)-flexible. Because of Lemma 3.2, it suffices to show that the split sequence on $\mathbb{R}^{2N}$

$$\{F^u_i \oplus F^{\mathrm{cs}}_i \xrightarrow{A_i} F^u_{i+1} \oplus F^{\mathrm{cs}}_{i+1}\}_{0 \leqslant i < m}, \quad \text{where } F^*_i = \mathcal{L}_i(E^*(f^iz))$$

is (ε', κ)-flexible.

The maps A_i are the identity on the plane $\mathbb{R}^2$ spanned by $\partial/\partial p_1$ and $\partial/\partial q_1$. So we can write $A_i(x, y) = (x, C_i(y))$ for $x \in \mathbb{R}^2$, $y \in (\mathbb{R}^2)^\omega$. Apply Lemma 5.7 with $\nu = 2$, $H_i = H$ for $0 \leqslant i < m$, and $\kappa/2$ in the place of κ. We obtain a cylinder $\hat{U}$, hamiltonians $\hat{H}_i$ that are constant outside $A_{i-1} \circ\circ\circ A_0(\hat{U})$ and satisfy $\|D^2\hat{H}_i\| < 2\delta$, and a set $\hat{G} \subset \hat{U}$ with measure greater than $(1 - \kappa/2)\mu(\hat{U})$ where

$$A_{n-1} \circ \varphi^1_{\hat{H}_{n-1}} \circ\circ\circ A_0 \circ \varphi^1_{\hat{H}_0}(x, y) = (\varphi^m_H(x), C_{n-1} \circ\circ\circ C_0(y)).$$

Let $g_i = A_i \circ \varphi^1_{\hat{H}_i}$. We check that the maps g_i have the properties demanded by flexibility (for any $\gamma > 0$, in fact).

(1) $g_i = A_i$ outside $A_{i-1} \circ\circ\circ A_0(\hat{U})$.

(2) By Lemma 5.6, $\|D(A_i^{-1} \circ g_i) - \mathrm{Id}\,\| < \mathrm{e}^{2\delta} - 1 = \varepsilon'$.

(3) The cylinder $\hat{U} \cap \{p_1^2 + q_1^2 < \sigma\}$ has measure $\sigma\mu(\hat{U})$; let G be its intersection with $\hat{G}$. Then $\mu(G)/\mu(\hat{U}) > \sigma - \kappa/2 = 1 - \kappa$. If $\xi = (x, y) \in G$ then

$$g_{m-1} \circ\circ\circ\, g_0(\xi) = (R_{\pi/2}(x), C_{n-1} \circ\circ\circ\, C_0(y))$$

and therefore $D(g_{m-1} \circ\circ\circ\, g_0)(\xi) \cdot (\partial/\partial p_1) = (\partial/\partial q_1)$. In particular the angle between $D(g_{m-1} \circ\circ\circ\, g_0)(\xi) \cdot F_0^u$ and F_n^{cs} is zero.

□

5.4. Dealing with case (IV)

As already mentioned, the proof of Lemma 5.4 will be essentially reduced to dimension 4. Let us fix some notation. For $t \in \mathbb{R}$, define the following symplectic linear map on $\mathbb{R}^4 = \{(p_1, p_2, q_1, q_2)\}$:

$$R_t = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}. \tag{5.2}$$

For t in the circle $\mathbb{R}/\pi\mathbb{Z}$, let us indicate $|||t||| = \min_{k\in\mathbb{Z}} |t - k\pi|$.

If $v = (p_1, p_2, q_1, q_2)$ is a vector in $\mathbb{R}^4$ such that $(p_1, p_2) \neq (0, 0)$ then we let $\Theta(v)$ be such that $(p_1, p_2) = \pm(r \cos \Theta(v), r \sin \Theta(v))$, where $r = (p_1^2 + p_2^2)^{1/2}$; thus $\Theta(v)$ is uniquely defined in $\mathbb{R}/\pi\mathbb{Z}$.

For $\beta > 0$, define cones

$$\mathcal{C}_\beta = \{(p_1, p_2, q_1, q_2) \in \mathbb{R}^4;\ \|(q_1, q_2)\| < \beta\|(p_1, p_2)\|\}.$$

Lemma 5.9. *For every $v \in \mathcal{C}_1$ there is a symplectic linear map $L_v : \mathbb{R}^4 \to \mathbb{R}^4$ such that:*

(1) *L_v preserves the plane spanned by $\partial/\partial q_1$ and $\partial/\partial q_2$;*

(2) *$L_v(\partial/\partial p_1)$ is collinear to v;*

(3) *$\Theta(L_v(w)) = \Theta(w) + \Theta(v)$ for all $v, w \in \mathcal{C}_1$;*

(4) *$\|L_v\| = \|L_v^{-1}\| \leqslant K_L$ for all $v \in \mathcal{C}_1$, where $K_L > 1$ is an absolute constant.*

Proof. Let $v = (p_1, p_2, q_1, q_2) \in \mathcal{C}_1$. Assume that $p_1^2 + p_2^2 = 1$. Let $\theta = \Theta(v)$. Then $R_{-\theta}(v) = (1, 0, a, b)$ for certain a and b with $a^2 + b^2 \leqslant 1$. (Recall (5.2).) The matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

is symplectic and preserves Θ. Then $L_v = R_\theta \circ M$ has the required properties. □

The following well-known fact about random walks will play an important role in the proof.

Lemma 5.10. *Let $X_0, X_1, \dots$ be independent identically distributed random variables, with $\mathbb{E}|X_0| < \infty$ and $0 < \mathbb{E}X_0^2 < \infty$. Let $S_n = X_0 + \cdots + X_{n-1}$. For any fixed $K > 0$, the probability that $|S_n| \leqslant K$ for all n is zero.*

Proof. Let a and σ be respectively the mean and the variance of X_0. Of course, $\sigma > 0$. By the Central Limit Theorem, $Y_n = (S_n - an)/(\sigma\sqrt{n})$ converges in distribution to a standard normal random variable. That is,

$$\lim_{n\to\infty} \mathbb{P}[\alpha \leqslant Y_n \leqslant \beta] = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \mathrm{e}^{-t^2/2}\,\mathrm{d}t \quad \forall \alpha < \beta.$$

Fix $K > 0$. If $a = 0$ then

$$\mathbb{P}[|S_n| \leqslant K] = \mathbb{P}\biggl[|Y_n| \leqslant \frac{K}{\sigma\sqrt{n}}\biggr].$$

If $a \neq 0$ then

$$\mathbb{P}[|S_n| \leqslant K] \leqslant \mathbb{P}\biggl[|Y_n| \geqslant \frac{|a|n - K}{\sigma\sqrt{n}}\biggr].$$

In either case, we have $\lim_{n\to\infty} \mathbb{P}[|S_n| \leqslant K] = 0$. In particular, $\mathbb{P}[|S_n| \leqslant K\ \forall n] = 0$. $\square$

Proof of Lemma 5.4.

Step 1: preparation. Let ε, κ, K_{IV}, τ be given. Let $\varepsilon' > 0$ be such that $\varepsilon' < (K_{\mathcal{A}} K_{\mathrm{IV}})^{-2}\varepsilon$ (recall the definition of $K_{\mathcal{A}}$ from § 3.2) and $M(\mathcal{C}_1) \subset \mathcal{C}_{\tau^2}$ for all linear $M : \mathbb{R}^4 \to \mathbb{R}^4$ with $\|M - \mathrm{Id}\| < \varepsilon'$. Let δ be given by $\mathrm{e}^{2K_L^2\delta} - 1 = \varepsilon'$ (where K_L comes from Lemma 5.9). Let $\alpha > 0$ be given by Lemma 5.5 applied with ε' and $\kappa/10$ in place of ε and κ, respectively.

Let $\mathbb{D}$ be the open unit ball in $\mathbb{R}^4$.* Let $\bar{\mu}$ be Lebesgue measure on $\mathbb{R}^4$ normalized so that $\bar{\mu}(\mathbb{D}) = 1$.

