CONJUNTOS EQUIDISTANTES
Y CÓNICAS GENERALIZADAS

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Introducción

El conjunto de puntos del plano que equidistan de dos conjuntos dados aparece naturalmente en muchas situaciones clásicas de la geometría. Por ejemplo, las cónicas clásicas, definidas como el conjunto de nivel de una ecuación polinomial de grado 2, pueden ser siempre realizadas como el conjunto equidistante a dos circunferencias (ver Sección 3.1). La importancia de las cónicas para el desarrollo e historia de la Matemática es indiscutible. Cada nuevo progreso en su estudio ha representado un verdadero adelanto: la determinación del área encerrada por Arquímides, su concepción como curvas planas por Apolonio, su aparición como la solución para el movimiento de los planetas por Kepler, el desarrollo de las geometrías proyectivas y analíticas por Desargues y Descartes, etc.

En otro campo, digamos menos académico, podemos hallar los conjuntos equidistantes como fronteras definidas convencionalmente en diferendos territoriales. Por ejemplo, la Convención de las Naciones Unidas sobre el Derecho del Mar, establece en su Artículo 15 que, en ausencia de un acuerdo previo entre las partes, la delimitación territorial del mar entre países en disputa ocurrirá en *... una línea media cuyos puntos sean equidistantes de los puntos más próximos de las líneas de base...* La importancia de esto evidencia la necesidad de comprender la estructura geométrica de los conjuntos equidistantes.

El estudio de conjuntos equidistantes (distintos de las cónicas), floreció varias décadas atrás con los trabajos de Wilker [11] y Loveland [4] (en la Sección 1.2 revisamos sus mayores contribuciones en lo que respecta a las propiedades topológicas de los conjuntos equidistantes).

Podemos encontrar en la literatura varias generalizaciones de cónicas. Por ejemplo, Groß y Strempel [3], basados en la definición usual de cónica como el conjunto de puntos del plano cuyas sumas ponderadas de distancias hasta dos puntos (los *puntos focales*), generalizan las cónicas admitiendo más de dos puntos focales, otros pesos distintos a ± 1, y puntos en mayores dimensiones. Recientemente, Vincze y Nagy [8] propusieron que una cónica generalizada es el conjunto de puntos cuya distancia promedio hasta un conjunto dado es prescrita. El estudio de estas generalizaciones de cónicas no está motivado solamente por razones teóricas, esto pues existen aplicaciones muy interesantes en teoría de aproximaciones, problemas de optimización, de tomografía geométrica (ver [5], [9]), etc.

En este curso presentaremos los conjuntos equidistantes como una generalización natural para las cónicas a partir de que estas pueden presentarse como conjuntos equidistantes a dos circuenferencias (*conjuntos focales*). Admitiendo conjuntos focales más complicados obtendremos conjuntos equidistantes más complicados (*cónicas generalizadas*). Nuestro propósito es mostrar que estas generalizaciones poseen muchas características de sus contrapartes clásicas. Por ejemplo, mostraremos que el conjunto equidistante a dos conjuntos disjuntos no vacíos, compactos y conexos, se parece mucho a una rama de una hipérbola en el sentido que se acerca a infinito de manera asintótica a dos rayos. Discutiremos también posibles generalizaciones de elipses y parábolas.

Al tratar con conjuntos equidistantes, las simulaciones computacionales aparecen como una herramienta muy poderosa, tanto para las aplicaciones como para detectar patrones y ganar intuición. Sin embargo, la validez de estas simulaciones se ve en entredicho por causa de dos situaciones bien específicas. La primera de ellas tiene que ver con el hecho que un computador manipula solo aproximaciones discretas de los conjuntos focales (cuya calidad está determinada por la memoria del computador, la recolección de datos reales, la resolución de la pantalla, etc.) En esta dirección, presentamos un resultado acerca de la continuidad en la topología Hausdorff de los conjuntos equidistantes (ver Teorema 2.1), que en palabras simples dice que al mejorar las aproximaciones a los conjuntos focales, los conjuntos equidistantes que se obtienen se parecen cada vez más al conjunto equidistante genuino. Este resultado es muy importante en aplicaciones computacioneles, pues implica en particular que al aumentar la memoria del computador (o la calidad de la recolección de datos), seremos capaces de calcular una mucho mejor aproximación del conjunto equidistante que queremos simular. La segunda situación referida tiene que ver con lo siguiente: la pantalla de un computador (o el arreglo de memoria en que se guardan los datos) es un objeto discreto formado por una cantidad finita de pixeles (o espacios de memoria). Para determinar cuales de estos pixeles serán considerados como parte del conjunto equidistante debemos, para cada uno de ellos, verificar que la diferencia de las distancias a los conjuntos focales se anula. Sin embargo, al trarse de un conjunto discreto, las funciones distancias asumen valores discretos y quizás esta diferencia de distancias nunca se anule. Con el objetivo de obtener una aproximación razonable del conjunto equidistante debemos entonces introducir un criterio más tolerante. La situación general es la siguiente: nos encontramos con un pixel para el cual la diferencia de distancias es muy pequeña y nos preguntamos si esto significa que hay puntos del plano dentro de la región representada por tal pixel que están efectivamente en el conjunto equidistante. El Teorema 2.2 nos entrega un criterio útil para responder afirmativamente a tal pregunta, pues nos provee de una estimación fina para la distancia desde un punto casi equidistante hacia el conjunto equidistante.

Este curso está basado en el artículo [7], el cual contiene varios resultados originados en la Memoria de Iniciación Científica de Patricio Santibáñez, dirigida por el autor en 2011, en la Facultad de Matemáticas de la Universidad Católica de Chile.

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Capítulo 1

Conjuntos equidistantes

En este Capítulo definiremos los conjuntos equidistantes en el espacio euclidiano $\mathbb{R}^n$ y mostraremos algunas de sus propiedades topológicas más interesantes.

