Spheres in \( \mathbb{R}^n \)

Let \( c_n \) be the volume in \( \mathbb{R}^n \) bounded by
\[
x_1^2 + \cdots + x_n^2 = 1.
\]

It is easy to see that
\[
c_{n+1} = 2c_n \int_0^1 (1 - x^2)^{n/2} dx,
\]
which implies that \( \lim_{n \to \infty} c_n = 0 \).

We want to study the behavior of the center of gravity of the region bounded by a hemisphere \( H_n \) as \( n \to \infty \). In \( \mathbb{R}^{n+1} \) the region is given by
\[
x_1^2 + \cdots + x_n^2 + x_{n+1}^2 \leq 1, \quad x_{n+1} \geq 0,
\]
and the center of gravity will be
\[
G_n = (0, \ldots, 0, h), \quad h = h(n) > 0.
\]

Intuition suggests \( h(n+1) < h(n) \), and we will furthermore show that

**Lemma 1:** The center of gravity \( G_n \) tends to the origin as \( n \to \infty \), that is,
\[
\lim_{n \to \infty} h(n) = 0.
\]

**Proof:** Let \( \epsilon > 0 \). We will show first that there exists \( n_0 \) so that \( h(n) < \epsilon \) if \( n \geq n_0 \). Let \( A = A(n, \epsilon) \) be the subregion of \( H_n \) where \( x_{n+1} \geq \epsilon \), and let \( B = B(n, \epsilon) \) be its complement in \( H_n \). Then
\[
\text{Vol}(A) = c_n \int_{\epsilon}^1 (1 - x^2)^{n/2} dx \leq c_n \left[ 1 - \epsilon^2 \right]^{n/2},
\]
while
\[
\text{Vol}(B) = c_n \int_0^\epsilon (1 - x^2)^{n/2} dx \geq c_n (\epsilon/2) \left[ 1 - (\epsilon/2)^2 \right]^{n/2}.
\]
The second bound follows from the fact that \( B \) contains a “disk” of height \( \epsilon/2 \) and radius \( \sqrt{1 - (\epsilon/2)^2} \). We will show that if \( n \) is sufficiently large, then the region \( A \) produces a momentum with respect to the hyperplane \( x_{n+1} = \epsilon \) that is smaller than that momentum of the region \( B \). This shows that for such large values of \( n \), the center of gravity must lie below \( \epsilon \).

The momentum produced by \( A \) is \( M_A = \text{Vol}(A) \cdot [a(n) - h(n)] \), where \( a(n) \) represents the position on its axis of symmetry of the center of gravity of \( A \). It follows that
\[
M_A \leq \text{Vol}(A) \leq c_n \left[ 1 - \epsilon^2 \right]^{n/2}.
\]
The momentum $M_B$ is larger than that of the aforementioned disk, that is,

$$M_B \geq c_n \frac{\epsilon}{2} \left[1 - (\epsilon/2)^2\right]^{n/2} \cdot \frac{3\epsilon}{4}.$$  \hfill (4)

It follows from (3) and (4) that, for $\epsilon$ fixed, $M_B > M_A$ for all large $n$.

**Lemma 2:** The heights $h(n)$ are decreasing with $n$.

**Proof:** The expression $h = h(n)$ is determined by the equation:

$$\int_0^h (h - x)(1 - x^2)^{n/2} dx = \int_1^1 (x - h)(1 - x^2)^{n/2} dx.$$  \hfill (5)

Let $\alpha(s), \beta(s)$ be defined by

$$\alpha(s) = \int_s^1 (x - s)(1 - x^2)^{n/2} dx, \quad \beta(s) = \int_0^s (s - x)(1 - x^2)^{n/2} dx.$$

It is easy to see that

$$\alpha'(s) = -\int_s^1 (1 - x^2)^{n/2} dx < 0, \quad \beta'(s) = \int_0^s (1 - x^2)^{n/2} dx > 0,$$

and that

$$\alpha(0) > 0, \quad \alpha(1) = 0, \quad \beta(0) = 0, \quad \beta(1) > 0.$$

With this, the value $h = h(n)$ for which (5) hold is unique. On the other hand,

$$\frac{\partial \alpha}{\partial n} = \frac{1}{2} \int_s^1 (x - s)(1 - x^2)^{n/2} \log(1 - x^2) dx,$$

and

$$\frac{\partial \beta}{\partial n} = \frac{1}{2} \int_0^s (s - x)(1 - x^2)^{n/2} \log(1 - x^2) dx.$$

Since (5) holds, one can see that

$$\int_0^h (h - x)(1 - x^2)^{n/2} \log(1 - x^2) dx > \int_1^1 (x - h)(1 - x^2)^{n/2} \log(1 - x^2) dx,$$

es decir,

$$\frac{\partial \beta}{\partial n}(h) > \frac{\partial \alpha}{\partial n}(h).$$

This ensure that the point of intersection of the graphs of $\alpha$ y $\beta$ moves to the left as $n$ increases.