Choose a smooth function $H : \mathbb{R}^4 \to \mathbb{R}$ not identically zero that vanishes outside of $\mathbb{D}$, and such that $\|D^2H\| < \delta$. Let $h : \mathbb{R}^4 \to \mathbb{R}^4$ be the associated time 1 map, that is, $h = \varphi_H^1$. Let ν be the probability measure on the circle $\mathbb{R}/\pi\mathbb{Z}$ defined by

$$\nu(A) = \bar{\mu}\biggl\{x \in \mathbb{D};\ \Theta\biggl(Dh(x) \cdot \frac{\partial}{\partial p_1}\biggr) \in A\biggr\}, \quad \text{for each Borel set } A \subset \frac{\mathbb{R}}{\pi\mathbb{Z}}.$$

We assume that H was chosen so that the support of ν is contained in the interval $\{t;\ |||t||| < \alpha/20\}$.

Let $X_0, X_1, \dots$ be independent circle-valued random variables, all distributed according to the measure ν.† Consider the random walk $S_n = X_0 + \cdots + X_{n-1}$. By Lemma 5.10, there exists m_1 such that

$$\text{the probability that } |||S_n - \tfrac{1}{2}\pi||| > \tfrac{1}{20}\alpha \text{ for all } n \leqslant m_1 \text{ is less than } \tfrac{1}{20}\kappa. \tag{5.3}$$

We will show that m_1 has the desired properties.

* A 'box' as in § 2.3 would work equally well.

† It is interesting, although unimportant, to see that $\mathbb{E}(\tan X_0) = 0$.

Take $m \geqslant m_1$ and assume that $\{z, \dots, f^m z\}$ is a segment of type IV with constants K_{IV}, τ. Let $\mathcal{L}_i : T_{f^i z}M \to \mathbb{R}^{2N}$ and $A_i = \mathcal{L}_{i+1} \circ Df(f^i z) \circ \mathcal{L}_i$ be as in the definition of type IV. We want to prove that the split sequence $Df(f^i z) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow (0 \leqslant i < m)$ is (ε, κ)-flexible. Bearing in mind Lemma 3.2, it suffices to show that the split sequence on $\mathbb{R}^{2N}$

$$\{F_i^u \oplus F_i^{\mathrm{cs}} \xrightarrow{A_i} F_{i+1}^u \oplus F_{i+1}^{\mathrm{cs}}\}_{0 \leqslant i \leqslant m}, \quad \text{where } F_i^* = \mathcal{L}_i(E^*(f^i z)), \tag{5.4}$$

is (ε', κ)-flexible.

By definition of type IV,

$$A_i(x, y) = (B_i(x), C_i(y)), \quad B_i(p_1, p_2, q_1, q_2) = (c_i p_1, c_i p_2, c_i^{-1} q_1, c_i^{-1} q_2), \quad c_i > \tau.$$

Also, for all i,

$$\frac{\partial}{\partial p_1} \in F_i^u, \qquad \frac{\partial}{\partial p_2}, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2} \in F_i^{\mathrm{cs}}. \tag{5.5}$$

Step 2: reduction to $\mathbb{R}^4$. Let $U_0 = \mathbb{D}$ and $U_n = B_{n-1} \circ\circ\circ B_0(\mathbb{D})$ for $1 \leqslant n \leqslant m$.

Sublemma 5.11. *There exist symplectomorphisms $g_n : \mathbb{R}^4 \to \mathbb{R}^4$, for $0 \leqslant n < m$, with the following properties:*

(1) *g_n equals B_n outside U_n and $\|D(B_n^{-1} \circ g_n) - \mathrm{Id}\,\| < \varepsilon'$ at each point;*

(2) *for each n, there is a smooth function $H_n : \mathbb{R}^4 \to \mathbb{R}$ constant outside U_n such that $\|D^2 H\| < K_L^2 \delta$ and the time 1 map $\varphi_{H_n}^1$ equals $B_n^{-1} \circ g_n$;*

(3) *there is a set $G \subset U_0$ with (normalized) measure $\bar{\mu}(G) > 1 - \kappa/10$ such that*

$$\left|\!\left|\!\left| \Theta\left(D(g_{m-1} \circ\circ\circ g_0)(x) \cdot \frac{\partial}{\partial p_1} \right) - \tfrac{1}{2}\pi \right|\!\right|\!\right| < \tfrac{1}{2}\alpha \quad \textit{for all } x \in G. \tag{5.6}$$

Let us assume the sublemma for a while and see how to conclude the proof of Lemma 5.4. Let $\gamma > 0$ be given (as in the definition of flexibility). We will assume $2N > 4$, leaving for the reader the easy adaptation for the four-dimensional case. Consider the hamiltonians H_n given by Sublemma 5.11, and apply Lemma 5.7 with $2\nu = 4$, $K_L^2 \delta$ in the place of δ, and $\kappa/10$ in the place of κ. We obtain a cylinder $\hat{U} \subset \mathbb{R}^{2N}$ and hamiltonians $\hat{H}_n : \mathbb{R}^{2N} \to \mathbb{R}$ such that writing $\hat{g}_n = A_n \circ \varphi_{\hat{H}_n}^1$ we have:

- $\hat{g}_n$ equals A_n outside of $A_{n-1} \circ\circ\circ A_0(\hat{U})$;
- $\|D^2 \hat{H}_n\| < 2K_L^2 \delta$ and hence, by Lemma 5.6, $\|D(A_n^{-1} \circ \hat{g}_n) - \mathrm{Id}\,\| < \varepsilon'$;
- there is a set $\hat{G} \subset \hat{U}$ with $\mu(\hat{G}) > (1 - \kappa/10)\mu(\hat{U})$ such that if $\xi = (x, y) \in \hat{G}$ then

$$\hat{g}_{m-1} \circ\circ\circ \hat{g}_0(\xi) = (g_{m-1} \circ\circ\circ g_0(x), C_{m-1} \circ\circ\circ C_0(y)).$$

Since $\hat{U}$ is a cylinder, the set $\{(x, y) \in \hat{U};\ x \in G\}$ has measure $> (1 - \kappa/10)\mu(\hat{U})$; let G_1 be its intersection with $\hat{G}$. Then $\mu(G_1) > (1 - 2\kappa/10)\mu(\hat{U})$. If $\xi = (x, y) \in G_1$ then by (5.6), the angle between the vector $D(g_{m-1} \circ\circ\circ\, g_0)(x) \cdot (\partial/\partial p_1)$ in $\mathbb{R}^4$ and the space spanned by $\partial/\partial p_2$, $\partial/\partial q_1$, $\partial/\partial q_2$ is at most α. Using (5.5) we conclude that

$$\measuredangle(D(\hat{g}_{m-1} \circ\circ\circ\, \hat{g}_0)(\xi) \cdot F_0^u, F_m^{\mathrm{cs}}) < \tfrac{1}{2}\alpha \quad \text{for all } \xi \in G_1.$$

We need to perform a last perturbation $\hat{g}_m$ to make the angle smaller than γ.

Let γ' be very small. By Vitali's Lemma, we can find a finite family of disjoint small euclidian balls D_ℓ contained in the open set G_1 and whose union leaves out a set of measure at most $(1-\kappa/10)\mu(\hat{U})$. In fact, the balls are taken small enough so that the variation of the angle $\measuredangle(D(\hat{g}_{m-1} \circ\circ\circ\, \hat{g}_0)(\xi) \cdot F_0^u, F_m^{\mathrm{cs}})$ is less than γ' when ξ runs over D_ℓ. For each ℓ, let ξ_ℓ be the centre of the ball D_ℓ, and let v_ℓ be the vector $D(\hat{g}_{m-1} \circ\circ\circ\, \hat{g}_0)(\xi_\ell) \cdot (\partial/\partial p_1)$.

