The purpose of this short note is to find estimates on the number of zeros of solutions of the equation
\[ u''(x) + \frac{C}{x^{2\alpha}} u(x) = 0, \tag{1}\]
for \( x \in [1, \infty) \) and \( \alpha > 1 \). We will establish that the number of zeros \( N = N(C, \alpha) \) satisfies
\[ N = O\left(\frac{\sqrt{C}}{\alpha - 1}\right). \tag{2}\]

When \( \alpha = 1 \) it is known that solutions of (1) have finitely (in fact, at most one) or infinitely many zeros depending on whether \( C \leq 1/4 \) or \( C > 1/4 \).

Let \( x_f \) be defined by the equation
\[ \frac{C}{x_f^{2\alpha}} = \frac{1}{4}, \]
that is
\[ x_f^{\alpha - 1} = 2\sqrt{C}. \]

It follows that \( C/x^{2\alpha} \leq (1/4)(1/x^2) \) exactly for \( x \geq x_f \), and thus by the comments above and Sturm comparison, the number of zeros of solutions of (1) in the interval \([x_f, \infty)\) is at most one. With this, let us define the recurrence relation
\[ x_{k+1} = x_k + \epsilon x_k^\alpha, \tag{3}\]
where \( \epsilon = \pi/\sqrt{C} \). Because in the interval \( I_k = [x_k, x_{k+1}] \) one has \( C/x^{2\alpha} \leq C/x_k^{2\alpha} \), it follows from Sturm comparison that the number of zeros contained in \( I_k \) of any solution of (1) cannot exceed 1. Hence we need to estimate the number \( m \) of iterations required to reach the point \( x_f \) under the recurrence relation (3) starting with \( x_0 = 1 \). To do this we do an area comparison. Let us consider the function \( y = 1/(\epsilon x^\alpha) \). The area under its graph in the interval \( I_k \), say \( A_k \), satisfies
\[ \frac{1}{\epsilon} \frac{x_{k+1} - x_k}{x_k^{\alpha+1}} \leq A_k \leq \frac{1}{\epsilon} \frac{x_{k+1} - x_k}{x_k^{\alpha}} = 1. \tag{4}\]

On the other hand,
\[ \frac{x_{k+1}}{x_k} = 1 + \epsilon x_k^{\alpha - 1} \leq 1 + \epsilon x_f^{\alpha - 1} = 1 + 2\pi, \]
which shows that the number \( m \) and of iterations, and thus \( N \), are bounded above by
\[ (1 + 2\pi)^\alpha \int_1^{x_f} \frac{dx}{\epsilon x^\alpha} = \frac{(1 + 2\pi)^\alpha}{\epsilon (\alpha - 1)} \left( 1 - \frac{1}{x_f^{\alpha-1}} \right) \leq \frac{(1 + 2\pi)^\alpha \sqrt{C}}{\pi (\alpha - 1)}. \tag{5}\]

In order to get a lower for \( N \) we argue as follows. Let \( 0 < r_0 < 1 \) be fixed. We will define \( 0 < r_{k+1} < r_k \) recursively in such a way that any solution of (1) is guaranteed to have at least one zero in the interval \( J_k = [r_k x_f, r_k x_f] \). Suppose \( 0 < r_k < 1 \) is defined. It is easy to see that for \( 1 \leq x \leq r_k x_f \) one has
\[ \frac{C}{x^{2\alpha}} \geq \left( \frac{1 + a_k}{4} \right) \frac{1}{x^2}, \tag{6}\]
where \( a_k \) is given by the equation
\[ \frac{1}{\sqrt{1 + a_k}} = x_f^{\alpha - 1}. \tag{7}\]
On $[1, r_k x_f]$ we compare equation (1) with
\[ v'' + \left( \frac{1 + a_k}{4} \right) \frac{1}{x^2} v = 0, \]
the solutions of which are given by linear combination of
\[ \sqrt{x} \sin \left( \frac{1}{2} \sqrt{a_k} \log x \right) \quad \text{and} \quad \sqrt{x} \cos \left( \frac{1}{2} \sqrt{a_k} \log x \right). \]
It follows that any solution of (8), and thus of (1), will have a zero in the interval $J_k = [r_{k+1} x_f, r_k x_f]$ provided
\[ \log r_k - \log r_{k+1} = \frac{2\pi}{\sqrt{a_k}}. \]
We use (9) to define the $r_k'$s recursively. Notice from (7) that the $a_k'$s will be increasing.

We now need to estimate how many iterations are required to bring $r_k x_f$ for the first time below the value 1, that is, roughly when $r_k = 1/x_f$. We will do this again by resorting to integrals. Let $s_k = -\log r_k$. Then (9) becomes
\[ s_{k+1} - s_k = \frac{2\pi}{\sqrt{e^{2(\alpha - 1)s_k}} - 1}. \]
We need to estimate how many steps are needed, roughly, to make $s_k = \log x_f$. Consider the function
\[ t = \frac{1}{2\pi} \sqrt{e^{2(\alpha - 1)s_k}} - 1. \]
On each interval $J_k$ the area $B_k$ under its graph satisfies
\[ 1 = \frac{1}{2\pi} \sqrt{e^{2(\alpha - 1)s_k}} - 1 (s_{k+1} - s_k) \leq B_k \leq \frac{1}{2\pi} \sqrt{e^{2(\alpha - 1)s_{k+1}} - 1} (s_{k+1} - s_k). \]
But
\[ \frac{e^{(\alpha - 1)s_{k+1}}}{e^{(\alpha - 1)s_k}} = e^{(\alpha - 1)(s_{k+1} - s_k)} = \left( \frac{r_k}{r_{k+1}} \right)^{\alpha - 1} = e^{\frac{2\pi(\alpha - 1)}{\sqrt{a_k}}} \leq e^{\frac{2\pi(\alpha - 1)}{\sqrt{a_k}}}, \]
and therefore
\[ \frac{1}{2\pi} \int_{s_0}^{\log x_f} \sqrt{e^{2(\alpha - 1)s} - 1} \, ds \]
is comparable to the number of iterations to be determined. For large $C$ and thus large $x_f$ this integral is comparable to
\[ \frac{1}{2\pi} \int_{s_0}^{\log x_f} e^{(\alpha - 1)s} \, ds, \]
which is easily computed and found to be less than $\sqrt{C}/(\alpha - 1)$. 

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