Ricci curvature and a criterion for simple-connectivity on the sphere
Martin Chuaqui
University of Pennsylvania

Abstract

From the recent work of Osgood and Stowe on the Schwarzian derivative for conformal maps between Riemannian manifolds we derive a sharp sufficient condition for a domain on the sphere to be simply-connected. We show further that a less restrictive form of the condition yields a uniform lower bound for the length of closed geodesics.

Introduction

Osgood and Stowe have recently defined a notion of Schwarzian derivative for conformal mappings of Riemannian manifolds which generalizes the classical operator for analytic functions in the plane [O-S 1]. As in complex analysis, where the Schwarzian derivative has been central as a means of characterizing conditions for global univalence, these authors establish in [O-S 2] an injectivity criterion for conformal local diffeomorphisms \( \psi \) of a Riemannian \( n \)-manifold \((M,g)\) to the standard sphere \( S^n \). The univalence of \( \psi \) follows from a bound on the norm of the Schwarzian derivative by geometric quantities of \( M \) (Theorem 1.1). This result allows a unified approach to a vast class of distortion theorems in the plane, as different criteria can be derived from it on a given domain just by changing the metric \( g \) conformally. Indeed, in [O-S 2] the authors obtain as corollaries with \( M \) the unit disc in the plane and \( g \) alternately the euclidean and hyperbolic metric, two classical conditions of Nehari. Most of the known criteria, including a recent injectivity result of Epstein [Ep], and some new conditions on the unit disc and simply-connected domains are derived in [Ch 1] from Theorem 1.1.

We shall show in this paper that a local diffeomorphism \( \psi \) as before satisfying a particular form of the criterion in [O-S 2] forces the manifold \( M \) to be simply-connected (Theorem 2.1). Our main result, a sharp criterion for simple-connectivity for domains in \( S^n \), will appear as a reformulation of Theorem 2.1 when using conformal invariance. This allows one to translate the existence of \( \psi \) to that of a conformal metric on a domain \( \Omega \subset S^n \) with the property that the norm of the trace free Ricci tensor is bounded above by a dimensional constant multiple \( c_n \) of the scalar curvature. The effect of changing the constant \( c_n \) to \( c \) is, in our opinion, quite remarkable. With \( c < c_n \) one can construct a reflection \( \Lambda : S^n \to S^n \) which maps \( \Omega \) to \( S^n - \overline{\Omega} \) and which fixes pointwise \( \partial \Omega \) [Ch 2]. The mapping \( \Lambda \) is quasiconformal in the sense of Ahlfors, i.e., the ratio of the largest and smallest eigenvalue of the symmetrized differential \( (\mathcal{D}\Lambda)(\mathcal{D}\Lambda)^t \) is uniformly bounded. Here we show that for \( c > c_n \) the criterion yields a uniform lower bound for the length of closed geodesics in \( \Omega \).

1. Preliminaries

We shall present in this section enough of the work in [O-S 1] so that we can state the injectivity criterion in [O-S 2]. We will omit proofs and refer the reader to the sources for more details.
Let $M$ be an $n$-dimensional Riemannian manifold with metric $g$. When $M = \mathbb{R}^n$, we will denote by $g_0$ the euclidean metric and $g_1$ will stand for the standard metric on the sphere $S^n$.

Given a conformal metric $\hat{g} = e^{2\varphi}g$ on $M$, Osgood and Stowe define the Schwarzian tensor of $\hat{g}$ with respect to $g$ as the symmetric, trace free (0,2)-tensor

$$B_g(\varphi) = \text{Hess}(\varphi) - d\varphi \otimes d\varphi - \frac{1}{n}(\Delta \varphi - |\text{grad} \varphi|^2)g,$$

where the metric dependent quantities on the right hand side are computed with respect to the metric $g$. We mention here that the tensor $B_g(\varphi)$ appears as the term by which the trace free part of the Ricci tensor changes under the conformal change of metric $g$ to $e^{2\varphi}g$. When $\psi$ is a conformal local diffeomorphism of $(M,g)$ to another Riemannian manifold $(N,g')$, then $\psi^*(g') = e^{2\varphi}g$ with $\varphi = \log |D\psi|$. The Schwarzian derivative of $\psi$ is defined by

$$S_g(\psi) = B_g(\varphi).$$

For an analytic map $\psi$ in the plane, with $g = g' = g_0$, then $\varphi = \log |\psi'|$ and computing in standard coordinates one gets

$$S_g(\psi) = \left(\begin{array}{cc} \text{Re}\{\psi, z\} & -\text{Im}\{\psi, z\} \\ -\text{Im}\{\psi, z\} & -\text{Re}\{\psi, z\} \end{array}\right),$$

where $\{\psi, z\} = (\psi''/\psi')' - 1/2(\psi'/\psi')^2$ is the classical Schwarzian derivative.