We now use the definition of α. For each ℓ, Lemma 5.5 applied to the set $D'_\ell = \hat{g}_{m-1} \circ\circ\circ\, \hat{g}_0(D_\ell)$ gives a symplectomorphism $h_\ell : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ with the following properties:

- h_ℓ equals the identity outside of D'_ℓ;
- $\|Dh_\ell - \mathrm{Id}\,\| < \varepsilon'$;
- there is a set $G'_\ell \subset D'_\ell$ with $\mu(G'_\ell) > (1-\kappa/10)\mu(D'_\ell)$ such that for every $\xi' \in G'_\ell$, the vector $Dh_\ell(\xi') \cdot v_\ell$ belongs to F_m^{cs}.

Let $G_0 = \bigsqcup_\ell (\hat{g}_{m-1} \circ\circ\circ\, \hat{g}_0)^{-1}(G'_\ell)$. Then $\mu(G_0) > (1-\kappa)\mu(\hat{U})$. Finally, define the perturbation $\hat{g}_m$ as equal to $A_m \circ h_\ell$ in each D'_ℓ, and equal to A_m outside. If γ' was chosen sufficiently small then for every $\xi \in G_0$ we have

$$\measuredangle(D(\hat{g}_m \circ\circ\circ\, \hat{g}_0)(\xi) \cdot F_0^u, F_{m+1}^{\mathrm{cs}}) < \gamma.$$

This shows that the split sequence (5.4) is (ε', κ)-flexible. Hence to complete the proof of Lemma 5.4 we are left to prove Sublemma 5.11.

Step 3: definition of perturbations in $\mathbb{R}^4$. Before starting the proof of the sublemma, notice the first condition there implies that

$$Dg_n(x)(\mathcal{C}_1) \subset \mathcal{C}_1 \quad \forall x, \tag{5.7}$$

due to the definition of ε' and the fact that $B_n(\mathcal{C}_{\tau^2}) \subset \mathcal{C}_1$.

Let $\mathcal{N}(v)$ indicate $v/\|v\|$. Fix a constant $K > 1$ such that for all unit vectors v, $w \in \mathcal{C}_1$ we have:

$$\left.\begin{aligned} |||\Theta(v) - \Theta(w)||| &\leqslant K\|v-w\|, \\ \|\mathcal{N}(Dg_n(x)\cdot v) - \mathcal{N}(Dg_n(x)\cdot w)\| &\leqslant K\|v-w\| \quad \forall x \end{aligned}\right\} \tag{5.8}$$

(provided g_n complies with the first condition in Sublemma 5.11). Let

$$\eta = \min\left(\frac{\alpha}{100K^2 m}, \frac{\kappa}{20m}\right). \tag{5.9}$$

For each $n = 0, \dots, m$, we are also going to define a finite family $\{D_i\}_{i \in I_n}$ of disjoint subsets of U_n. Also, the sets of indices $I_0, \dots, I_m$ will be disjoint, and each I_n will be partitioned as $I_n = I_n^{\text{arrived}} \sqcup I_n^{\text{not yet}}$.

Start defining $g_0 = B_0 \circ h$ (recall the definition of h in Step 1). Then, by Lemma 5.6, $\|D(B_0^{-1} \circ g_0) - \mathrm{Id}\,\| < \mathrm{e}^{\delta} - 1 < \varepsilon'$, as required. Also define $I_0 = I_0^{\text{not yet}} = \{0\}$, $D_0 = \mathbb{D}$.

By induction, assume that $g_0, \ldots, g_{n-1}$ and $\{D_i\}_{i \in I_{n-1}}$ are already defined, for some n with $0 < n \leqslant m$, and let us proceed to define g_n (if $n < m$) and $\{D_i\}_{i \in I_n}$. First define a vector field $\boldsymbol{v}_n$ on $\mathbb{R}^4$ by

$$\boldsymbol{v}_n(g_{n-1} \circ\circ\circ\, g_0(x)) = \mathcal{N}\left(D(g_{n-1} \circ\circ\circ\, g_0)(x) \cdot \frac{\partial}{\partial p_1}\right).$$

Then $\boldsymbol{v}_n$ takes values on the cone $\mathcal{C}_1$, because (5.7) holds for $g_0, \ldots, g_{n-1}$.

Let $V_{n-1} = \bigsqcup_{i \in I_{n-1}} D_i \subset U_{n-1}$, so that $g_{n-1}(V_{n-1}) \subset U_n$. For $x \in g_{n-1}(V_{n-1})$ and $r > 0$, define a neighbourhood of x by

$$\tilde{D}(x, r, n) = \{x + rL_{\boldsymbol{v}_n(x)}(y);\ y \in \mathbb{D}\}$$

(where the Ls come from Lemma 5.9). These neighbourhoods are 'quasi-round', in the sense that $B_{K_L^{-1} r}(x) \subset \tilde{D}(x, r, n) \subset B_{K_L r}(x)$. Now consider the family of sets $\tilde{D}(x, r, n)$ with r sufficiently small so that the variation of $\boldsymbol{v}_n$ in each $\tilde{D}(x, r, n)$ is less than η. This family constitutes a Vitali cover of the set $g_{n-1}(V_{n-1})$. Therefore, we can find a finite subfamily $\{D_i = \tilde{D}(\xi_i, r_i, n)\}_{i \in I_n}$ whose disjoint union covers most of the set, that is,

$$\bar{\mu}(g_{n-1}(V_{n-1}) \smallsetminus V_n) < \eta, \quad \text{where } V_n = \bigsqcup_{i \in I_n} D_i. \tag{5.10}$$

So we have defined the set of indices I_n and the family of sets $\{D_i\}_{i \in I_n}$. Let I_n^{arrived} be the set of $i \in I_n$ such that at least one of the following two properties is satisfied:

- $|||\varTheta(\boldsymbol{v}_n(\xi_i)) - \frac{1}{2}\pi||| < \frac{1}{10}\alpha$;
- if i' denotes the unique index in I_{n-1} such that $D_i \subset g_{n-1}(D_{i'})$ then i' already belongs to I_{n-1}^{arrived}.

Let $I_n^{\text{not yet}} = I_n \smallsetminus I_n^{\text{arrived}}$.

Next we define g_n (in the case $n < m$). Let g_n be equal to B_n outside of $\bigsqcup_{i \in I_n^{\text{not yet}}} D_i$. Inside each domain D_i with $i \in I_n^{\text{not yet}}$, let $g_n = B_n \circ T_i^{-1} \circ h \circ T_i$, where

$$T_i : D_i \to \mathbb{D} \quad \text{is given by } T_i(x) = L_{\boldsymbol{v}_n(\xi_i)}^{-1}((x - \xi_i)/r_i).$$

Since T_i is an affine map that expands the symplectic form by a constant factor, g_n is a well-defined symplectomorphism of $\mathbb{R}^4$.

Let us see that g_n satisfies parts (1) and (2) from Sublemma 5.11. Let

$$H_n(x) = \begin{cases} r_i^{-2} H(T_i(x)) & \text{if } x \in D_i \text{ with } i \in I_n^{\text{not yet}}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

It follows from Lemma 5.8 that the time 1 map $\varphi_{H_n}^1$ is precisely $B_n^{-1} \circ g_n$. The lemma also gives that $\|D^2 H_n\| \leqslant K_L^2 \|D^2 H\| < K_L^2 \delta$. This shows part (2) of Sublemma 5.11. Recalling Lemma 5.6, one sees that the first part follows from the second.

To summarize, we have defined the maps g_n (together with other objects) and have verified that they satisfy properties (1) and (2) of Sublemma 5.11. Next we will show that property (3) also holds.

Step 4: random walk behaviour. Recall that we have defined in Step 1 circle-valued random variables X_n. We will only be interested in the first m of them. Let us choose a probability space for these variables (as well as their sums $S_n = X_0 + \cdots + X_{n-1}$) to 'live in': it is $(\Omega, \mathbb{P})$, where $\Omega = \mathbb{D}^m$ and $\mathbb{P} = \bar{\mu}^m$. Let now each random variable X_n be the function

$$X_n : \Omega \to \mathbb{R}/\pi\mathbb{Z} \quad \text{given by } X_n(\omega_0, \ldots, \omega_{m-1}) = \Theta\left(Dh(\omega_n) \cdot \frac{\partial}{\partial p_1} \right).$$

In imprecise words, we will see that the angles $\Theta(\boldsymbol{v}_n(\cdot))$ behave approximately like the random walk S_n, with an absorbing barrier around $\pi/2$. This and (5.3) will permit us to show the third part of Sublemma 5.11.

