# Flexibility of Lyapunov exponents

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- $M$ = compact manifold of dimension $d \geq 2$.
- $m$ = normalized volume measure on $M$.

If $f : M \to M$ is a conservative (i.e., $m$-preserving) ergodic diffeomorphism, the *Lyapunov exponents* are:

$$\boxed{\lambda_i(f) := \lim_{n\to+\infty} \frac{1}{n} \log\big(i\text{-th singular value of } Df^n(x)\big)}$$

(for $m$-a.e. $x \in M$).

Note: $\boxed{\lambda_1(f) \geq \cdots \geq \lambda_d(f)}$ and $\boxed{\sum_{i=1}^{d} \lambda_i(f) = 0}$.

- *Lyapunov spectrum* $\vec{\lambda}(f) = \big(\lambda_1(f), \ldots, \lambda_d(f)\big)$.
- The Lyapunov spectrum is called *simple* if these numbers are all different.

### Problem

*Which Lyapunov spectra* $\vec{\lambda}(f) = \big(\lambda_1(f), \ldots, \lambda_d(f)\big)$ *may appear for* $C^\infty$ *ergodic diffeomorphisms* $f$*?*

Apart from the obvious restrictions that the $\lambda_i$'s should be ordered and their sum should be zero, no other conditions are known.

### Conjecture (Flexibility of L.E.)

*These are the only restrictions.*

Best result in this direction:

### Theorem (Dolgopyat–Pesin, 2002)

*There exists a* $C^\infty$ *ergodic diffeomorphism* $f$ *such that* $\lambda_i(f) \neq 0$ *for every* $i$*.*

### Problem

*Continuity of* $f \mapsto \vec{\lambda}(f)$*???*

Let us work with a more manageable class of (still $C^\infty$ conservative) diffeos $f$:

- $f$ is an *Anosov diffeomorphism*, and so:
  - ergodicity is automatic
  - $\lambda_i(f) \neq 0 \ \forall\, i$.
- $f$ has a *simple dominated splitting*:

$$TM = \underbrace{E_1 \oplus \cdots \oplus E_k}_{E^u} \oplus \underbrace{E_{k+1} \oplus \cdots \oplus E_d}_{E^s}$$

  and so:
  - the Lyapunov spectrum is simple: $\lambda_1(f) > \lambda_2(f) > \cdots$

These conditions define a (maybe empty) open subset $\mathcal{A} \subset \mathrm{Diff}^\infty_m(M)$, and $f \in \mathcal{A} \mapsto \vec{\lambda}(f)$ is continuous.

# Our first result

**Theorem (Flexibility of L.E. for the torus – B,K,RH)**

*If $M = \mathbb{T}^d$ then for all nonzero numbers $\xi_1 > \xi_2 > \cdots > \xi_d$ whose sum is $0$, there exists a $C^\infty$ conservative Anosov diffeo $f: \mathbb{T}^d \to \mathbb{T}^d$ with simple dominated splitting such that $\vec{\lambda}(f) = \vec{\xi} := (\xi_1, \ldots, \xi_d)$.*

**Question**

Explicit construction? (Formulas for $f$?)

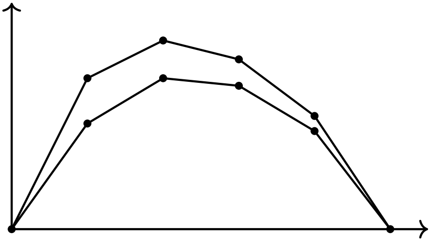
The theorem follows from a more abstract result.

