Extremal norms for fiber-bunched cocycles

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General setting for the talk

- $X =$ compact metric space
- $T\colon X \to X$ continuous map
- $\mathcal{M}_T :=$ set of T-invariant Borel probability measures (compact convex)
- $\mathcal{M}_T^{\mathrm{erg}} :=$ subset of ergodic measures $= \mathrm{ext}(\mathcal{M}_T)$.

Part 1

Commutative ergodic optimization: Birkhoff averages

References: Surveys by O. Jenkinson.

- *Ergodic Optimization*, Discrete and Cont. Dyn. Sys. A, vol. 15 (2006), pp. 197–224.
- *Ergodic Optimization in Dynamical Systems*, Ergodic Theory Dynam. Systems (2018; online)

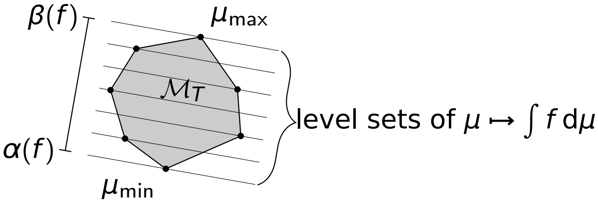
Apology / Disclaimer: I won't discuss relations with Lagrangian Mechanics, nor Thermodynamical Formalism.

Ergodic optimization of Birkhoff averages

Given a continuous function $f\colon X \to \mathbb{R}$ (“potential”),

$$\left\{ \int f \, \mathrm{d}\mu \; ; \mu \in \mathcal{M}_T \right\} =: [\alpha(f), \beta(f)]$$

$\mu \in \mathcal{M}_T$ s.t. $\int f \, \mathrm{d}\mu = \beta(f)$ is called a **maximizing measure**.



Note: **Ergodic** maximizing measures always exist. In particular, uniqueness ⇒ ergodicity.

Expressing $\beta(f)$ in terms of Birkhoff averages

Birkhoff sum $f^{(n)} := f + f \circ T + \cdots + f \circ T^{n-1}$

$$\beta(f) = \sup_{x \in X} \limsup_{n \to \infty} \frac{f^{(n)}(x)}{n}$$
$$= \lim_{n \to \infty} \sup_{x \in X} \frac{f^{(n)}(x)}{n}$$

Ergodic optimization of Birkhoff averages

Meta-Problem

Describe maximizing measures.

Maximizing measures: Generic uniqueness

Theorem (Conze–Guivarch, Jenkinson, . . .)

Let $\mathcal{F}$ be any "reasonable"(*) space $\mathcal{F}$ of continuous functions.
For generic f in the maximizing measure is ***unique****.*

(*) a vector space $\mathcal{F}$ continuously and densely embedded in $C^0(X)$.

Generic set: intersection of a countable family of open and dense sets.

The inverse problem

Theorem (Jenkinson)

Given $\mu \in \mathcal{M}_T^{\text{erg}}$, there exists $f \in C^0(X)$ such that μ is the unique maximizing measure for f.

If μ has finite support then f can be taken C^∞.

How regular f can be taken, in general? Not much:
As we will see later, if T is "hyperbolic" and $\operatorname{supp} \mu$ is not uniquely ergodic, then f cannot be Hölder.

Maximizing measures should be simple

Meta-Conjecture (∼ Hunt–Ott, Phys. Rev. 1996)

Suppose $T : X \to X$ is chaotic. Then for typical regular functions $f : X \to \mathbb{R}$, the maximizing measure has low complexity.

Many results (including Yuan, Hunt'99; Contreras, Lopes, Thieullen'01; Bousch'01; Morris'08; Quas, Siefken'12); the best one is:

Theorem (Contreras'16)

*T unif. expanding ⇒ for generic Lipschitz f's (actually all f's in an **open** and dense subset), the maximizing measure is supported on a periodic orbit.*

Only result with a probabilistic notion of typicality (**prevalence**): Bochi–Zhang'16.

A nice example

Conze–Guivarch'93, Hunt–Ott'96, Jenkinson'96, Bousch'00

$T(x) = 2x \bmod 2\pi$ on the circle $X := \mathbb{R}/2\pi\mathbb{Z}$

f = trigonometric polynomial of deg. 1
WLOG, $f(x) = f_\theta(x) = \cos(x - \theta)$

Theorem (Bousch'00)

For every $\theta \in [0, 2\pi]$, the function f_θ has a unique maximizing measure μ_θ, and it has zero entropy (actually, Sturmian).
Furthermore, for Lebesgue-a.e. θ (actually, all θ outside a set of Hausdorff dim. 0), μ_θ is supported on a periodic orbit.