In what follows, let $\boldsymbol{L}(c)$ stand for an unspecified $t \in \mathbb{R}/\pi\mathbb{Z}$ with $|||t||| < c$. By construction, if x and x' both belong to the same D_i with $i \in I_n$ then $\|\boldsymbol{v}_n(x_n) - \boldsymbol{v}_n(x'_n)\| < \eta$ and so (5.8) implies $\Theta(\boldsymbol{v}_n(x)) = \Theta(\boldsymbol{v}_n(x')) + \boldsymbol{L}(K\eta)$.

An *itinerary* is a sequence $\vec{\imath} = (i_0, i_1, \ldots, i_m) \in I_0 \times \cdots \times I_m$ such that $D_{i_{n+1}} \subset g_n(D_{i_n})$ for $0 \leqslant n < m$. (In fact, $\vec{\imath}$ is uniquely determined by i_m.) A *pseudo-orbit with itinerary* $\vec{\imath} = (i_n)$ is a sequence $(x_1, \ldots, x_m)$ such that $x_n \in D_{i_n}$ for each n. One example is the orbit $(x_n) = (g_{n-1} \circ\circ\circ\, g_0(x_0))$ of a point x_0 in $(g_{m-1} \circ\circ\circ\, g_0)^{-1}(D_{i_m})$. Other example of pseudo-orbit is $(\xi_{i_1}, \ldots, \xi_{i_m})$. (Recall ξ_i is the 'centre' of D_i.)

All pseudo-orbits with itinerary $\vec{\imath} = (i_n)$ are of the form

$$(x_1, \ldots, x_m) = (g_0(\omega_0), g_1(T_{i_1}^{-1}(\omega_1)), \ldots, g_{m-1}(T_{i_{m-1}}^{-1}(\omega_{m-1}))) \tag{5.12}$$

for some $\omega = (\omega_n) \in \mathbb{D}^m = \Omega$. With this writing, we claim that

$$\Theta(\boldsymbol{v}_1(x_1)) = X_0(\omega), \tag{5.13}$$

$$\Theta(\boldsymbol{v}_{n+1}(x_{n+1})) = \begin{cases} \Theta(\boldsymbol{v}_n(x_n)) + \boldsymbol{L}(K\eta) & \text{if } i_n \in I_n^{\text{arrived}}, \\ \Theta(\boldsymbol{v}_n(x_n)) + X_n(\omega) + \boldsymbol{L}(2K^2\eta) & \text{if } i_n \in I_n^{\text{not yet}}. \end{cases} \tag{5.14}$$

The proof of (5.13) is immediate:

$$\Theta(\boldsymbol{v}_1(x_1)) = \Theta\left(Dg_0(g_0^{-1}(x_1)) \cdot \frac{\partial}{\partial p_1} \right) = \Theta\left(Dg_0(\omega_0) \cdot \frac{\partial}{\partial p_1} \right) = X_0(\omega).$$

Now take n with $1 \leqslant n \leqslant m-1$. We have

$$\boldsymbol{v}_{n+1}(x_{n+1}) = \mathcal{N}(Dg_n(g_n^{-1}(x_{n+1})) \cdot \boldsymbol{v}_n(g_n^{-1}(x_{n+1}))).$$

Notice that the point $g_n^{-1}(x_{n+1})$ belongs to D_{i_n}. If $i_n \in I_n^{\text{arrived}}$ then g_n restricted to D_{i_n} equals B_n, which preserves Θ, therefore

$$\Theta(\boldsymbol{v}_{n+1}(x_{n+1})) = \Theta(\boldsymbol{v}_n(g_n^{-1}(x_{n+1}))) = \Theta(\boldsymbol{v}_n(x_n)) + \boldsymbol{L}(K\eta),$$

proving the first part of (5.14). For $i_n \in I_n^{\text{not yet}}$ we have

$$Dg_n(g_n^{-1}(x_{n+1})) = B_n \circ L_{\boldsymbol{v}_n(\xi_{i_n})} \circ Dh(\omega_n) \circ L_{\boldsymbol{v}_n(\xi_{i_n})}^{-1}.$$

Lemma 5.9 leads therefore to

$$\Theta(Dg_n(g_n^{-1}(x_{n+1}))\cdot \boldsymbol{v}_n(\xi_{i_n})) = \Theta\left(Dh(\omega_n)\cdot \frac{\partial}{\partial p_1}\right) + \Theta(\boldsymbol{v}_n(\xi_{i_n})) = X_n(\omega) + \Theta(\boldsymbol{v}_n(\xi_{i_n})).$$

Therefore, using that the points $g_n^{-1}(x_{n+1})$, ξ_{i_n}, and x_n belong to the same D_{i_n}, we can write

$$\begin{aligned}\Theta(\boldsymbol{v}_{n+1}(x_{n+1})) &= \Theta(Dg_n(g_n^{-1}(x_{n+1}))\cdot \boldsymbol{v}_n(g_n^{-1}(x_{n+1})))\\ &= \Theta(Dg_n(g_n^{-1}(x_{n+1}))\cdot \boldsymbol{v}_n(\xi_{i_n})) + \boldsymbol{L}(K^2\eta)\\ &= X_n(\omega) + \Theta(\boldsymbol{v}_n(\xi_{i_n})) + \boldsymbol{L}(K^2\eta)\\ &= X_n(\omega) + \Theta(\boldsymbol{v}_n(x_n)) + \boldsymbol{L}(2K^2\eta).\end{aligned}$$

This completes the proof of the claim (5.14).

Still assuming (x_n) and (ω_n) as in (5.12), we now claim that

$$\text{if } i_m \in I_m^{\text{arrived}} \quad \text{then } \Theta(\boldsymbol{v}_m(x_m)) = \tfrac{1}{2}\pi + \boldsymbol{L}(\tfrac{1}{2}\alpha), \tag{5.15}$$

$$\text{else } |||S_n(\omega) - \tfrac{1}{2}\pi||| > \tfrac{1}{20}\alpha \text{ for all } n. \tag{5.16}$$

If $i_m \in I_m^{\text{arrived}}$ then let n_0 be the least such that $i_{n_0} \in I_{n_0}^{\text{arrived}}$. It follows from the definitions that

$$\Theta(\boldsymbol{v}_{n_0}(\xi_{i_{n_0}})) = \tfrac{1}{2}\pi + \boldsymbol{L}(\tfrac{1}{10}\alpha) \quad \text{and} \quad i_n \in I_n^{\text{arrived}} \text{ for all } n \geqslant n_0.$$

Using repeatedly (5.14), together with (5.9), the claim (5.15) follows. On the other hand, if $i_m \in I_m^{\text{not yet}}$ then $i_n \in I_n^{\text{not yet}}$ for all n. Using (5.13) and (5.14) for the pseudo-orbit (ξ_n), and also (5.9), we obtain

$$\Theta(\boldsymbol{v}_n(\xi_{i_n})) = S_n(\omega) + \boldsymbol{L}(\tfrac{1}{50}\alpha).$$

The fact that $i_n \in I_n^{\text{not yet}}$ also implies that $|||\Theta(\boldsymbol{v}_n(\xi_{i_n})) - \frac{1}{2}\pi||| \geqslant \frac{1}{10}\alpha$, so (5.16) follows.