On $M$, the conformal metric $\hat{g} = e^{2\varphi}g$ is called Möbius with respect to $g$ if $B_g(\varphi) = 0$, and so a conformal local diffeomorphism $\psi$ is said to be Möbius if $S_g(\psi) = 0$. If $\varphi$ and $\sigma$ are smooth functions on $M$, then there is an important identity:

$$B_g(\varphi + \sigma) = B_g(\varphi) + B_{\hat{g}}(\sigma), \quad (1.1)$$

where $\hat{g} = e^{2\varphi}g$. In a chain of conformal local diffeomorphisms $\psi_1 : (M, g) \to (N_1, g')$ and $\psi_2 : (N_1, g') \to (N_2, g'')$, equation (1.1) can be formulated as

$$S_g(\psi_2 \circ \psi_1) = S_g(\psi_1) + \psi_1^*(S_{g'}(\psi_2)). \quad (1.2)$$

This recovers the classical formula of the Schwarzian derivative of a composition of analytic maps in the plane.

By $||B_g(\varphi)||$ we mean the norm of the Schwarzian tensor $B_g(\varphi)$ with respect to $g$, as a bilinear form on each tangent space, that is,

$$||B_g(\varphi)|| = \max\{|B_g(\varphi)(X,Y)| : |X| = |Y| = 1\}.$$  

In cases, we will need to consider the norm of $B_g(\varphi)$ in a metric $\hat{g} = e^{2\varphi}g$ conformal to $g$. Then

$$||B_g(\varphi)||_{\hat{g}} = e^{-2\sigma}||B_g(\varphi)||.$$

With this, we present the theorem in [O-S 2].
Theorem 1.1 Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 2\) and \(\psi : (M, g) \to (S^n, g_1)\) a conformal local diffeomorphism. Suppose that the scalar curvature of \(M\) is bounded above by \(n(n-1)K\) for some \(K \in \mathbb{R}\), and that any two points in \(M\) can be joined by a geodesic of length \(< \delta\) for some \(0 < \delta \leq \infty\). If

\[ ||S_g(\psi)|| \leq \frac{2\pi^2}{\delta^2} - \frac{1}{2}K \]

then \(\psi\) is injective.

With \(M\) the unit disc in the plane and \(g\) alternately the euclidean and hyperbolic metric, Osgood and Stowe derive from this theorem the classical criteria of Nehari, namely that

\[ |\{\psi, z\}| \leq \frac{\pi^2}{2} \quad \text{or} \quad |\{\psi, z\}| \leq \frac{2}{(1-|z|^2)^2}, \quad \text{all } |z| < 1 \]

imply that \(\psi\) is injective.

We point out that in Theorem 1.1, the target \((S^n, g_1)\) can be replaced by the standard hyperbolic space \(H^n\) or \((R^n, g_0)\). This follows from the transformation law (1.1) and the fact that both \(g_1\) and the hyperbolic metric are Möbius with respect to the euclidean metric. Finally, let \(\text{scal}(g)\) be the scalar curvature of \(g\). It is easy to see that the proof given by Osgood and Stowe works equally well when assuming that at each point in \(M\) the norm of the Schwarzian derivative of \(\psi\) is bounded above by

\[ \frac{2\pi^2}{\delta^2} - \frac{\text{scal}(g)}{2n(n-1)}. \]

2. A criterion for simple-connectivity

To begin with, we note the following consequence of Theorem 1.1.

Theorem 2.1 Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\) and \(\psi : (M, g) \to (S^n, g_1)\) a conformal local diffeomorphism. If

\[ ||S_g(\psi)|| \leq -\frac{\text{scal}(g)}{2n(n-1)} \]

then \(M\) is simply-connected.

