Discontinuity of the Lyapunov exponent for non-hyperbolic cocycles

Jairo Bochi

December 7, 1999

Abstract

For fixed ergodic dynamical systems over a compact space, we show that there is a residual set of continuous $SL(2,\mathbb{R})$-cocycles which are either hyperbolic or have Lyapunov exponent zero.

1. Introduction

Let $T : (X, \mu) \hookleftarrow$ be an ergodic automorphism of a standard probability space. To a given bounded measurable mapping $A : X \to SL(2,\mathbb{R})$, which we call a cocycle, we can associate its upper Lyapunov exponent

$$\Lambda(A) = \lim_{n\to+\infty} \frac{1}{n} \log \left\| A(T^{n-1}x) \cdots A(x) \right\| \geq 0.$$

We are interested in continuity (or discontinuity) properties of the function $\Lambda : L^\infty(X, SL(2,\mathbb{R})) \to [0,\infty)$, where the transformation T is kept fixed. We consider also the situation where X is a compact space X. Then we study the function $\Lambda : C(X, SL(2,\mathbb{R})) \to [0,\infty)$ over the continuous cocycles.

It is easy to see that Λ is upper-semicontinuous. However, Knill [K] showed that $\Lambda : L^\infty(X, SL(2,\mathbb{R})) \to \mathbb{R}$ is never continuous if T is aperiodic (i.e., periodic points have measure zero). Actually, he proves that the subset of $L^\infty(X, SL(2,\mathbb{R}))$ consisting of the cocycles with positive exponent is not open, thus Λ can drop to zero for small perturbations. An example a similar situation is constructed in [T], where perturbations with small exponent are obtained by multiplying a cocycle by some constant matrices.

On the other hand, if a cocycle is hyperbolic (meaning that the product matrices grow uniformly) then it has positive exponent by definition and this positivity is robust by perturbations. Moreover, as Ruelle proved [R2], the function Λ is even real-analytic (in Banach-algebra sense) in the open set of hyperbolic cocycles.

Our main result is the following: *If T his ergodic then the set of cocycles that are either hyperbolic or have zero Lyapunov exponent is a residual set in $L^\infty(X, SL(2,\mathbb{R}))$ or $C(X, SL(2,\mathbb{R}))$, according to the case considered.* In other words, if a cocycle with positive exponent is not hyperbolic then its exponent can drop to zero if we perturb the cocycle. The techniques used in the proof are similar to those of [K]. The basic idea for vanishing the exponent is to exchange the expanding and contracting directions, noticing that this can be done in the absence of hyperbolicity.

This fact leads us to ask whether or not do non-hyperbolic cocycles with positive exponent exist for a given continuous dynamical system (X, T, μ). The answer is easily seem (see section 6) to be "yes" if T is not uniquely ergodic and is unknown in the general case. Though, if T is an irrational translation of the torus $\mathbb{T}^n$, $n \geq 1$, the answer is positive, due to Herman's examples in [H], §4. The papers [W] and [F] deal with cocycles over uniquely ergodic transformations.

We remark that the theorem does not extend to the C^1 topology. Young exhibits [Y] open subsets of $C^1(X, SL(2,\mathbb{R}))$ made up of non-hyperbolic cocycles with positive exponent, where the base transformations are automorphisms of the two-torus.

Our results should be compared to Mañé's claim that in the space of C^1 area preserving diffeomorfisms of a compact surface there is a residual subset consisting of diffeomorfisms that either are Anosov or have zero Lyapunov exponent. The posthumous article [M] is a sketch of a possible proof. Clearly Mañé's considerations are more delicate than ours.

It is a common belief that positive exponents are prevalent even among non-hyperbolic systems, but there is no general theorem in this direction. Therefore our set of cocycles with exponent zero, although residual on the non-hyperbolic part, is probably "thin" in some measure sense. The extreme discontinuity of the Lyapunov exponent must be one of the reasons why it is hard to prove that specific systems have positive exponent.

2. Preliminaries

Let (X, μ) be a Lebesgue probability space and $T : X \hookleftarrow$ an automorphism of it. Denote

$$L^\infty_M = \{A : X \to SL(2,\mathbb{R}) \text{ measurable; } \|A\| \text{ is essencially bounded}\}$$

$$\text{and } \|A\|_\infty = \operatorname{ess\,sup} \|Ax\| .$$

Given $A \in L^\infty_M$, we denote

$$A^n(x) = \begin{cases} A(T^{n-1}x) \cdots A(x) & \text{if } n > 0 \\ I & \text{if } n = 0 \\ A^{-1}(T^n x) \cdots A^{-1}(T^{-1}x) & \text{if } n > 0 \end{cases}$$