Next, for each itinerary $\vec{\imath} = (i_n)$, define the following subset of Ω:

$$W_{\vec{\imath}} = g_0^{-1}(D_{i_1}) \times T_{i_1}(g_1^{-1}(D_{i_2})) \times \cdots \times T_{i_{m-1}}(g_{m-1}^{-1}(D_{i_m})).$$

Let us evaluate its probability. Using the fact that g_ns preserve $\bar{\mu}$ and that the affine maps $T_i : D_i \to \mathbb{D}$ expand $\bar{\mu}$ by the factor $\det T_i = 1/\bar{\mu}(D_i)$, we get

$$\mathbb{P}(W_{\vec{\imath}}) = \bar{\mu}(D_{i_1})\det(T_{i_1})\bar{\mu}(D_{i_2})\cdots\det(T_{i_{m-1}})\bar{\mu}(D_{i_m}) = \bar{\mu}(D_{i_m}).$$

Summing over the itineraries such that $i_m \in I_m^{\text{not yet}}$, using (5.16) and (5.3), we obtain

$$\begin{aligned}\sum_{i_m \in I_m^{\text{not yet}}} \bar{\mu}(D_{i_m}) &= \mathbb{P}\Bigg(\bigsqcup_{i_m \in I_m^{\text{not yet}}} W_{\vec{\imath}}\Bigg)\\ &\leqslant \mathbb{P}[|||S_n - \tfrac{1}{2}\pi||| > \tfrac{1}{20}\alpha \text{ for all} n \leqslant m]\\ &< \tfrac{1}{20}\kappa.\end{aligned} \tag{5.17}$$

Consider the union of all D_{i_m} with $i_m \in I_m$, that is, the set V_m. It follows from (5.10) and (5.9) that

$$\bar{\mu}(V_m) > 1 - m\eta > 1 - \tfrac{1}{20}\kappa.$$

Hence (5.17) implies that the union G' of all D_{i_m} with $i_m \in I_m^{\text{arrived}}$ has measure $\bar{\mu}(G') > 1 - \frac{1}{10}\kappa$. Let

$$G = (g_{m-1} \cdots g_0)^{-1}(G'), \quad \text{so that } \bar{\mu}(G) = \bar{\mu}(G') > 1 - \tfrac{1}{10}\kappa.$$

If $x \in G$ then (5.15) applied to the orbit $(x_n) = (g_{n-1} \circ\circ\circ g_0)(x)$ gives $|||\Theta(\boldsymbol{v}_m(x_m)) - \frac{1}{2}\pi||| < \frac{1}{2}\alpha$, which is precisely (5.6). This proves part (3) of Sublemma 5.11 and hence Lemma 5.4 (and the Main Lemma). □

6. Exploiting flexibility

With the Main Lemma, Theorem 2.4 is proven following [**13**]. For the first part of the proof, we explain in § 6.1 how the arguments from [**13**] can be adapted. The second part could be done repeating parts of [**13**] almost word for word. However, we present (§ 6.2) a new and significantly simpler proof, following suggestions by A. Avila.

Given $f \in \mathrm{Diff}^1_\omega(M)$, $p \in \{1, \ldots, N\}$ and $m \in \mathbb{N}$, let $\Gamma_p(f, m)$ be the (open) set of points x such that there is no m-dominated splitting of index p along the orbit of x.

The symplectomorphism f is called *aperiodic* if the measure of the set of its periodic points is zero. By Robinson's [**30**] symplectic version of the Kupka–Smale Theorem, the generic f has countably many periodic points and in particular is aperiodic.

Let

$$\Lambda_p(f, x) = \sum_{i=i}^{p} \lambda_i(f, x) = \lim_{n\to\infty} \frac{1}{n} \log \|\wedge^p(Df^n(x))\|.$$

(The reader should recall relations between exterior products and Lyapunov exponents: see, for example, [**13**, § 2.1.2].)

6.1. Lowering the norm along an orbit segment

As consequence of the Main Lemma, we can perturb the map f on a neighbourhood of an orbit segment of length n in such a way that $\|\wedge^p Df^n\|$ drops. In precise terms, we have the following lemma.

Lemma 6.1. *Let $f \in \mathrm{Diff}^1_\omega(M)$ be aperiodic, $\mathcal{V}$ be a neighbourhood of f, $\delta > 0$, and $0 < \kappa < 1$. If $m \in \mathbb{N}$ is sufficiently large, then there exists a measurable function $N : \Gamma_p(f, m) \to \mathbb{N}$ with the following properties.*

For almost every $x \in \Gamma_p(f, m)$ and every $n \geqslant N(x)$, there exists $r = r(x, n) > 0$ such that the following holds. First, the iterates $f^j(\bar{B}_r(x))$, for $0 \leqslant j \leqslant n$, are pairwise disjoint. Second, for any $0 < r' < r$ there exists $g \in \mathcal{V}$ such that

(1) *g equals f outside $\bigsqcup_{j=0}^{n-1} f^j(B_{r'}(x))$;*

(2) *there is a set $G \subset B_{r'}(x)$ such that $\mu(G) > (1 - \kappa)\mu(B_{r'}(x))$ and*

$$\frac{1}{n} \log \|\wedge^p(Dg^n(y))\| \leqslant \frac{\Lambda_{p-1}(f, x) + \Lambda_{p+1}(f, x)}{2} + \delta \quad \textit{for all } y \in G.$$

We remark that the lemma corresponds to [**13**, Proposition 4.2], giving at the same stroke the conclusions of [**13**, Lemma 4.13].

Proof. Define

$$\Phi(x) = \frac{\Lambda_{p-1}(f,x) + \Lambda_{p+1}(f,x)}{2}. \tag{6.1}$$

Let $\varepsilon = \varepsilon(f, \mathcal{V})$ be given by Lemma 3.5. Let $m \in \mathbb{N}$ be sufficiently large so that the conclusion of the Main Lemma holds (with $\kappa/2$ in place of κ).

For the points $x \in \Gamma_p(f,m)$ that are non periodic, Oseledets regular, and have $\lambda_p(f,x) = \lambda_{p+1}(f,x)$, the conclusion of the lemma is trivial: first take $N(x)$ large so that if $n \geqslant N(x)$ then $(1/n)\log\|\wedge^p(Df^n(x))\|$ is $\delta/2$-close to $\Lambda_p(f,x) = \Phi(x)$. Then for each $n \geqslant N(x)$, take $r = r(x,n)$ small so that the ball $\bar{B}_r(x)$ is disjoint from its n first iterates and $Df^n(y)$ is close to $Df^n(x)$ for all $y \in B_r(x)$. Letting $g = f$, all the desired conclusions of the lemma hold.

Next consider the set Γ formed by the points $x \in \Gamma_p(f,m)$ that are non-periodic, Oseledets regular, and such that $\lambda_p(f,x) > \lambda_{p+1}(f,x)$. That is, Γ is the intersection of $\Gamma_p(f,m)$ with the set $\Sigma_p(f)$ introduced in § 3.3. Assume that $\mu(\Gamma) > 0$, otherwise there is nothing left to prove. Let $A \subset \Sigma_p(f)$ be the set of points such that the non-domination condition (3.6) holds. Then $\Gamma = \bigcup_{n\in\mathbb{Z}} f^n(A)$ (because the splitting $E^{\mathrm{u}} \oplus E^{\mathrm{cs}}$ over the set $\Sigma_p(f) \setminus \bigcup_{n\in\mathbb{Z}} f^n(A)$ is m-dominated of index p). Fix $C > \sup_{g\in\mathcal{V},\, x\in M} \|Dg(x)^{\pm 1}\|$.