Even though one can give an independent proof of this theorem, it will follow from Theorem 2.2. Let us consider the image \(\Omega = \psi(M)\). Under the hypotheses of Theorem 2.1, \(\psi\) is injective and we let \(g_2 = e^{2\rho}g_1 = \phi^*(g)\), where \(\phi = \psi^{-1}\). The addition formula (1.2) implies

\[ S_g(\psi) = -\psi^*(S_{g_1}(\phi)) = -\psi^*(B_{g_1}(\rho)) \]

and therefore

\[ ||S_g(\psi)|| = ||B_{g_1}(\rho)||_{g_2}. \]

Our main result is
Theorem 2.2 Let $\Omega \subset S^n$ be a domain with a complete metric $g_2 = e^{2\rho}g_1$. If

$$||B_{g_1}(\rho)||_{g_2} \leq \frac{-\text{scal}(g_2)}{2n(n-1)}$$

then $\Omega$ is simply-connected.

Proof: Let $\tilde{\Omega}$ be the universal cover of $\Omega$ with covering map $\pi$ and metric $\tilde{g} = \pi^*(g_2)$. We consider $\pi$ as a conformal map from $(\tilde{\Omega}, \tilde{g})$ into $(S^n, g_1)$. We shall show that

$$||S_{\tilde{g}}(\pi)|| = ||B_{g_1}(\rho)||_{g_2},$$

which by Theorem 1.1 implies the univalence of $\pi$ and consequently, the theorem.

We have

$$\pi^*(g_1) = \pi^*(e^{-2\rho}g_2) = e^{-2(\rho \circ \pi)}\tilde{g},$$

hence

$$S_{\tilde{g}}(\pi) = B_{\tilde{g}}(-\rho \circ \pi) = \pi^*(B_{g_2}(-\rho)) = -\pi^*(B_{g_1}(\rho)).$$

This proves (2.1).

Remarks

(1) Theorem 2.2 can be stated as well for domains in $R^n$ with $g_0$ as the background metric. The hypotheses of the theorem implicitly require that $\text{scal}(g_2) \leq 0$. Moreover one can show that $g_2$ has nonpositive curvature. Indeed, since $g_2$ is conformally flat, its Weyl tensor vanishes, and now a classical decomposition of the Riemann curvature tensor allows us to compute sectional curvatures solely in terms of $\text{scal}(g_2)$ and the trace free part of the Ricci tensor. If $X,Y$ are orthonormal tangent vectors in the metric $g_2$, then the sectional curvature $K(X,Y)$ of $g_2$ is given by

$$K(X,Y) = \frac{\text{scal}(g_2)}{n(n-1)} + B_{g_1}(\rho)(X,X) + B_{g_1}(\rho)(Y,Y).$$

Therefore $K(X,Y) \leq 0$ and so $\Omega$ as in the theorem is actually diffeomorphic to $R^n$.

(2) The condition that $g_2$ be complete can be relaxed, in that, what one really needs is that any two points in $\tilde{\Omega}$ can be joined by some geodesic in the metric $\tilde{g}$. Also, the theorem can be stated slightly more generally: without the assumption of completeness, if

$$||B_{g_1}(\rho)||_{g_2} \leq \frac{2\pi^2}{\delta^2} - \frac{\text{scal}(g)}{2n(n-1)}$$

then there are no closed (not even nonsmoothly closing) geodesics in the metric $g_2$ of length $< \delta$.

(3) Theorem 2.2 is sharp. For $n = 2$ this can be verified by taking in the plane the ring $R_1 < |z| < R_2$ with its Poincaré metric $g = e^{2\varphi}g_0$. This metric satisfies the inequality

$$||B_{g_0}(\varphi)||_g \leq -(1 + \epsilon)\frac{\text{scal}(g)}{4}$$
where \( \epsilon = \left( \frac{\log(R_2/R_1)}{\pi} \right)^2 \) can be made arbitrarily small.

In higher dimensions we consider a similar example: a hyperbolic solid torus. Let \( n \geq 2 \) and let \( \Omega \) be the domain in \( \mathbb{R}^{n+1} \) given by

\[
\left\{ (x_1 \cos \theta, x_1 \sin \theta, x_2, \ldots, x_n) : (x_1 - a)^2 + x_2^2 + \cdots + x_n^2 < 1 \right\},
\]

where \( a > 1 \). To simplify notation, we write \( r^2 = (x_1 - a)^2 + x_2^2 + \cdots + x_n^2 \). Let \( g = e^{2\varphi} g_0 \) with \( \varphi = -\log(1 - r^2) \). This metric is complete and we will show that given \( \epsilon > 0 \), the inequality

