ANOTHER PROOF OF THE SPECTRAL RADIUS FORMULA

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Abstract. This note gives an elementary proof of the spectral radius formula in finite dimension, inspired by ideas from ergodic theory.

1. Introduction

Let $A\colon V \to V$ be a linear operator on a complex finite-dimensional vector space $V \neq \{0\}$. The *spectral radius* of A is defined as:

$$r(A) := \max\{|\lambda| \ ; \ \lambda \in \mathbb{C} \text{ is an eigenvalue of } A\} .$$

Let $\|\cdot\|$ be a norm on V. Then the *operator norm* of A is defined as:

$$\|A\| := \sup_{v \in V \smallsetminus \{0\}} \frac{\|Av\|}{\|v\|} .$$

Note the submultiplicativity property $\|A^{n+m}\| \leqslant \|A^n\| \, \|A^m\|$, which by the easy and well-known Fekete Lemma (see e.g. [Mo, Lemma A.1]) implies that the limit

$$\rho(A) := \lim_{n\to\infty} \|A^n\|^{1/n} \quad \text{exists and equals} \quad \inf_{n \in \mathbb{N}} \|A^n\|^{1/n} .$$

It is clear that $r(A) \leqslant \|A\|$. Since $r(A^n) = r(A)^n$ for every $n \in \mathbb{N}$, it follows that $r(A) \leqslant \rho(A)$. A much less trivial fact is that equality holds, that is,

$$r(A) = \rho(A) ;$$

this is called the *spectral radius formula*. The usual proofs rely either on complex analysis or on normal forms of matrices: see [La, Appendix 10]. There is also a short proof based on the Cayley–Hamilton theorem: see [Bo]. All these proofs consist in bounding the growth of the powers of A in terms of its eigenvalues.

In this note we prove the spectral radius formula following a different route: we start from the growth rate $\rho(A)$ and then show the existence of an eigenvalue λ such that $|\lambda| = \rho(A)$. The idea essentially comes from multiplicative ergodic theory. Nevertheless, the proof only uses basics of linear algebra and real analysis, and has no matrix calculations.

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2. Proof of the Spectral Radius Formula

We need to show that $r(A) \geqslant \rho(A)$, since the reverse inequality was already established. Let us assume that $\rho(A) > 0$, otherwise we are already done. In particular, no power of A is zero, that is, none of the subspaces

$$V \supset A(V) \supset A^2(V) \supset \cdots$$

is $\{0\}$. Since $\dim V < \infty$, this nested sequence necessarily stabilizes at some subspace $A^{n_0}(V) =: W$. Let $B\colon W \to W$ be the restricted operator, which is invertible. Note that for all $n \geqslant n_0$,

$$\|B^n\| \leqslant \|A^n\| \leqslant \|B^{n-n_0}\| \, \|A^{n_0}\| \, .$$

Taking n-th roots and making $n \to \infty$ we obtain the equality $\rho(B) = \rho(A)$. On the other hand, it is clear that $r(B) \leqslant r(A)$. Therefore, in order to conclude that $r(A) \geqslant \rho(A)$ it is sufficient to prove that $r(B) \geqslant \rho(B)$. So we assume from now on that $A\colon V \to V$ is invertible, without loss of generality.

Consider the unit sphere $S := \{v \in V \; ; \; \|v\| = 1\}$. Note that for all $n \in \mathbb{N}$,

$$\inf_{v \in S} \|A^{-n} v\| = \|A^n\|^{-1} \leqslant \rho(A)^{-n} \, . \tag{1}$$

Actually a stronger fact holds:

Lemma 1. *There exists $v_0 \in S$ such that $\|A^{-n} v_0\| \leqslant \rho(A)^{-n}$ for all $n \in \mathbb{N}$.*

Proof. We argue by contradiction: let us assume that for every $v \in S$ there exists $i = i(v) \in \mathbb{N}$ and $a = a(v) > \rho(A)^{-1}$ such that $\|A^{-n} v\| > a^n$. By continuity of A and compactness of S, we can find constants $m \in \mathbb{N}$ and $b > \rho(A)^{-1}$ such that $S = O_1 \cup \cdots \cup O_m$, where $O_i := \left\{v \in S \; ; \; \|A^{-i} v\| > b^i\right\}$. In particular, $V = C_1 \cup \cdots \cup C_m$, where $C_i := \left\{v \in V \; ; \; \|A^{-i} v\| \geqslant b^i \|v\|\right\}$.

