CANTOR SPECTRUM FOR SCHRÖDINGER OPERATORS WITH POTENTIALS ARISING FROM GENERALIZED SKEW-SHIFTS

ARTUR AVILA, JAIRO BOCHI, and DAVID DAMANIK

Abstract

We consider continuous $\mathrm{SL}(2, \mathbb{R})$*-cocycles over a strictly ergodic homeomorphism that fibers over an almost periodic dynamical system (generalized skew-shifts). We prove that any cocycle that is not uniformly hyperbolic can be approximated by one that is conjugate to an* $\mathrm{SO}(2, \mathbb{R})$*-cocycle. Using this, we show that if a cocycle's homotopy class does not display a certain obstruction to uniform hyperbolicity, then it can be* C^0*-perturbed to become uniformly hyperbolic. For cocycles arising from Schrödinger operators, the obstruction vanishes, and we conclude that uniform hyperbolicity is dense, which implies that for a generic continuous potential, the spectrum of the corresponding Schrödinger operator is a Cantor set.*

1. Statement of the results

Throughout this article, we let X be a compact metric space. Furthermore, unless specified otherwise, $f : X \to X$ is understood to be a strictly ergodic homeomorphism (i.e., f is minimal and uniquely ergodic) which fibers over an almost periodic dynamical system. This means that there exists an infinite compact abelian group $\mathbb{G}$ and an onto continuous map $h : X \to \mathbb{G}$ such that $h(f(x)) = h(x) + \alpha$ for some $\alpha \in \mathbb{G}$. Examples of particular interest include

- minimal translations of the d-torus $\mathbb{T}^d$ for any $d \geq 1$;
- the skew-shift $(x, y) \mapsto (x + \alpha, y + x)$ on $\mathbb{T}^2$, where α is irrational.

1.1. Results for $\mathrm{SL}(2, \mathbb{R})$*-cocycles*

Given a continuous map $A : X \to \mathrm{SL}(2, \mathbb{R})$, we consider the skew-product $X \times \mathrm{SL}(2, \mathbb{R}) \to X \times \mathrm{SL}(2, \mathbb{R})$ given by $(x, g) \mapsto (f(x), A(x) \cdot g)$. This map is called the *cocycle* (f, A). For $n \in \mathbb{Z}$, A^n is defined by $(f, A)^n = (f^n, A^n)$.

DUKE MATHEMATICAL JOURNAL
Vol. 146, No. 2, © 2009 DOI 10.1215/00127094-2008-065
Received 17 September 2007. Revision received 22 April 2008.
2000 *Mathematics Subject Classification.* Primary 37D; Secondary 47B36, 47B80, 81Q10.
Avila's work partially supported by a Clay Research Fellowship.
Bochi's work partially supported by a grant from the Conselho Nacional de Desenvolvimento Científico e Tecnológico–Brazil.
Damanik's work partially supported by National Science Foundation grant DMS-0653720.

We say that a cocycle (f, A) is *uniformly hyperbolic** if there exist constants $c > 0$, $\lambda > 1$ such that $\|A^n(x)\| > c\lambda^n$ for every $x \in X$ and $n > 0$.† This is equivalent to the usual hyperbolic splitting condition (see [Y]). Recall that uniform hyperbolicity is an open condition in $C^0(X, \mathrm{SL}(2, \mathbb{R}))$.

We say that two cocycles (f, A) and $(f, \tilde{A})$ are *conjugate* if there exists a *conjugacy* $B \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ such that $\tilde{A}(x) = B(f(x))A(x)B(x)^{-1}$.

Our first result is the following.

THEOREM 1
Let f be as above. If $A : X \to \mathrm{SL}(2, \mathbb{R})$ is a continuous map such that the cocycle (f, A) is not uniformly hyperbolic, then there exists a continuous $\tilde{A} : X \to \mathrm{SL}(2, \mathbb{R})$, arbitrarily C^0-close to A, such that the cocycle $(f, \tilde{A})$ is conjugate to an $\mathrm{SO}(2, \mathbb{R})$-valued cocycle.

Remark 1
A cocycle (f, A) is conjugate to a cocycle of rotations if and only if there exists $C > 1$ such that $\|A^n(x)\| \leq C$ for every $x \in X$ and $n \in \mathbb{Z}$ (here, it is enough to assume that f is minimal) (see [C], [EJ, Section 2], [Y, Proposition 1]).

Remark 2
In Theorem 1, one can drop the hypothesis of unique ergodicity of f (still asking f to be minimal and to fiber over an almost periodic dynamics) as long as X is finite-dimensional (see Remark 8).

Next, we focus on the opposite problem of approximating a cocycle by one that is uniformly hyperbolic. As we will see, this problem is related to the important concept of *reducibility*.

To define reducibility, we need a slight variation of the notion of conjugacy. Let us say that two cocycles (f, A) and $(f, \tilde{A})$ are $\mathrm{PSL}(2, \mathbb{R})$-*conjugate* if there exists $B \in C^0(X, \mathrm{PSL}(2, \mathbb{R}))$ such that $\tilde{A}(x) = B(f(x))A(x)B(x)^{-1}$ (the equality being considered in $\mathrm{PSL}(2, \mathbb{R})$). We say that (f, A) is *reducible* if it is $\mathrm{PSL}(2, \mathbb{R})$-conjugate to a constant cocycle.

Remark 3
Reducibility does not imply, in general, that (f, A) is conjugate to a constant cocycle, which would correspond to taking $B \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$. For example, let $X = \mathbb{T}^1$, and let $f(x) = x + \alpha$. Let $H = \mathrm{diag}(2, 1/2)$, and define $A(x) = R_{-\pi(x+\alpha)} H R_{\pi x}$.** Notice that A is continuous, and notice that (f, A) is $\mathrm{PSL}(2, \mathbb{R})$-conjugate but not

*Some authors say that the cocycle has an *exponential dichotomy*.
†If A is a real (2×2)-matrix, then $\|A\| = \sup_{\|v\| \neq 0} \|A(v)\|/\|v\|$, where $\|v\|$ is the Euclidean norm of $v \in \mathbb{R}^2$.
**R_θ indicates the rotation of angle θ.

$\mathrm{SL}(2, \mathbb{R})$-conjugate to a constant (for an example in which (f, A) is not uniformly hyperbolic, see Remark 9).

Let us say that an $\mathrm{SL}(2, \mathbb{R})$-cocycle (f, A) is *reducible up to homotopy* if there exists a reducible cocycle $(f, \tilde{A})$ such that the maps A and $\tilde{A} : X \to \mathrm{SL}(2, \mathbb{R})$ are homotopic. Let *Ruth* be the set of all A such that (f, A) is reducible up to homotopy.

Remark 4
In the case where f is homotopic to the identity map, it is easy to see that *Ruth* coincides with the set of maps $A : X \to \mathrm{SL}(2, \mathbb{R})$ which are homotopic to a constant.

It is well known that there exists an obstruction to approximating a cocycle by a uniformly hyperbolic one: a uniformly hyperbolic cocycle is always reducible up to homotopy (see Lemma 4). Our next result shows that up to this obstruction, uniform hyperbolicity is dense.

THEOREM 2
Uniform hyperbolicity is dense in Ruth.

This result is obtained as a consequence of a detailed investigation of the problem of denseness of reducibility.

THEOREM 3
Reducibility is dense in Ruth. More precisely,
(a) *if (f, A) is uniformly hyperbolic, then it can be approximated by a reducible cocycle that is uniformly hyperbolic;*
(b) *if $A \in$ Ruth, but (f, A) is not uniformly hyperbolic and $A_* \in \mathrm{SL}(2, \mathbb{R})$ is nonhyperbolic (i.e., $|\operatorname{tr} A_*| \leq 2$), then (f, A) lies in the closure of the* $\mathrm{PSL}(2, \mathbb{R})$*-conjugacy class of (f, A_*).*

Proof of Theorem 2
The closure of the set of uniformly hyperbolic cocycles is obviously invariant under $\mathrm{PSL}(2, \mathbb{R})$-conjugacies and clearly contains all constant cocycles (f, A_*) with $\operatorname{tr} A_* = 2$. The result follows from Theorem 3(b). □

Remark 5
It would be interesting to investigate also the closure of an arbitrary $\mathrm{PSL}(2, \mathbb{R})$-conjugacy class. Even the case of the $\mathrm{PSL}(2, \mathbb{R})$-conjugacy class of a constant hyperbolic cocycle already escapes our methods.

Let us say a few words about the proofs of the results and their relation with the literature. In the diffeomorphism and flow settings, Smale conjectured in the 1960s that hyperbolic dynamical systems are dense. This turned out to be false in general.

However, there are situations where denseness of hyperbolicity holds (see, e.g., the recent work [KSV] in the context of one-dimensional dynamics).

Dinh Cong [D] proved that uniform hyperbolicity is (open and) dense in the space of $L^\infty(X, \mathrm{SL}(2,\mathbb{R}))$-cocycles for any base dynamics f. So our Theorem 2 can be seen as a continuous version of his result. Dinh Cong's proof involves a tower argument to perturb the cocycle and produce an invariant section for its action on the circle $\mathbb{P}^1$. We develop a somewhat similar technique, replacing $\mathbb{P}^1$ with other spaces. Special care is needed in order to ensure that perturbations and sections be continuous.

Another related result was obtained by Fabbri and Johnson, who considered continuous-time systems over translation flows on $\mathbb{T}^d$ and proved for a generic translation vector that uniform hyperbolicity occurs for an open and dense set of cocycles (see [FJ, Theorem 1.2]).

1.2. Results for Schrödinger cocycles

We say that (f, A) is a *Schrödinger cocycle* when A takes its values in the set

$$S = \left\{ \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}, \ t \in \mathbb{R} \right\} \subset \mathrm{SL}(2,\mathbb{R}). \tag{1}$$

The matrices A^n arising in the iterates of a Schrödinger cocycle are the so-called transfer matrices associated with a discrete one-dimensional Schrödinger operator.

