

ROBUST TRANSITIVITY AND TOPOLOGICAL MIXING FOR C^1 -FLOWS

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ABSTRACT. We prove that non-trivial homoclinic classes of C^r -generic flows are topologically mixing. This implies that given Λ a non-trivial C^1 -robustly transitive set of a vector field X , there is a C^1 -perturbation Y of X such that the continuation Λ_Y of Λ is a topologically mixing set for Y . In particular, robustly transitive flows become topologically mixing after C^1 -perturbations. These results generalize a theorem by Bowen on the basic sets of generic Axiom A flows. We also show that the set of flows whose non-trivial homoclinic classes are topologically mixing is *not* open and dense, in general.

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1. STATEMENT OF THE RESULTS

Throughout this paper M denotes a compact d -dimensional boundaryless manifold, $d \geq 3$, and $\mathfrak{X}^r(M)$ is the space of C^r vector fields on M endowed with the usual C^r topology, where $r \geq 1$. We shall prove that, generically (residually) in $\mathfrak{X}^r(M)$, nontrivial homoclinic classes are topologically mixing. As a consequence, nontrivial C^1 -robustly transitive sets (and C^1 -robustly transitive flows in particular) become topologically mixing after arbitrarily small C^1 -perturbations of the flow.

These results generalize the following theorem by Bowen [B]: non-trivial basic sets of C^r -generic Axiom A flows are topologically mixing. Note that C^1 -robustly transitive sets are a natural generalization of hyperbolic basic sets; they are the subject of several recent papers, such as [BD1] and [BDP].

In order to announce precisely our results, let us introduce some notations and definitions.

Given $t \in \mathbb{R}$ and $X \in \mathfrak{X}^r(M)$, we shall denote by X^t the induced time t map. A subset \mathcal{R} of $\mathfrak{X}^r(M)$ is *residual* if it contains the intersection of a countable number of open dense subsets of $\mathfrak{X}^r(M)$. Residual subsets of $\mathfrak{X}^r(M)$ are dense. Given an open subset U of $\mathfrak{X}^r(M)$, then property (P) is *generic in U* if it holds for all flows in a residual subset \mathcal{R} of U ; (P) is *generic* if it is generic in all of $\mathfrak{X}^r(M)$.

A compact invariant set for X is *non-trivial* if it is neither a periodic orbit nor a single point. A compact invariant set Λ of X is *transitive* if there is some point $x \in \Lambda$ such that the future orbit $\{X^t(x) : t > 0\}$ of x is dense in Λ ; Λ is *topologically mixing* for X if given any nonempty open subsets U and V of Λ then there is some $t_0 > 0$ such that $X^t(U) \cap V \neq \emptyset$ for all $t \geq t_0$. A non-trivial X -invariant transitive set Λ is *Ω -isolated* if there is some open neighborhood U of Λ such that $U \cap \Omega(X) = \Lambda$. Furthermore, Λ is *isolated* if there is a neighborhood U of Λ (called an *isolating block*) such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} X^t(U).$$

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Given a hyperbolic closed orbit γ of X , the *homoclinic class* of γ relative to X is given by

$$H_X(\gamma) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)},$$

where \pitchfork denotes points of transverse intersection of the invariant manifolds. $H_X(\gamma)$ is a transitive compact X -invariant subset of the non-wandering set $\Omega(X)$. Moreover, if γ is a closed orbit of index i , then the set $P_i(H_X(\gamma)) \equiv \{p \in H_X(\gamma) \cap \text{Per}(X) : p \text{ is hyperbolic with index } i\}$ is dense in $H_X(\gamma)$. (See [BDP]). $H_X(\gamma)$ is not necessarily hyperbolic, but if X is Axiom A then its basic sets are hyperbolic homoclinic classes. In the absence of ambiguity, we may write $H(\gamma)$ for $H_X(\gamma)$.