1.1. Definitions

We consider $\mathbb{R}^n$ endowed with the classical Euclidean distance

$$dist(x, y) := \sqrt{\sum_{j=1}^{n} (y_j - x_j)^2},$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. One easily extends this definition to admit the distance between a point $\tilde{x}$ and a set $X \subset \mathbb{R}^n$ as

$$dist(\tilde{x}, X) = \inf_{x \in X} dist(\tilde{x}, x).$$

Given two nonempty sets $A, B \subset \mathbb{R}^n$ we define the *equidistant set* to A and B as

$$\{A = B\} := \{x \in \mathbb{R}^n \ : \ dist(x, A) = dist(x, B)\}.$$

This notation is due to Wilker [11]. We also use the word *midset* as proposed by Loveland [4]. We say that A and B are the *focal sets* of the midset $\{A = B\}$. For $x \in \mathbb{R}^n$ we write $\mathcal{P}_x(A) = \{p \in A \ : \ dist(x, A) = dist(x, p)\}$ the set of *foot points from x to A.*

Next we define some notation. Given two points $x, y \in \mathbb{R}^n$ we write

$$[x, y] := \{tx + (1-t)y \ : \ 0 \leq t \leq 1\},$$

and we call it the *closed segment* between x and y (analogously for $[x, y), (x, y]$, and (x, y)). For $r > 0$, we write $\overline{B}(x, r), B(x, r)$, and $C(x, r)$, to represent the closed ball, the open ball, and the sphere centered at x with radius r, respectively. For $v \neq 0$ in $\mathbb{R}^n$ we write

$$[x, \infty)_v := \{x + tv \ : \ t \geq 0\}$$

for the *infinite ray starting at* x *in the direction of* v. We write $l_{a,v} := [a, \infty)_{-v} \cup [a, \infty)_v$ for the entire straight line passing through a in the direction v.

Exercise Compute explicitly the midset $\{A = B\} \subset \mathbb{R}^2$ in the following cases

- $A = \{(-1, 0)\}, B = \{(1, 0)\}$.
- $A = [0, 1] \subset \mathbb{R}^2$ and $B = [(0, 1), (\frac{1}{3}, 1)] \cup [(\frac{2}{3}, 1), (1, 1)]$.
- $A = [0, 1] \subset \mathbb{R}^2$ and $B = \mathcal{C}_{1/3} \times \{1\}$, where $\mathcal{C}_{1/3}$ is the classical Cantor Set.

Exercise Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for every $x \in [0, 1]$. $A = [0, 1] \subset \mathbb{R}^2$ and $B = graph(f)$. Show that $\{A = B\} \cap \{(x, y) \mid 0 \leq x \leq y\}$ is the graph of a continuous function $M_f : [0, 1] \to \mathbb{R}$. Under what conditions one can claim that M_f is differentiable ?

Exercise Let $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}^n$ be an euclidian isometry, that is $dist(x, y) = dist(\mathcal{I}(x), \mathcal{I}(y))$ for every $x, y \in \mathbb{R}^n$. Let $A \subset \mathbb{R}^n$ be a closed nonempty set. What can you say about $\{A = \mathcal{I}(A)\}$?

1.2. Topological properties of midsets.

Since the closure $\overline{A}$ of a nonempty set $A \subset \mathbb{R}^n$ satisfies $dist(x, A) = dist(x, \overline{A})$ for every $x \in \mathbb{R}^n$, we can easily conclude that $\{A = B\} = \{\overline{A} = \overline{B}\}$, for every $A, B \subset \mathbb{R}^n$. Hence we can consider closed sets as focal sets of midsets, and this will be assumed in the remaining part of these notes.

Lemma 1.1 *Midsets are always closed sets.*

Proof. The function

$$\begin{array}{rcl} d_A : \mathbb{R}^n & \longrightarrow & \mathbb{R}, \\ x & \longmapsto & d_A(x) := dist(x, A) \end{array}$$

is continuous and

$$\{A = B\} = d_{A,B}^{-1}(0),$$

where $d_{A,B}(x) := d_A(x) - d_B(x)$. □

Lemma 1.2 *Midsets are non-empty sets.*

Proof. We can compute the function $d_{A,B}$ over a continuous path joining A and B in order to obtain a zero for $d_{A,B}$ (in fact, it can be shown that every midset is nonempty if and only if the ambient space is connected). □

The main theorem in [11] is the connectivity of midsets for connected focal sets. We state this result without proof.

Theorem 1.3 (see Theorem 4 in [11]) *If A and B are nonempty connected sets, then $\{A = B\}$ is connected.* □

Example The above result can not be improved to path connectedness. Indeed, in the plane consider

$$\begin{aligned} A &= \{(x,1) \mid x \geq 0\} \cup \left\{(x,y) \mid x = \frac{1}{n}, -1 + \frac{1}{n} \leq y \leq 1, n = 1,3,5,\dots\right\}, \\ B &= \{(x,-1) \mid x \geq 0\} \cup \left\{(x,y) \mid x = \frac{1}{n}, -1 \leq y \leq \frac{1}{n} - 1, n = 1,3,5,\dots\right\}. \end{aligned}$$

The sets $\overline{A}, \overline{B}$ are path connected, while the midset $\{A = B\}$ is not path connected (notice that it is composed by the semi space $\{(x,y) \mid x \leq 0\}$ and the graph of a very fast oscillating function that resembles the graph of $\sin \frac{1}{x}$). △

The following is a simple property that, at least from the point of view of applications to sea frontiers, provides the politically correct fact that there is no region inside the UN's definition of the sea boundary between two disjoint countries.

Proposition 1.4 (see Theorem 2 in [11]) *Let $A, B \subset \mathbb{R}^n$ be two disjoint nonempty closed sets. Then the midset $\{A = B\}$ has empty interior.*