Oseledec theorem states that the limit

$$\lambda(A, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|$$

exists for μ-a.e. $x \in X$. (It does not depend on x if T is ergodic). We define the Lyapunov exponent as

$$\Lambda(A) = \int_X \lambda(A, x) d\mu(x)$$

(Maybe a better name would be "integrated LE". Besides, one usually says that there are two exponents, $\lambda(A,x)$ and $-\lambda(A,x)$.) If $\Lambda(A) > 0$ then there exists a splitting $\mathbb{R}^2 = E^u(x) \oplus E^s(x)$, where the spaces E^u and E^s are one-dimensional and depend measurably on x, such that for μ-a.e. $x \in X$ and for $v \in \mathbb{R}^2 - \{0\}$,

$$\begin{aligned}
\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x).v\| &= \begin{cases} \Lambda(A) \text{ if } v \notin E^u(x) \\ -\Lambda(A) \text{ if } v \in E^u(x) \end{cases}, \\
\lim_{n\to+\infty} \frac{1}{n} \log \|A^{-n}(x).v\| &= \begin{cases} \Lambda(A) \text{ if } v \notin E^s(x) \\ -\Lambda(A) \text{ if } v \in E^s(x) \end{cases},
\end{aligned}$$

$$\lim_{n\to+\infty} \frac{1}{n} \log \sin \measuredangle(E^u(T^n(x)), E^s(T^n(x))) = 0.$$

Now we will define the notion of hyperbolicity for cocycles.

Definition 1. *A cocycle $A \in L^\infty_M$ over $T : (X,\mu) \hookleftarrow$ is called* hyperbolic *if the two conditions below hold:*

1. *Uniform growth of the products: there exist constants $C > 0$ and $\tau > 1$ such that $\|A^n(x)\| > C\tau^n$ for every $n > 0$ and a.e. $x \in X$.*
2. *Bounded angles: there exist $\delta > 0$ such that $\measuredangle(E^u(x), E^s(x)) > \delta$ for a.e. $x \in X$.*

Remark. The first condition implies $\Lambda(A) > 0$, hence the second one makes sense.

Remark. One can show that the first condition does not imply the second one.

Our main result is

Theorem 2.1. *If T is ergodic then the set of the cocycles $A \in L^\infty_M$ such that either A is hyperbolic or $\Lambda(A) = 0$ is residual in L^∞_M.*

From the above theorem we will deduce its continuous version. Now we suppose that X is a compact Hausdorff space and μ is a regular Borel measure on X which is invariant for T. (T is not assumed to be continuous). In this setting, we denote

$$C_M = \{A : X \to SL(2,\mathbb{R}) \text{ continuous}\}.$$

Theorem 2.2. *Let X, T, μ be as above. If T is ergodic then the set of the cocycles $A \in C_M$ such that either A is hyperbolic or $\Lambda(A) = 0$ is residual in C_M.*

We remark that the sequence $a_n = \int \log \|A^n\| \, d\mu$ is subadditive ($a_{n+m} \le a_n + a_m$). Therefore the integrated Lyapunov exponent satisfy the formula

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \int_X \log \|A^n\| \, d\mu = \inf_n \frac{1}{n} \int_X \log \|A^n\| \, d\mu.$$

The following lemma, adapted from [T], gives a property of upper semi-continuity of Λ and will be essential in the proofs of theorems 1 and 2.

Lemma 1. *Given $A \in L_M^\infty$, $\varepsilon > 0$ and $M > 0$ there exists $\delta > 0$ such that*

$$\|B\|_\infty \le M, \int_X \|B(x) - A(x)\| \, d\mu(x) < \delta \Rightarrow \Lambda(B) < \Lambda(A) + \varepsilon.$$

Proof. Let n be such that

$$\frac{1}{n} \int_X \log \|A^n(x)\| \, d\mu(x) < \Lambda(A) + \varepsilon.$$

For given $\nu > 0$, we define the sets $R = \{x \in X; \|B(x) - A(x)\| > \nu\}$ and $S = R \cup T^{-1}R \cup \cdots \cup T^{-n+1}R$. We have $\mu(R) \le \frac{\delta}{\nu}$ and $\mu(S) \le \frac{n\delta}{\nu}$. Without loss of generality, we can assume that $M \ge \|A\|_\infty$. For $x \notin S$, we claim that

$$\|B^n(x) - A^n(x)\| \le n\nu M^{n-1}.$$

This fact can be proved by induction as following:

$$\begin{aligned} \left\|B^{j+1}(x) - A^{j+1}(x)\right\| &\le \left\|B(T^j x).(B^j(x) - A^j(x))\right\| + \left\|(B(T^j x) - A(T^j x))A^j(x)\right\| \le \\ &\le M.j\nu M^{j-1} + \nu.M^j = (j+1)\nu M^j. \end{aligned}$$