An *attractor* is a transitive set Λ of X that admits a neighborhood U such that

$$X^t(U) \subset U \text{ for all } t > 0, \text{ and } \bigcap_{t \in \mathbb{R}} X^t(U) = \Lambda.$$

A *repeller* is an attractor for $-X$. Clearly any attractor or repeller is Ω -isolated.

An isolated X -invariant compact set Λ is C^1 -*robustly transitive* if there is some open neighborhood \mathcal{V} of X in $\mathfrak{X}^1(M)$ and some isolating block U of Λ such that given any $Y \in \mathcal{V}$, then

$$\Lambda_Y \equiv \bigcap_{t \in \mathbb{R}} Y_t(U)$$

is a compact transitive non-trivial set of Y .

Finally, a flow X is C^1 -*robustly transitive* if there is some open neighborhood \mathcal{W} of X in $\mathfrak{X}^1(M)$ such that given any $Y \in \mathcal{W}$ then Y is transitive.

Our main result is the following:

Theorem A. *There is a residual subset \mathcal{R} of $\mathfrak{X}^1(M)$ such that if Λ is an isolated non-trivial transitive set of $X \in \mathcal{R}$, then Λ is topologically mixing for X .*

Theorem A has the following immediate consequence for robustly transitive sets or flows:

Corollary A'. *Let Λ be a non-trivial robustly transitive set, with \mathcal{V} and U as in the definition above. Then there is some residual subset \mathcal{R} of \mathcal{V} such that if $Y \in \mathcal{R}$ then Λ_Y is topologically mixing for Y . In particular, given an open set $\mathcal{W} \subset \mathfrak{X}^1(M)$ of transitive flows, then there is some residual subset \mathcal{R} of \mathcal{W} such that any $Y \in \mathcal{R}$ is topologically mixing.*

Theorem A is very much a nonhyperbolic, C^1 version of Bowen's aforementioned result. It is actually a consequence of the proof of the following result:

Theorem B. *Given any $r \in \mathbb{N}$, there is a residual subset \mathcal{R} of $\mathfrak{X}^r(M)$ such that if $Y \in \mathcal{R}$ and $H(\gamma)$ is a non-trivial homoclinic class of Y , then $H(\gamma)$ is topologically mixing for Y .*

Theorem B follows from general properties of homoclinic classes combined with simple topological arguments. All of the arguments in the proof of Theorem B hold in any C^r topology with $r \geq 1$, whereas Theorem A requires the use of C^1 -generic properties which are not known in finer topologies.

Pugh's General Density Theorem [Pu] and Theorem B of [BD2] (which is stated for diffeomorphisms but holds for flows via the same arguments) imply that C^1 -generically any Ω -isolated transitive set coincides with some homoclinic class. Therefore Theorem B implies the following corollary:

Corollary B'. *There is a residual subset \mathcal{R} of $\mathfrak{X}^1(M)$ such that if $Y \in \mathcal{R}$ and Λ is a non-trivial transitive Ω -isolated set of Y , then Λ is topologically mixing for Y . In particular, C^1 -generically any non-trivial attractor/repeller is topologically mixing.*

Note that Corollary B' generalizes the “mixing” aspect of [MP]. We remark that the dependence of our proofs on the (C^1) Closing and Connecting Lemmas means that extending our results (with the exception of Theorem B) to finer topologies is probably very difficult.

Of course, in general not every non-trivial homoclinic class is topologically mixing: the basic sets of suspensions of Axiom A diffeomorphisms, for example, are not mixing. One may therefore ask how large is the set of the flows that have a non-trivial homoclinic class which is *not* mixing. A partial answer to this question is given by:

Theorem C. *There exists a 4-manifold M and an open set $\mathcal{U} \subset \mathfrak{X}^1(M)$ such that each flow X in a dense subset $\mathcal{D} \subset \mathcal{U}$ has a non-trivial homoclinic class which is not topologically mixing for X .*

Theorem C shows that the residual set \mathcal{R} of Theorem B is, in general, not open. The construction in Theorem C relies on the wild diffeomorphisms from [BD2] and [BD3].