\[
||B_{g_0}(\varphi)|| g \leq -(1 + \epsilon) \frac{\text{scal}(g)}{2n(n+1)}
\]

will hold throughout \( \Omega \) provided the constant \( a \) is sufficiently large. With respect to the coordinates \( x_1, \ldots, x_n, \theta \) the tensor \( B_{g_0}(\varphi) \) is a diagonal matrix with eigenvalues \( \lambda = \lambda_1 = \cdots = \lambda_n \) and \( \lambda_{n+1} \) given by

\[
\lambda = \frac{2a}{(n+1)(1 - r^2)x_1}
\]

and

\[
\lambda_{n+1} = \frac{-2na x_1}{(n+1)(1 - r^2)}.
\]

A standard formula gives

\[
-e^{2\varphi} \frac{\text{scal}(g)}{n(n+1)} = \frac{2}{n+1} \Delta \varphi + \frac{n-1}{n+1} |\nabla \varphi|^2
\]

\[
= \frac{4}{(1 - r^2)^2} - \frac{4a}{(n+1)(1 - r^2)x_1}.
\]

The vector fields \( \frac{\partial}{\partial x_i} \) have euclidean length 1 while \( \frac{\partial}{\partial \theta} \) has length \( x_1 \). It follows that

\[
||B_{g_0}(\varphi)||_g = e^{-2\varphi} x_1^2 \lambda_{n+1}.
\]

If \( c = 1 + \epsilon \) then the sought inequality is

\[
\frac{2na}{(n+1)(1 - r^2)x_1} \leq (1 + \epsilon)\left\{ \frac{2}{(1 - r^2)^2} - \frac{2a}{(n+1)(1 - r^2)x_1} \right\}
\]

which simplifies to

\[
\frac{(n+1+\epsilon)a}{(n+1)x_1} \leq \frac{1 + \epsilon}{1 - r^2}.
\]

Since \( a - 1 < x_1 \) the last inequality will hold if \( a \geq \frac{(n+1)(1+\epsilon)}{n\epsilon} \).

### 3. Short geodesics

The proof of Theorem 1.1 relies on translating the given inequality on \( \psi \) to a differential inequality along geodesics of a suitably chosen test function \( w \). To be more precise, let \( \gamma \) be a geodesic joining two given points \( x, y \in M \). The function \( w \) is nonnegative and constructed so that it vanishes at \( p \) if and only if \( \psi(x) = \psi(p) \) \cite{O-S 2}. Along \( \gamma \)

\[
w'' \geq -w(||S_g(\psi)|| + \frac{\text{scal}(g)}{2n(n-1)}) + \frac{(w')^2}{2w}
\]
whenever \( w > 0 \). The estimate on \(||S_g(\psi)|||\) as in Theorem 2.2 implies that \((w^\frac{1}{2})'' \geq 0\) and therefore \( \psi(x) \neq \psi(y) \).

We assume now that \( g \) is complete and consider the case when \( \psi \) satisfies the estimate

\[
||S_g(\psi)|| \leq -c \frac{\text{scal}(g)}{2n(n-1)}
\]

for some \( c > 1 \). Suppose also that \( \frac{\text{scal}(g)}{n(n-1)} \geq -s > -\infty \). Then

\[
(w^\frac{1}{2})'' \geq -\frac{(c-1)s}{4} w^\frac{1}{2}.
\]

A standard Sturm comparison theorem guarantees that \( w \) cannot vanish again before time

\[
d = \frac{2\pi}{\sqrt{s(c-1)}}.
\]

In other words, if \( \psi(x) = \psi(y) \) for \( x \neq y \), then the distance between these two points is at least equal to \( d \). We reformulate this as:

**Theorem 3.1** Let \( \Omega \subset S^n \) be a domain with a complete metric \( g_2 = e^{2\varphi}g_1 \). Assume that

\[-\infty < -s \leq \frac{\text{scal}(g)}{n(n-1)} \leq 0 \]

If for some \( c > 1 \)

\[
||B_{g_1}(\varphi)||_{g_2} \leq -c \frac{\text{scal}(g)}{2n(n-1)} \quad (3.1)
\]

then any closed geodesic in \( \Omega \) has length at least \( d \).

The ring domain \( R_1 < |z| < R_2 \) with its hyperbolic metric shows that (3.1) is sharp.

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**References**


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