In particular, for $x \notin S$,

$$\frac{\|B^n x\|}{\|A^n x\|} \le 1 + \frac{\|B^n x - A^n x\|}{\|A^n x\|} \le 1 + \|B^n x - A^n x\| \le 1 + n\nu M^{n-1}.$$

Choose $\nu > 0$ such that $\log\left(1 + n\nu M^{n-1}\right) < \varepsilon$ and then choose $\delta > 0$ such that $\mu(S) < \frac{\varepsilon}{\log M}$. Hence we have

$$\begin{aligned} \Lambda(B) &\le \frac{1}{n} \int_X \log \|B^n\| \, d\mu = \frac{1}{n} \int_{X-S} \log \|B^n\| \, d\mu + \frac{1}{n} \int_S \log \|B^n\| \, d\mu \le \\ &\le \frac{1}{n} \int_{X-S} (\log \|A^n\| + \varepsilon) d\mu + \frac{1}{n} \mu(S) \log M^n < \\ &< \Lambda(A) + \varepsilon + \frac{\varepsilon}{n} + \varepsilon. \blacksquare \end{aligned}$$

For any two by two real matrix $B = (b_{ij})$, we denote

$$\|B\|_{\max} = \max\{|b_{11}|, |b_{12}|, |b_{21}|, |b_{22}|\}.$$

Let $K > 1$ be such that $K^{-1}\|B\| \le \|B\|_{\max} \le K\|B\|$ for every B. As always, $\|\cdot\|$ denotes the usual operator norm, induced by the euclidean norm in $\mathbb{R}^2$.

Lemma 2. *If $\Lambda(A) > 0$ then there exists a measurable conjugacy $C : X \to SL(2, \mathbb{R})$ satisfying $\lim\limits_{n \to \pm\infty} \frac{1}{|n|} \log \|C(T^n x)\| = 0$ (we call that such a C* tempered*) such that the matrix $D(x) = C(Tx)^{-1} A(x) C(x)$ is diagonal. Moreover, if the angle between E_A^u and E_A^s is bounded from zero, then C can be chosen in L_M^∞.*

Proof. Let $w^u(x) \in E^u(x), w^s(x) \in E^s(x)$ be unitary vectors such that $\{w^u(x), w^s(x)\}$ is a positive basis of $\mathbb{R}^2$. Let $\widetilde{C}(x) : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $\widetilde{C}(x).(1,0) = w^u(x)$ and $\widetilde{C}(x).(0,1) = w^s(x)$. Hence $\widetilde{C}(x)$ has positive determinant, $C(x) = \left(\det \widetilde{C}(x)\right)^{-1/2} \widetilde{C}(x)$ has determinant 1 and $C(Tx)^{-1}A(x)C(x)$ is a diagonal matrix. In order to estimate $\|C(x)\|$, we can suppose $w^u(x) = (1,0)$. Hence, if $0 < \theta < \pi$ denotes the angle between $w^u(x)$ and $w^s(x)$, we have

$$\widetilde{C} = \begin{pmatrix} 1 & \cos\theta \\ 0 & \operatorname{sen}\theta \end{pmatrix} \text{ and } C = \begin{pmatrix} \operatorname{sen}^{-1/2}\theta & \cos\theta \operatorname{sen}^{-1/2}\theta \\ 0 & \operatorname{sen}^{1/2}\theta \end{pmatrix}.$$

Thus $\|C\| \le K \|C\|_{\max} = K \operatorname{sen}^{-1/2}\theta$. Oseledec theorem informs that

$$\frac{-1}{n} \log \sin \theta(T^n x) \to 0$$

and the proof is finished. ■

Lemma 3. *If T is an aperiodic invertible transformation, U is a measurable set with $\mu(U) > 0$ and $n \ge 1$, then there exists $V \subset U$ with $\mu(V) > 0$ and such that $V, TV, ..., T^{n-1}V$ are disjoint sets.*

Proof. It follows from Rokhlin-Kakutani lemma.

3. The main lemma (lemma 5)

Definition 2. *A measurable set $Z \subset X$ is called* a coboundary *if there exists a measurable set $W \subset X$ such that $Z = W \Delta TW$. (Δ denotes symmetric difference, $=$ means differing by a measure zero set).*

Lemma 4. *Given a set $F \subset X$ with positive measure, there exists a set $Z \subset F$ with positive measure which is not a coboundary.*

Proof. See [K].

Remark. Here the assumption that X is a Lebesgue space is needed.

The following lemma, which is essentially due to Knill[K], will be the basic tool in the proof of theorem 2.1.