Part 2
Mañé–type Lemmas

Coboundaries and $\beta(\cdot)$

$f \in C^0(X)$ is a **coboundary** if $f = h \circ T - h$ for some $h \in C^0(X)$. Notation: $f \sim 0$.

$f, g \in C^0(X)$ are **cohomologous** if $f - g$ is a coboundary. Notation: $f \sim g$.
Note:

$$\begin{aligned} f \sim g \quad &\Rightarrow \quad \int f \, d\mu = \int g \, d\mu \quad \forall \mu \in \mathcal{M}_T \\ &\Rightarrow \quad \beta(f) = \beta(g). \end{aligned}$$

Note:

$$\beta(f) \leq \max(f) \, .$$

Consequence:

$$\beta(f) \leq \max(g) \quad \forall g \sim f$$
$$\beta(f) \leq \inf_{g \sim f} \max(g)$$

Coboundaries and $\beta(\cdot)$

Proposition (Duality formula; Furstenberg, Kifer'83 (?))

$\forall f \in C^0(X)$ we have $\beta(f) = \inf_{g\sim f} \max(g)$.

Lemma (Folklore)

$\forall f \in C^0(X)$ and $n \geq 1$ we have $\frac{f^{(n)}}{n} \sim f$.

Proof.

$$h := \frac{1}{n}\sum_{i=1}^{n} f^{(i)} \quad \Rightarrow \quad f + h\circ T - h = \frac{f^{(n)}}{n}.$$

□

Proof of the duality formula.

$$\inf_{g\sim f} \max(g) \geq \beta(f) = \inf_{n} \max\left(\frac{f^{(n)}}{n}\right) \geq \inf_{g\sim f} \max(g).$$

Reformulation of duality formula

Proposition

Suppose $T: X \to X$ and $f: X \to \mathbb{R}$ are continuous. Then for every $\epsilon > 0$, there exists $g \sim f$ taking values in the interval $[\alpha(f) - \epsilon, \beta(f) + \epsilon]$. Actually, $g = \frac{f^{(n)}}{n}$ for some large n.

Remark. This proposition can be extended in several ways:

1. Optimization of Birkhoff averages of vector-valued functions. – same proof.
2. Optimization of Lyapunov exponents: we will see later.

Mañé Lemma

Theorem (Mañé Lemma or Revelation Lemma)

Suppose:

- $T : X \to X$ *is "**hyperbolic**" (e.g. uniformly expanding, SFT, Anosov);*
- $f : X \to \mathbb{R}$ *is **Hölder-continuous**.*

Then the inf in the duality formula is attained: there exists $g \sim f$ *such that*

$$\beta(f) = \max(g).$$

Furthermore, $g = f + h \circ T - h$ *with* h *Hölder.*

Several formulations (and proofs): Mañé'92, Conze–Guivarc'h'93, Fathi'97, Savchenko'99, Bousch'00, Contreras–Lopes–Thieullen'01, Lopes–Thieullen'03, Pollicott–Sharp'04, Bousch'11).

Mañé Lemma = Non-positive Livsic

Theorem (Livsic Lemma)

Suppose $T : X \to X$ is hyperbolic and $f : X \to \mathbb{R}$ is Hölder.

$\forall \mu \in \mathcal{M}_T, \ \int f \, d\mu = 0 \ \Rightarrow \ \exists h$ *Hölder such that* $f = h \circ T - h$.

Theorem (Mañé Lemma (equivalent formulation))

Suppose $T : X \to X$ is hyperbolic and $f : X \to \mathbb{R}$ is Hölder.

$\forall \mu \in \mathcal{M}_T, \ \int f \, d\mu \leq 0 \ \Rightarrow \ \exists h$ *Hölder such that* $f \leq h \circ T - h$.

Maximizing sets

Proposition (Subordination principle)

Suppose $T\colon X \to X$ is hyperbolic and $f\colon X \to \mathbb{R}$ is Hölder. Then there exists a T-invariant compact set $K \subseteq X$ such that $\mu \in \mathcal{M}_T$ is maximizing iff $\operatorname{supp}\mu \subseteq K$.