Proof. Let $x \in \{A = B\}$ and let $p_A \neq p_B$ be foot points in $\mathcal{P}_x(A), \mathcal{P}_x(B)$ respectively. We claim that for any $\tilde{x} \in [p_A, x)$ we have

$$dist(\tilde{x}, A) < dist(\tilde{x}, B). \tag{1.1}$$

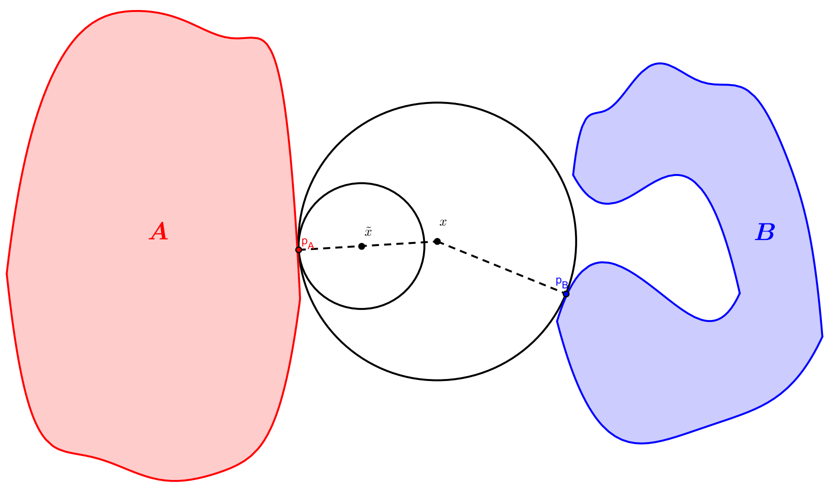


Figura 1.1:

Indeed, the closed ball $\overline{B}(x, dist(x, A))$ strictly contains the closed ball $\overline{B}(\tilde{x}, dist(\tilde{x}, A))$, except for the foot point p_A (see Figure 1.2). But $dist(\tilde{x}, p_A) = dist(\tilde{x}, A)$, which implies that there is no point of B in $\overline{B}(\tilde{x}, dist(\tilde{x}, A))$; this gives us (1.1). The inequality (1.1) tells us in particular that $\tilde{x} \notin \{A = B\}$, and the proposition follows by picking $\tilde{x}$ as close to x as necessary. □

Remark Notice above that $dist(\tilde{x}, A) = dist(x, A) - dist(x, \tilde{x})$ and (1.1) can be improved to

$$dist(\tilde{x}, B) > dist(x, A) - dist(x, \tilde{x}). \tag{1.2}$$

(Counter)Example Consider $A = \{(0, y) \mid y \geq 0\}, B = \{(x, 0) \mid x \leq 0\}$. It is easy to see that

$$\{A = B\} = \{(-y, y) \mid y \geq 0\} \cup \{(x, y) \mid x \geq 0 \text{ and } y \leq 0\}.$$

Notice that $A \cap B = \{(0, 0)\}$ and $\{A = B\}$ has no empty interior. △

Continuing with the topological properties of midsets, we concentrate on the case when the focal sets A, B are disjoint compact connected nonempty sets. In that case one has the following.

Theorem 1.5 (see Theorem 3.2 in [4]**)** *Let $A, B \subset \mathbb{R}^n$ be two disjoint compact connected nonempty sets.*

i) *If $n = 2$ then the midset $\{A = B\}$ is a topological $1-$manifold.*

ii) *For $n > 2$ the above result is no longer true in general. However, for every n, if A is convex then $\{A = B\}$ is topologically equivalent to an open set of the sphere $\mathbb{S}^{n-1}$. Furthermore, the midset $\{A = B\}$ is homeomorphic to the sphere $\mathbb{S}^{n-1}$ if and only if A is convex and lies in the interior of the convex hull of B.* □

Capítulo 2

Resultados de aproximación

En este Capítulo nos concentramos en los dos problemas computacionales planteados en la Introducción. Mostraremos por una parte que los conjuntos equidistantes dependen continuamente de sus conjuntos focales en una topología adecuada. En una siguiente Sección mostramos una estimación fina de manera de asegurar la presencia de un punto del conjunto equidistante cerca de un punto para el cual la diferencia de distancias a los conjuntos focales es muy pequeña.

2.1. Continuity of midsets.

Let $(X, dist_X)$ be a compact metric space. For $A \subset X$ and $\varepsilon > 0$ we denote by

$$B(A, \varepsilon) := \{x \in X \ : \ dist_X(x, A) < \varepsilon\},$$

the ε-neighborhood of A. The *Hausdorff distance* between two compact sets $K_1, K_2 \subset X$ is

$$dist_{\mathcal{H}}(K_1, K_2) := \text{ínf}\{\varepsilon > 0 \ : \ K_1 \subset B(K_2, \varepsilon) \text{ and } K_2 \subset B(K_1, \varepsilon)\}.$$

This distance defines a topology on the space $\mathcal{K}(X)$, of compact subsets of X. With this topology, $\mathcal{K}(X)$ is itself a compact space (see for instance [6]). Given a convergent sequence $A_n \in \mathcal{K}(X)$, the Hausdorff limit is characterized as the set of points that are limits of sequences $x_n \in A_n$.

In general equidistant sets are closed but not necessarily bounded sets. In order to treat with compact sets and use the Hausdorff topology, we are going to consider restrictions of equidistant sets to a large enough ball containing both focal sets. Let R be a large positive number and A, B be compact sets such that $A \cup B \subset B(0,R)$. We write

$$\{A = B\}_R := \{A = B\} \cap \overline{B}(0,R).$$

We are interested in the continuity of the mapping

$$\begin{array}{rcl} \mathcal{M}id_R : \mathcal{K}(\overline{B}(0,R)) \times \mathcal{K}(\overline{B}(0,R)) & \longrightarrow & \mathcal{K}(\overline{B}(0,R)), \\ (A,B) & \longmapsto & \{A = B\}_R. \end{array}$$

Theorem 2.1 *If $A \cap B = \emptyset$ then $\mathcal{M}id_R$ is continuous at (A,B).*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathcal{K}(\overline{B}(0,R))$ so that

$$A_n \to A \quad \text{and} \quad B_n \to B.$$

Define $E_n := \{A_n = B_n\}_R \in \mathcal{K}(\overline{B}(0,R))$. A compactness argument (and a suitable subsequence), allows us to assume that there exists $E \in \mathcal{K}(\overline{B}(0,R))$ such that $E_n \to E$. We affirm that $\{A = B\}_R = E$.