More explicitly, given $V \in C^0(X, \mathbb{R})$ (called the *potential*) and $x \in X$, we consider the operator

$$(H_x \psi)_n = \psi_{n+1} + \psi_{n-1} + V(f^n x)\psi_n \tag{2}$$

in $\ell^2(\mathbb{Z})$. Notice that u solves the difference equation

$$u_{n+1} + u_{n-1} + V(f^n x) u_n = E u_n \tag{3}$$

if and only if

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A^n_{E,V} \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}, \tag{4}$$

where $(f, A_{E,V})$ is the Schrödinger cocycle with

$$A_{E,V}(x) = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

Properties of the spectrum and the spectral measures of the operator (2) can be studied by looking at the solutions to (3) and hence, by virtue of (4), the one-parameter family of Schrödinger cocycles $(f, A_{E,V})$. Using minimality of f, it follows quickly by strong operator convergence that the spectrum of H_x is independent of $x \in X$, and we may therefore denote it by Σ. It is well known that Σ is a perfect set; as a spectrum, it is closed and there are no isolated points by ergodicity of f and finite-dimensionality of the solution space of (3) for fixed E. Johnson [J, Section 3] (see also Lenz [L, Theorem 3]) showed that Σ consists of those energies E for which $(f, A_{E,V})$ is not uniformly hyperbolic:

$$\mathbb{R} \setminus \Sigma = \{E \in \mathbb{R}; (f, A_{E,V}) \text{ is uniformly hyperbolic}\}. \tag{5}$$

Our results have natural versions for Schrödinger cocycles, with the added simplification that all such cocycles are homotopic to a constant. Simple repetition of the proofs leads to difficulties in the construction of perturbations (since there are fewer parameters to vary). We prove instead a general reduction result, which is of independent interest. Recall definition (1) of the set S.

THEOREM 4
Let $f : X \to X$ be a minimal homeomorphism of a compact metric space. Let P be any conjugacy-invariant property of $\mathrm{SL}(2, \mathbb{R})$-valued cocycles over f. If $A \in C^0(X, S)$ can be approximated by $\mathrm{SL}(2, \mathbb{R})$-valued cocycles with property P, then A can be approximated by S-valued cocycles with property P.

We are even able to treat the case of more regular cocycles.

THEOREM 5
Let $1 \leq r \leq \infty$, and let $0 \leq s \leq r$. Let X be a C^r compact manifold, and let $f : X \to X$ be a minimal C^r-diffeomorphism. Let P be any property of $\mathrm{SL}(2, \mathbb{R})$-valued C^r-cocycles over f which is invariant by C^r-conjugacy. If $A \in C^s(X, S)$ can be C^s-approximated by $\mathrm{SL}(2, \mathbb{R})$-valued cocycles with property P, then A can be C^s-approximated by S-valued cocycles with property P.

Remark 6
Having in mind applications to other types of difference equations, it would be interesting to investigate the validity of the results of Theorems 4 and 5 for more general classes of sets.

It follows from Theorems 2 and 4 that uniformly hyperbolic Schrödinger cocycles are C^0-dense. This has the following corollary.

COROLLARY 6
For a generic $V \in C^0(X, \mathbb{R})$, we have that $\mathbb{R} \setminus \Sigma$ is dense; that is, the associated Schrödinger operators have Cantor spectrum.

Proof
For $E \in \mathbb{R}$, consider the set

$$UH_E = \{V \in C^0(X, \mathbb{R}); (f, A_{E,V}) \text{ is uniformly hyperbolic}\}.$$

By Theorems 2 and 4, UH_E is (open and) dense. Thus we may choose a countable dense subset $\{E_n\}$ of $\mathbb{R}$ and then use (5) to conclude that for $V \in \bigcap_n UH_{E_n}$, the set $\mathbb{R} \setminus \Sigma$ is dense. □

Let us discuss this result in the two particular cases of interest, translations and skew-shifts on the torus.

If the base dynamics is given by a translation on the torus, that is, for quasi-periodic Schrödinger operators, then Cantor spectrum is widely expected to occur generically. Corollary 6 proves this statement in the C^0-topology. There are other related results that also establish a genericity statement of this kind. Dinh Cong and Fabbri [DF] considered bounded measurable potentials V. Fabbri, Johnson, and Pavani [FJP, Theorem 2.2] studied quasi-periodic Schrödinger operators in the continuum case, that is, acting in $L^2(\mathbb{R})$; they proved for generic translation vector that Cantor spectrum is C^0-generic. More recently, generic Cantor spectrum for almost periodic Schrödinger operators in the continuum was established by Gordon and Jitomirskaya [GJ].

On the other hand, Corollary 6 is rather surprising in the case of the skew-shift. Although few results are known, it is often assumed that in many respects the skew-shift behaves similarly to a Bernoulli shift, and Schrödinger operators associated to Bernoulli shifts never have Cantor spectrum. More precisely, the following is expected for Schrödinger operators defined by the skew-shift and a sufficiently regular nonconstant potential function $V : \mathbb{T}^2 \to \mathbb{R}$ (cf. [Bo3, page 114]). The (top) Lyapunov exponent of $(f, A_{E,V})$ is strictly positive for almost every $E \in \mathbb{R}$, the operator $H_{(x,y)}$ has pure point spectrum with exponentially decaying eigenfunctions for almost every $(x, y) \in \mathbb{T}^2$, and the spectrum Σ is not a Cantor set. Some partial affirmative results concerning the first two statements can be found in [Bo1, Proposition 5], [Bo2, Theorem 2], [Bo3, Chapter 15], and [BGS, Proposition 2.11, Theorem 3.7], whereas Corollary 6 above shows that the third expected property fails generically in the C^0-category.* It is natural to pose the question of whether our result is an artifact of weak regularity: can the spectrum of a Schrödinger operator associated to the skew-shift with analytic potential ever be a Cantor set?

The following result follows quickly from the results described above and the standard Kolmogorov-Arnold-Moser (KAM) theorem.

*A result of a similar flavor was recently obtained in [BD]; if α is not badly approximable, then the second expected property also fails generically in the C^0-category.

THEOREM 7
Assume that f is a Diophantine translation of the d-torus. Then the set of $V \in C^0(X, \mathbb{R})$ for which the corresponding Schrödinger operators have some absolutely continuous spectrum is dense.

This should be compared with [AD, Theorem 1], which showed that singular spectrum is C^0-generic in the more general context of ergodic Schrödinger operators.

2. Proof of the results for $\mathrm{SL}(2, \mathbb{R})$-cocycles

Our goal is to prove Theorems 1 and 3.

LEMMA 1
There are two possibilities about the group $\mathbb{G}$:
(a) (circle case) *either there is an onto continuous homomorphism $s : \mathbb{G} \to \mathbb{T}^1$;*
(b) (Cantor case) *or $\mathbb{G}$ is a Cantor set.*
In the second alternative, there exist continuous homomorphisms from $\mathbb{G}$ onto finite cyclic groups of arbitrarily large order.

In view of the lack of an exact reference, a proof of Lemma 1 is given in Appendix A.

We first work out the arguments for the more difficult circle case. By assumption, f fibers over a translation on $\mathbb{G}$ and hence also over a translation of the circle. That translation is minimal because so is f. Therefore, *in the circle case, we can and do assume that $\mathbb{G} = \mathbb{T}^1$.*

The proofs then go as follows. In Section 2.1, we explain a construction of almost-invariant sections for skew-products. This is used in Section 2.2 to prove Lemma 3, which says that functions that are cohomologous to constant are dense in $C^0(X, \mathbb{R})$. Using Lemma 3, the first case of Theorem 3 is easily proved in Section 2.3. In Section 2.4, we establish some lemmas about the action of $\mathrm{SL}(2, \mathbb{R})$ on hyperbolic space. The proof of Theorem 1 is given in Section 2.5. It is similar to the proof of Lemma 3, with additional ingredients including results on Lyapunov exponents from [B] and [F] and the material from Section 2.4. To prove the second case of Theorem 3 in Section 2.6, we employ Theorem 1 and Lemma 3 again.

In Section 2.7, we discuss the Cantor case, which is obtained by a simplification of the arguments (because then no gluing considerations are needed).

2.1. Almost-invariant sections

From here until Section 2.6, we consider only the circle case.

A continuous *skew-product* over f is a continuous map $F : X \times Y \to X \times Y$ (where Y is some topological space) of the form $F(x, y) = (f(x), F_x(y))$. F is called

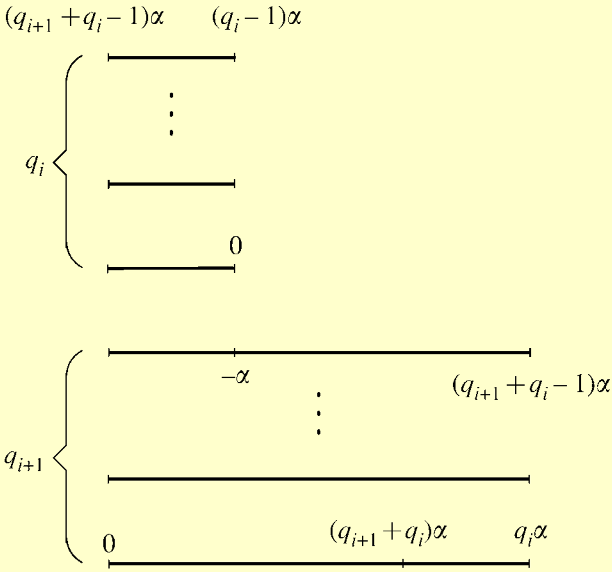

Figure 1. Castle with base I_i

invertible if it is a homeomorphism of $X \times Y$. In this case, we write $F^n(x, y) = (f^n(x), F_x^n(y))$ for $n \in \mathbb{Z}$. An *invariant section* for F is a continuous map $x \mapsto y(x)$ whose graph is F-invariant.

Let p_i/q_i be the ith continued fraction approximation of α. We recall that $q_i\alpha$ is closer to zero than any $n\alpha$ with $1 \le n < q_i$; moreover, the points $q_i\alpha$ alternate sides around zero.

Let $I_i \subset \mathbb{T}^1$ be the shortest closed interval containing zero and $q_i\alpha$. Notice that the first $n > 0$ for which $I_i + n\alpha$ intersects the interior of $I_i \cup I_{i+1}$ is $n = q_{i+1}$. Moreover, $(I_i + n\alpha) \smallsetminus I_i$ coincides (modulo a point) with I_{i+1}. Also, notice that $I_{i+1} + q_i\alpha$ is contained in I_i.

Let i be fixed. The above remarks show that the family of intervals

$$I_i,\ I_i+\alpha, \dots, I_i+(q_{i+1}-1)\alpha,\ I_{i+1},\ I_{i+1}+\alpha, \dots, I_{i+1}+(q_i-1)\alpha \qquad (6)$$

has these properties:

- the union of the intervals is the whole circle;
- the interiors of the intervals are two-by-two disjoint.

(Another way to obtain the family (6) is to cut the circle along the points $n\alpha$ with $0 \le n \le q_{i+1} + q_i - 1$.) We draw the intervals from (6) from bottom to top, as in Figure 1. Then each point is mapped by the α-rotation to the point directly above it or else to somewhere in the bottom floor I_i.

The following lemma (and its proof) is used in several situations (namely, Sections 2.2, 2.5).