# Majorization (Hardy, Littlewood, Pólya)

Let $\vec{\xi} = (\xi_1, \ldots, \xi_d)$ and $\vec{\eta} = (\eta_1, \ldots, \eta_d)$ be ordered vectors ($\xi_i \geq \xi_{i+1}$) whose entries have zero sums. Define a partial order $\vec{\xi} \succcurlyeq \vec{\eta}$ ( $\vec{\xi}$ *majorizes* $\vec{\eta}$ ) if, for every $i \in \{1, \ldots, d-1\}$,

$$\xi_1 + \xi_2 + \cdots + \xi_i \geq \eta_1 + \eta_2 + \cdots + \eta_i \,.$$

Two concave graphs, one above the other:



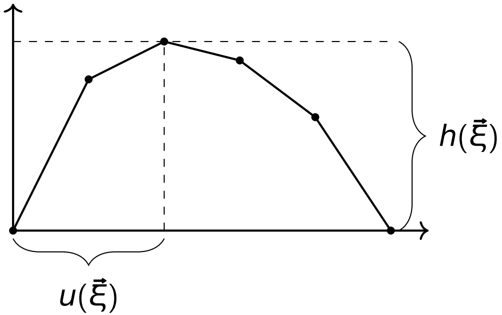
$\vec{\xi} \succ \vec{\eta}$ ( $\vec{\xi}$ *strictly majorizes* $\vec{\eta}$ ) if the inequalities are strict.

Let $\vec{\xi} = (\xi_1, \ldots, \xi_d)$ be a ordered vector whose entries have zero sum.

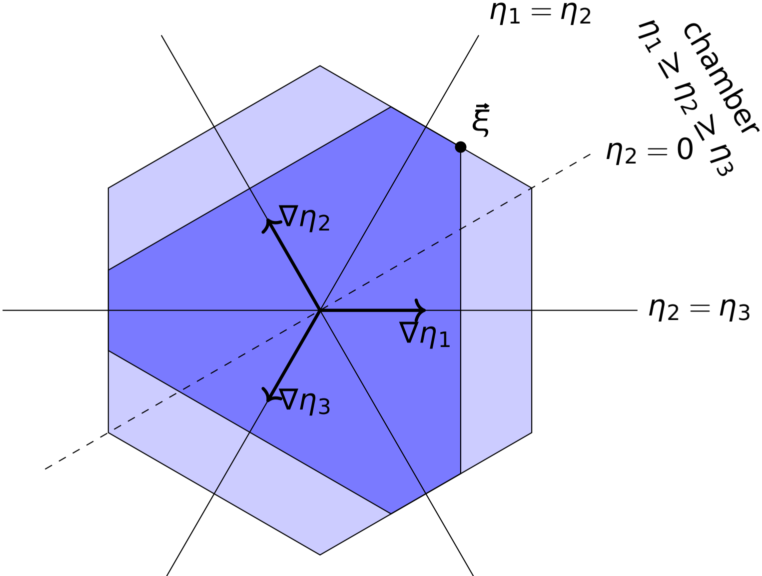
- *unstable index* of $\vec{\xi}$ is the number of positive entries. Notation: $u(\vec{\xi})$.
- The *entropy* of $\vec{\xi}$ is:

$$h(\vec{\xi}) := \xi_1 + \xi_2 + \cdots + \xi_u, \quad u = u(\vec{\xi})$$

i.e. the *height* of the graph of $i \mapsto \xi_1 + \cdots + \xi_i$:



Let $d = 3$ and fix $\vec{\xi}$.
Visualization of the sets $\{\vec{\eta}\,;\,\vec{\eta} \preccurlyeq \vec{\xi}\} \subset \{\vec{\eta}\,;\,h(\vec{\eta}) \leq h(\vec{\xi})\}$:



# Our main result

Let $M$ be a compact manifold. Let $\mathcal{A}$ be the set of $C^\infty$ conservative Anosov diffeomorphisms $f : M \to M$ with simple dominated splitting.