Sublemma 6.2. *There exists a measurable function $N : \Gamma \to \mathbb{N}$ such that for almost every $x \in \Gamma$ and for every $n \geqslant N(x)$, there exists ℓ with $0 < \ell < n - m$ such that $z = f^\ell x$ belongs to A and the following holds. If $L_i : T_{f^i z}M \to T_{f^{i+1}z}M$, where $0 \leqslant i \leqslant m-1$, are linear maps such that $\|L_i^{\pm 1}\| \leqslant C$ and*

$$L_{m-1} \cdots L_0 \cdot E^{\mathrm{u}}(z) \cap E^{\mathrm{cs}}(f^m z) \neq \{0\}, \tag{6.2}$$

then

$$\frac{1}{n} \log \|\wedge^p[Df^{n-\ell-m}(f^{\ell+m}x) L_{m-1} \cdots L_0 Df^\ell(x)]\| < \Phi(x) + \tfrac{1}{2}\delta.$$

Proof. It is contained in the proof of [**13**, Proposition 4.2]. □

Let $x \in \Gamma$ be fixed from now on, and let $n \geqslant N(x)$, $\ell = \ell(x,n)$, and $z = f^\ell x$ be as in Sublemma 6.2. By mere continuity, we can weaken the requirement (6.2) to a small angle condition. More precisely, there exists $\gamma = \gamma(x,n) > 0$ with the following properties. Given points $y_0, \dots, y_n \in M$ such that

$$d(y_i, f^i x) < \gamma \ \forall i \quad \text{and} \quad f(y_i) = y_{i+1} \ \forall i \in \{0, \dots, n-1\} \setminus \{\ell, \dots, \ell+m-1\},$$

and given linear maps $\tilde{L}_i : T_{y_{\ell+i}}M \to T_{y_{\ell+i+1}}M$, for $0 \leqslant i \leqslant m-1$, such that $\|\tilde{L}_i^{\pm 1}\| \leqslant C$ and

$$\measuredangle(\tilde{L}_{m-1} \cdots \tilde{L}_0 \cdot i^z_{y_\ell} \cdot E^{\mathrm{u}}(z), i^{f^m z}_{y_{\ell+m}} \cdot E^{\mathrm{cs}}(f^m z)) < \gamma \tag{6.3}$$

(recall (3.1)), then

$$\frac{1}{n} \log \|\wedge^p[Df^{n-\ell-m}(y_{\ell+m}) \tilde{L}_{m-1} \cdots \tilde{L}_0 Df^\ell(y_0)]\| < \Phi(x) + \delta. \tag{6.4}$$

Since z belongs to A, the Main Lemma says that the split sequence $Df(f^i z) : E^{\mathrm{u}} \oplus E^{\mathrm{cs}} \hookleftarrow$ $(0 \leqslant i < m)$ is (ε, κ)-flexible. Let r_0 be the radius $r(z, \gamma)$ given by Lemma 3.5. Since z is not periodic, there is $r > 0$ be such that for $0 \leqslant j \leqslant n$, $f^j(\bar{B}_r(x))$ is contained in $B_\gamma(f^i x)$ and does not intersect $\bar{B}_r(x)$. Let us see that r has the required properties.

Given r' with $0 < r' < r$, let $U = f^\ell(B_{r'}(x))$. By Lemma 3.5, there exist $g \in \mathcal{V}$ and $\hat{G} \subset U$ such that

(1) g equals f outside $\bigsqcup_{j=0}^{n-1} f^j(U)$;

(2) $\mu(\hat{G}) > (1 - \kappa)\mu(U)$;

(3) $\measuredangle(Dg^n(\xi) \cdot i^z_\xi \cdot E^{\mathrm{u}}(z), i^{f^n z}_{g^n \xi} \cdot E^{\mathrm{cs}}(f^m z)) < \gamma$ for every $\xi \in \hat{G}$.

Now let $G = f^{-\ell}(\hat{G}) \subset B_{r'}(x)$. For any $y \in G$, if we define $y_i = g^i y$ for $0 \leqslant i \leqslant n$, and $\tilde{L}_i = Dg(y_{\ell+i})$ for $0 \leqslant i < m$ then relation (6.3) holds. Therefore, so does (6.4), that is, $(1/n) \log \|\wedge^p(Dg^n(y))\| \leqslant \Phi(x) + \delta$, as we wanted to show. □

6.2. Globalization

The next step in the proof is to construct a global perturbation of f that exhibits a drop in some integrated exponent $\mathrm{LE}_p(f) = \int \Lambda_p(f)$. Let $\Gamma_p(f, \infty)$ be the set of points x such that there is no dominated splitting of index p along the orbit of x; that is, $\Gamma_p(f, \infty) = \bigcap_{m \in \mathbb{N}} \Gamma_p(f, m)$.

Proposition 6.3. *Given an aperiodic diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$, let*

$$J_p(f) = \int_{\Gamma_p(f,\infty)} \frac{\lambda_p(f,x) - \lambda_{p+1}(f,x)}{2} \, \mathrm{d}\mu(x).$$

Then for any neighbourhood $\mathcal{V}$ of f and any $\delta > 0$, there exists $g \in \mathcal{V}$ such that

$$\mathrm{LE}_p(g) < \mathrm{LE}_p(f) - J_p(f) + \delta. \tag{6.5}$$

Proof. Let f and δ be given. Let Φ be given by (6.1). We are going to show that there exists $m \in \mathbb{N}$ and g arbitrarily C^1-close to f that equals f outside the open set $\Gamma_p(f, m)$ and such that

$$\int_{\Gamma_p(f,m)} \Lambda_p(g) < \delta + \int_{\Gamma_p(f,m)} \Phi. \tag{6.6}$$

Let us postpone the proof and see how (6.6) implies the proposition. We have

$$\begin{aligned}
\int \Lambda_p(g) &= \int_{\Gamma_p(f,m)} \Lambda_p(g) + \int_{M \smallsetminus \Gamma_p(f,m)} \Lambda_p(f) \quad \text{(because } g = f \text{ outside } \Gamma_p(f,m)\text{)} \\
&\leqslant \delta + \int_{\Gamma_p(f,m)} \Phi + \int_{M \smallsetminus \Gamma_p(f,m)} \Lambda_p(f) \quad \text{(by (6.6))} \\
&\leqslant \delta + \int_{\Gamma_p(f,\infty)} \Phi + \int_{M \smallsetminus \Gamma_p(f,\infty)} \Lambda_p(f) \quad \text{(since } \Gamma_p(f,m) \supset \Gamma_p(f,\infty) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \Phi \leqslant \Lambda_p(f)\text{)} \\
&= \delta - J_p(f) + \int_M \Lambda_p(f),
\end{aligned}$$

which is (6.5).

Let us see how to construct g. Let $\kappa = \delta$. Take $m \in \mathbb{N}$ large enough so that Lemma 6.1 applies and gives a function $N : \mathit{\Gamma}_p(f, m) \to \mathbb{N}$. For simplicity, write $\mathit{\Gamma} = \mathit{\Gamma}_p(f, m)$.

Sublemma 6.4. *There is a measurable set $B \subset \mathit{\Gamma}$ such that*

- *the orbit of almost every point in $\mathit{\Gamma}$ visits B;*
- *for each $x \in B$ and j with $1 \leqslant j \leqslant N(x)$ we have $f^j(x) \notin B$.*

Proof. Take some positive measure set $C^{(0)}$ of $\mathit{\Gamma}^{(0)} = \mathit{\Gamma}$ where N is constant, say equal to n_0. Since f is aperiodic, we can select a positive measure subset $B^{(0)}$ of $C^{(0)}$ that is disjoint from its first n_0 iterates. Next consider the (invariant) set $\mathit{\Gamma}^{(1)}$ of points in $\mathit{\Gamma}^{(0)}$ whose f-orbits never visit $B^{(0)}$. If $\mathit{\Gamma}^{(1)}$ has zero measure, then we take $B = B^{(0)}$ and we are done. Otherwise we take a positive measure subset $C^{(1)}$ of $\mathit{\Gamma}^{(1)}$ where N is constant, and choose $B^{(1)} \subset C^{(1)}$ of positive measure that is disjoint from its first $n_1 = N|C^{(1)}$ iterates. If the set $\mathit{\Gamma}^{(2)}$ formed by the points that never visit $B^{(1)}$ has zero measure then we take $B = B^{(0)} \cup B^{(1)}$ and stop; otherwise we continue analogously and define $B^{(2)}$, etc. If this process does not end after finitely many steps then we define $\mathit{\Gamma}^{(\omega)} = \bigcap_{n<\omega} \mathit{\Gamma}^{(n)}$ and proceed as before, using transfinite induction. Since a disjoint class of positive measure sets is countable, the process will terminate at some countable ordinal. Taking a union, we find the desired measurable set B. $\square$

Let B be given by the sublemma. For $x \in B$, let $H(x)$ be the minimal positive integer n such that $f^n(x) \in B$. Then for almost every $x \in B$ we have $N(x) < H(x) < \infty$.