Lemma 5. *Suppose T is ergodic. Let $Z \subset X$ be a positive measure set which is not a coboundary. Suppose that $n \ge 1$ is such that $Z, TZ, ..., T^{n-1}Z$ are disjoint. Take cocycles $A \in L^\infty_M$ with $\Lambda(A) > 0$ and $J \in L^\infty_M$ equal to I in the complementary of $Z \cup TZ \cup \cdots \cup T^{n-1}Z$. Suppose that $(AJ)^n(x).E^s_A(x) = E^u_A(x)$ and $(AJ)^n(x).E^u_A(x) = E^s_A(x)$ $\forall x \in Z$. Then $\Lambda(AJ) = 0$.*

Proof. We analyze two cases separately.

3.1. First case: $n = 1$

First we need to make some general considerations. One can define the skew-product

$$T \times A : X \times \mathbb{P}^1 \hookleftarrow$$

as

$$(T \times A)(x, \overline{v}) = (T(x), \overline{A(x).v}).$$

If $\Lambda(A) > 0$ then there are two measures μ^u and μ^s that are invariant for $T \times A$, given by

$$\mu^{u,s}(B) = \mu \{x \in X; (x, E^{u,s}(x)) \in B\}$$

If $\pi : X \times \mathbb{P}^1 \to X$ denotes the obvious projection then $\pi_*(\mu^u) = \pi_*(\mu^s) = \mu$ and we say that μ^u and μ^s project on μ.

Claim 1. If μ is ergodic and $\Lambda(A) > 0$ then there are only two ergodic measures for $T \times A$ which project on μ, namely μ^u and μ^s.

Proof. Let η be an ergodic measure for $T \times A$ which projects on μ. Let us define a function $f : X \times \mathbb{P}^1 \to \mathbb{R}$ by

$$f(x, \overline{v}) = \log \frac{\|A(x).v\|}{\|v\|}$$

For μ-a.e. $x \in X$ and all $v \in \mathbb{R}^2 - \{0\}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ (T \times A)^j (x, \overline{v}) = \begin{cases} \Lambda(A) & \text{if } v \notin E^s(x) \\ -\Lambda(A) & \text{if } v \in E^s(x) \end{cases}$$

Therefore, by Birkoff's theorem,

$$\eta \left\{(x, \overline{v}) \in X \times \mathbb{P}^1; v \notin E^s(x)\right\} = 0 \text{ or } 1.$$

By the same reasoning,

$$\eta \left\{(x, \overline{v}) \in X \times \mathbb{P}^1; v \notin E^u(x)\right\} = 0 \text{ or } 1.$$

Thus the only possibilities are $\eta = \mu^s$ or $\eta = \mu^u$ and the claim is proved.

We now return to the proof of the main lemma. The skew-product $T \times AJ$ has the invariant measure

$$\widehat{\mu} = \frac{1}{2}(\mu^u + \mu^s).$$

Claim 2. $\widehat{\mu}$ is an ergodic measure for $T \times AJ$.

Proof. Assume that there exists a measurable set $Q \subset X \times \mathbb{P}^1$ with $0 < \widehat{\mu}(Q) < 1$ witch is invariant for $T \times AJ$. For each $x \in X$, denote $Q_x = \{\overline{v} \in \mathbb{P}^1; (x, \overline{v}) \in Q\}$. By definition of $\widehat{\mu}$ we can suppose that $Q_x \subset \{E^u(x), E^s(x)\}$ for every x. Since $\pi(Q)$ is T-invariant, we have $Q_x \neq \emptyset$. Further, the T-invariant set $\{x; Q_x = \{E^u(x), E^s(x)\}\}$ must have μ-measure zero. To simplify notations, let us write $Q_x = u$ or s, with the obvious meanings. Let

$$W = \{x \in X; Q_x = u\}.$$

Then

$$W\Delta T^{-1}W = \{x \in X; Q_x = u,\ Q_{Tx} = s\} \cup \{x \in X; Q_x = s,\ Q_{Tx} = u\} = Z.$$

This contradicts the assumption that Z is not a coboundary.

Finally, if we had $\Lambda(AJ) > 0$ then $\widehat{\mu}$ would be a measure of the type given by the first claim. This is clearly impossible.

3.2. Second case: $n > 1$

We will reduce this case to the first one, but again some background information is needed. Given a measurable set, one can define the induced first-return system (U, T_U, μ_U) as follows

$$x \in U \Rightarrow r(x) = \min\{n \geq 1; T^n x \in U\} \text{ and } T_U(x) = T^{r(x)}x,$$

$$V \subset U \Rightarrow \mu_U(V) = \frac{\mu(V)}{\mu(U)}.$$

We can also define a induced cocycle A_U over this system as

$$x \in U \Rightarrow A_U(x) = A^{r(x)}(x).$$

Claims. The system (U, T_U, μ_U) is ergodic and

$$\Lambda(A_U) = \frac{\Lambda(A_U)}{\mu(U)}.$$

Proofs. See [K], lemma 2.2.