On the other hand, robustly transitive flows have relatively tame dynamics. We pose the following:

Question. Is the set of (C^1 -)robustly topologically mixing flows dense in the set of robustly transitive flows?

In [AA] the first two authors prove analogues of Theorems A, B, and C for diffeomorphisms. In addition, a robustly transitive but non-mixing diffeomorphism is constructed.

The next section first lists some definitions and properties needed for the proofs and then sets out the proofs themselves.

2. THE PROOFS

Given a hyperbolic periodic point p , let $\gamma = \gamma(p)$ be its orbit and $\Pi_X(p) = \Pi_X(\gamma)$ be its period. Set also

$$\begin{aligned} W^s(p) &= \{x \in M : d(X^t(x), \gamma(p)) \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ W^{ss}(p) &= \{x \in M : d(X^t(x), X^t(p)) \rightarrow 0 \text{ as } t \rightarrow +\infty\}. \end{aligned}$$

We define $W^u(p)$ and $W^{uu}(p)$ as the corresponding sets for $-X$. Note that the set $W^s(p)$ is X^t -invariant for all $t \in \mathbb{R}$, whereas $W^{ss}(p)$ is X^t -invariant only for $t \in \Pi_X(p) \cdot \mathbb{Z}$. The *index* of γ is the dimension of the stable manifold $W^s(\gamma) = W^s(p)$.

Lemma 1. *Given any $r \in \mathbb{N}$, there exists a residual subset \mathcal{R}_1 of $\mathfrak{X}^r(M)$ such that if $X \in \mathcal{R}_1$ then given any distinct closed orbits γ, γ' , we have that*

$$\frac{\Pi_X(\gamma)}{\Pi_X(\gamma')} \in \mathbb{R} \setminus \mathbb{Q}.$$

Proof. For $N \in \mathbb{N}$, let $A_N \subset \mathfrak{X}^r(M)$ be the set of vector fields X such that all singularities of X are hyperbolic and all closed orbits with periods less than N are hyperbolic. It follows from the standard proof of the Kupka–Smale theorem that the set A_N is open and dense in $\mathfrak{X}^r(M)$.

Now let a_1, a_2, \dots be an enumeration of the positive rational numbers and let $B_N \subset \mathfrak{X}^r(M)$ be the set of vector fields $X \in A_N$ such that if γ, γ' are distinct closed orbits with periods less than N then $\Pi_X(\gamma)/\Pi_X(\gamma')$ does not belong to $\{a_1, \dots, a_N\}$.

If $X \in A_N$ then the number of orbits with periods less than N is finite. Moreover, each of these orbits has a continuation and the period varies continuously. It follows that the set B_N is open.

Let us show that B_N is also dense, so we can define $\mathcal{R}_1 = \bigcap_N B_N$. Given any $X_0 \in \mathfrak{X}^1(M)$, first approximate it by $X_1 \in A_N$. Let $\gamma_1, \dots, \gamma_k$ be the X_1 -orbits with periods less than N . Let $\delta > 0$ be small enough such that the neighborhoods $B(\gamma_i, \delta)$ are disjoint.

Take C^r functions $\psi_i : M \rightarrow [0, 1]$ such that ψ_i equals 1 in γ_i and equals 0 outside $B(\gamma_i, \delta)$. For $s \in \mathbb{R}_+^k$ close to 0, let

$$Y_s = \left(\prod_{i=1}^k (1 + s_i \psi_i)^{-1} \right) X_1$$

Then Y_s has the same orbits as X_1 and Y_s converges to X_1 in the C^r topology as $s \rightarrow 0$. Moreover,

$$\Pi_{Y_s}(\gamma_i) = (1 + s_i) \Pi_{X_1}(\gamma_i),$$

so we can find $s \in \mathbb{R}_+^k$ close to 0 such that $Y_s \in A_N$ and the quotients $\Pi_{Y_s}(\gamma_i)/\Pi_{Y_s}(\gamma_j)$, $i \neq j$, do not intersect $\{a_1, \dots, a_N\}$. If γ is another closed orbit of Y_s , then $\Pi_{Y_s}(\gamma) \geq \Pi_{X_1}(\gamma) \geq N$. This proves that $Y_s \in B_N$. \square