- Let $e \in E$. There exist sequences $e_n \in E_n$, $a_n \in A_n$, $b_n \in B_n$ and two points $a \in A$, $b \in B$ such that

$$dist(e_n, a_n) = dist(e_n, A_n) = dist(e_n, B_n) = dist(e_n, b_n) \tag{2.1}$$

  with $e_n \to e$, $a_n \to a$ and $b_n \to b$. We claim that $dist(e,A) = dist(e,a)$. Assume otherwise that there is a point $\tilde{a} \in A$ such that $dist(e,\tilde{a}) < dist(e,a)$. There exists a sequence $\tilde{a}_n \in A_n$ with $\tilde{a}_n \to \tilde{a}$. But (2.1) implies

$$dist(e_n, a_n) \le dist(e_n, \tilde{a}_n),$$

  which leads to $dist(e,a) \le dist(e,\tilde{a})$. In a similar way one shows $dist(e,B) = dist(e,b)$. Taking the limit in (2.1), we get $dist(e,A) = dist(e,B)$ and then $E \subset \{A = B\}_R$.

- Let $m \in \{A = B\}_R$ and $a_n \in A_n$, $b_n \in B_n$ satisfying

$$dist(m, A_n) = dist(m, a_n), \text{ and} \tag{2.2}$$
$$dist(m, B_n) = dist(m, b_n). \tag{2.3}$$

Passing to a subsequence if necessary there exist $a \in A$, $b \in B$ such that $a_n \to a$ and $b_n \to b$. Then

$$\begin{aligned} dist(m, a_n) = dist(m, A_n) &\to dist(m, a), \text{ and} \quad (2.4) \\ dist(m, b_n) = dist(m, B_n) &\to dist(m, b). \quad (2.5) \end{aligned}$$

From (2.2), (2.3), (2.4), (2.5) one has

$$\lim_{n\to\infty} dist(m, A_n) - dist(m, B_n) = 0.$$

Passing to a subsequence (or interchanging the roles of A_n and B_n) we can assume that $dist(m, A_n) - dist(m, B_n)$ increases to zero. Let $t \geq 0$. We define $m_t \in [m, b]$ so that $dist(m, m_t) = t$. Define $f_n(t) = dist(m_t, A_n) - dist(m_t, B_n)$. Let $\varepsilon > 0$. We claim that there exists $\tilde{n} \in \mathbb{N}$ such that $f_n(\varepsilon) > 0$ for every $n \geq \tilde{n}$. Indeed, we know that

$$\begin{aligned} dist(m_\varepsilon, B) &= dist(m, B) - \varepsilon, \text{ and} \\ dist(m_\varepsilon, A) &> dist(m, A) - \varepsilon. \end{aligned}$$

Notice that the second inequality above follows from (1.2) (here we use the hypothesis $A \cap B = \emptyset$). From these facts we obtain $dist(m_\varepsilon, A) - dist(m_\varepsilon, B) > 0$. Since $f_n(\varepsilon) \to dist(m_\varepsilon, A) - dist(m_\varepsilon, B) > 0$ our claim holds. Using that $f_n(0) \leq 0$ for every $n \geq \tilde{n}$ we can pick $m_n \in [m, m_\varepsilon]$ such that $f_n(m_n) = 0$, that is $m_n \in \{A_n = B_n\}$. This construction holds for every $\varepsilon > 0$. A diagonal sequence argument then allows one to construct a sequence $m_n \in E_n$ with $m_n \to m$. That is, $m \in \lim E_n = E$, and finally $\{A = B\}_R \subset E$. □

2.2. Error estimates for quasi-equidistant points.

Given two nonempty closed disjoint sets A, B and $\varepsilon > 0$, we define the set of $\varepsilon-$equidistant points to A and B as

$$\{|A - B| < \varepsilon\} := \{x \in \mathbb{R}^n \ : \ |dist(x, A) - dist(x, B)| < \varepsilon\}.$$

This notion is crucial when we deal with computer simulations. Recall that finding an equidistant point is equivalent to finding a zero of a continuous function. In the case of computer simulations, this function is no longer continuous since it is evaluated in pixels (a discrete set). In fact, this function in general may have no zeros at all. Then, in order to draw a good picture of the equidistant set, we need to check for points (pixels) such that the difference between the distances to the focal sets is small enough to guarantee that inside a small neighborhood there is a zero for the continuos function that defines the midset. In conclusion, we look for a set $\{|A - B| < \varepsilon\}$ for some positive ε that depends on the screen resolution, computer capabilities, etc. As we will see, the theorem we present here requires a very specific configuration of the focal sets. Nevertheless, the reader should notice that the result can be applied to more general situations.

Let $x \notin A \cup B$. We say that A *and* B *are separated by an angle of measure* α *at* x if there exist two supporting lines l_A, l_B passing through x such that

1. l_A is a supporting line for A, and B lies in a different half-plane than A,
2. l_B is a supporting line for B, and A lies in a different half-plane than B, and
3. the angle formed at x by l_A and l_B measures α.

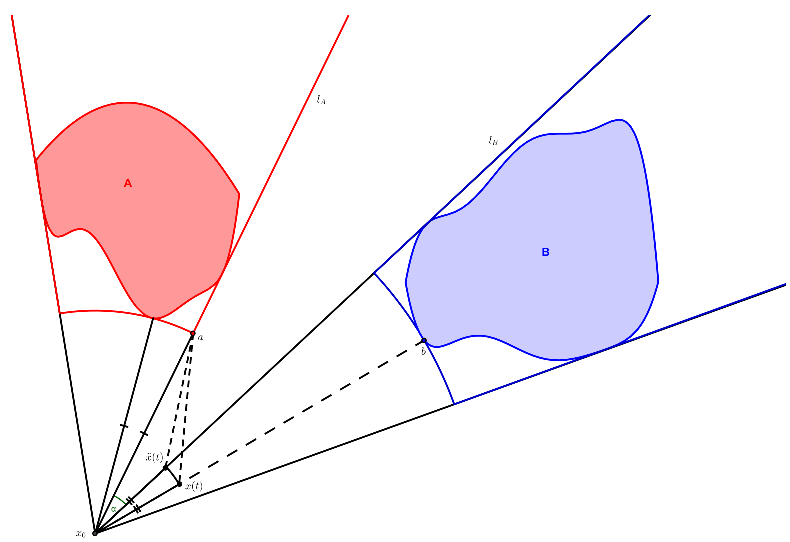


Figura 2.1: Construction for the proof of Theorem 2.2.