## Theorem (B,K,RH)

*Let $f \in \mathcal{A}$. Let $\vec{\eta}$ be a ordered vector whose entries are distinct, nonzero, have sum zero, and such that*

$$\vec{\eta} \prec \vec{\lambda}(f) \text{ (strict majorization)} \quad \text{and} \quad u(\vec{\eta}) = u(\vec{\lambda}(f)).$$

*Then there exists a continuous path*

$$t \in [0, 1] \mapsto f_t \in \mathcal{A}$$

*such that*

$$f_0 = f \quad \text{and} \quad \vec{\lambda}(f_1) = \vec{\eta}.$$

# Keywords of the proof

The proof is essentially a streamlined **Baraviera–Bonatti** perturbation method, which needs:

- special adapted metrics (a la Gourmelon);
- careful linear algebra (in order to mix several exponents simultaneously);
- tower methods (Rokhlin $+$ Vitali).

(Details later.)

# Corollary for the torus

The first theorem that we stated was:

**Theorem**

*For all nonzero numbers $\xi_1 > \cdots > \xi_d$ whose sum is $0$, there exists a $C^\infty$ conservative Anosov diffeo $f : \mathbb{T}^d \to \mathbb{T}^d$ with simple dominated splitting such that $\vec{\lambda}(f) = \vec{\xi} := (\xi_1, \ldots, \xi_d)$.*

**Proof.**

Given $\vec{\xi}$, take a linear Anosov $L \in \mathrm{SL}(d, \mathbb{Z})$ "large" enough so that:

$$\vec{\lambda}(L) \succ \vec{\xi}$$

and apply the Main Theorem. $\square$

Note that we don't control the *homotopy class* of $f$.

## Problem

*What Lyapunov spectra can be achieved* ***inside a given homotopy class*** *of conservative Anosov diffeos of* $\mathbb{T}^d$ *(not necessarily with simple dom. spl.)?*

In that case, there another obstruction:

If $f$ is a conservative Anosov diffeo of $\mathbb{T}^d$ and $L = H_1(f) \in \mathrm{SL}(d, \mathbb{Z})$ is the linear Anosov in its homotopy class then by the Variational Principle:

$$h_m(f) \leq h_{\mathrm{top}}(f) = h_m(L).$$

I.e., by Pesin formula (recall our definition of $h(\vec{\xi})$):

$$\boxed{h(\vec{\lambda}(f)) \leq h(\vec{\lambda}(L))}$$

Consider the following restricted version of the previous problem:

## Problem

*What Lyapunov spectra can be achieved inside a given homotopy class of conservative Anosov diffeos of* $\mathbb{T}^d$ *with simple dominated splitting?*

Then there is another obstruction, at least if $d = 3$:

## Theorem

*If* $f$ *is a conservative Anosov diffeo of* $\mathbb{T}^3$ *with simple dominated splitting and* $L = H_1(f) \in \mathrm{SL}(3, \mathbb{Z})$ *is the linear Anosov in its homotopy class then* $\boxed{\vec{\lambda}(f) \preccurlyeq \vec{\lambda}(L)}$.

## Proof.

- Let $f\colon \mathbb{T}^3 \to \mathbb{T}^3$ be a conservative Anosov diffeo, and $L = H_1(f)$.
- WLOG, $\lambda_2(f) > 0$. By the entropy obstruction, $\lambda_1(f) + \lambda_2(f) \leq \lambda_1(L) + \lambda_2(L)$.
- By contradiction, suppose $\vec{\lambda}(f) \not\preceq \vec{\lambda}(L)$, i.e., $\lambda_1(f) > \lambda_1(L)$.
- For a.e. $x$, and $n \gg 1$ the curve $\Gamma = f^n(W^{uu}_{\mathrm{loc}}(f^{-n}x))$ has length $\sim e^{\lambda_1(f)n}$.
- The distance between the endpoints of the lifted curve $\tilde{\Gamma} \subset \mathbb{R}^3$ is $\sim e^{\lambda_1(L)n}$ (much smaller).
- This contradicts Brin–Burago–Ivanov'09 ($\widetilde{W}^{uu}$ leaves are quasi-isometric).