Take $\ell_0 \in \mathbb{N}$ large, and for $1 \leqslant n \leqslant \ell_0$, take compact sets $K_n \subset \{x \in B;\ H(x) = n\}$ in a way such that the set $\mathit{\Gamma} \setminus \bigsqcup_{n=1}^{\ell_0} \bigsqcup_{j=0}^{n-1} f^j(K_n)$ has measure less than δ. Take open sets $U_n \supset K_n$, all contained in the open set $\mathit{\Gamma}$, and such that the union $\bigsqcup_{n=1}^{\ell_0} \bigsqcup_{j=0}^{n-1} f^j(U_n)$ is still disjoint.

Let $K = \bigcup_{n=1}^{\ell_0} K_n$. For each $x \in K$, say with $x \in K_n$, since $n > N(x)$ we can apply Lemma 6.1 and get a radius $r = r(x) > 0$. If necessary, we reduce $r(x)$ so that $\bar{B}_{r(x)}(x)$ is contained in the open set U_n. Since $\mathit{\Phi}$ is a measurable f-invariant function, for almost every x, we can reduce $r(x)$ further and ensure that

$$\left.\begin{array}{r} 0 < r < r(x) \\ 0 \leqslant j < H(x) \end{array}\right\} \;\Rightarrow\; \frac{1}{\mu(B_r(x))}\mu(\{y \in B_r(x);\ |\mathit{\Phi}(f^j y) - \mathit{\Phi}(x)| \geqslant \delta\}) < \frac{\delta}{H(x)}. \tag{6.7}$$

Consider the Vitali cover of K by the balls $\bar{B}_{r'}(x)$, with $0 < r' < r(x)$. By the Vitali Covering Lemma, there is a countable family of disjoint balls $\bar{B}_{r_i}(x_i)$ with $0 < r_i < r(x_i)$ that covers the set K mod 0. Write $n_i = H(x_i)$. By construction, the union $\bigsqcup_i \bigsqcup_{j=0}^{n_i-1} f^j(\bar{B}_{r_i}(x_i))$ is still disjoint.

Applying Lemma 6.1 for each ball $B_{r_i}(x_i)$, we get a diffeomorphism g_i close to f such that

- g_i equals f outside $\bigsqcup_{j=0}^{n_i-1} f^j(B_{r_i}(x_i))$;
- there is a set $G_i \subset B_{r_i}(x_i)$ such that $\mu(G_i) > (1-\delta)\mu(B_{r_i}(x_i))$ and
$$\frac{1}{n_i} \log \|\wedge^p(Dg_i^{n_i}(y))\| \leqslant \mathit{\Phi}(x_i) + \delta \quad \text{for all } y \in G_i.$$

Let us define the global perturbation g of f as follows: g is equal to g_i in each corresponding $\bigsqcup_{j=0}^{n_i-1} f^j(B_{r_i}(x_i))$, and equal to f outside. Then g is a symplectomorphism C^1-close to f. We will prove that g has the required properties.

By (6.7), for each $j = 0, \dots, n_i - 1$,

$$\mu\{y \in B_{r_i}(x_i);\ |\Phi(g^j y) - \Phi(x_i)| \geqslant \delta\} = \mu\{y \in B_{r_i}(x_i);\ |\Phi(f^j y) - \Phi(x_i)| \geqslant \delta\} < \frac{\delta}{n_i}\mu(B_{r_i}(x_i)).$$

Let G_i' be the set of $y \in G_i$ such that $|\Phi(g^j(y)) - \Phi(x_i)| < \delta$ for all j with $0 \leqslant j < n_i$. Then $\mu(G_i') > (1-2\delta)\mu(B_{r_i}(x_i))$. If $y \in G_i'$ then

$$\frac{1}{n_i} \log \|\wedge^p(Dg^{n_i}(y))\| \leqslant \sum_{j=0}^{n_i-1} \Phi(g^j y) + 2\delta. \tag{6.8}$$

Define sets

$$D_{\mathrm{b}} = \bigsqcup_i G_i' \quad \text{and} \quad D = \bigsqcup_i \bigsqcup_{j=0}^{n_i-1} g^j(G_i').$$

(The set D is called a castle with base D_{b}.) Let us see that D covers most of Γ. Indeed, $\Gamma \smallsetminus D$ is contained mod 0 in

$$\left(\Gamma \smallsetminus \bigsqcup_{n=1}^{\ell_0} \bigsqcup_{j=0}^{n-1} f^j(K_n)\right) \cup \left(\bigsqcup_i \bigsqcup_{j=0}^{n_i-1} f^j(B_{r_i}(x_i)) \smallsetminus D\right) = (\mathrm{I}) \cup (\mathrm{II}).$$

Recall that $\mu(\mathrm{I}) < \delta$. On the other hand, $(\mathrm{II}) = \bigsqcup_i \bigsqcup_{j=0}^{n_i-1} g^j(B_{r_i}(x_i) \smallsetminus G_i')$, therefore $\mu(\mathrm{II}) < 2\delta$. This shows that $\mu(\Gamma \smallsetminus D) < 3\delta$.

Let $\hat{D} = \bigcup_{n \geqslant 0} g^{-n}(D)$. Almost every $x \in \hat{D}$ visits D_{b} infinitely many times. Fix one such point x, and let

$$\{m_1 < m_2 < \cdots\} = \{n \geqslant 0;\ g^n(x) \in D_{\mathrm{b}}\}.$$

Each $g^{m_j}(x)$ belongs to some ball $B_{r_i}(x_i)$; consider the corresponding n_i and let $m_j' = m_j + n_i$. So we have defined numbers $m_1 < m_1' \leqslant m_2 < m_2' \leqslant \cdots$ such that $g^n(x)$ is in D if $m_j \leqslant n < m_j'$, and is not in D if $m_j' \leqslant n < m_{j+1}$.

Given a (large) integer n, let $k = k(n)$ be the biggest index such that $m_k' \leqslant n$. We want to estimate $\|\wedge^p Dg^n(x)\|$; we start with the following upper bound:

$$\|\wedge^p Dg^{m_1}(x)\|\, \|\wedge^p Dg^{m_1'-m_1}(g^{m_1}x)\|\, \|\wedge^p Dg^{m_2-m_1'}(g^{m_1'}x)\| \cdots \|\wedge^p Dg^{n-m_k'}(g^{m_k'}x)\|.$$

To estimate some of these factors, we use (6.8):

$$\log \|\wedge^p Dg^{m_j'-m_j}(g^{m_j}x)\| \leqslant \sum_{i=m_j}^{m_j'-1} \Phi(g^i x) = \sum_{i=m_j}^{m_j'-1} (\Phi 1_D)(g^i x) \quad \text{for each } j = 1, \dots, k-1.$$

To estimate the other factors, let e^C be an upper bound for $\|\wedge^p Dg\|$; then:

$$\log\|\wedge^p Dg^{m_1}(x)\| \leqslant Cm_1,$$

$$\log\|\wedge^p Dg^{m_{j+1}-m'_j}(g^{m'_j}x)\| \leqslant C(m_{j+1}-m'_j) = \sum_{i=m_j}^{m'_j-1}(C1_{\Gamma\setminus D})(g^i x),$$

$$\log\|\wedge^p Dg^{n-n'_k}(x)\| \leqslant C(n-m'_k).$$

Putting things together,

$$\log\|\wedge^p(Dg^n(x))\| \leqslant C(n-m'_k+m_1) + \sum_{i=m_1}^{m'_k}(C1_{\Gamma\setminus D}+\Phi 1_D)(g^i x). \tag{6.9}$$

Now we use the following sublemma.