Returning to the proof, let $U = X - (TZ \cup \cdots \cup T^{n-1}Z) \supset Z$

Claim. Z is not a coboundary for (U, T_U, μ_U).

Proof. See [K], lemma 3.4.

We have

$$x \in U \Rightarrow (AJ)_U(x) = \begin{cases} (AJ)^n(x) & \text{if } x \in Z \\ A(x) & \text{otherwise} \end{cases}$$

Hence the first case guarantees that $\Lambda((AJ)_U) = 0$ and therefore $\Lambda(AJ) = 0$.

4. Proof of theorem 1

By lemma 1, $\forall\lambda > 0$, the set $\{A \in L^\infty_M; \Lambda(A) < \lambda\}$ is open. We will then prove that the set $\{A \in L^\infty_M; \Lambda(A) = 0\}$ is dense in the complementary set of hyperbolic cocycles. We can suppose that T is aperiodic.

Take $A \in L^\infty_M$ non-hyperbolic with $\Lambda(A) > 0$. Given $\varepsilon > 0$, we will construct Z and J as is lemma 5 and with $\|J - I\|_\infty < \varepsilon$. The wanted perturbation with zero exponent will be AJ. Denote by $E^u(x) \oplus E^s(x)$ the Oseledec splitting associated to A. The proof now bifurcates in two cases, depending on whether or not the angles $\measuredangle(E^u(x), E^s(x))$ are essentially bounded from below.

4.1. First case: Bounded angles

By lemma 2, we can assume that A is diagonal, with $E^u(x) = e_1$ and $E^s(x) = e_2$, where we denote by e_1, e_2 the directions associated to the vectors $(1,0)$ and $(0,1)$ of $\mathbb{R}^2$.

The idea to exchange the directions is: Choose a place where the expansion is weak for some amount of time. At first, take J as to displace $E^u = e_1$ but keeping e_2 fixed. Now we have to "row against the tide", thus we set J to be an hyperbolic matrix expanding e_2 and contracting e_1. We do this until the displaced direction gets close to e_2 and then, using a rotation, we send it to e_2. Now $E^s = e_2$ was displaced, but we wait until it drifts back near e_1. Eventually, we send this direction to e_1 keeping e_2 fixed.

Let $\rho(x)$ be such that

$$A(x) = \begin{pmatrix} \rho(x) & 0 \\ 0 & \rho(x)^{-1} \end{pmatrix}.$$

We can suppose that $\rho(x) > 0$. Since A is not hyperbolic, there are integers k arbitrarily large such that the set

$$F = \left\{ x \in X; \left\| A^k(x) \right\| < (1+\varepsilon)^{k/2} \right\}$$

has positive measure. Fix some k such that $(1+\varepsilon)^{1-k/2} < \varepsilon$ and $\mu(F) > 0$.

Since $\frac{1}{n} \log \left\| A^n(x)|_{E^u(x)} \right\|$ converges to $\Lambda(A)$ a.e. when $n \to \infty$, convergence in measure also holds. Therefore if n is large enough then

$$\mu \left\{ x \in X; \frac{1}{n} \log \left\| A^n(x)|_{E^u(x)} \right\| > \Lambda(A) - \varepsilon \right\} > 1 - \mu(F).$$

Hence the set

$$\begin{aligned} F' &= F \cap \left\{ x \in X; \frac{1}{n} \log \left\| A^n(x)|_{E^u(x)} \right\| > \Lambda(A) - \varepsilon \right\} = \\ &= \left\{ x \in X; \left\| A^k(x) \right\| < (1+\varepsilon)^{k/2} \text{ and } \left\| A^n(x)|_{E^u(x)} \right\| > e^{n(\Lambda(A)-\varepsilon)} \right\} \end{aligned}$$

has positive measure. By lemma 3, there is $F'' \subset F'$ with positive measure and such that $F'', TF'', \ldots, T^n F''$ are disjoint. Finally, take a set $Z \subset F''$ with positive measure that is not a coboundary.