We shall use the following simple fact, whose proof is omitted (it follows easily from the transitivity of the future orbits of irrational rotations of the circle):

Lemma 2. *Given numbers $a > 0$, $b > 0$ and $\varepsilon > 0$, with a/b irrational, the set*

$$\{ma + nb + s : m, n \in \mathbb{N}, |s| < \varepsilon\}$$

contains an interval of the form $[T, +\infty)$.

We may now prove Theorem B:

Proof of Theorem B. Let $\mathcal{R}_1 \subset \mathcal{X}^r(M)$ be the residual set given by Lemma 1, and let $H = H_X(\gamma_0)$ be a non-trivial homoclinic class of some $f \in \mathcal{R}_1$. Take two nonempty open sets U, V intersecting H . We shall prove that there exists $t_0 > 0$ such that $X^t(U) \cap V \neq \emptyset$ for every $t \geq t_0$.

Let γ and γ' be distinct periodic orbits in H with same index, such that $\gamma \cap U \neq \emptyset$ and $\gamma' \cap V \neq \emptyset$. In order to simplify the notation, let $a = \Pi_X(\gamma)$ and $b = \Pi_X(\gamma')$. Recall that $a/b \in \mathbb{R} \setminus \mathbb{Q}$.

Take $p \in \gamma \cap U$ and $q \in \gamma' \cap V$. Notice that $W^u(p) \cap W^s(q)$ is non-empty. Fix a point y in this intersection. There exists $\tau_1 > 0$ such that $X^{-\tau_1}(y) \in W^{uu}(p)$ and, consequently, the sequence $\{X^{(-\tau_1 - ma)}(y)\}_{m \in \mathbb{N}}$ is contained in $W^{uu}(p)$ and converges to p . Therefore we can find $t_1 > 0$ such that

$$X^{(-t_1 - ma)}(y) \in U \text{ for every } m \in \mathbb{N}.$$

Analogously, there exist $t_2 > 0$ and $\varepsilon > 0$ such that

$$X^{(t_2 + nb + s)}(y) \in V \text{ for every } m \in \mathbb{N} \text{ and } |s| < \varepsilon.$$

Let $T > 0$, depending on a, b and ε , be given by Lemma 2. Set $t_0 = t_1 + t_2 + T$. Then, for any $t \geq t_0$, there exist numbers $m, n \in \mathbb{N}$ and $|s| < \varepsilon$ such that $t = t_1 + t_2 + ma + nb + s$. So $X^t(U) \cap V$ contains the point $X^{(t_2 + nb + s)}(y)$. This concludes the proof. \square

Now we explain how Theorem A follows from Theorem B. We first need a couple of definitions:

Definition 1. Let Λ be a compact invariant set of $X \in \mathcal{X}^1(M)$. Then we set $P_i(\Lambda) \equiv \{p \in \Lambda : p \text{ is a hyperbolic periodic point of } X \text{ with index } i\}$.

The next definition comes from [BDP]:

Definition 2. Let p be a periodic point of a flow $X \in \mathcal{X}^1(M)$ and U be a neighborhood of p in M . Then the *homoclinic class of p relative to U* is given by

$$HR_X(p, U) \equiv \text{cl} \{q \in H_X(p) \cap \text{Per}(X) : \text{the orbit } \gamma(q) \text{ is contained in } U\}.$$

It is easily seen that $HR_X(p, U)$ is a compact transitive invariant set. Moreover, if $\text{ind}(p) = i$, then $P_i(HR_X(p, U))$ is dense in $HR_X(p, U)$.