LEMMA 2

Let $F : X \times Y \to X \times Y$ be a continuous invertible skew-product over f. Fix any $i \in \mathbb{N}$, and let $I = I_i$. Given any map $y_0 : h^{-1}(I) \to Y$, there exists a unique map $y_1 : X \to Y$ which extends y_0 and such that

$$F\big(x, y_1(x)\big) = \big(f(x), y_1(f(x))\big) \quad \text{for all } x \in X \setminus h^{-1}(I). \tag{7}$$

If, in addition, y_0 is continuous and satisfies

$$F^n\big(x, y_0(x)\big) = \big(f^n(x), y_0(f^n(x))\big) \quad \text{for all } x \in h^{-1}(0),\ n \in \{q_i, q_{i+1} + q_i\}, \tag{8}$$

then y_1 is continuous.

Proof

For each $x \in X$, let

$$\tau(x) = \min\big\{n \geq 0,\ f^n(x) \in h^{-1}(I)\big\}. \tag{9}$$

Given $y_0 : h^{-1}(I) \to Y$, then y_1 must be given by

$$y_1(x) = (F_x^{\tau(x)})^{-1}\big(y_0(f^{\tau(x)}(x))\big). \tag{10}$$

Now, assume that y_0 is continuous, and assume that (8) holds. We need only check that y is continuous at each point x where τ is not. Fix such an x, and let $k = \tau(x)$. Then either (i) $f^k(x) \in h^{-1}(0)$ or (ii) $f^k(x) \in h^{-1}(q_i\alpha)$. Let $\ell = \tau(f^k(x))$; that is, $\ell = q_i$ in case (i) and $\ell = q_{i+1}$ in case (ii). Due to the definition of y_0, we have $F^{\ell}_{f^k(x)}\big(y_0(f^k(x))\big) = y_0(f^{k+\ell}(x))$ in both cases. Therefore

$$(F_x^k)^{-1}\big(y_0(f^k(x))\big) = (F_x^{k+\ell})^{-1}\big(y_0(f^{k+\ell}(x))\big).$$

The set of the possible limits of $\tau(x_j)$, where $x_j \to x$, is precisely $\{k, k+\ell\}$. It follows that y is continuous at x. □

2.2. *The cohomological equation*

LEMMA 3

For every $\phi \in C^0(X, \mathbb{R})$ and every $\delta > 0$, there exists $\tilde{\phi} \in C^0(X, \mathbb{R})$ such that $\|\phi - \tilde{\phi}\|_{C^0} < \delta$ and $\tilde{\phi}$ is C^0-cohomologous to a constant; there exist $w \in C^0(X, \mathbb{R})$ and $a_0 \in \mathbb{R}$ such that

$$\tilde{\phi} = w \circ f - w + a_0.$$

Remark 7
In the case where $X = \mathbb{T}^1$, there is a quick proof of Lemma 3: approximate ϕ by a (real) trigonometric polynomial $\tilde{\phi}(z) = \sum_{|n|\leq m} a_n z^n$, and let $w(z) = \sum_{0<|n|\leq m}(e^{2\pi i n\alpha} - 1)^{-1} a_n z^n$.

The following proof contains a construction that appears again in the (harder) proof of Theorem 1, so it may also be useful as a warm-up.

Proof of Lemma 3
Fix $\phi \in C^0(X, \mathbb{R})$, and fix $\delta > 0$ small. Let a_0 be the integral of ϕ with respect to the unique f-invariant probability measure. Without loss of generality, assume that $a_0 = 0$. Write $S_n = \sum_{j=0}^{n-1} \phi \circ f^j$. Let n_0 be such that $|S_n/n| < \delta$ uniformly for every $n \geq n_0$.

Choose and fix i such that

$$q_i > \max(n_0, \delta^{-1}\|\phi\|_{C^0}).$$

Let $I = I_i$. The rest of the proof is divided into these three steps.

Step 1: Finding an almost-invariant section $w_1 : X \to \mathbb{R}$. First, define a real function w_0 on $h^{-1}(\{0, q_i\alpha, (q_{i+1} + q_i)\alpha\})$ by

$$w_0\big(f^n(x)\big) = S_n(x) \quad \text{for } x \in h^{-1}(0) \text{ and } n = 0, q_i, \text{or } q_{i+1} + q_i.$$

Using Tietze's extension theorem, we extend continuously w_0 to $h^{-1}(I)$ so that

$$\sup_{h^{-1}(I)} |w_0| = \sup_{h^{-1}(\{0, q_i\alpha, (q_{i+1}+q_i)\alpha\})} |w_0|.$$

Now, we consider the skew-product

$$F : X \times \mathbb{R} \to X \times \mathbb{R}, \qquad F(x, w) = \big(f(x), w + \phi(x)\big).$$

Applying Lemma 2 to F and w_0, we find a continuous function $w_1 : X \to \mathbb{R}$ which extends w_0 and such that $w_1(f(x)) = w_1(x) + \phi(x)$ if $x \notin h^{-1}(\operatorname{int} I)$.

Step 2: Definition of functions $\tilde{\phi}$, $w : X \to \mathbb{R}$. Define $\tilde{\phi}$ by $\tilde{\phi} = \phi$ outside of $\bigsqcup_{n=0}^{q_{i+1}-1} f^n(h^{-1}(I))$, and

$$\begin{aligned}\tilde{\phi}&\big(f^n(x)\big)\\ &= \phi\big(f^n(x)\big) + \frac{w_1(f^{q_{i+1}}(x)) - w_1(x) - S_{q_{i+1}}(x)}{q_{i+1}} \quad \text{if } x \in h^{-1}(I), 0 \leq n < q_{i+1}.\end{aligned}$$

Define w by $w = w_1$ outside of $\bigsqcup_{n=1}^{q_{i+1}-1} f^n(h^{-1}(I))$, and

$$w\big(f^n(x)\big) = w(x) + \sum_{j=0}^{n-1} \tilde{\phi}\big(f^j(x)\big) \quad \text{if } x \in h^{-1}(I), 0 \leq n < q_{i+1}.$$

Then $\tilde{\phi}$ and w are continuous functions satisfying $\tilde{\phi} = w \circ f - w$.

Step 3: Distance estimate. Let $x \in h^{-1}(I)$ be fixed. Due to the definition of w_0, we have

$$|w_0(x)| \leq (q_{i+1} + q_i)\delta.$$

Recalling (9), we see that $\tau(f^{q_{i+1}}(x))$ equals either 1 or $q_i + 1$ (see Figure 1). In any case, $|S_{\tau(x)}(x)| \leq (q_i + 1)\delta$, and therefore, by (10),

$$\left|w_1\big(f^{q_{i+1}}(x)\big)\right| \leq \left|w_0\big(f^{\tau(x)+q_{i+1}}(x)\big)\right| + |S_{\tau(x)}(x)| \leq (q_{i+1} + 2q_i + 1)\delta.$$

Hence

$$\Big|\frac{w_1(f^{q_{i+1}}(x)) - w_1(x)}{q_{i+1}}\Big| \leq \frac{(3q_{i+1} + 3q_i + 1)\delta}{q_{i+1}} < 7\delta;$$

that is, the C^0-distance between $\tilde{\phi}$ and ϕ is less than 7δ. □

2.3. Denseness of reducibility in the uniformly hyperbolic case

First, let us note the following basic fact.

LEMMA 4

For any homeomorphism $f : X \to X$, if (f, A) is uniformly hyperbolic, then $A \in \mathcal{R}uth$.

Proof

By uniform hyperbolicity, for each $x \in X$ there exists a splitting $\mathbb{R}^2 = E^u(x) \oplus E^s(x)$ which depends continuously on x and is left invariant by the cocycle; that is, $A(x) \cdot E^{u,s}(x) = E^{u,s}(f(x))$.

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$. For each $x \in X$, let $e^u(x) \in E^u(x)$ and $e^s(x) \in E^s(x)$ be unit vectors so that $\{e^u(x), e^s(x)\}$ is a positively oriented basis. Define a matrix $B(x)$ by putting $B(x) \cdot e_1 = ce^u(x)$ and $B(x) \cdot e_2 = ce^s(x)$, where $c = [\sin \sphericalangle(e^u(x), e^s(x))]^{-1/2}$ is chosen so that $\det B(x) = 1$. Then $B(x)$ is uniquely defined as an element of $\mathrm{PSL}(2, \mathbb{R})$ and depends continuously on x.

Let $D(x)$ be given by $A(x) = B(f(x))D(x)B(x)^{-1}$. Then $D(x)$ is a diagonal "matrix." Therefore $D : X \to \mathrm{PSL}(2, \mathbb{R})$ is homotopic to a constant and A is homotopic to a reducible cocycle. □

Proof of Theorem 3(a)
Let us write, for $t \in \mathbb{R}$,

$$D_t = \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R}).$$

By the proof of Lemma 4, there exist $B \in C^0(X, \mathrm{PSL}(2, \mathbb{R}))$ and $\phi \in C^0(X, \mathbb{R})$ such that $A(x) = B(f(x))D_{\phi(x)}B(x)^{-1}$. By Lemma 3, we can perturb ϕ (and hence A) in the C^0-topology so that $\phi = w \circ f - w + a_0$ for some $w \in C^0(X, \mathbb{R})$ and $a_0 \in \mathbb{R}$. We can assume that $a_0 \neq 0$. Then $\hat{B}(x) = B(x)D_{w(x)}$ is a conjugacy between A and the constant D_{a_0}. □

2.4. Disk adjustment lemma
The aim here is to establish Lemma 6, which is used in the proof of Theorem 1. First, we need to recall some facts about hyperbolic geometry.

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the upper half-plane $\mathbb{H} = \{w \in \mathbb{C},\ \operatorname{Im} w > 0\}$ as follows:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \ \Rightarrow\ A \cdot w = \frac{aw + b}{cw + d}.$$

(In fact, the action factors through $\mathrm{PSL}(2, \mathbb{R})$.) We endow the half-plane with the Riemannian metric (of curvature -1)

$$v \in T_w\mathbb{H} \ \Rightarrow\ \|v\|_w = \frac{|v|}{\operatorname{Im} w}.$$

Then $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{H}$ by isometries.

We fix the following conformal equivalence between $\mathbb{H}$ and the unit disk $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$:

$$w = \frac{-iz - i}{z - 1} \in \mathbb{H} \longleftrightarrow\ z = \frac{w - i}{w + i} \in \mathbb{D}.$$

We take on the disk the Riemannian metric that makes the map above an isometry, namely, $\|v\|_z = 2(1 - |z|^2)^{-2}|v|$. By conjugating, we get an action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{D}$ by isometries, which we also denote as $(A, z) \mapsto A \cdot z$.