$\square$

# An exotic Anosov diffeomorphism?

### Problem

*Find a $C^\infty$ conservative Anosov diffeo of $\mathbb{T}^3$ with 2-dimensional unstable bundle such that $\lambda_1(f) > \lambda_1(L)$, where $L = H_1(f)$.*

As explained before the Pesin 1-dim manifolds $W^{uu}(x)$ should be very twisted inside the 2-dim leaves $W^u(x)$.

# Review of Baraviera–Bonatti

As the proof of our main result relies on the Baraviera–Bonatti strategy, let us recall (a particular case of) their result:

### Theorem (Baraviera, Bonatti 2003)

*Let $f$ be a stably ergodic $C^\infty$ conservative diffeomorphism with a simple dominated splitting. Then, for each $i \in \{1, \ldots, d\}$, there exists $C^\infty$ conservative diffeomorphism $\tilde{f}$ arbitrarily $C^1$-close to $f$ such that $\lambda_i(\tilde{f}) \neq \lambda_i(f)$.*

### Remark

Related work: Shub–Wilkinson'2000.

# Construction of the Baraviera–Bonatti perturbation

- Consider e.g. $d = 3$, $i = 1$.
- Take a small ball $B$ centered at a non-periodic point.
- Perturb $f$ inside $B$ in a conservative way, approximately preserving and rotating the $E_1 \oplus E_2$ planes, obtaining some $\tilde{f}$.
  (See fig. next slide)
- Then one can show that the fist two exponents "mix" a little (while the third almost doesn't move); in particular, $\lambda_1(\tilde{f}) < \lambda_1(f)$.

# Rotating the $E_1 \oplus E_2$ planes

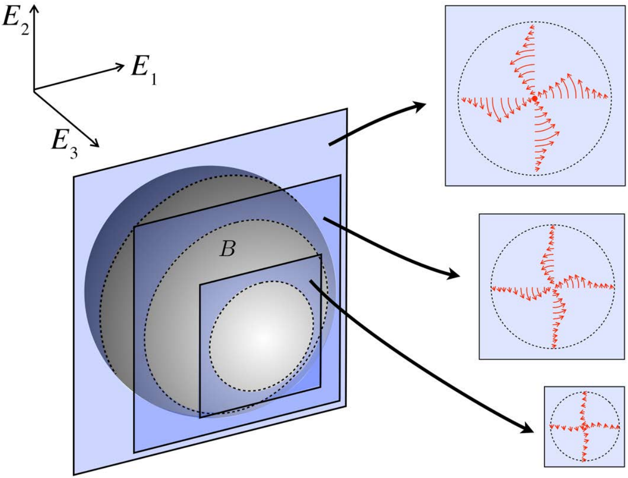
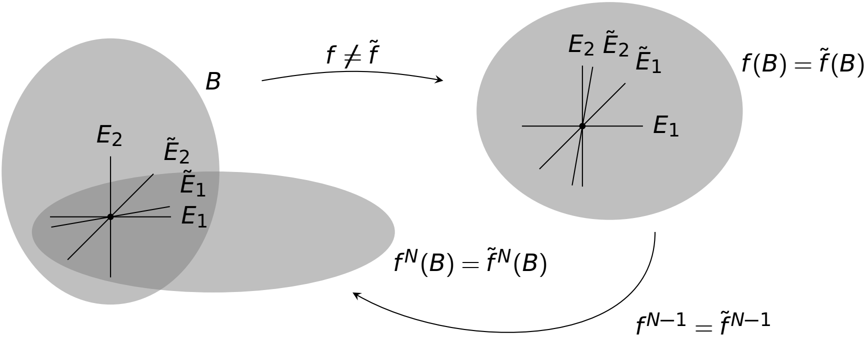



Figure by Avila–Crovisier–Wilkinson
(who extend the BB method to non-ergodic $f$)

# Why does $\lambda_1$ drop?

The new bundle $\tilde{E}_3$ is very close to the original $E_3$.
The other bundles move as follows:




So $N \gg 1 \Rightarrow \measuredangle(\tilde{E}_1, E_1) \ll 1$ on $B$.