Theorem 2.2 (Error estimates) *Let A, B be two disjoint nonempty closed sets. Let $\varepsilon > 0$ and $x_0 \in \{|A - B| < \varepsilon\}$ such that A and B are separated by an angle of measure α at x_0. Then there exists $x_1 \in \{A = B\}$ verifying*

$$dist(x_1, x_0) < \frac{\varepsilon}{2}\left(\frac{\varepsilon + 2d}{\varepsilon + d - d\cos\alpha}\right),$$

where $d = \text{mín}\{dist(x_0, A), dist(x_0, B)\}$.

Proof. Consider the function $f(x) := d_{B,A}(x) = dist(x, B) - dist(x, A)$ and assume $d = dist(x_0, A)$, that is

$$0 \leq f(x_0) < \varepsilon.$$

We look for x_1 such that $f(x_1) = 0$. Let $b \in \mathcal{P}_{x_0}(B)$. We write $x(t)$ as the point on $[x_0, b]$ such that $dist(x_0, x(t)) = t$. Finally we write

$$g(t) := f(x(t)) = dist(x(t), B) - dist(x(t), A).$$

Since $f(x_0) = g(0)$, we have $0 \leq g(0) < \varepsilon$ and $g(\overline{d}) < 0$ where $\overline{d} = dist(x_0, B)$. Although we know that there exists $\overline{t} \in (0, \overline{d})$ that satisfies $g(\overline{t}) = 0$, the function g is not differentiable in general and we cannot directly estimate the size of $\overline{t}$. We are going to construct an upper bound for g in order to get a good estimate. Let a be the intersection of the circle centered at x_0 and radius d with l_A as shown in Figure 2.1. For every t we have

$$dist(x(t), a) \leq dist(x(t), A). \tag{2.6}$$

Define $\tilde{x}(t) \in l_B$ so that $dist(x_0, \tilde{x}(t)) = t$ (see Figure 2.1.). Thus we have

$$dist(\tilde{x}(t), a) \leq dist(x(t), A). \tag{2.7}$$

The left term above can be explicitly computed using elementary Euclidean geometry as

$$dist(\tilde{x}(t), a)^2 = d^2 + t^2 - 2dt\cos(\alpha). \tag{2.8}$$

Moreover, we know that

$$\begin{aligned} dist(x(t), B) &= dist(x_0, B) - t \\ &< d + \varepsilon - t. \end{aligned} \tag{2.9}$$

Using (2.6, 2.7, 2.8, 2.9) one gets (and defines $\hat{g}$ by)

$$g(t) < d + \varepsilon - t - \sqrt{d^2 + t^2 - 2dt\cos(\alpha)} := \hat{g}(t). \tag{2.10}$$

Notice that $\hat{g}(0) = \varepsilon$ and

$$\hat{t} = \frac{\varepsilon}{2}\left(\frac{\varepsilon + 2d}{\varepsilon + d - d\cos\alpha}\right)$$

verifies $\hat{g}(\hat{t}) = 0$. Finally, the inequality (2.10) helps us to find a point $\bar{t} \in (0, \hat{t})$ such that $f(\bar{t}) = 0$. $\square$

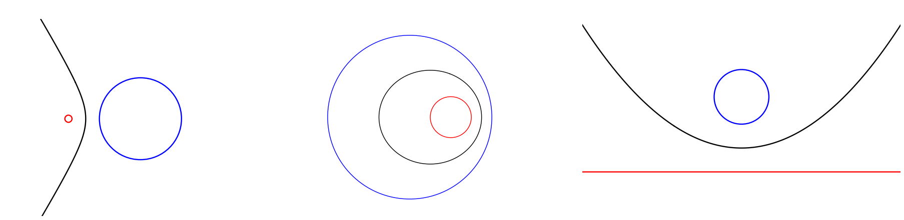
Capítulo 3

Cónicas generalizadas

En este Capítulo propondremos los conjuntos equidistantes como candidatos naturales para generalizaciones de las cónicas en el plano $\mathbb{R}^2$. Mostraremos que varias de las propiedades clásicas de las cónicas también aparecen en estas cónicas generalizadas.

3.1. Conics as midsets.

In this section we review the definition of the classical conics as the equidistant set to two circles (possibly degenerating into points or straight lines), as illustrated by Figure 3.1. In the sequel, we use complex notation for points in the plane.



(a) Hyperbola (b) Ellipse (c) Parabola

Figura 3.1: Classical conics

Hyperbola. Let $A = C(0, R)$ and $B = C(1, r)$ with $0 \leq r, R$ and $R < 1 - r$ (this implies $A \cap B = \emptyset$). The midset $\{A = B\}$ is composed of points $z \in \mathbb{C}$

such that

$$\begin{aligned} dist(z,A) &= dist(z,B), \\ |z|-R &= |z-1|-r, \text{ and} \\ |z-0|-|z-1| &= R-r. \end{aligned}$$

Thus, the midset $\{A = B\}$ is exactly the locus of points z in the plane such that the difference of the distance from z to 0 and 1 is constant, that is, the branch of a hyperbola. In the case $R = r$ we obtain a straight line.

Ellipse. We now consider two circles $A = C(0, R)$ and $B = C(1, r)$ with $R > 1+r$ (this implies that B lies inside A). The midset $\{A = B\}$ is composed of points $z \in \mathbb{C}$ such that

$$\begin{aligned} dist(z,A) &= dist(z,B), \\ R-|z| &= |z-1|-r, \text{ and} \\ |z-0|+|z-1| &= R+r. \end{aligned}$$

Thus, the midset $\{A = B\}$ is an ellipse with focal sets $\{0\}$ and $\{1\}$.

Parabola. The intermediate construction when one of the circles degenerates into a straight line and the other into a point is one of the most classical examples of an equidistant set. Namely, a parabola is the locus of points from where the distances to a fixed point (focus) and to a fixed line (directrix) are equal. Let us carry out the explicit computations for a simple example.