Let $d(\cdot, \cdot)$ denote the distance function induced on $\mathbb{D}$ by the Riemannian metric. We claim that

$$\|A\| = e^{d(A\cdot 0, 0)/2} \quad \text{for all } A \in \mathrm{SL}(2, \mathbb{R}). \tag{11}$$

Proof of (11)
It is sufficient to prove the corresponding fact that $\|A\| = e^{d(A\cdot i,i)/2}$ on the half-plane $\mathbb{H}$. We first check the case where A is a diagonal matrix $H_\lambda = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ with $\lambda > 1$:

$$d(A\cdot i, i) = d(\lambda^2 i, i) = \int_1^{\lambda^2} \frac{dy}{y} = 2\log\lambda = 2\log\|A\|.$$

Next, if A is a rotation R_θ, then its action on $\mathbb{H}$ fixes the point i, so the claim also holds. Finally, a general matrix can be written as $A = R_\beta H_\lambda R_\alpha$, so (11) follows. □

We now prove two lemmas.

LEMMA 5
There exists a continuous map $\Phi : \mathbb{D}\times\mathbb{D} \to \mathrm{SL}(2,\mathbb{R})$ *such that* $\Phi(p_1, p_2)\cdot p_1 = p_2$ *and* $\|\Phi(p_1, p_2) - \mathrm{Id}\| \le e^{d(p_1,p_2)/2} - 1$.

Let us recall a few more facts about the half-plane and the disk models which we use in the proof of the lemma. An *extended circle* means either a Euclidean circle or a Euclidean line in the complex plane.

- The geodesics on $\mathbb{H}$ (resp., $\mathbb{D}$) are arcs of extended circles which orthogonally meet the real axis $\partial\mathbb{H}$ (resp., the unit circle $\partial\mathbb{D}$) at the endpoints (called *points at infinity*).
- The points lying at a fixed positive distance from a geodesic γ form two arcs of extended circles γ_1 and γ_2, which have the same points at infinity as γ; see Figure 2 (left). We say that γ and γ_1 are *equidistant curves*.
- A quadrilateral $p_1q_1q_2p_2$ is called a *Saccheri quadrilateral* with *base* q_1q_2 and *summit* p_1p_2 if the angles at vertices q_1 and q_2 are right and the sides p_1q_1 and p_2q_2 (called the *legs*) have the same length; see Figure 2 (right). The fact is that the summit is necessarily longer than the base.

Proof
We define the matrix $\Phi(p_1, p_2)$ explicitly. It is the identity if $p_1 = p_2$. Next, consider the case $p_1 \ne p_2$.

We first consider a particular case where we rewrite the two points as q_1, q_2. Assume that the (whole) geodesic γ containing q_1 and q_2 also contains zero; that is, γ is a piece of Euclidean line. Let u be the point in the circle $\{|z| = 1\}$ such that the line contains $-u$, q_1, q_2, u, in that order. Consider the hyperbolic isometry that preserves the geodesic γ, translating it and taking q_1 to q_2. That isometry corresponds

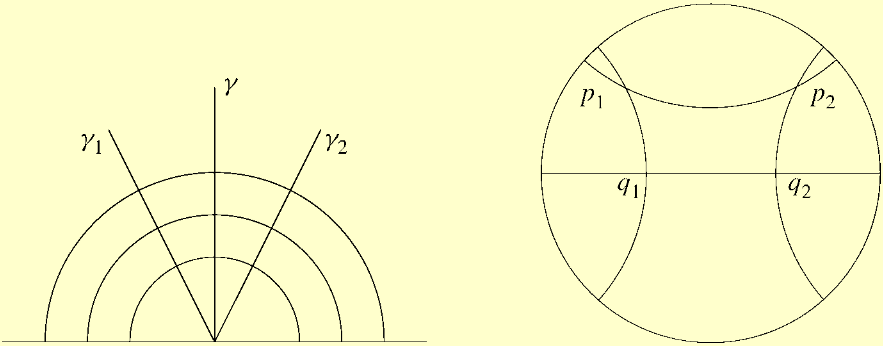

Figure 2. Left: equidistant curves on $\mathbb{H}$; right: a Saccheri quadrilateral $p_1q_1q_2p_2$ on $\mathbb{D}$

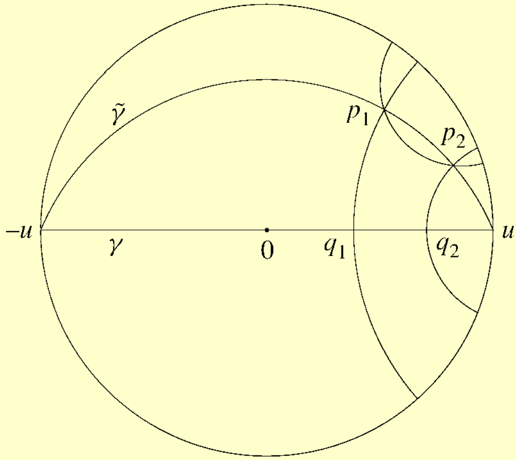

Figure 3. Proof of Lemma 5

to a matrix of the form

$$A = R_\theta H_\lambda R_{-\theta}, \quad \text{where } R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ H_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\theta \in \mathbb{R}$ (in fact, $e^{-2i\theta} = u$) and $\lambda > 1$. Since the isometry translates γ, we have $d(0, A\cdot 0) = d(q_1, A\cdot q_1) = d(q_1, q_2)$. Therefore (11) gives $\lambda = \|A\| = e^{d(q_1,q_2)/2}$. On the other hand, $\|A - \mathrm{Id}\| = \|H_\lambda - \mathrm{Id}\| = \lambda - 1$, so we can define $\Phi(q_1, q_2) = A$ and the bound claimed in the statement of the lemma becomes an equality.

Next, let us consider the general case. Given p_1 and p_2, consider the family of extended circles that contain p_1 and p_2. There exists a unique C in this family which intersects the circle $\{|z| = 1\}$ in two antipodal points u and $-u$ (see Figure 3). Let $\tilde{\gamma} = C \cap \mathbb{D}$, and let γ be the geodesic whose points at infinity are u and $-u$; so γ and

$\tilde{\gamma}$ are equidistant curves. Notice that the case already treated corresponds to the case where $\tilde{\gamma} = \gamma$ is a geodesic. Let q_1 (resp., q_2) be the point in γ which has the least hyperbolic distance to p_1 (resp., p_2). Notice that $\Phi(q_1, q_2)$ is already defined. We set $\Phi(p_1, p_2) = \Phi(q_1, q_2)$.

Because γ and $\tilde{\gamma}$ are equidistant, we have $d(p_1, q_1) = d(p_2, q_2)$. It follows that $p_1q_1q_2p_2$ is a Saccheri quadrilateral. In particular, $d(q_1, q_2) < d(p_1, p_2)$, and hence $\|\Phi(p_1, p_2)\| < e^{d(p_1,p_2)/2}$. Also, since $\Phi(q_1, q_2)$ translates the geodesic γ sending q_1 to q_2, it sends the leg q_1p_1 to the leg q_2p_2, and, in particular, sends p_1 to p_2, as desired.

This completes the definition of Φ; continuity is evident. □

LEMMA 6

For every $n \geq 1$, there exists a continuous map $\Psi_n : \mathrm{SL}(2, \mathbb{R})^n \times \mathbb{D}^2 \to \mathrm{SL}(2, \mathbb{R})^n$ such that if $\Psi_n(A_1, \ldots, A_n, p, q) = (\tilde{A}_1, \ldots, \tilde{A}_n)$, then

(a) $\tilde{A}_n \cdots \tilde{A}_1 \cdot p = q$, *and*

(b) $\|\tilde{A}_i A_i^{-1} - \mathrm{Id}\| \leq \exp\big((1/(2n))d(A_n \cdots A_1 \cdot p, q)\big) - 1$ *for* $1 \leq i \leq n$.

Proof

Let $w_0 = A_n \cdots A_1 \cdot p$, and let $L = d(w_0, q)$. For $1 \leq i \leq n$, let w_i be the point in the hyperbolic geodesic segment joining w_0 and q which is at distance iL/n of w_0. Also, let $z_i = (A_n \cdots A_{n-i+1})^{-1} \cdot w_i$ for $0 \leq i \leq n$. Then $z_0 = p$, $z_n = q$, and $d(A_i \cdot z_{i-1}, z_i) = L/n$. Let $\tilde{A}_i = \Phi(A_i \cdot z_{i-1}, z_i)A_i$, where Φ is as in Lemma 5.

2.5. Proof of Theorem 1

Let $A : X \to \mathrm{SL}(2, \mathbb{R})$ be such that (f, A) is not uniformly hyperbolic, and let $\delta_0 > 0$ be given. We want to find $\tilde{A} \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ and a continuous function $z : X \to \mathbb{D}$ so that $\|\tilde{A} - A\|_{C^0} < \delta_0$ and $\tilde{A}(x)\cdot z(x) = z(f(x))$. Accomplishing this, we simply set $B(x) = \Phi(z(x), 0)$ (where Φ is given by Lemma 5), and then $B(f(x))A(x)B(x)^{-1}$ are rotations.

Because the cocycle is not uniformly hyperbolic, a theorem by Bochi [B, Theorem C] gives a C^0-perturbation of A whose upper Lyapunov exponent (with respect to the unique invariant probability) is zero. For simplicity of writing, let A denote this perturbation. Since f is uniquely ergodic, a result due to Furman [F, Theorem 1] gives that (f, A) has *uniform subexponential growth*; that is,

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)\| = 0 \quad \text{uniformly on } x \in X. \tag{12}$$

Let $\delta > 0$ be such that $(e^{7\delta} - 1)\|A\|_{C^0} < \delta_0$. Let n_0 be such that $n \geq n_0$ implies that $\|A^n(x)\| \leq e^{n\delta}$ for every x.

The rest of the argument is analogous to the corresponding steps in the proof of Lemma 3, with the disk playing the role of the line. Let p_i/q_i be the ith continued fraction approximation of α. Choose and fix i large so that

$$q_i > \max(n_0, \delta^{-1} \log\|A\|_{C^0}).$$

Let $I \subset \mathbb{T}^1$ be the shortest closed interval containing zero and $q_i\alpha$. The rest of the proof is divided into three steps.