Now let's construct J. Let $x \in Z$. Define $J(x) = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$. Denoting

$$D[a] = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

set $J(T^i x) = D[(1+\varepsilon)^{-1}]$ for $1 \le i < j(x) \le k$. We have

$$(AJ)^{j(x)} = D[(1+\varepsilon)^{-j(x)+1} \rho(x) \ldots \rho(T^{j(x)-1} x)] \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}.$$

We want $(AJ)^{j(x)}(x).(1,0)$ to fall in the cone $\{(x,y); \frac{|x|}{|y|} < \varepsilon\}$. For this we need

$$(1+\varepsilon)^{-j(x)+1}\rho(x)...\rho(T^{j(x)-1}x) < \varepsilon.$$

Since $x \in F$,

$$(1+\varepsilon)^{-k+1}\rho(x)...\rho(T^{k-1}x) < (1+\varepsilon)^{-k+1}(1+\varepsilon)^{k/2} = (1+\varepsilon)^{1-k/2} < \varepsilon$$

and therefore we can take

$$j(x) = \min\{j; (1+\varepsilon)^{-j(x)+1}\rho(x)...\rho(T^{j(x)-1}x) < \varepsilon\} \leq k.$$

Denote

$$R[a] = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}$$

and set $J(T^{j(x)}x) = R[\alpha(x)]$, where $\sin\alpha(x) < \varepsilon$ is chosen so that

$$J(T^{j(x)}x).(AJ)^{j(x)}(x).e_1 = e_2.$$

We will also require that $\sin\alpha(x) \geq \frac{\varepsilon}{2}$. This can be done by weakening $J(T^{j(x)-1}x)$, if needed.

Now set $J(T^i x) = I$ for $j(x) < i < m(x)$. We want

$$A^{m(x)-j(x)}(T^{j(x)}x).(-\sin\alpha(x), \cos\alpha(x))$$

to fall in the cone $\{(x,y); \frac{|y|}{|x|} < \varepsilon\}$, that is, we want that

$$\left[\rho(T^{j(x)}x)...\rho(T^{m(x)-1}x)\right]^{-2} < \varepsilon \tan\alpha(x).$$

Since

$$\rho(T^{j(x)}x)...\rho(T^{n-1}x) > \frac{e^{n(\Lambda(A)-\varepsilon)}}{(\sup\rho)^k},$$

if n is chosen large enough the desired $m(x)$ will exist and will be less than n.

At the end, set $J(T^{m(x)}x) = \begin{pmatrix} 1 & 0 \\ \beta(x) & 1 \end{pmatrix}$, where $\beta(x)$ is chosen so that $(AJ)^{m(x)}(x)e_2 = e_1$.

All the $J(\cdot)$ matrices employed have distance to I less than, say, 10ε.

4.2. Second case: Unbounded angles

The idea now is: Choose a place where E^s and E^u are close together and then take for J a rotation sending E^u to E^s. Then E^s will be displaced and (setting $J = I$) we wait until this displaced E^s is sent close to E^u (and much closer to E^u than E^s is). Then we take J as to send this direction to E^u and fixing the direction E^s.

Denote $\psi(x) = \measuredangle(E^u(x), E^s(x))$. We have $0 < \psi(x) < \pi$. We can assume that the set $F = \{x \in X; \psi(x) < \varepsilon\}$ has positive measure. By Oseledec theorem,

$$\frac{-1}{n}\log\sin\psi(T^n x) \to 0 \text{ a.e. when } n \to \infty,$$

therefore convergence in measure also holds. The same applies to

$$\frac{1}{n}\log\frac{\left\|A^n(x)|_{E^u(x)}\right\|}{\left\|A^n(x)|_{E^s(x)}\right\|}\to 2\Lambda(A)$$

Hence the measure of the set

$$F_n=\left\{x\in X;\frac{-1}{n}\log\sin\psi(T^nx)>\varepsilon \text{ and } \frac{1}{n}\log\frac{\left\|A^n(x)|_{E^u(x)}\right\|}{\left\|A^n(x)|_{E^s(x)}\right\|}>2\Lambda(A)-\varepsilon\right\}$$

tends to zero when $n\to\infty$. We fix some n satisfying

$$e^{\varepsilon n}\left(\tfrac{1}{2}e^{(2\Lambda(A)-\varepsilon)n}-1\right)^{-1}<\varepsilon \text{ and } \mu(F_n)>1-\mu(F).$$

(Of course, we can assume that $\varepsilon<\Lambda(A)$.) Thus the set $F'=F\cap F_n$ has positive measure. By lemma 3, there exist $F''\subset F'$ of positive measure such that $F'',TF'',...,T^{n+1}F''$ are disjoint. Eventually, we take a set $Z\subset F''$ of positive measure which is not a coboundary.