Proof.

By Mañé Lemma, replacing f by some function $\sim f$, we can assume that $f \leq \beta = \beta(f)$. Let $K := f^{-1}(\beta)$. Then:

$$\int f \, d\mu = \beta \quad \Leftrightarrow \quad \mu(K) = 1 \quad \Leftrightarrow \quad \operatorname{supp}\mu \subseteq K\,. \qquad \square$$

Corollary

Suppose $T\colon X \to X$ is hyperbolic and $f\colon X \to \mathbb{R}$ is Hölder. If the maximizing measure is unique then its support is uniquely ergodic.

Bilateral Mañé Lemma

Theorem (Bilateral Mañé Lemma; Bousch'02)

Suppose $T : X \to X$ is hyperbolic and $f : X \to \mathbb{R}$ is Hölder. Then there exists $g \overset{\text{Hölder}}{\sim} f$ taking values in the interval

$$[\alpha(f), \beta(f)] =: \left\{ \int f \, d\mu \, ; \mu \in \mathcal{M}_T \right\} .$$

Remark: The corresponding statement in higher dimension ("vectorial Mañé Lemma") is false – J.B., Vicent Delecroix.

Details: See J.B., ArXiv 1712.01612

Part 3

Non-commutative ergodic optimization: Lyapunov exponent

Replace the scalar function f by a (continuous) matrix-valued function:

$$F\colon X \to \mathrm{Mat}(d \times d, \mathbb{R}) \text{ or } \mathrm{GL}(d, \mathbb{R}) \quad (\text{“cocycle”}).$$

The Birkhoff sums are replaced by products:

$$F^{(n)}(x) := F(T^{n-1}x) \cdots F(Tx)F(x)\,.$$

Top Lyapunov exponent:

$$\lambda_1(F, x) := \lim_{n\to\infty} \frac{1}{n} \log \|F^{(n)}(x)\| \quad \text{(if it exists)}$$

For any $\mu \in \mathcal{M}_T$, the limit exists for μ-a.e. $x \in X$.

$$\lambda_1(F, \mu) := \int \lambda_1(F, x)\, \mathrm{d}\mu(x)$$

Optimization of the top Lyapunov exponent

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$
$$\beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$

For “one-step cocycles”:

- $e^{\beta(F)}$ is called **joint spectral radius** (Rota, Strang’60; Daubechies, Lagarias’92, ...)
- $e^{\alpha(F)}$ is called **joint spectral subradius** (Gurvits’95).

λ_1-minimizing/maximizing measures?

Basic difficulty:
$\mu \in \mathcal{M}_T \mapsto \lambda_1(F,\mu)$ is **not continuous**, in general.
It is **upper semi-continuous**, at least.

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F,\mu)$$ ☹ not necessarily attained

$$\beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F,\mu)$$ ☺ always attained

Let us forget about $\alpha(F)$ and focus on $\beta(F)$ and the corresponding Lyapunov-maximizing measures.

Another characterization:

$$\beta(F) = \lim_{n\to\infty} \sup_{x\in X} \frac{1}{n} \log \|F^{(n)}(x)\| .$$

Expected panorama for λ_1-maximization

Meta-Conjecture

Suppose $T\colon X \to X$ is hyperbolic.
Then for typical regular cocycles $F\colon X \to \mathrm{GL}(d, \mathbb{R})$, the Laypunov-maximizing measure is unique and low complexity.

A result of this type: Bochi–Rams'16.

But let's go back to basics. . .

Conjugacy

Two cocycles F, G are called **conjugate** if there is a continuous $H\colon X \to \mathrm{GL}(d,\mathbb{R})$ such that:

$$G(x) = H(Tx)F(x)H(x)^{-1}.$$

Notation: $G \sim F$.

By "telescopic multiplication":

$$G^{(n)}(x) = H(T^n x)F^{(n)}(x)H(x)^{-1}.$$

Therefore $\boxed{\beta(\mathrm{G}) = \beta(\mathrm{F})}$.

"Duality"

$$G \sim F \quad \Rightarrow \beta(G) = \beta(F)$$

Trivial estimate: $\beta(F) \le \max_{x \in X} \log \|F(x)\|$.