# Why does $\lambda_1$ drop?

WLOG, the unperturbed bundles are orthogonal:
$E_1 \perp E_2$ etc
Let $v$, $\tilde{v}$ be unit-norm vector fields tangent to $E_1$, $\tilde{E}_1$, respectively.

$$\lambda_1(f) = \int_M \log \frac{\langle Df(x)v(x), v(fx)\rangle}{\langle v(x), v(x)\rangle}\, dm(x)$$

$$\lambda_1(\tilde{f}) = \int_M \log \frac{\langle D\tilde{f}(x)\tilde{v}(x), v(fx)\rangle}{\langle \tilde{v}(x), v(x)\rangle}\, dm(x)$$

The two integrands are everywhere equal, except on $B$.
On $B$ we use that $\tilde{v} \simeq v$ to compare the integrals.
Jensen inequality $\Rightarrow \lambda_1(\tilde{f}) < \lambda_1(f)$.

# Our proof

We rotate several $E_i \oplus E_{i+1}$ planes so to manipulate (i.e., "mix") all the Lyapunov exponents simultaneously (careful Linear Algebra).

In order to maximize the effect of the Baraviera–Bonatti-like perturbations, it will be fundamental to specially **adapted coordinates**.

# Adapted metrics for dominated splitting

Given the dominated splitting $TM = E_1 \oplus \cdots \oplus E_d$ into 1-dim bundles, and a Riemannian norm $|||\cdot|||$, define *expansion functions* $\rho_1, \ldots, \rho_d \colon M \to \mathbb{R}$:

$$\rho_j(x) := \log \frac{|||Df(x)v|||}{|||v|||} \quad \text{(arbitrary nonzero } v \in E_j(x)\text{)}.$$

Each $\rho_j$ is continuous and its integral is $\lambda_j(f)$. We say that the Riemannian metric is *adapted* if:

$$\rho_1(x) > \rho_2(x) > \cdots > \rho_d(x)\,.$$

and $E_i \perp E_j \; \forall i \neq j$.

## Proposition (Adapted metric with $L^1$ estimate)

*Given $\varepsilon > 0$, we can choose an adpated metric such that $\int_M |\rho_i(x) - \lambda_i(f)| dm(x) < \varepsilon$ for every $i$.*

# Our proof

We must be able to change (i.e., "mix") the Lyapunov spectrum $\vec{\lambda}(f)$ of $f$ by some small but constant amount that depends **not on $f$ but only on $\vec{\lambda}(f)$ itself.**

- We take a disjoint family of small "good" balls $B_i$ (in the adapted coordinates) whose union has $N \gg 1$ disjoint iterates from itself (a tower).
- On each of these balls, we do Baraviera–Bonatti-like perturbations (rotating several planes).
- By Rokhlin Lemma, we can take $m\left(\bigsqcup B_i\right)$ approximately equal to $1/N$.

# Our proof

- Actually we will take height $\boxed{N \simeq \varepsilon/\mathrm{GAP}}$, where $\varepsilon \ll 1$ is fixed and $\mathrm{GAP} \coloneqq \min_j \lambda_j(f) - \lambda_{j+1}(f)$. **Using the $L^1$ estimate for the adapted metrics**, we see that for most points, **time $N$ is sufficient for cones to contract** and therefore for the Baraviera–Bonatti perturbation to have a controllable and significant effect on the Lyapunov exponents.
- More precisely, the effect on the Lyapunov exponents is approximately proportional to

$$m\left(\bigsqcup B_i\right) \sim \frac{1}{N} \sim O(\mathrm{GAP}).$$

- So we are able to change the Lyapunov spectrum by some small amount that depends **not on $f$ but only on $\vec{\lambda}(f)$ itself.** Done!

# Further results?

Our (as well as Baraviera–Bonatti's) method is very adaptable.

Combining it with Katok, Brin, and Dolgopyat–Pesin constructions, one should be able to obtain some versions of the flexibility conjecture, even on manifold that do not support global dominated splittings.