Step 1: Finding an almost-invariant section $z_1 : X \to \mathbb{R}$. First, we define z_0 on $h^{-1}(\{0, q_i\alpha, (q_{i+1} + q_i)\alpha\})$ by

$$z_0\big(f^n(x)\big) = A^n(x) \cdot 0 \quad \text{for } x \in h^{-1}(0) \text{ and } n = 0, q_i, \text{ or } q_{i+1} + q_i.$$

Then we extend continuously z_0 to $h^{-1}(I)$ in a way such that

$$\sup_{x \in h^{-1}(I)} d\big(z_0(x), 0\big) = \sup_{x \in h^{-1}(\{0, q_i\alpha, (q_{i+1}+q_i)\alpha\})} d\big(z_0(x), 0\big).$$

Consider the skew-product

$$F : X \times \mathbb{D} \to X \times \mathbb{D}, \qquad F(x, z) = \big(f(x), A(x) \cdot z\big).$$

Applying Lemma 2 to F and z_0, we find a continuous map $z_1 : X \to \mathbb{D}$ which extends z_0 and such that $z_1(f(x)) = A(x) \cdot z_1(x)$ if $x \notin h^{-1}(\operatorname{int} I)$.

Step 2: Definition of maps $\tilde{A} : X \to \mathrm{SL}(2, \mathbb{R})$ and $z : X \to \mathbb{D}$. Let $\Psi_{q_{i+1}}$ be given by Lemma 6, and put

$$\begin{aligned}
&\big(\tilde{A}(x), \tilde{A}(f(x)), \ldots, \tilde{A}(f^{q_{i+1}-1}(x))\big) \\
&\quad = \Psi_{q_{i+1}}\big(A(x), A(f(x)), \ldots, A(f^{q_{i+1}-1}(x)), z_1(x), z_1(f^{q_{i+1}}(x))\big)
\end{aligned}$$

for each $x \in h^{-1}(I)$. This defines $\tilde{A}$ on $\bigsqcup_{n=0}^{q_{i+1}-1} f^n(h^{-1}(I))$. Let $\tilde{A}$ equal A on the rest of X.

For each $x \in h^{-1}(I)$ and $1 \le n \le q_{i+1} - 1$, let $z(f^n(x)) = \tilde{A}^n(x) \cdot z_1(x)$. This defines z on $\bigsqcup_{n=1}^{q_{i+1}-q_i-1} f^n(h^{-1}(I))$. Let z equal z_1 on the rest of X.

It is easy to see that both maps $\tilde{A}$ and z are continuous on the whole X and satisfy $\tilde{A}(x) \cdot z(x) = z(f(x))$.

Step 3: Distance estimate. To complete the proof, we need to check that $\tilde{A}$ is C^0-close to A. Begin by noticing that, by (11),

$$B \in \mathrm{SL}(2, \mathbb{R}),\ w \in \mathbb{D} \ \Rightarrow\ d(B \cdot w, 0) \le d(w, 0) + 2 \log\|B\|.$$

Now, fix $x \in h^{-1}(I)$. By the definition of z_0, we have

$$d\big(z_0(x), 0\big) \leq 2(q_{i+i} + q_i)\delta.$$

If $y = f^{q_{i+1}}(x)$, then $\tau(y)$ equals 1 or $q_i + 1$. In either case, $\|[A^{\tau(y)}(y)]^{-1}\| = \|A^{\tau(y)}(y)\| \leq e^{(q_i+1)\delta}$. Since $z_1(y) = [A^{\tau(y)}(y)]^{-1} \cdot z_0(f^{\tau(y)}(y))$, we get

$$d\big(z_1(y), 0\big) \leq d\big(z_0(f^{\tau(y)}(y)), 0\big) + 2\log\|A^{\tau(y)}(y)\| \leq 2(q_{i+1} + 2q_i + 1)\delta.$$

Putting things together,

$$\begin{aligned} d\big(A^{q_{i+1}}(x) \cdot z_0(x), z_1(f^{q_{i+1}}(x))\big) &\leq d\big(A^{q_{i+1}}(x) \cdot z_0(x), 0\big) + d\big(0, z_1(f^{q_{i+1}}(x))\big) \\ &\leq d\big(z_0(x), 0\big) + 2\log\|A^{q_{i+1}}(x)\| + d\big(0, z_1(f^{q_{i+1}}(x))\big) \\ &\leq 2(3q_{i+1} + 3q_i + 1)\delta. \end{aligned}$$

By Lemma 6,

$$\|\tilde{A}A^{-1} - \mathrm{Id}\|_{C^0} \leq \exp\left[\frac{1}{2q_{i+1}} d\big(A^{q_{i+1}}(x) \cdot z_0(x), z_1(f^{q_{i+1}}(x))\big)\right] - 1 < e^{7\delta} - 1.$$

So $\|\tilde{A} - A\|_{C^0} \leq \delta_0$, as desired. □

Remark 8

A result by Avila and Bochi [AB] says that a generic $\mathrm{SL}(2, \mathbb{R})$-cocycle over a *minimal* homeomorphism either is uniformly hyperbolic or has uniform subexponential growth (12) (which is equivalent to the simultaneous vanishing of the Lyapunov exponent for *all* f-invariant measures), provided that the space X is compact with finite dimension. Using this in place of the aforementioned results from [B] and [F], we obtain the generalization claimed in Remark 2; the rest of the proof is the same.

2.6. *Completion of the proof of Theorem 3*

We first prove two lemmas.

LEMMA 7

Assume that $A : X \to \mathrm{SL}(2, \mathbb{R})$ is homotopic to a constant, and assume that (f, A) is not uniformly hyperbolic. Then there exist $\tilde{A}$ arbitrarily C^0-close to A and $B \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ such that $B(f(x))\tilde{A}(x)B(x)^{-1}$ is a constant in $\mathrm{SO}(2, \mathbb{R})$.

Proof

By Theorem 1, we can perturb A so that there exist $A_1 \in C^0(X, \mathrm{SO}(2, \mathbb{R}))$ and $B_1 \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ such that $A(x) = B_1(f(x))A_1(x)B_1(x)^{-1}$.

Let $r : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SO}(2, \mathbb{R})$ be a deformation retract. Let $B_2(x) = r(B_1(x))$ and $A_2(x) = B_2(f(x))A_1(x)B_2(x)^{-1}$. Then $A_2(x)$ is
(i) $\mathrm{SO}(2, \mathbb{R})$-valued,
(ii) conjugate to $A(x)$,
(iii) homotopic to $A(x)$ and therefore homotopic to a constant.
Due to the existence of the deformation retract r, A_2 is also homotopic to a constant as an $X \to \mathrm{SO}(2, \mathbb{R})$ map. Consider the covering map $\mathbb{R} \to \mathrm{SO}(2, \mathbb{R})$ given by $\theta \mapsto R_\theta$. Let $\phi : X \to \mathbb{R}$ be a lift of A_2, that is, $A_2(x) = R_{\phi(x)}$.

By Lemma 3, there exists $\tilde{\phi}$ very close to ϕ such that $\tilde{\phi} = w \circ f - w + a_0$ for some $w \in C^0(X, \mathbb{R})$ and $a_0 \in \mathbb{R}$. So the map $\tilde{A}_2 = R_{\tilde{\phi}(x)}$ is close to A_2 and conjugate to the constant R_{a_0}.

Since A and A_2 are conjugate, there exists $\tilde{A}$ close to A and conjugate to $\tilde{A}_2$ (and therefore to the constant). Then $\tilde{A}$ is the map we were looking for. □

LEMMA 8
If $A \in$ Ruth, then (f, A) is $\mathrm{PSL}(2, \mathbb{R})$*-conjugate to a cocycle that is homotopic to a constant.*

Proof
Let $\pi : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ be the quotient map. Since $A \in$ *Ruth*, there exist $B : X \to \mathrm{SL}(2, \mathbb{R})$ homotopic to A, $D : X \to \mathrm{PSL}(2, \mathbb{R})$, and $C \in \mathrm{PSL}(2, \mathbb{R})$ such that $\pi(B(x)) = D(f(x))CD(x)^{-1}$. The map $x \in X \mapsto D(f(x))^{-1}\pi(A(x))D(x) \in \mathrm{PSL}(2, \mathbb{R})$ is homotopic to a constant; therefore, it can be lifted to a map $\tilde{A} : X \to \mathrm{SL}(2, \mathbb{R})$, which is itself homotopic to a constant. Thus D is a $\mathrm{PSL}(2, \mathbb{R})$-conjugacy between A and $\tilde{A}$. □

Proof of Theorem 3(b)
We have already treated Theorem 3(a), so we restrict ourselves here to Theorem 3(b).

Fix A and A_* as in the statement of Theorem 3. By Lemma 8, (f, A) is $\mathrm{PSL}(2, \mathbb{R})$-conjugate to a cocycle that is homotopic to a constant. Since the closure of a $\mathrm{PSL}(2, \mathbb{R})$-conjugacy class is invariant under $\mathrm{PSL}(2, \mathbb{R})$-conjugacies, it is enough to consider the case where A is homotopic to a constant. We are going to show that in this case, (f, A) lies in the closure of the $\mathrm{SL}(2, \mathbb{R})$-conjugacy class of (f, A_*).

By Lemma 7, A can be perturbed to become conjugate to a constant $C_* = R_\theta$ in $\mathrm{SO}(2, \mathbb{R})$. We explain how to perturb C_* (and hence A because the conjugacy between C_* and A is fixed) in order that (f, C_*) becomes conjugate to (f, A_*). There are two cases, depending on A_*.

In the case where $A_* = \mathrm{Id}$ or A_* is elliptic (i.e., $|\operatorname{tr} A_*| < 2$), there exist $B_* \in \mathrm{SL}(2, \mathbb{R})$ and $\beta \in \mathbb{R}$ such that $A_* = B_*^{-1} R_\beta B_*$. Since α is irrational, we can choose $k \in \mathbb{Z}$ such that $2\pi k\alpha + \theta$ is very close to β (modulo $2\pi\mathbb{Z}$). We still have the right to perturb θ, so we can assume that $2\pi k\alpha + \theta = \beta$. Letting $B(x) = B_* R_{2\pi k h(x)}$, we see that $B(f(x))R_\theta B(x)^{-1}$ is precisely A_*. So (f, R_θ) is conjugate to (f, A_*), as desired.

In the remaining case, A_* is parabolic (i.e., $\operatorname{tr} A_* = \pm 2$) with $A_* \neq \pm\mathrm{Id}$. By the previous case, we can assume that $C_* = \pm\mathrm{Id}$ to start. In fact, we can perturb further and assume that C_* is a parabolic matrix with $C_* \neq \pm\mathrm{Id}$. Then C_* and A_* are automatically conjugate in the group $\mathrm{SL}(2, \mathbb{R})$, and the theorem is proved. □

Remark 9

Assuming that A is homotopic to some cocycle that is conjugate to a constant, "$\mathrm{PSL}(2, \mathbb{R})$-conjugacy class" can be replaced by "$\mathrm{SL}(2, \mathbb{R})$-conjugacy class" in Theorem 3(b). We give an example showing that the stronger conclusion does not hold without additional hypotheses. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be given by $(x, y) \mapsto (x+\alpha, y+2x)$. Let $A(x, y) = R_{2\pi x}$. We claim that

(a) (f, A) is reducible;

(b) for any $\tilde{A}$ close to A, $(f, \tilde{A})$ is not conjugate to a constant $\mathrm{SL}(2, \mathbb{R})$-cocycle.