We can "optimize" this estimate:

Proposition ("Duality formula" for β)

Suppose $T \colon X \to X$ and $F \colon X \to \mathrm{GL}(d, \mathbb{R})$ are continuous. Then

$$\beta(F) = \inf_{G \sim F} \max_{x \in X} \log \|G(x)\| \, .$$

Proof: Lyapunov–Pesin norms trick.

Remark: There is a generalization of the Proposition that takes into account **all** Lyapunov exponents: J.B. ArXiv 1712.01612, Prop 4.1, using **averaging in a symmetric space** of nonpositive curvature (∼B.–Navas'15)

A Mañé Lemma for $\beta(F)$?

Question

Suppose $T\colon X \to X$ is hyperbolic and $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ is Hölder. Is there a cocycle G conjugate to F such that

$$\beta(F) = \max_{x\in X} \log \|G(x)\| \, ?$$

The answer is **NO**! Cheap example: $F = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ constant. Then:

$$\beta(F) = 0 \text{ but } \nexists\, G \sim F \text{ s.t. } \|G\| \le 1 \text{ everywhere.} \qquad (\star)$$

A honest (irreducible and fiber-bunched) example (B., Garibaldi):
One-step cocycle: $T\colon \{0,1\}^{\mathbb{N}} \hookleftarrow$ shift, $F(x) = A_{x_0}$ where $A_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_1 = \left(\begin{smallmatrix} 0.8 & -0.1 \\ 0.8 & 0.1 \end{smallmatrix}\right)$. Then $(\star)$.

Let us insist anyway

A **Riemannian norm** is a **continuous** choice of inner products $\langle \cdot, \cdot \rangle_x$ (and so of Euclidian norms $\|\cdot\|_x$) on $\mathbb{R}^d_x$ ($x \in X$).

Remark

Given $T \colon X \to X$ and $F \colon X \to \mathrm{GL}(d, \mathbb{R})$, the following are equivalent:

1. $\exists\, G \sim F$ such that $e^{\beta(F)} = \max_{x \in X} \|G(x)\|_{\text{eucl}}$.
2. $\exists$ a Riemannian norm such that $\|F(x)v\|_{Tx} \leq e^{\beta(F)} \|v\|_x$, $\forall x \in X$, $\forall v \in \mathbb{R}^d_x$.

Proof.

$G(x) = H(Tx)^{-1} F(x) H(x)$ where $H(x)$ takes the euclidian unit ball on $\mathbb{R}^d_x$ to the unit ball w.r.t. the Riemannian norm $\|\cdot\|_x$. □

What about Finsler?

Consider instead **Finsler** norms $|||\cdot|||_x$, $x \in X$.

Theorem (Mañé Lemma for Cocycles; B., Garibaldi)

Suppose $T: X \to X$ is hyperbolic and $F: X \to \mathrm{GL}(d, \mathbb{R})$ is Hölder. Then, under two natural conditions, there exists a Finsler norm $|||\cdot|||_x$, $x \in X$, such that:

$$|||F(x)v|||_{Tx} \le e^{\beta(F)} |||v|||_x \qquad \forall x \in X, \ \forall v \in \mathbb{R}^d_x. \qquad (\star)$$

Furthermore, the norm can be taken Hölder continuous.

Any norm satisfying $(\star)$ is called an **extremal norm**.

Motivation: Barabanov norms

Fix a tuple $(A_1, \ldots, A_k)$ of $d \times d$ matrices.
One-step cocycle: $T\colon \{1, \ldots, k\}^{\mathbb{N}} \hookleftarrow$ shift, $F(x) := A_{x_0}$.

The tuple is called **irreducible** if there is no nontrivial subspace $V \subset \mathbb{R}^d$ such that $A_i(V) \subseteq V$, $\forall i$.

Theorem (Barabanov'88)

If the tuple is irreducible then the cocycle admits an extremal norm, i.e., $|||A_{x_0} v|||_{Tx} \le e^{\beta(F)} |||v|||_x$.
Actually, the norm is constant (does not depend on x), and satisfies the stronger ***calibration property****:*
$\forall v \in \mathbb{R}^d$,

$$\max_{i \in \{1,\ldots,k\}} |||A_i v||| = e^{\beta(F)} |||v||| \,.$$

Existence of extremal norm fails for reducible tuples: $A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$.