Sublemma 6.5. *For almost every $x \in \hat{D}$, we have that $m'_{k(n)}/n \to 1$ as $n \to \infty$.*

Proof. It suffices to consider points $x \in D_{\mathrm{b}}$. Let $\hat{g} : D_{\mathrm{b}} \to D_{\mathrm{b}}$ be the first return map, and let $T : D_{\mathrm{b}} \to \mathbb{N}$ be the return time. Then $\hat{g}$ preserves μ restricted to D_{b} and T is integrable. Since m_{j+1} equals the Birkhoff sum $\sum_{i=0}^{j-1} T(\hat{g}^i x)$, for almost every $x \in D_{\mathrm{b}}$, the $\lim_{j\to\infty} m_j/j$ exists and is positive. Now, if $k = k(n)$ then $n < m'_{k+1} < m_{k+2}$. Hence

$$\frac{m_k}{m_{k+2}} \leqslant \frac{m'_k}{n} \leqslant 1.$$

As n goes to infinity, $k = k(n) \to \infty$ and $m_k/m_{k+2} \to 1$. This proves the sublemma. □

It follows from (6.9) and the sublemma that for almost every $x \in \hat{D}$,

$$\Lambda_p(g,x) = \lim_{n\to\infty}\frac{1}{n}\log\|\wedge^p(Dg^n(x))\| \leqslant \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(C1_{\Gamma\setminus D}+\Phi 1_D)(g^i x).$$

The same is obviously true if $x \in \Gamma \setminus \hat{D}$. Integrating over $x \in \Gamma$, we obtain

$$\int_\Gamma \Lambda_p(g) \leqslant \int(C1_{\Gamma\setminus D}+\Phi 1_D) \leqslant 3C\delta + \int_D \Phi \leqslant 3C\delta + \int_\Gamma \Phi.$$

This gives (6.6) (replace δ with $\delta/(3C)$ everywhere), and therefore the proposition is proved. □

Remark 6.6. With some additional work one can show that the aperiodicity hypothesis is not necessary for the validity of Proposition 6.3; indeed it does not appear in [**13**].

Proof of Theorem 2.4. Let $\mathcal{A}$ be the residual subset of $\mathrm{Diff}^1_\omega(M)$ formed by aperiodic diffeomorphisms. Consider the semi-continuous maps $\mathrm{LE}_p : \mathcal{A} \to \mathbb{R}$. Since $\mathcal{A}$ is also a Baire space, it follows that there is a residual subset $\mathcal{R}$ of $\mathcal{A}$ (and hence also residual as a subset $\mathrm{Diff}^1_\omega(M)$) such that every $f \in \mathcal{R}$ is a point of continuity of each LE_p, with

$p = 1, \ldots, N$. Fix one such f; by Proposition 6.3, each $J_p(f)$ vanishes. This implies that for almost every regular point $x \in M$, if $p \leqslant N$ is such that $\lambda_p(f,x) > \lambda_{p+1}(f,x)$ then x does not belong to $\Gamma(f,\infty)$. That is, there is a dominated splitting $E^{\mathrm{u}} \oplus F$ of index p along the orbit of x. Theorem 2.2 implies that $E^{\mathrm{u}} \oplus F$ can be refined to a partially hyperbolic splitting $E^{\mathrm{u}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$, with $\dim E^{\mathrm{s}} = \dim E^{\mathrm{u}} = p$. Thus E^{u}, E^{c} and E^{s} must be the sum of the Oseledets spaces associated to the Lyapunov exponents $\lambda_i(f,x)$ respectively with

$$1 \leqslant i \leqslant p, \quad p < i \leqslant 2N - p, \quad 2N - p < i \leqslant 2N.$$

All this holds whenever $\lambda_p(f,x) > \lambda_{p+1}(f,x)$, so proving that the Oseledets splitting is dominated along the orbit of x. □

Theorem 1.1 is an immediate consequence of Theorem 2.4.

7. Results for partially hyperbolic maps

We will obtain Theorem 1.3 as a corollary of the slightly more technical Theorem 7.5 below.

First of all, we need the following two results about the well-known accessibility property from partially hyperbolic theory.

Theorem 7.1 (Dolgopyat and Wilkinson [21]). *There is an open and dense set $\mathcal{A} \subset \mathrm{PH}^1_\omega(M)$ formed by accessible symplectomorphisms.*

Theorem 7.2 (Brin [16]). *If f is a C^2 volume-preserving partially hyperbolic diffeomorphism with the accessibility property then almost every point has a dense orbit.*

In fact, Brin proved the result for *absolute* partially hyperbolic maps (recall Remark 2.1). Another proof was given by Burns, Dolgopyat and Pesin (see [**18**, Lemma 5] or [**23**, § 7.2]). Their proof also applies to *relative* partially hyperbolic maps: the only necessary modification is to use the property of absolute continuity of stable and unstable foliations in the relative case, which is proven by Abdenur and Viana in [**1**].

In order to extract from Theorem 7.2 consequences for C^1 maps, we need the following well-known result.

Theorem 7.3 (Zehnder [35]). *C^∞ diffeomorphisms form a dense subset of $\mathrm{Diff}^1_\omega(M)$.*

We remark that the volume-preserving analogue of Theorem 7.3 was recently obtained by Avila [**5**].

As a consequence of the above theorems, we obtain the following proposition.

Proposition 7.4. *For a generic f in $\mathrm{PH}^1_\omega(M)$, the orbit of almost every point is dense in M.*

Proof. Given $f \in \mathrm{PH}^1_\omega(M)$, let $D(f)$ be the set of points in M whose f-orbits are dense. Let $\mathcal{R}$ be set of $f \in \mathrm{PH}^1_\omega(M)$ such that $m(D(f)) = 1$. Theorems 7.1–7.3 together imply that $\mathcal{R}$ is dense in $\mathrm{PH}^1_\omega(M)$. We will complete the proof showing that $\mathcal{R}$ is a G_δ set.

Let $\mathcal{B}$ be a countable basis of (non-empty) open sets of M. Then

$$D(f) = \bigcap_{U,V\in\mathcal{B}} G(U,V,f), \quad \text{where } G(U,V,f) = (M \setminus U) \cup \bigcup_{n\in\mathbb{N}} f^{-n}(V).$$

For $k \in \mathbb{N}$, let $\mathcal{A}(U,V,k)$ be the set of $f \in \mathrm{PH}^1_\omega(M)$ such that $m(G(U,V,f)) > 1 - 1/k$. Then each $\mathcal{A}(U,V,k)$ is open. Their intersection is precisely the set of $f \in \mathrm{PH}^1_\omega(M)$ such that $m(D(f)) = 1$, that is, $\mathcal{R}$. $\square$

A dominated splitting $TM = E^1 \oplus \cdots \oplus E^k$ (into non-zero bundles) for a diffeomorphism $f : M \to M$ is called the *finest dominated splitting* if there is no dominated splitting defined over all M with more than k (non-zero) bundles. For any f, either there is no dominated splitting over M, or there is a unique finest dominated splitting (and moreover it refines every dominated splitting on M). See [**15**].

Now we can state and prove the following theorem.

Theorem 7.5. *For a generic f in $\mathrm{PH}^1_\omega(M)$, the Oseledets splitting at almost every point coincides with the finest dominated splitting of f. In particular, the multiplicities of the Lyapunov exponents are almost everywhere constant.*

Proof. Let $k(f)$ denote the number of bundles in the finest dominated splitting of a map $f : M \to M$. Then the Oseledets splitting at any regular point for f has at least $k(f)$ bundles. Now let $f \in \mathrm{PH}^1_\omega(M)$ satisfy the generic properties from Proposition 7.4 and Theorem 1.1. That is, for almost every $x \in M$, the orbit of x is dense and the Oseledets splitting along it is (non-trivial and) dominated. The Oseledets splitting along the orbit of any such point extends to a dominated splitting over M, and hence must have exactly $k(f)$ bundles. $\square$

Proof of Theorem 1.3. If f belongs to the residual set given by Theorem 7.5 then the Oseledets space corresponding to zero exponents (if they exist) coincides almost everywhere with the 'middle' bundle of the finest dominated splitting, which by Theorem 2.2 is the centre bundle of a partially hyperbolic splitting. $\square$

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