We need the following lemma, which is a consequence of a theorem by Arnaud [Ar] together with an argument from [BDP]:

Lemma 3. *There is a residual subset \mathcal{R} of $\mathfrak{X}^1(M)$ such that if Λ is an isolated transitive set of $X \in \mathcal{R}$, then $\Lambda = HR_X(p, U)$ for some periodic point $p \in \Lambda$.*

Proof. Let \mathcal{R}_2 be as in Theorem 1 of [Ar] and let \mathcal{R}_3 be as in Theorem B of [BD2], and set $\mathcal{R} \equiv \mathcal{R}_2 \cap \mathcal{R}_3$. Let Λ be an isolated transitive set of $X \in \mathcal{R}$, with U an isolating block of Λ .

By Theorem 1 of [Ar], there is a sequence of periodic orbits γ_k which converge to Λ in the Hausdorff topology. The orbit γ_k is contained in U for k sufficiently large. Since Λ is the maximal invariant set of U , it follows that for large k the orbit γ_k is contained in Λ . Since the sequence $\{\gamma_k\}$ converges to Λ in the Hausdorff topology, the set of periodic points contained in Λ must be a dense subset of Λ .

Now, since Λ is transitive and has a dense subset of periodic points, we apply an argument from [BDP] which uses Theorem B of [BD2] to conclude that given any periodic point $p \in \Lambda$ then

$$\Lambda = HR_X(p, U).$$

□

We are now ready to prove Theorem A:

Proof of Theorem A. It is easy to see that the proof of Theorem B actually implies the following result:

Theorem D. *There is a residual subset \mathcal{R} of $\mathfrak{X}^1(M)$ such that if $Y \in \mathcal{R}$ and Λ is a non-trivial transitive set of Y such that for some $i \in \{1, \dots, d-1\}$ the set $P_i(\Lambda)$ is dense in Λ , then Λ is topologically mixing for Y .*

Now by Lemma 3 above we have that Λ coincides with some relative homoclinic class $HR_X(p, U)$. Let i be the index of the periodic point p . Since $P_i(HR_X(p, U))$ is dense in $HR_X(p, U)$, we conclude that Λ satisfies the hypotheses of Theorem D above, and therefore that Λ is a mixing set for X . □

At last, we give the:

Proof of Theorem C. Let S be a compact 3-manifold and let $\text{Diff}^1(S)$ be the set of C^1 diffeomorphisms of S endowed with the C^1 -topology. The key of the construction is the following result of Bonatti and Díaz ([BD3, Theorem 3.2]): *There exist an open set $\mathcal{U}_0 \subset \text{Diff}^1(S)$ and a dense subset $\mathcal{D}_0 \subset \mathcal{U}_0$ such that for every $f \in \mathcal{D}_0$ there are an open set $B \subset S$ and an integer $n \in \mathbb{N}$ such that every $x \in B$ is a periodic point of f of (prime) period n .*

Let $f_0 : S \rightarrow S$ be a diffeomorphism from the set \mathcal{D}_0 above. Let $X_0^t : M \rightarrow M$ be the suspension flow. As usual, M is the 4-manifold obtained from $S \times [0, 1]$ by gluing points $(x, 1)$ and $(f_0(x), 0)$. We will identify S with the submanifold $\{(x, 0) \in M; x \in S\}$ of M .

Let $\mathcal{U} \subset \mathfrak{X}^1(M)$ be a small neighborhood of X_0 such that every vector field $X \in \mathcal{U}$ is transverse to S and, moreover, the first-return map $f_X \in \text{Diff}^1(S)$ belongs to \mathcal{U}_0 . For $X \in \mathcal{U}$, we let $\tau_X : S \rightarrow \mathbb{R}_+$ be the return-time map, which is a C^1 -smooth function depending continuously (in the C^1 topology) on $X \in \mathcal{U}$.

We will omit the proof of the following:

Lemma 4. *For every $X \in \mathcal{U}$ and every neighborhood $\mathcal{V} \ni X$, if \tilde{f} is a small perturbation of f_X and $\tilde{\tau}$ is a small perturbation of τ_X then there is $\tilde{X} \in \mathcal{V}$ such that $f_{\tilde{X}} = \tilde{f}$ and $\tau_{\tilde{X}} = \tilde{\tau}$.*

Now let $X_1 \in \mathcal{U}$. We shall prove that there exists X_4 arbitrarily close to X_1 which has a non-trivial homoclinic class which is not topologically mixing.