Precise statement

Theorem (Mañé Lemma for Cocycles; B., Garibaldi)

Let $T\colon X \to X$ be hyperbolic and $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ be θ-Hölder. Suppose:

1. *F is irreducible;*
2. *F is strongly fiber bunched;*

Then there exists a (Hölder-continuous) extremal norm, i.e. a Finsler norm $|||\cdot|||_x$, $x \in X$, such that:

$$|||F(x)v|||_{Tx} \le e^{\beta(F)} |||v|||_x \qquad \forall x \in X,\ \forall v \in \mathbb{R}^d_x .$$

Furthermore, if T is a shift then the norm is "Barabanov-like".

Remark: Irreducibility is open and dense (and prevalent) among fiber-bunched cocycles.

The first condition: irreducibility

Suppose $T\colon X \to X$ is hyperbolic and $F\colon X \to \mathrm{GL}(d, \mathbb{R})$ is θ-Hölder.

We say that F is **irreducible** if it admits no θ-Hölder invariant proper subbundle.

Note: It is perfectly ok that F admits a **continuous** (or even θ'-Hölder, $\theta' < \theta$) invariant proper subbundle: indeed this happens if F admits a **dominated splitting**.

Bolicity

The **bolicity** of a matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is:

$$\mathrm{bol}(A) := \|A\|_{\mathrm{eucl}} \, \|A^{-1}\|_{\mathrm{eucl}} \, .$$

Notes:

- $\mathrm{bol}(A) \geq 1$;
- $\mathrm{bol}(A) = 1$ iff A is conformal (angle preserving);
- $\mathrm{bol}(A) \gg 1$ iff distorts angles very much.

The second condition: fiber-bunching

Let $T\colon X \to X$ be a **hyperbolic homeomorphism**.

Hyperbolicity rate $\boxed{\tau > 0}$: T contracts local stable sets by factor $e^{-\tau}$; similarly for T^{-1}.

A cocycle $F\colon X \to \mathrm{GL}(d, \mathbb{R})$ is **fiber-bunched** if it is θ-Hölder and, $\forall x \in X$,

$$\boxed{\mathrm{bol}(F(x)) < e^{\tau\theta}}$$

(A sort of partial hyperbolicity for the projective skew-product).

When $d > 2$, our main results actually need **strong fiber-bunched** (smaller bolicity) – details omitted.

Example: One-step cocycles are (strongly) fiber-bunched, because we can take $\theta \gg 1$ (they are locally constant).

Subordination principle for λ_1

Corollary

Suppose T is a hyperbolic homeomorphism, and that F is a strongly fiber-bunched cocycle. Then there exists a ***maximizing set****: a T-invariant compact set $K \subsetneq X$ such that:*

$$\mu \text{ is } \lambda_1\text{-maximizing} \iff \operatorname{supp}\mu \subseteq K$$

Proof.

Induction on dimension . . . □

Related work: Morris'13.

Holonomies

Proposition

If (T, F) is fiber-bunched then there exist ***stable holonomies****: linear maps $H^s_{y\leftarrow x} : \mathbb{R}^d_x \to \mathbb{R}^d_y$, defined whenever $y \in W^s(x)$, such that:*

1. $H^s_{x\leftarrow x} = \mathrm{id}$.
2. $H^s_{z\leftarrow y} \circ H^s_{y\leftarrow x} = H^s_{z\leftarrow x}$.
3. $F(y) \circ H^s_{y\leftarrow x} = H^s_{Ty\leftarrow Tx} \circ F(x)$.
4. *(Hölder-)continuity properties . . .*

Likewise for ***unstable holonomies*** *H^u.*

Proof.

$H^s_{y\leftarrow x} := \lim_{n\to+\infty} \left[F^{(n)}(y)\right]^{-1} \circ F^{(n)}(x)$. □

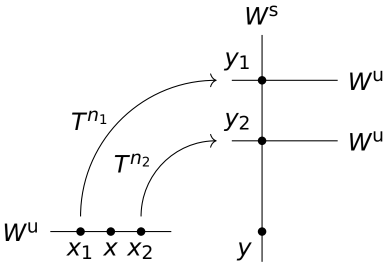
Spannability

A fiber-bunched cocycle (T, F) is called **spannable** if for all $x, y \in X$, and all nonzero $u \in \mathbb{R}^d_x$, there exist:

- points $x_1, \ldots, x_d \in W^{\mathrm{u}}(x)$;
- integers $n_1, \ldots, n_d \geq 0$ s.t. each $y_i := T^{n_i} x_i \in W^{\mathrm{s}}(y)$;

in such a way that $\{v_1, \ldots, v_d\}$ is a basis for $\mathbb{R}^d_y$, where:

$$v_i := H^{\mathrm{s}}_{y \leftarrow y_i} \circ F^{(n_i)}(x_i) \circ H^{\mathrm{u}}_{x_i \leftarrow x}(u)$$



Irreducibility vs Spannability

Assume $\boxed{(T, F) \text{ is fiber bunched}}$.