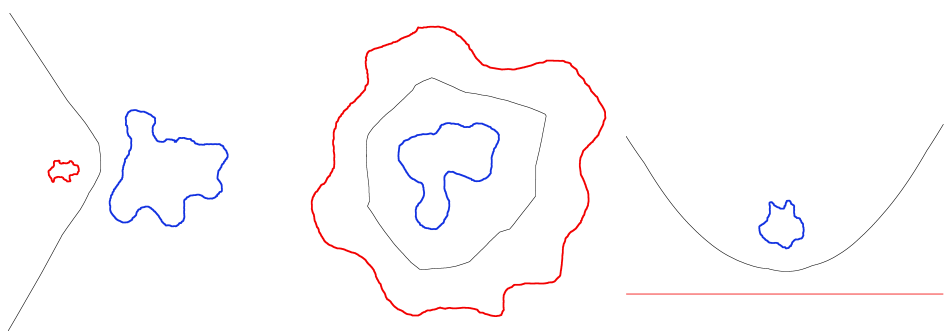
Indeed, we construct a parabola as the equidistant set between the line $y = -1$ (the set A), and the circle with center in $(0, 2)$ and radius 1 (the set B). Hence, a point (x, y) belongs to the equidistant set if and only if

$$\begin{aligned} dist((x,y),A) &= dist((x,y),B), \\ y+1 &= \sqrt{x^2+(2-y)^2}-1, \\ (y+2)^2 &= x^2+(2-y)^2, \text{ and} \\ y &= \frac{x^2}{8}. \end{aligned}$$

Conversely, we leave it as an exercise to the reader to show that every ellipse or branch of a hyperbola can be constructed as the midset of two conveniently chosen circles.

3.2. Midsets as generalized conics.

In section 3.1 we have seen how the classical conics can be realized as equidistant sets with circular focal sets. In this section, we want to interpret equidistant sets as natural generalizations of conics when admitting focal sets that are more complicated than circles. We concentrate on recovering geometric properties from conics for more general midsets. Figure 2 shows three (approximative versions of) midsets obtained by using an exhaustive algorithm for checking every pixel on the screen.



(a) Hyperbola (b) Ellipse (c) Parabola

Figura 3.2: Generalized conics

Generalized hyperbolas. (see Fig. 3.2(a)) In Section 3.1 we have seen that a branch of a hyperbola can be realized as the midset of two disjoint circles. In this section, we show that replacing these two discs by two disjoint compact connected sets we recover a midset that asymptotically resembles a branch of a hyperbola. Indeed, we show that far enough from these focal sets the midset consists of two disjoint continuous curves that asymptotically approach two different directions in the plane in infinity. This is the content of Theorem 3.5 below.

We require some additional definitions and notation. Let $\vec{r} = [a, \infty)_v$ be a ray starting at $a \in \mathbb{R}^2$ in direction $v \in \mathbb{R}^2$, with $\|v\| = 1$. Pick $v^\perp$ such that $\{v, v^\perp\}$ is a positive orthonormal basis for $\mathbb{R}^2$. For $\varepsilon > 0$, we define the *tube* of width ε around $\vec{r}$ as

$$tub_\varepsilon(\vec{r}) := \left\{ a + tv + sv^\perp \ : \ t \geq 0, |s| \leq \varepsilon \right\}.$$

We say that a set M has an *asymptotic end in the direction of* $\vec{r}$ if there exists $\varepsilon > 0$ such that the set $M_{\varepsilon,\vec{r}} = M \cap tub_\varepsilon(\vec{r})$ verifies the two following conditions.

i) The orthogonal projection from $M_{\varepsilon,\vec{r}}$ to $\vec{r}$ is a bijection.

ii) If we write $M_{\varepsilon,\vec{r}}$ using the parameters (t, s) of the tube $tub_\varepsilon(\vec{r})$, then the point (i) above yields a function

$$\begin{array}{rcl} s : [0,\infty) & \longrightarrow & [-\varepsilon,\varepsilon], \\ t & \longmapsto & s(t) \end{array}$$

in such a way that $M_{\varepsilon,\vec{r}}$ coincides with the graph of s. The second requirement is that

$$\lim_{t\to\infty} s(t) = 0.$$

Remark

i) Notice that the function s defined above is continuous since its graph is a closed set.

ii) The reader can notice that every ray $\vec{p} \subset \vec{r}$ induces an asymptotic end just by considering the suitable restriction. Even though one can properly formalize using an equivalence relation, we are going to consider all these ends as equal.

Let $K \subset \mathbb{R}^2$ be a compact set. We say that the straight line $l = l_{b,w}$ is a *supporting line* for K if $l \cap K \neq \emptyset$ and K is located entirely in one of the two half-planes defined by l. We say that $b \in l \cap K$ is a *right extreme point with respect to* l if $l \cap K$ is contained in $[b,\infty)_{-w}$ (analogously we define a *left extreme point*). A supporting line always has both types of extreme points, and they coincide if and only if the intersection $l \cap K$ contains only one point.

Lemma 3.1 *Let $\varepsilon > 0$. Assume that $K \subset \{(x, y) \mid x \leq \varepsilon\ ,\ y \leq 0\}$. For $h > 0$ we define $f_h(x) = dist\,((x, h), K)$. The function f_h is strictly increasing for $x \geq \varepsilon$.*

Proof. Let $x_2 > x_1 \geq \varepsilon$ and let $p_2 \in \mathcal{P}_{(x_2,h)}(K)$ be a foot point. We have

$$f_h(x_1) \leq dist((x_1, h), p_2) < dist((x_2, h), p_2) = f_h(x_2).$$

Indeed, the first inequality comes from the definition of $f_h(x_1)$ and the strict inequality is due to $x_2 > x_1$. □

In what follows, we are going to consider two disjoint compact sets A, B and a common supporting line l such that both sets are located in the same half-plane determined by l. For simplicity we assume that l is the real line and $(-1, 0)$ is the right extreme point of A and $(1, 0)$ is the left extreme point of B. Let $\varepsilon > 0$ be small enough. We assume that

$$A \subset \{(x, y) \ : \ x \leq -1 + \varepsilon \ , \ y \leq 0\}, \text{ and} \tag{3.1}$$
$$B \subset \{(x, y) \ : \ x \geq 1 - \varepsilon \ , \ y \leq 0\}. \tag{3.2}$$

Lemma 3.2 *Under the above hypotheses, for every $h > 0$ there exists a unique $x(h) \in [-1, 1]$ such that*

$$dist((x(h), h), A) = dist((x(h), h), B). \tag{3.3}$$

Moreover, $x(h)$ belongs to $(-\varepsilon, \varepsilon)$.