Now let's construct J. We can assume that $E^u(x)=e_1$ $\forall x\in X$. Let $x\in Z$. Set $J(x)=R[\psi(x)]$, a rotation through angle $\psi(x)$. Then set $J(T^ix)=I$ para $1\le i<m(x)$. We want the direction $A^{m(x)-1}(x).J(x).E^s(x)$ to get very close to e_1, actually, much closer to e_1 than $E^s(T^{m(x)-1}x)$ is. Moreover, we want that $m(x)\le n$. At the end we set $J(T^{m(x)}x)$ as a parabolic matrix that fixes $E^s(T^{m(x)-1}x)$ and sends $A^{m(x)-1}(x).J(x).E^s(x)$ to $E^u(T^{m(x)-1}x)$. To this aim we will need the following estimate

Claim. Let $0<\phi<\pi$, $\theta\ll\min\{\phi,\pi-\phi\}$ and $u,v,w\in\mathbb{R}^2-\{0\}$ be such that

$$\measuredangle(u,v)=\phi,\quad \measuredangle(v,w)=\theta,\quad \measuredangle(u,w)=\theta+\phi.$$

Then the matrix J with $\det J=1$ that fixes u and sends w to a multiple of v satisfies

$$\|J-I\|<K_1\frac{|\tan\theta|}{\sin^2\phi},$$

where K_1 is a constant.

Proof of the claim. Replacing, if necessary, u by $-u$, we can suppose that $\phi\le\frac{\pi}{2}$. We can also suppose that

$$u=(1,0),\quad v=(\cos\phi,\sin\phi),\quad w=(\cos(\phi+\theta),\sin(\phi+\theta)).$$

Thus the matrix is

$$J=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$$

where

$$b=\operatorname{cotg}\phi-\operatorname{cotg}(\phi+\theta)<\frac{\theta}{\sin^2\phi}<\frac{|\tan\theta|}{\sin^2\phi}$$

and the claim follows.

Now denote

$$\psi_j = \psi(T^j x),$$
$$w_0 = J(x).E^s(x) = (\cos 2\psi_0, \sin 2\psi_0),$$
$$w_j = A^j(x).w_0 = w_j^u(1,0) + w_j^s(\cos\psi_j, \sin\psi_j),$$

and

$$\theta_j = \measuredangle(w_j, (1,0)).$$

We have

$$w_0^s = \frac{\sin 2\psi_0}{\sin\psi_0} < 2 \text{ and } w_0^u = w_0^s \cos\psi_0 - \cos 2\psi_0 = -1.$$

Also

$$\begin{aligned}
|\tan\theta_j| &= \frac{|w_j^s| \sin\psi_j}{|w_j^u + w_j^s \cos\psi_j|} \leq \\
&\leq (\sin\psi_j)\left(\frac{|w_j^u|}{|w_j^s|} - 1\right)^{-1} = \\
&= (\sin\psi_j)\left(\frac{\left\|A^j(x)|_{E^u(x)}\right\|}{\left\|A^j(x)|_{E^s(x)}\right\|}\frac{|w_0^u|}{|w_0^s|} - 1\right)^{-1} < \\
&< (\sin\psi_j)\left(\frac{1}{2}\frac{\left\|A^j(x)|_{E^u(x)}\right\|}{\left\|A^j(x)|_{E^s(x)}\right\|} - 1\right)^{-1}.
\end{aligned}$$

For $j = n$ we have

$$\begin{aligned}
\frac{|\tan\theta_n|}{\sin^2\psi_n} &\leq (\sin\psi_n)^{-1}\left(\frac{1}{2}\frac{\left\|A^n(x)|_{E^u(x)}\right\|}{\left\|A^n(x)|_{E^s(x)}\right\|} - 1\right)^{-1} < \\
&< e^{\varepsilon n}\left(\tfrac{1}{2}e^{(2\Lambda(A)-\varepsilon)n} - 1\right)^{-1} < \varepsilon.
\end{aligned}$$

This guarantees that if we set

$$m(x) = \min\left\{j; \frac{|\tan\theta_j|}{\sin^2\psi_j} < \varepsilon\right\}$$

then $m(x) \leq n$ and we can find the wanted parabolic matrix $J(T^{m(x)}x)$ with

$$\left\|J(T^{m(x)}) - I\right\| < K_1\varepsilon. \blacksquare$$

5. Proof of theorem 2

By lemma 1, the set

$$E_\lambda = \{A \in C_M; \Lambda(A) < \lambda\}$$

is open $\forall \lambda > 0$. It remains to show that it is dense. Take $A \in C_M$ and $\varepsilon > 0$. By theorem 2.1, there is $\widetilde{A} \in L^\infty_M$ near A with $\Lambda(\widetilde{A}) = 0$. Write $\widetilde{A} = A.(I + J)$ with