Remark

Spannable $\Rightarrow$ Irreducible

Theorem (B., Garibaldi)

Irreducible + strongly bunched $\Rightarrow$ Spannable

Theorem (Clark Butler; personal comm.)

Pinching & Twisting $\Rightarrow$ Spannable

Pinching & Twisting is a strong form of irreducibility used by Bonatti-Viana and Avila-Viana to get simplicity of Lyapunov spectrum (w.r.t. to certain "good" invariant measures).

Spannability: to-do-list

Assume $\boxed{(T, F) \text{ is fiber bunched}}$.

Problem

Characterize spannability "geometrically". Is it equivalent to (strong?) irreducibility?

Potential application of spannability: existence and uniqueness of **equilibrium states** with Gibbs property for the **subadditive pressure**

$$P_t(F, \mu) := h(F, \mu) + t\lambda_1(F, \mu).$$

The idea is that spannability should imply a cocycle version of the "quasi-multiplicativity property"...

An even more precise statement

Theorem (Mañé Lemma for Cocycles; B., Garibaldi)

Suppose *(T, F)* ***is spannable.*** *Then there exists an extremal norm, i.e. a Finsler norm $|||\cdot|||_x$, $x \in X$, such that:*

$$|||F(x)u|||_{Tx} \le e^{\beta(F)} |||u|||_x \qquad \forall x \in X,\ \forall u \in \mathbb{R}^d_x,$$

and this norm is Hölder-continuous.

Furthermore, if T is a shift then the norm is ***"Barabanov-like"****:*

1. *local H^{u}-invariance:* $\forall x \in X$, $\forall u \in \mathbb{R}^d_x$, $\forall y \in W^{\mathrm{u}}_{\mathrm{loc}}(x)$,
$$|||u|||_x = |||H^{\mathrm{u}}_{y \leftarrow x}(u)|||_y;$$
2. *calibration:* $\forall x \in X$, $\forall u \in \mathbb{R}^d_x$, $\exists y \in W^{\mathrm{u}}_{\mathrm{loc}}(x)$ *s.t.*
$$v := H^{\mathrm{u}}_{y \leftarrow x}(u) \qquad \Rightarrow \qquad |||F(y)v|||_{Ty} = e^{\beta(F)} |||v|||_y\,.$$

Construction of extremal norms (shift case)

Suppose $T =$ shift. Our norm is given by en explicit formula:

$$|||u|||_x := \limsup_{n\to\infty} e^{-\beta(F)n} \sup_{y\in W^{\mathrm{u}}_{\mathrm{loc}}(x)} \|F^{(n)}(y) \circ H^{\mathrm{u}}_{y\leftarrow x}(u)\|$$

- Compactness argument $\Rightarrow |||u_0|||_{x_0} < \infty$ for some (x_0, u_0) with $u_0 \neq 0$.
- Spannability $\Rightarrow |||u|||_x < \infty$ for all (x, u).
- Verifications. . .

Case $T \neq$ shift: use bump functions.

Applications

Assuming fiber-bunching:

1. Subordination principle (and therefore Mather sets).
2. $\beta(\cdot)$ is locally Lipschitz among irreducible cocycles [extending Wirth'02]
3. $e^{-n\beta(F)}\|F^{(n)}\|$ is either bounded (irreducible case) or grows polynomially.
4. Extra structure for the Mather sets (dominated splittings) [extending Morris'10].
5. $\beta(F)$ can be approximated by $\lambda_1(F,\mu)$ with μ supported on periodic orbits, and the quality of the approximation is super-polynomial w.r.t. the period of the orbit. [extending Bressaud, Quas'07; Morris'10]
6. Meta-conjecture (typical λ_1-maximizing measures should have low complexity)?? – OPEN