Let $f_1 = f_{X_1}$, and $\tau_1 = \tau_{X_1}$. Take $f_2 \in \mathcal{D}_0$ close to f_1 . Since $f_2 \in \mathcal{D}_0$, there is a ball $B \subset S$ of points that are f_2 -periodic, of period n . Let $\tau_1^n : S \rightarrow \mathbb{R}_+$ be defined by $\tau_1^n = \sum_{j=0}^{n-1} \tau_1 \circ f_1^j$.

Using a chart, we identify B with a ball $B(0, r) \subset \mathbb{R}^3$, in such a way that the kernel of the differential $D\tau_1^n(0)$ contains the plane xy .

Let $f_3 \in \text{Diff}^1(S)$ be a perturbation of f_2 such that:

- f_3 equals f_2 outside $\cup_{j=0}^{n-1} f_2^j(B)$;
- there exists a ball $B_1 = B(0, r_1)$, with $0 < r_1 < r$, such that $f_3^n(B_1) = B_1$;
- f_3^n restricted to B_1 is a orthogonal rotation (indicated by R) of angle $2\pi/m$, where $m \in \mathbb{N}$, along the axis y .

It is easy to construct a map $\tau_3 : S \rightarrow \mathbb{R}$ C^1 close to τ_1 , such that $\tau_3^n = \sum_{j=0}^{n-1} \tau_3 \circ f_3^j$ is an affine map in a smaller ball B_2 around 0 and such that $D\tau_3^n(0) = D\tau_1^n(0)$. That is, if $x \in B_2$ then $\tau_3^n(x) = \tau_3^n(0) + D\tau_3^n(0) \cdot x$.

Using Lemma 4, we find a flow X_3 close to X_1 and such that $f_{X_3} = f_3$ and $\tau_{X_3} = \tau_3$.

Let $x \in B_2 \setminus \{0\}$. Its successive returns to B_1 under the flow X_3 are $R(x), \dots, R^{m-1}(x), R^m(x) = x$. In particular, x is a periodic point. Summing the respective return times we get that the period of x is $\sum_{j=0}^{m-1} \tau_3^n(R^j(x)) = m\tau_3^n(0)$, since $\sum_{j=0}^{m-1} R^j(x)$ belongs to the y axis. That is, all points in $B_2 \setminus \{0\}$ are periodic under X_3 of (prime) period $m\tau_3^n(0)$.

Let $B_3 \subset B_2$ be a ball (not centered in 0) such that $\text{cl } B_3, \text{cl } R(B_3), \dots, \text{cl } R^{m-1}(B_3)$ are pairwise disjoint.

Choose now some $f_4 : S \rightarrow S$ which is C^1 close to f_3 and such that:

- f_4 equals f_3 outside B_3 ;
- f_4^{nm} restricted to B_3 has a non-trivial homoclinic class (say, a solenoid attractor).

Let also $\tau_4 : S \rightarrow \mathbb{R}_+$ be given by $\tau_4 = \tau_3 \circ f_3^{-1} \circ f_4$. Then τ_4 is C^1 close to τ_3 . Using Lemma 4 again, we obtain X_4 C^1 close to X_3 such that $f_{X_4} = f_4$ and $\tau_{X_4} = \tau_4$. The return time of a point $x \in B_3$ to B_3 under X_4 is $\tau_4^{nm}(x) = \tau_3^{nm}(f_3^{-1} \circ f_4(x))$, which independes of x (where, as usual, we let $\tau_4^{nm} = \sum_{j=0}^{nm-1} \tau_4 \circ f_4^j$ and $\tau_3^{nm} = \sum_{j=0}^{nm-1} \tau_3 \circ f_3^j$). Therefore X_4^t has a non-trivial homoclinic class which is not topologically mixing. \square

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