Proof. Since $(-1, 0) \in A$, we know that for every $(x, y) \in \{x \leq -\varepsilon \ , \ y \geq 0\}$ one has

$$dist((x, y), A) < dist((x, y), B).$$

Similarly we obtain that for every $(x, y) \in \{x \geq \varepsilon \ , \ y \geq 0\}$ one has

$$dist((x, y), A) > dist((x, y), B).$$

The continuity of the function f_h defined in Lemma 3.1 then gives at least one point $x(h) \in (-\varepsilon, \varepsilon)$ satisfying the equality (3.3). Applying the conclusion of Lemma 3.1, we see that the function

$$x \longmapsto dist((x, h), A) - dist((x, h), B)$$

is strictly increasing for $x \in [-1+\varepsilon, 1-\varepsilon]$. We then deduce the unicity of $x(h)$ as required. □

We apply the above Lemma in order to characterize asymptotically the midset of two focal sets with a common supporting line. Notice that in the hypotheses of the next Proposition we drop conditions (3.1) and (3.2).

Proposition 3.3 *Consider two disjoint compact sets A, B and a common supporting line l such that both sets are located in the same half-plane determined by l. For simplicity we assume that l is the real line and $(-1,0)$ is the right extreme point of A and $(1,0)$ is the left extreme point of B. For every $\varepsilon > 0$ there exists $\tilde{h} = \tilde{h}(\varepsilon) > 0$ such that for every $h > \tilde{h}$ there exists $x(h) \in (-\varepsilon, \varepsilon)$ such that the following holds*

$$\{A = B\} \cap \{(x,h) \ : \ x \in [-1,1]\} = \{(x(h),h)\}.$$

Proof. In order to apply Lemma 3.2, we need to show that we can recover conditions (3.1), (3.2). Since $(-1,0)$ is in A, for every $h > 0$ the foot points $\mathcal{P}_{(0,h)}(A)$ belong to the closed ball D_h centered at $(0,h)$ and passing through $(-1,0)$ (the same happens for $\mathcal{P}_{(0,h)}(B)$, for the same ball D_h since it also passes through $(1,0)$). In other words, one has

$$\mathcal{P}_{(0,h)}(A) \cup \mathcal{P}_{(0,h)}(B) \subset D_h \cap \{(x,y) \mid y \leq 0\}.$$

We define

$$A_h := D_h \cap A \quad , \quad B_h := D_h \cap B.$$

With these definitions it is clear that

$$\begin{aligned} dist((0,h),A) &= dist((0,h),A_h), \text{ and} \\ dist((0,h),B) &= dist((0,h),B_h). \end{aligned}$$

We claim that for every $\varepsilon > 0$ there exists $\tilde{h} > 0$ such that for every $h > \tilde{h}$ one has

$$\begin{aligned} A_h &\subset \{(x,y) \mid x \leq -1+\varepsilon \ , \ y \leq 0\}, \text{ and} \\ B_h &\subset \{(x,y) \mid x \geq 1-\varepsilon \ , \ y \leq 0\}. \end{aligned}$$

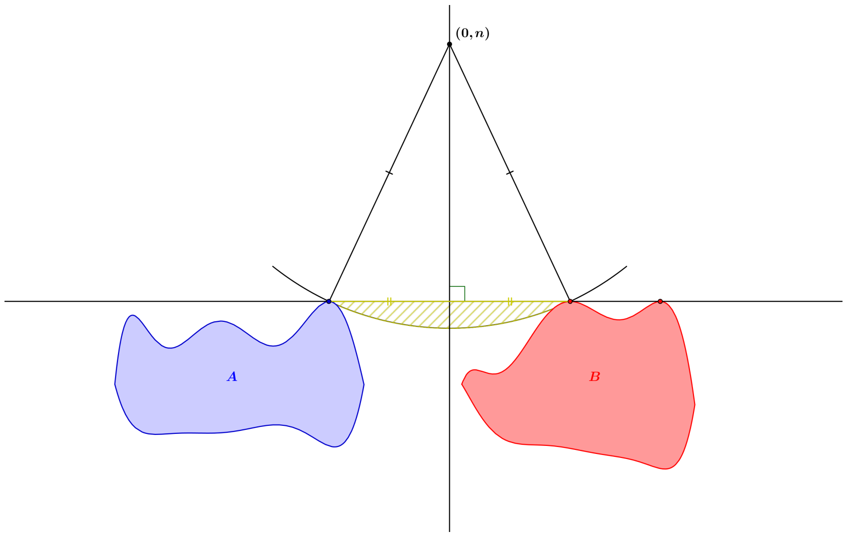


Figura 3.3: Foot points to $(0, n)$ lie inside the small shaded region.

Assume on the contrary that there exists $\tilde{\varepsilon} > 0$ and a sequence $(x_n, y_n) \in A_n$ with $x_n > -1 + \tilde{\varepsilon}$. Notice that since $(x_n, y_n) \in D_n$, we then have

$$-y_n + n \leq \sqrt{n^2 + 1}.$$

This, and the classical undergraduate limit $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$, implies that $y_n \to 0$ (see Fig. 3.3 for a geometric meaning). Since A is a compact set, there exists a subsequence (x_n, y_n) converging to a point $(\tilde{x}, 0) \in A$, with $\tilde{x} \geq -1 + \tilde{\varepsilon} > -1$. This contradicts the fact that $(-1, 0)$ is the right extreme point of A. We then apply Lemma 3.2 in order to find $x(h) \in (-\varepsilon, \varepsilon)$ in the midset $\{A = B\}$. It is easy to see for fixed $\varepsilon > 0$ and h large enough that $(-\varepsilon, h)$ is closer to A and (ε, h) is closer to B, thus concluding the proof. $\square$

Given two disjoint nonempty compact connected sets A, B, we want to discuss the existence of a common supporting line leaving both sets in the same half-plane. For this we need to remember the concept of the convex hull $ch(K)$ of a compact set $K \subset \mathbb{R}^2$, defined as the smallest convex set containing K. The convex hull $ch(K)$ is a convex compact set. Given two disjoint compact convex sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^2$, it is an interesting exercise to show that there exist four common supporting lines. Two of them are called *interior common tangents* and each one leaves the sets $\mathcal{A}, \mathcal{B}$ in a different half-plane. The remaining two supporting lines are called *exterior common tangents* and each one leaves both sets in the same half-plane.