To prove (a), let $D(x, y) = R_{\pi y}$ (which is well defined in $\mathrm{PSL}(2, \mathbb{R})$), and notice that $D(f(x, y))D(x, y)^{-1} = A(x, y)$. To prove (b), we show that for any continuous $B : \mathbb{T}^2 \to \mathrm{SL}(2, \mathbb{R})$, the map $C(x, y) = B(f(x, y))A(x, y)B(x, y)^{-1}$ is not homotopic to a constant. Consider the homology groups $H_1(\mathbb{T}^2) = \mathbb{Z}^2$ and $H_1(\mathrm{SL}(2, \mathbb{R})) = \mathbb{Z}$ and the induced homomorphisms

$$f_* : (m, n) \mapsto (m, 2m + n), \qquad A_* : (m, n) \mapsto m, \qquad B_* : (m, n) \mapsto km + \ell n.$$

We have* $C_* = B_* \circ f_* + A_* - B_*$; therefore $C_* : (m, n) \mapsto (2\ell + 1)m$ cannot be the zero homomorphism.

2.7. The case of Cantor groups

Now, assume the second case in Lemma 1. So there are integers $q_i \to \infty$ and continuous homomorphisms $h_i : \mathbb{G} \to \mathbb{T}^1$ such that the image of h_i is the (cyclic) subgroup of $\mathbb{T}^1$ of order q_i. Notice that the level sets of h_i are compact, open, and cyclically permuted by f. They form a tower of height q_i which replaces the more complicated castle of Figure 1 in our arguments. Changing the definition of h_i, we can assume that $h_i(f(x)) = h_i(x) + (1/q_i) \pmod 1$.

There are only three proofs that need modification.

Proof of Lemma 3 in the Cantor case

Fix $\phi \in C^0(X, \mathbb{R})$ with mean zero, and let $\delta > 0$ small. Let n_0 be such that $n \geq n_0 \Rightarrow |S_n/n| < \delta$ uniformly, where S_n is the nth Birkhoff sum of f. Choose i such that

*Recall that if G is a path-connected topological group and $\gamma_1, \gamma_2, \gamma : [0, 1] \to G$ are such that $\gamma(t) = \gamma_1(t)\gamma_2(t)$, then the 1-chains γ and $\gamma_1 + \gamma_2$ are homologous.

$q_i > n_0$. Define $\tilde{\phi}$ and $w : X \to \mathbb{R}$ by

$$x \in h_i^{-1}(0),\, 0 \le n < q_i$$
$$\Rightarrow \tilde{\phi}\big(f^n(x)\big) = \phi\big(f^n(x)\big) - \frac{S_{q_i}(x)}{q_i},\ \ w\big(f^n(x)\big) = S_n(x) - \frac{nS_{q_i}(x)}{q_i}.$$

Then $\tilde{\phi}$ and w are continuous, $\tilde{\phi} = w \circ f - w$, and $|\tilde{\phi} - \phi| < \delta$. □

Proof of Theorem 1 in the Cantor case

Assume that (f, A) is not uniformly hyperbolic. Given $\delta_0 > 0$, let $\delta > 0$ be such that $(e^\delta - 1)\|A\|_{C^0} < \delta_0$. Perturbing A, we can assume that $\|A^n(x)\| < e^{n\delta}$ for every $n \ge n_0 = n_0(\delta)$. Fix $q_i > n_0$. Define $\tilde{A} : X \to \mathrm{SL}(2, \mathbb{R})$ so that

$$\big(\tilde{A}(x), \tilde{A}(f(x)), \dots, \tilde{A}(f^{q_i-1}(x))\big)$$
$$= \Psi_{q_i}\big(A(x), A(f(x)), \dots, A(f^{q_{i+1}-1}(x)), 0, 0\big), \quad \forall x \in h_i^{-1}(0),$$

where Ψ_{q_i} is given by Lemma 6. Define $z : X \to \mathbb{D}$ by $z(f^n(x)) = \tilde{A}^n(x) \cdot 0$ for $x \in h_i^{-1}(0)$ and $0 \le n < q_i$. Then $\tilde{A}$ and z are continuous and satisfy $\tilde{A}(x) \cdot z(x) = \tilde{A}(f(x))$. Moreover, since $d(A^{q_i}(0), 0) < q_i\delta$, we have $\|\tilde{A} - A\| < (e^\delta - 1)\|A\| < \delta_0$. □

Proof of Theorem 3(b) in the Cantor case

We need only show that the closure of the $\mathrm{SO}(2, \mathbb{R})$-conjugacy class of a constant $\mathrm{SO}(2, \mathbb{R})$-valued cocycle contains all constant $\mathrm{SO}(2, \mathbb{R})$-valued cocycles. Given a constant cocycle R_θ, $i \in \mathbb{N}$, and $k \in \mathbb{Z}$, let $B(x) = R_{2\pi k h_i(x)}$. Then $B(f(x)) R_\theta B(x)^{-1} = R_{\theta + 2\pi k/q_i}$. So the claim follows. □

This completes the proofs of Theorems 1 and 3.

3. Proof of the results for Schrödinger cocycles

In this section, we prove Theorems 4, 5, and 7.

3.1. Projection lemma

The proof of Theorems 4 and 5 is based on the following projection lemma.

LEMMA 9

Let $0 \le r \le \infty$. Let $f : X \to X$ be a minimal homeomorphism of a compact metric space with at least three points (if $r = 0$) or a minimal C^r diffeomorphism of a C^r

compact manifold. Let $A \in C^r(X, S)$ be a map whose trace is not identically zero. Then there exist a neighborhood $\mathcal{W} \subset C^0(X, \mathrm{SL}(2, \mathbb{R}))$ of A and continuous maps

$$\Phi = \Phi_A : \mathcal{W} \to C^0(X, S) \qquad \text{and} \qquad \Psi = \Psi_A : \mathcal{W} \to C^0\big(X, \mathrm{SL}(2, \mathbb{R})\big)$$

satisfying

$$\Phi \text{ and } \Psi \text{ restrict to continuous maps } \mathcal{W} \cap C^s \to C^s \quad \text{for } 0 \le s \le r, \tag{13}$$

$$\Psi(B)\big(f(x)\big) \cdot B(x) \cdot [\Psi(B)(x)]^{-1} = \Phi(B)(x), \tag{14}$$

$$\Phi(A) = A \qquad \text{and} \qquad \Psi(A) = \mathrm{id}. \tag{15}$$

Proof of Theorems 4 and 5

The result is easy if $\#X \le 2$, so we assume that $\#X \ge 3$. In this case, Lemma 9 implies the result unless $\operatorname{tr} A$ is identically zero.

Assume that $\operatorname{tr} A$ is identically zero. Let $V \subset X$ be an open set such that $\overline{V} \cap f(\overline{V}) = \varnothing$ and $\overline{V} \cap f^2(\overline{V}) = \varnothing$. Let $\tilde{A} \in C^r(X, S)$ be C^r-close to A such that $\operatorname{tr} \tilde{A}$ is supported in $V \cup f^2(V)$ and, moreover, $\operatorname{tr} \tilde{A}(z) + \operatorname{tr} \tilde{A}(f^2(z)) = 0$ for $z \in V$. Then $(f, \tilde{A})$ is C^r-conjugate to A; letting $B(x) = \mathrm{id}$ for $x \notin f(V) \cup f^2(V)$, $B(x) = \tilde{A}(f^{-1}(x))R_{-\pi/2}$ for $x \in f(V)$, and $B(x) = R_{-\pi/2}\tilde{A}(f^{-2}(x))$ for $x \in f^2(V)$, we have $B(f(x))A(x)B(x)^{-1} = \tilde{A}(x)$. If A can be C^s-approximated by $\mathrm{SL}(2, \mathbb{R})$-valued cocycles with property P, then so can $\tilde{A}$. Since $\operatorname{tr} \tilde{A}$ does not vanish identically, $\tilde{A}$ can be C^s-approximated by S-valued cocycles with property P. Since $\tilde{A}$ can be chosen arbitrarily C^r-close to A, we conclude that A can be C^s-approximated by S-valued cocycles with property P. □

The proof of Lemma 9 has two distinct steps. First, we show that $\mathrm{SL}(2, \mathbb{R})$-perturbations can be conjugated to localized $\mathrm{SL}(2, \mathbb{R})$-perturbations, and then we show how to conjugate localized perturbations to Schrödinger perturbations.

In order to be precise, the following definition is useful. Given $A \in C^r(X, \mathrm{SL}(2, \mathbb{R}))$ and $K \subset X$ compact, let $C^r_{A,K}(X, \mathrm{SL}(2, \mathbb{R})) \subset C^r(X, \mathrm{SL}(2, \mathbb{R}))$ be the set of all B such that $B(x) = A(x)$ for $x \notin K$. The two steps that we described correspond to Lemmas 10 and 11.

LEMMA 10

Let $V \subset X$ be any nonempty open set, and let $A \in C^r(X, \mathrm{SL}(2, \mathbb{R}))$ be arbitrary. Then there exist an open neighborhood $\mathcal{W}_{A,V} \subset C^0(X, \mathrm{SL}(2, \mathbb{R}))$ of A and continuous maps

$$\Phi = \Phi_{A,V} : \mathcal{W}_{A,V} \to C^0_{A,\overline{V}}\big(X, \mathrm{SL}(2, \mathbb{R})\big)$$

and

$$\Psi = \Psi_{A,V} : \mathcal{W}_{A,V} \to C^0\big(X, \mathrm{SL}(2,\mathbb{R})\big)$$

satisfying (13), (14), and (15).

LEMMA 11
Let $K \subset X$ be a compact set such that $K \cap f(K) = \varnothing$ and $K \cap f^2(K) = \varnothing$. Let $A \in C^r(X, S)$ be such that for every $z \in K$, we have $\operatorname{tr} A(z) \neq 0$. *Then there exist an open neighborhood $\mathcal{W}_{A,K} \subset C^0_{A,K}(X, \mathrm{SL}(2,\mathbb{R}))$ of A and continuous maps*

$$\Phi = \Phi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, S) \qquad \textit{and} \qquad \Psi = \Psi_{A,K} : \mathcal{W}_{A,K} \to C^0\big(X, \mathrm{SL}(2,\mathbb{R})\big)$$

satisfying (13), (14), and (15).

Before proving the two lemmas, let us show how they imply Lemma 9.