$$J \in L^\infty(X, M(2, \mathbb{R})), \ \|J\|_\infty < \varepsilon.$$

By Lusin's theorem (see [R1], for instance), there is $J' \in C(X, M(2, \mathbb{R}))$ with $\mu[J' \neq J] < \delta$ (δ will be specified later) and

$$\sup_x \|J'(x)\|_{\max} \leq \sup_x \|J(x)\|_{\max}$$

(hence $\|J'\|_\infty \leq$constant.$\|J\|_\infty$). Replacing J' by

$$\frac{J' + I}{\sqrt{\det(J' + I)}} - I$$

we can assume that $\det(J' + I) = 1$. Let $B = A.(I + J')$. Thus

$$\|B - A\|_\infty = \|J'\|_\infty \leq \text{constant}.\varepsilon$$

and

$$\int \left\| B - \widetilde{A} \right\| d\mu \leq \|A\|_\infty \int \left\| J - J^{'} \right\| d\mu < \text{constant}. \|A\|_\infty \varepsilon\delta.$$

For δ suitably chosen, this implies $\Lambda(B) < \lambda$. ■

Remark. I believe that this theorem can be extended to the continuous bundle case, but I did not check the details.

6. Appendix. Existence of discontinuity points

It is easy to construct non-hyperbolic cocycles with positive exponent if the system is not uniquely ergodic, for one can even choose matrices that commute. This is clearly impossible in the uniquely ergodic case.

Proposition 1. *If (X, T, μ) is ergodic but not uniquely ergodic then the function $\Lambda : C_M \to \mathbb{R}$ is discontinuous.*

Proof. Assume, without loss of generality, that the support of μ is X. Take another invariant measure ν. Take a continuous function $f : X \to \mathbb{R}$ such that $\int f d\mu \neq 0$ but $\int f d\nu = 0$. Define the cocycle

$$A(x) = \begin{pmatrix} e^{f(x)} & 0 \\ 0 & e^{-f(x)} \end{pmatrix}.$$

We have $\Lambda(A) = \left|\int f d\mu\right|$. Besides, for every $\varepsilon > 0$ and $n_0 > 0$ there is $n > n_0$ such that the (open) set

$$\left\{ x \in X; \frac{1}{n} \left| \sum_{j=0}^{n-1} f(T^j x) \right| < \varepsilon \right\}$$

is not empty and thus its μ-measure is positive. This shows that A is not hyperbolic. ■

Remark. If the support of μ is not a minimal set for T then one can even take $f \geq 0$ in the proof.

As a consequence of Herman examples (see [H], §4) we have

Proposition 2. *For every irrational rotation of the torus $\mathbb{T}^n$, the function*

$$\Lambda : C(\mathbb{T}^n, SL(2,\mathbb{R})) \to \mathbb{R}$$

is discontinuous.

This theorem generalizes a previous result of Furman [F], which says that there is some irrational rotation such that Λ is discontinuous.

References

[F] Furman, A., On the multiplicative ergodic theorem for uniquely ergodic systems, *Ann. Inst. Henri Poincaré* (1997), **33**, n.6, pp. 797-815.

[H] Herman, M. R., Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, *Comment. Math. Helvetici* (1983), **58**, pp. 453-502.

[K] Knill, O., The upper Lyapunov exponent of Sl(2,R) cocycles: Discontinuity and the problem of positivity, *Lecture Notes in Math.*(1991), **1486**, *Lyapunov Exponents (Oberwolfach, 1990)*, pp. 86-97, Springer.

[M] Mañé, R., The Lyapunov exponents of generic area preserving diffeomorfisms, in *International Conference on Dynamical Systems (Montevideo, 1995), Pitman Res. Notes Math. Ser.* (1996), vol. **362**, editors Ledrappier, Lewowicz and Newhouse, pp. 110-119.

[R1] Rudin, W., *Real and Complex Analysis*, 3rd. ed., 1987, McGraw-Hill.

[R2] Ruelle, D., Analycity Properties of the Characteristic Exponents of Random Matrix Products, *Advances in Math.* (1979), **32**, pp. 68-80.

[T] Thouvenot, J.-P., An Example of Discontinuity in the Computation of the Lyapunov Exponents, *Proceedings of the Steklov Institute of Math.* (1997), **216**, pp. 366-369.

[W] Walters, P., Unique ergodicity and matrix products, *Lecture Notes in Math.* (1984), **1186**, *Lyapunov Exponents*, pp. 37-55, Springer.

[Y] Young, L.-S., Some open sets of nonuniformly hyperbolic cocycles, *Ergod. Th. Dynam. Sys.* (1993), **13**, pp. 409-415.

Jairo Bochi (bochi@impa.br)
IMPA - Rio de Janeiro, Brazil.