Two disjoint nonempty compact connected sets A, B are called *ch-disjoint* if $ch(A) \cap ch(B) = \emptyset$. It is easy to see that supporting lines and common supporting lines of $ch(A), ch(B)$ are also supporting lines and common supporting lines of A, B respectively. The above discussion directly yields the following, which we present without proof.

Lemma 3.4 *Two nonempty compact connected sets A, B that are ch-disjoint have two distinct common supporting lines, each of which leaves both sets A, B in the same half-plane.* □

Now we can state the main theorem of this section.

Theorem 3.5 (Generalized hyperbola) *Let A, B be two nonempty compact connected sets that are ch-disjoint. There exists $R > 0$ and two disjoint rays $\vec{r}_1, \vec{r}_2$ such that*

$$\{A = B\} \cap B(0, R)^c$$

consists of exactly two asymptotic ends in the directions $\vec{r}_1$ and $\vec{r}_2$ respectively.

Proof. The existence of the two different asymptotic ends is due to Proposition 3.3 and Lemma 3.4. The remaining part of the proof consists of showing that there is no other piece of the midset going to infinity. This can be directly deduced from Theorem 1.5 part (i) which ensures that the midset $\{A = B\}$ is homeomorphic to the real line. We also present a self-contained proof. Assume that $\vec{r}_1, \vec{r}_2$ are not parallel and suppose that there exists a sequence $x_n \in \{A = B\}$ with $|x_n| > n$. Let $x_n = |x_n|e^{i\theta_n}$ be the complex notation for x_n. Taking a subsequence if needed, we can assume that there exists $\tilde{\theta} \in [0, 2\pi]$ such that $\theta_n \to \tilde{\theta}$. Let $l_{\tilde{\theta}^\perp}$ be a supporting line for $A \cup B$ that is orthogonal to the direction $\tilde{\theta}$ such that $A \cup B$ and infinitely many elements from $\{x_n\}_{n \in \mathbb{N}}$ are located in different half-planes. We claim that $l_{\tilde{\theta}^\perp}$ is a common supporting line for A and B. Indeed, assume for instance that $l_{\tilde{\theta}^\perp} \cap A = \emptyset$. In this case it is easy to see that for n large enough we should have $dist(x_n, A) > dist(x_n, B)$, which is impossible since x_n belongs to $\{A = B\}$. Hence, $\tilde{\theta}$ coincides with the direction of $\vec{r}_1$ (or $\vec{r}_2$) and one deduces that $\{x_n\}$ is necessarily a subset of the union of the two asymptotic ends. In order to prove the existence of exactly two asymptotic ends in case the rays $\vec{r}_1, \vec{r}_2$ are parallel, we can consider a slight perturbation A_ε of A in the Hausdorff topology in such a way that the above arguments can be applied to A_ε and B. We can conclude using Theorem 2.1 from Section 2.1. □

Remark For simplicity we stated this theorem for *ch*-disjoint sets, even though it holds for every pair of compact connected disjoint sets having two supporting lines, each of wich leaves both sets in the same half-plane.

Generalized parabolas. (see Fig. 3.2(c)) A remarkable geometric property of parabolas is that they are *strictly convex* in the sense that for any supporting line the parabola becomes asymptotically more and more separated from the supporting line. In other words, the parabola can be seen as the graph of a continuous function over the supporting line (a tangent) such that the values of this function tend to infinity with the parameter of the line (check for instance the parabola $y = x^2$ and see how the derivatives grow to infinity). We do not give a definition for generalized parabolas, but just say that midsets sharing various properties like *strict convexity* should be considered as some kind of generalization for parabolas.

Along the lines of the generalized hyperbolas treated in the previous paragraphs, we want to consider a midset defined by a compact connected focal set A (instead of the classical focus point) and some disjoint unbounded closed set B playing the role of the directrix. We also need to require some additional properties such as: $ch(B)$ does not intersect A (in order to obtain an unbounded midset); there is no common supporting line for A and B (in order to avoid the existence of an asymptotic ray), etc. For simplicity we are going to keep B as a straight line, even though the reader will be able to treat more general situations.

Proposition 3.6 *Let $A \subset \mathbb{R}^2$ be a nonempty connected compact set and B be a disjoint straight line. There exists $\overline{R} > 0$ so that for every $R \geq \overline{R}$ and every supporting line l for $\{A = B\} \cap B(0, R)^c$ one has*

$$\lim_{s \to \infty} dist\,(l, \{A = B\} \cap B(0, s)^c) = \infty.$$

Sketch of the proof. For this special case where B is a straight line, the proof can be easily obtained from the fact that the midset $\{A = B\}$ is actually the graph of a continuous function over B. We leave as an exercise to the reader to show that this function grows faster than any linear map.

We now outline a proof that fits more general situations. Assume for simplicity that B is the real line. The idea is to truncate B and consider the midset $\{A = B_R\}$, where $B_R = [-R, R] \subset \mathbb{R}$. As seen before, this is a generalized hyperbola that is asymptotic (lets say, to the right) to a ray $\vec{r}_R$ that is perpendicular to a segment $[a_R, (R, 0)]$ for some point $a_R \in A$ and passes through its midpoint. Since A is compact, the slope of $\vec{r}_R$ grows to infinity with R and the reader can easily complete the details. □

Exercise Study the midset that appears if instead of considering the directrix B as the real line, we consider other unbounded sets such as

- An angle such that A belongs to the convex region defined by this angle.
- An angle such that A belongs to the concave region defined by this angle.
- A classical parabola such that A belongs to the (convex) epigraph.

Generalized ellipses. (see Fig. 3.2(b)) In this case we don't have much to say. However, part (*ii*) of Theorem 1.5 (see Section 1.2) serves to recognize some topological reminiscences of ellipses when the focal sets of the midset are a convex compact set *inside* a compact set. Indeed, given a connected compact set $B \subset \mathbb{R}^2$ and a convex compact set $A \subset \mathbb{R}^2$ that lies *inside* B, the midset is homeomorphic to $\mathbb{S}^1$.

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