Now let $v \in S$ be arbitrary. We recursively find numbers $i_1, i_2, \ldots \in \{1, 2, \ldots, m\}$ such that:

$$v \in C_{i_1}, \quad A^{-i_1} v \in C_{i_2}, \quad A^{-i_2 - i_1} v \in C_{i_3}, \quad \ldots$$

Therefore for every $k \geqslant 1$, we have:

$$\|A^{-i_k - i_{k-1} - \cdots - i_1} v\| \geqslant b^{i_k} \|A^{-i_{k-1} - \cdots - i_1} v\| \geqslant \cdots \geqslant b^{i_k + \cdots + i_1} \, .$$

Given $n \in \mathbb{N}$, decompose it as $n = i_1 + \cdots + i_k + r$ for some $k = k(n) \geqslant 1$ and $r = r(n) \in \{0, 1, \ldots, m-1\}$. Then:

$$\|A^{-n}(v)\| = \|A^{-r} A^{-i_k - \cdots - i_1} v\| \geqslant \|A^r\|^{-1} \, \|A^{-i_k - \cdots - i_1} v\| \geqslant \|A\|^{-r} \, b^{n-r} \, ,$$

and since $b > \rho(A)^{-1} \geqslant \|A\|^{-1}$, we obtain the inequality:

$$\|A^{-n}(v)\| \geqslant \|A\|^{-m} \, b^{n-m} \, ,$$

which holds for every $v \in S$ and $n \in \mathbb{N}$. Now (1) yields:

$$\rho(A)^{-n} \geqslant \|A\|^{-m} \, b^{n-m}$$

Taking the n-th root and making $n \to \infty$, we obtain $\rho(A)^{-1} \geqslant b$. This is a contradiction, and the lemma is proved. $\square$

For each $c > 0$, consider the following set of vectors:

$$U_c := \left\{ v \in V \; ; \; \limsup_{n\to\infty} \|A^{-n}v\|^{1/n} \leqslant c \right\} .$$

We claim that U_c is an A-invariant subspace. The proof is straightforward; for example, to check that U_c is closed under sums, note that $\|A^{-n}(v+w)\| \leqslant 2\max\left\{\|A^{-n}v\|, \|A^{-n}w\|\right\}$ and so

$$\limsup_{n\to\infty} \|A^{-n}(v+w)\|^{1/n} \leqslant \max\left\{ \limsup_{n\to\infty} \|A^{-n}v\|^{1/n}, \; \limsup_{n\to\infty} \|A^{-n}w\|^{1/n} \right\} .$$

As a consequence of Lemma 1, the A-invariant space $U_{\rho(A)^{-1}}$ is not $\{0\}$. In particular, it must contain an eigenvector u. Let $\lambda \in \mathbb{C} \smallsetminus \{0\}$ be the corresponding eigenvalue; then:

$$\rho(A)^{-1} \geqslant \limsup_{n\to\infty} \|A^{-n}u\|^{1/n} = \limsup_{n\to\infty} |\lambda|^{-1} \|u\|^{1/n} = |\lambda|^{-1} ,$$

so $\rho(A) \leqslant |\lambda| \leqslant r(A)$. This concludes the proof of the spectral radius formula.

3. Concluding Remarks

Let us reassess what we have done. The space $U_{\rho(A)^{-1}}$ is a clear candidate for the eigenspace associated to the dominating eigenvector(s), and the real issue is to show that it is nontrivial. This is accomplished by Lemma 1, whose proof relies on a compactness argument.

In a more sophisticated setting, a similar strategy was used by Walters [Wa] to prove the multiplicative ergodic theorem. Our Lemma 1 is a particular case of an abstract result of Morris [Mo, Proposition A.7], which in turn is related to a ergodic-theoretical lemma of Peres [Pe, Lemma 2].

References

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