Proof of Lemma 9
Let $z \in X$ be such that $\operatorname{tr} A(z) \neq 0$. Let V be an open neighborhood of z such that with $K = \overline{V}$, we have $\operatorname{tr} A(x) \neq 0$ for $x \in K$, $K \cap f(K) = \varnothing$, and $K \cap f^2(K) = \varnothing$. Let $\Phi_{A,V} : \mathcal{W}_{A,V} \to C^0_{A,K}(X, \mathrm{SL}(2,\mathbb{R}))$ and $\Psi_{A,V} : \mathcal{W}_{A,V} \to C^0(X, \mathrm{SL}(2,\mathbb{R}))$ be given by Lemma 10. Let $\Phi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, S)$ and $\Psi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, \mathrm{SL}(2,\mathbb{R}))$ be given by Lemma 11. Let $\mathcal{W}$ be the domain of $\Phi = \Phi_{A,K} \circ \Phi_{A,V}$, and let $\Psi = (\Psi_{A,K} \circ \Phi_{A,V}) \cdot \Psi_{A,V}$. The result follows. □

Proof of Lemma 10
For every $x \in X$, let $y = y_x \in V$ and $n = n_x \geq 0$ be such that $f^n(y) = x$ but $f^{-i}(x) \notin V$ for $0 \leq i \leq n-1$. Let $W = W_x \subset V$ be an open neighborhood of y such that $W \cap f^i(W) = \varnothing$ for $1 \leq i \leq n$. Let $K = K_x \subset W$ be a compact neighborhood of y. Let $U = U_x \subset f^n(K_x)$ be an open neighborhood of x. Let $\phi = \phi_x : W \to [0,1]$ be a C^r-map such that $\phi(z) = 0$ for $z \in W \setminus K$, while $\phi(z) = 1$ for $z \in f^{-n}(U)$.

Define maps $\Phi_x, \Psi_x : \mathcal{W}_x \to C^0(X, \mathrm{SL}(2,\mathbb{R}))$ on some open neighborhood $\mathcal{W}_x$ of A as follows. Let $\Pi : \mathrm{GL}^+(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ be given by $\Pi(M) = (\det M)^{-1/2} M$. Let $\Phi_x(B)(z) = B(z)$ for $z \notin \bigcup_{i=0}^n f^i(W)$, let $\Phi_x(B)(z) = \Pi\big(B(z) + \phi(f^{-j}(z))(A(z) - B(z))\big)$ for $z \in f^j(W)$ and $1 \leq j \leq n$, and let

$$\Phi_x(B)(z) = \big[\Phi_x(B)\big(f(z)\big)\big]^{-1} \cdots \big[\Phi_x(B)\big(f^n(z)\big)\big]^{-1} \cdot B\big(f^n(z)\big) \cdots B(z) \quad \text{for } z \in W.$$

Let $\Psi_x(B)(z) = \mathrm{id}$ for $x \notin \bigcup_{i=1}^n f^i(W)$, and let $\Psi_x(B)(z) = \Phi_x(B)(f^{-1}(z)) \cdots \Phi_x(B)(f^{-j}(z)) \cdot B(f^{-j}(z))^{-1} \cdots B(f^{-1}(z))^{-1}$ for $z \in f^j(W)$ and $1 \leq j \leq n$. Then Φ_x and Ψ_x are continuous and have the following properties:

(a) for every $z \in X$, we have $\Psi_x(B)(f(z)) \cdot B(z) \cdot [\Psi_x(B)(z)]^{-1} = \Phi_x(z)$;
(b) the set $\{z \in X \setminus V;\ \Phi_x(B)(z) = A(z)\}$ contains $\{z \in X \setminus V;\ B(z) = A(z)\} \cup (U_x \setminus V)$;
(c) $\Phi_x(A) = A$ and $\Psi_x(A) = \mathrm{id}$.

Choose a finite sequence $x_1, \dots, x_k$ such that $X = \bigcup_{i=1}^{k} U_{x_i}$. Let $\mathcal{W}_{A,V} \subset C^0(X, \mathrm{SL}(2,\mathbb{R}))$ be an open neighborhood of A such that $\Phi_i = \Phi_{x_i} \circ \cdots \circ \Phi_{x_1}$ is well defined for $1 \le i \le k$. Let $\Psi_i = \Psi_{x_i} \circ \Phi_{i-1}$ for $2 \le i \le k$, and let $\Psi_1 = \Psi_{x_1}$. The result follows with $\Phi = \Phi_k$ and $\Psi = \Psi_k \cdots \Psi_1$. □

Proof of Lemma 11

Let $Z \subset S^3$ be the set of all (B_1, B_2, B_3) such that $\operatorname{tr} B_2 \neq 0$. One easily checks that $(B_1, B_2, B_3) \mapsto B_3 B_2 B_1$ is an analytic diffeomorphism between Z and

$$L = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\ d \neq 0 \right\}.$$

Let $\eta : L \to Z$ be the inverse map.

Let $\Phi(B)(x) = A(x)$ if $z \notin \bigcup_{i=-1}^{1} f^i(K)$, and for $z \in K$, let

$$\big(\Phi(B)(f^{-1}(z)), \Phi(B)(z), \Phi(B)(f(z))\big) = \eta\big(B(f^{-1}(z)), B(z), B(f(z))\big).$$

Let $\Psi(B)(z) = \mathrm{id}$ for $z \notin K \cup f(K)$, let $\Psi(z) = \Phi(B)(f^{-1}(z)) \cdot [B(f^{-1}(z))]^{-1}$ for $z \in K$, and let $\Psi(z) = \Phi(B)(f^{-1}(z)) \cdot \Phi(B)(f^{-2}(z)) \cdot [B(f^{-2}(z))]^{-1} \cdot [B(f^{-1}(z))]^{-1}$ for $z \in f(K)$. All properties are easy to check. □

Remark 10

Let $f : X \to X$ be a homeomorphism of a compact metric space, and let $N \ge n \ge 1$. Let us say that a compact set K is (n, N)-*good* if $K \cap f^k(K) = \emptyset$ for $1 \le k \le n-1$ and $\bigcup_{k=0}^{N-1} f^k(K) = X$. Then Lemma 9 holds under the weaker (than minimality of f) hypothesis that there exist $N \ge 3$ and a $(3, N)$-good compact set K such that $\operatorname{tr} A(x) \neq 0$ for every $x \in K$.

3.2. Dense absolutely continuous spectrum

To prove Theorem 7, we use the following standard result.

THEOREM 8

Let f be a Diophantine translation of the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Then there exists a set $\Theta \subset \mathbb{R}$ of full Lebesgue measure such that if $V \in C^\infty(\mathbb{T}^d, \mathbb{R})$ and $E_0 \in \mathbb{R}$ are such that $(f, A_{E_0,V})$ is C^∞ $\mathrm{PSL}(2,\mathbb{R})$-conjugate to $(f, R_{\pi\theta})$ for some $\theta \in \Theta$, then the associated Schrödinger operator has some absolutely continuous spectrum.

For completeness, we discuss the reduction of this result to the standard KAM theorem in more detail in Appendix B.

Proof of Theorem 7
Let $V \in C^0(\mathbb{T}^d, \mathbb{R})$ be nonconstant, and let E be in the spectrum of the associated Schrödinger operator. By Lemma 7, there exists $\tilde{A} \in C^0(\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$ arbitrarily close to $A_{E,V}$ such that $(f, \tilde{A})$ is conjugate to $(f, R_{2\pi\theta})$ for some $\theta \in \Theta$. Approximating the conjugacy by a C^∞-map, we may assume that $\tilde{A}$ is C^∞. Applying Lemma 9 with $r = \infty$, we find a C^∞-function $\tilde{V}$ which is C^0-close to V such that $(f, A_{E,\tilde{V}})$ is conjugate to (f, A) and hence to $(f, R_{2\pi\theta})$. The result now follows from Theorem 8. □

Appendices

A. Topological groups

We quickly review some material that can be found in [R]. Let G be a topological group. If G is locally compact and abelian, one defines the *dual group* $\Gamma = \hat{G}$; it consists of all characters of G (i.e., continuous homomorphisms $\gamma : G \to \mathbb{T}^1$). Then Γ is an abelian group, and with the suitable topology, it is also locally compact. Some important facts are: (1) G is compact if and only if Γ is discrete; (2) $\hat{\Gamma} = G$ (Pontryagin duality).*

Proof of Lemma 1
Since $\mathbb{G}$ is compact and infinite, the dual group Γ is discrete and infinite.

First, assume that Γ contains an element of infinite order; thus we can assume that $\mathbb{Z}$ is a closed subgroup of Γ. Let $\iota : \mathbb{Z} \to \Gamma$ be the inclusion homomorphism, and let $s : \hat{\Gamma} \to \hat{\mathbb{Z}}$ be its dual.† Then s is onto: every character on $\mathbb{Z}$ can be extended to a character on Γ (see [R, Section 2.1.4]). Since $\hat{\Gamma} = \mathbb{G}$ and $\hat{\mathbb{Z}} = \mathbb{T}^1$, alternative (a) holds.

Now, assume that all elements of Γ are of finite order; then $\mathbb{G}$ is a Cantor set (see [R, Section 2.5.6]). There is a translation $x \mapsto x + \alpha$ of $\mathbb{G}$ which is a factor of the minimal homeomorphism f, and so it is itself minimal. Therefore Γ is a subgroup of $\mathbb{T}^1_d$, the circle group with the discrete topology (see [R, Section 2.3.3]). So there exist cyclic subgroups $\Lambda_i \subset \Gamma$ with $|\Lambda_i| \to \infty$. Let $H_i \subset \mathbb{G}$ be the annihilator (see [R, Section 2.1]) of Λ_i. Then $\mathbb{G}/H_i = \Lambda_i$, and the quotient homomorphism $\mathbb{G} \to \mathbb{G}/H_i$ is continuous. So we are in case (b). □

*Here, "=" means isomorphic and homeomorphic.
†The dual of a continuous homomorphism $h : G_1 \to G_2$ is the continuous homomorphism $\hat{h} : \hat{G}_2 \to \hat{G}_1$ defined by $\langle x_1, \hat{h}(\gamma_2)\rangle = \langle h(x_1), \gamma_2\rangle$, where $x_1 \in G_1$, $\gamma_2 \in \hat{G}_2$.

B. Absolutely continuous spectrum via KAM

In this section, $f : \mathbb{T}^d \to \mathbb{T}^d$ is a minimal translation of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We show how Theorem 8 reduces to a result of [H], based on the usual KAM theorem. We use the following criterion for absolutely continuous spectrum (see, e.g., [S, proof of Theorem 1] for a simple proof).

LEMMA 12
Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a homeomorphism, and let $V \in C^0(\mathbb{T}^d, \mathbb{R})$. If $\Sigma_b = \{E \in \mathbb{R},\ (f, A_{E,V})$ is conjugate to a cocycle of rotations$\}$ has positive Lebesgue measure, then the associated Schrödinger operators have some absolutely continuous spectrum.

Recall that if $A : \mathbb{T}^d \to \mathrm{SL}(2, \mathbb{R})$ is homotopic to a constant, one can define a fibered rotation number $\rho(f, A) \in \mathbb{R}/\mathbb{Z}$ (see [H, Section 5]). The following properties of the fibered rotation number are easy to check.

(a) If A is a constant rotation of angle $\pi\theta$, then $\rho(f, A) = \theta$.

(b) If B is a $\mathrm{PSL}(2, \mathbb{R})$-conjugacy between (f, A) and (f, A'), then $\rho(f, A) = \rho(f, A') + k\alpha$, where $k = k(B) \in \mathbb{Z}$ only depends on the homotopy class of B.

(c) The fibered rotation number $\rho(f, A)$ is a continuous function of $A \in C^0(\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$.

(d) If $A \in C^0(\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$ is $\mathrm{PSL}(2, \mathbb{R})$-conjugate to a constant rotation, then the fibered rotation number (as a function of $C^0(\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$) is K-Lipschitz at A for some $K > 0$.*

In [H], it is shown how the KAM theorem implies reducibility for cocycles close to constant, under a Diophantine assumption on f and on the fibered rotation number. To state it precisely, it is convenient to introduce some notation.

Let $n \geq 1$, and let $\kappa, \tau > 0$. Let $\mathrm{DC}_{n,\kappa,\tau}$ be the set of all $\alpha \in \mathbb{R}^n$ such that there exists $\kappa, \tau > 0$ such that for every $k_0 \in \mathbb{Z}$, $k \in \mathbb{Z}^n \setminus \{0\}$, we have $\left|k_0 + \sum_{i=1}^n k_i\alpha_i\right| > \kappa\left(\sum_{i=1}^n |k_i|\right)^{-\tau}$. Notice that $\mathrm{DC}_{n,\kappa,\tau}$ is $\mathbb{Z}^d$-invariant. Let $\mathrm{DC}_n = \bigcup_{\kappa,\tau>0} \mathrm{DC}_{n,\kappa,\tau}$. We say that f is a *Diophantine translation* if $f(x) = x + \alpha_f$ for some $\alpha_f \in \mathrm{DC}_d$.

For every $\alpha \in \mathbb{R}^d$, let $\Theta_{\alpha,\kappa,\tau}$ be the set of all $\theta \in \mathbb{R}$ such that $(\alpha, \theta) \in \mathrm{DC}_{d+1,\kappa,\tau}$. Notice that $\Theta_{\alpha,\kappa,\tau}$ is $\mathbb{Z}$-invariant. Let $\Theta_\alpha = \bigcup_{\kappa,\tau>0} \Theta_{\alpha,\kappa,\tau}$. Then $\Theta_\alpha = \emptyset$ if $\alpha \notin \mathrm{DC}_d$, and Θ_α has full Lebesgue measure if $\alpha \in \mathrm{DC}_d$. Moreover, every $\theta \in \Theta_\alpha$ is a Lebesgue density point of $\Theta_{\alpha,\kappa,\tau}$ for some $\kappa, \tau > 0$.

*Let $\phi : Y \to Z$ be a function between metric spaces, and let $K > 0$. We say that ϕ is K-Lipschitz at $y \in Y$ if there exists a neighborhood $\mathcal{V} \subset Y$ of y such that for every $z \in \mathcal{V}$, we have $d_Z(\phi(z), \phi(y)) \leq K d_Y(z, y)$.

THEOREM 9 ([H, corollaire 3, remarque 1])
For every $\theta \in \mathbb{R}$ and $\kappa, \tau > 0$, there exists a neighborhood $\mathcal{W} \subset C^\infty(\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$ of R_θ such that if $A \in \mathcal{W}$ and $\rho(f, A) \in \Theta_{\alpha_f,\kappa,\tau}/\mathbb{Z}$, then (f, A) is C^∞-conjugate to a constant rotation.

Proof of Theorem 8
Let $\Theta = \Theta_{\alpha_f}$. Since $\alpha_f \in \mathrm{DC}_d$, Θ has full Lebesgue measure. Let V, E_0, and θ be as in the statement of the theorem. Let $\kappa, \tau > 0$ be such that θ is a Lebesgue density point of $\Theta_{\alpha_f,\kappa,\tau}$.

Let Σ_r be the set of all $E \in \mathbb{R}$ such that $(f, A_{E,V})$ is C^∞-conjugate to a constant rotation.

Let $k \in \mathbb{Z}$ be such that $\rho(f, A_{E_0,V}) = \theta + k\alpha$. By Theorem 9, there exists an open interval I containing E_0 such that if $E' \in I$ and $\rho(f, A_{E',V}) - k\alpha \in \Theta_{\alpha_f,\kappa,\tau}$, then $E' \in \Sigma_r$. Let $\rho : I \to \mathbb{R}/\mathbb{Z}$ be given by $\rho(E) = \rho(f, A_{E,V}) - k\alpha$. If $\rho(I) = \{\theta\}$ (this cannot really happen, but we do not need this fact), then $I \subset \Sigma_r$. Otherwise, by continuity of the fibered rotation number, $\rho(I \cap \Sigma_r) \supset \rho(I) \cap \Theta_{\alpha,\kappa,\tau}/\mathbb{Z}$ has positive Lebesgue measure. Since ρ is $K(E)$-Lipschitz at every $E \in \Sigma_r$, we conclude in any case that Σ_r has positive Lebesgue measure. The result follows by Lemma 12. □

Acknowledgments. We thank the referees for several suggestions that improved the exposition, and we thank the Conselho Nacional de Desenvolvimento Científico e Tecnológico and the Programa Nacional de Cooperação Acadêmica–Brazil for travel assistance.

References

[AB] A. AVILA and J. BOCHI, *A uniform dichotomy for generic* SL(2, $\mathbb{R}$) *cocycles over a minimal base*, Bull. Soc. Math. France **135** (2007), 407–417. MR 2430187

[AD] A. AVILA and D. DAMANIK, *Generic singular spectrum for ergodic Schrödinger operators*, Duke Math. J. **130** (2005), 393–400. MR 2181094 269 259

[B] J. BOCHI, *Genericity of zero Lyapunov exponents*, Ergodic Theory Dynam. Systems **22** (2002), 1667–1696. MR 1944399 259, 267, 269

[BD] M. BOSHERNITZAN and D. DAMANIK, *Generic continuous spectrum for ergodic Schrödinger operators*, Comm. Math. Phys. **283** (2008), 647–662. MR 2434741 258

[Bo1] J. BOURGAIN, "Positive Lyapounov exponents for most energies" in *Geometric Aspects of Functional Analysis*, Lecture Notes in Math. **1745**, Springer, Berlin, 2000, 37–66. MR 1797971 258

[Bo2] ———, *On the spectrum of lattice Schrödinger operators with deterministic potential*, J. Anal. Math. **87** (2002), 37–75. MR 1945277 258

[Bo3] ———, *Green's Function Estimates for Lattice Schrödinger Operators and Applications*, Ann. of Math. Stud. **158**, Princeton Univ. Press, Princeton, 2005. MR 2100420 258

[BGS] J. BOURGAIN, M. GOLDSTEIN, and W. SCHLAG, *Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by the skew-shift*, Comm. Math. Phys. **220** (2001), 583 – 621. MR 1843776 258

[C] R. H. CAMERON, *Almost periodic properties of bounded solutions of linear differential equations with almost periodic coefficients*, J. Math. Phys. **15** (1936), 73 – 81. 254

[D] N. DINH CONG, *A generic bounded linear cocycle has simple Lyapunov spectrum*, Ergodic Theory Dynam. Systems **25** (2005), 1775 – 1797. MR 2183293 256

[DF] N. DINH CONG and R. FABBRI, *On the spectrum of the one-dimensional Schrödinger operator*, Discrete Contin. Dyn. Syst. Ser. B **9** (2008), 541 – 554. MR 2379426 258

[EJ] R. ELLIS and R. A. JOHNSON, *Topological dynamics and linear differential systems*, J. Differential Equations **44** (1982), 21 – 39. MR 0651685 254

[FJ] R. FABBRI and R. JOHNSON, *Genericity of exponential dichotomy for two-dimensional differential systems*, Ann. Mat. Pura Appl. (4) **178** (2000), 175 – 193. MR 1849385 256

[FJP] R. FABBRI, R. JOHNSON, and R. PAVANI, *On the nature of the spectrum of the quasi-periodic Schrödinger operator*, Nonlinear Anal. Real World Appl. **3** (2002), 37 – 59. MR 1941947 258

[F] A. FURMAN, *On the multiplicative ergodic theorem for uniquely ergodic systems*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), 797 – 815. MR 1484541 259, 267, 269

[GJ] A. GORDON and S. JITOMIRSKAYA, in preparation. 258

[H] M.-R. HERMAN, *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension* 2, Comment. Math. Helv. **58** (1983), 453 – 502. MR 0727713 277, 278

[J] R. A. JOHNSON, *Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients*, J. Differential Equations **61** (1986), 54 – 78. MR 0818861 257

[KSV] O. KOZLOVSKI, W. SHEN, and S. VAN STRIEN, *Density of hyperbolicity in dimension one*, Ann. of Math. (2) **166** (2007), 145 – 182. MR 2342693 256

[L] D. LENZ, *Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals*, Comm. Math. Phys. **227** (2002), 119 – 130. MR 1903841 257

[R] W. RUDIN, *Fourier Analysis on Groups*, Interscience Tracts in Pure and Applied Mathematics **12**, Interscience, New York, 1962. MR 0152834 276

[S] B. SIMON, *Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators*, Proc. Amer. Math. Soc. **124** (1996), 3361 – 3369. MR 1350963 277

[Y] J.-C. YOCCOZ, "Some questions and remarks about SL(2, $\mathbb{R}$) cocycles" in *Modern Dynamical Systems and Applications*, Cambridge Univ. Press, Cambridge, 2004, 447 – 458. MR 2093316 254

Avila
Centre National de la Recherche Scientifique, Laboratoire de Probabilités et Modèles Aléatoires, 75252 Paris CEDEX 05, France; Instituto Nacional de Matemática Pura e Aplicada, 22460-320 Rio de Janeiro, RJ, Brazil; artur@math.sunysb.edu

Bochi
Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, 22453-900 Rio de Janeiro, RJ, Brazil; jairo@mat.puc-rio.br

Damanik
Department of Mathematics, Rice University, Houston, Texas 77005, USA; damanik@rice.edu