How circular are generalized circles

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A plane circle is defined as the locus of points that have constant distance (*radius*) from a distinguished point (*center*). In this short note we treat with a natural generalization of circles consisting in to replace the center point by a more complicated compact set. More precisely, given a non-empty compact set $K \subset \mathbb{R}^2$ and $r > 0$, we define the *generalized circle* centered at K and radius r as

$$\mathcal{C}_K(r) := \{x \in \mathbb{R}^2 \ : \ dist(x, K) = r\}.$$

The distance $dist(\cdot, \cdot)$ is the euclidean one and the distance from a point to a plane set is defined as usual taking the infimum over the distances to the points in the set. We will denote $C_y(r)$ the classical circle centered at $y \in \mathbb{R}^2$ and radius r.

The main goal of this note is to discuss in what measure these sets inherit geometric properties of classical circles. The study of *generalized circles* started many decades ago. Let us make a quick survey of the known topological properties of these sets.

Theorem 1 (see H. Federer [2]) *If K a compact convex subset of the plane $\mathbb{R}^2$ and $r > 0$, then $\mathcal{C}_K(r)$ is a closed curve of class C^1.* ■

Theorem 2 (see M. Brown [1], S. Ferry [3]) *Let $K \subset \mathbb{R}^2$ be a compact set and $r > 0$. Then*

- (i) *$\mathcal{C}_K(r)$ is the union of a finite collection of simple closed curves minus the union of their interiors.*
- (ii) *Each component of $\mathcal{C}_K(r)$ is locally connected.*
- (iii) *For almost all r the set $\mathcal{C}_K(r)$ is a $1-$manifold.* ■

Theorem 3 (see P. Pikuta [4]) *Let K a compact subset of the plane $\mathbb{R}^2$. There exists $r_0 > 0$ so that for every $r > r_0$ the set $\mathcal{C}_K(r)$ is a closed absolutely continuous curve.* ■

We are interested in the asymptotic behavior of generalized circles, in the sense that we look for $\mathcal{C}_K(r)$ for large r. Nevertheless, our methods also allow to recover

Theorems 1 and 3 above.

Every concept we will use is invariant under plane isometries, hence we will always assume that the compact set K to be used as *center*, is located in such way that the origin of the plane coincides exactly with the center of the smallest closed disc containing K (the so called *Chebyshev center*). Notice that in this case the real number $|K| := \max\{|y| \ : \ y \in K\}$ is the radius of this *Chebyshev* disc. Let us warm up with a direct result:

Lemma 4 *The set $\mathcal{C}_K(r)$ is a compact set for every $r > 0$.*

Proof. If $x \in \mathcal{C}_K(r)$ then $|x| \leq |K| + r$, hence $\mathcal{C}_K(r)$ is bounded. Moreover, the distance map $x \mapsto dist(x, K)$ is continuous and $\mathcal{C}_K(r)$ is exactly the pre-image of $\{r\}$ by this map, hence it is closed. $\square$

The reader can get in shape proving the following useful characterization of $\mathcal{C}_K(r)$:

Lemma 5 *The set $\mathcal{C}_K(r)$ is the boundary of the set*

$$\mathcal{D}_K(r) := \bigcup_{y \in K} B(y, r),$$

that is, $\mathcal{C}_K(r) = \partial \mathcal{D}_K(r)$. $\blacksquare$

The next result, and more precisely, the construction performed in its proof, constitutes the core of this note.

Proposition 6 *For every $r > |K|$ there exists a continuous function $\eta_{K,r} : \mathcal{S}^1 \to \mathbb{R}_+$ such that $\mathcal{C}_K(r)$ coincides with the graph of the map*

$$\theta \longmapsto \eta_{K,r}(\theta) e^{i\theta}.$$

In other words, for r large enough, the generalized circle $\mathcal{C}_K(r)$ is composed by one point for each direction in the plane.

Proof. Fix $\theta \in \mathcal{S}^1$. Since K is compact we can define

$$\eta_{K,r}(\theta) = \max\{\eta > 0 \ : \ \overline{B}(\eta e^{i\theta}, r) \cap K \neq \emptyset\}.$$

We can interpret this definition in the following way: think about a closed disc with radius r and with center moving along the ray starting at the origin in direction of θ. Since $r > |K|$, this disc meets K for positions of the center close to the origin. Moving away the center along the ray, the disc meets K for a last center, that we call $\eta_{K,r}(\theta)$. Clearly we have $dist(\eta_{K,r}(\theta)e^{i\theta}, K) = r$ and then $\eta_{K,r}(\theta)e^{i\theta} \in \mathcal{C}_K(r)$. For $\eta > \eta_{K,r}(\theta)$ the disc $\overline{B}(\eta e^{i\theta}, r)$ does not meet K, then $dist(\eta e^{i\theta}, K) > r$ and $\eta e^{i\theta} \notin \mathcal{C}_K(r)$. Notice that for $0 \leq \eta < r - |K|$ the open disc $B(\eta e^{i\theta}, r)$ contains K, then $dist(\eta e^{i\theta}, K) < r$ and $\eta e^{i\theta} \notin \mathcal{C}_K(r)$. Define the arc $\Gamma := \overline{B}(0, |K|) \cap C_{\eta_{K,r}(\theta)e^{i\theta}}(r)$.

Pick $\bar{x} \in \Gamma \cap K$. For $\eta \in [r - |K|, \eta_{K,r}(\theta)]$, the open disc $B(\eta e^{i\theta}, r)$ contains Γ, in particular $\bar{x} \in B(\eta e^{i\theta}, r)$ and $\eta e^{i\theta} \notin \mathcal{C}_K(r)$. Summarizing, we have shown that every ray starting at the origin intersects $\mathcal{C}_K(r)$ at one and only one point. Hence the closed set $\mathcal{C}_K(r)$ is the graph of the map $\theta \mapsto \eta_{K,r}(\theta)e^{i\theta}$, that is continuos since its graph is closed. This implies that $\theta \mapsto \eta_{K,r}(\theta)$ is continuous too. □

Corollary 7 *For $r > |K|$ the set $\mathcal{C}_K(r)$ is connected.* □

Scaling generalized circles. Let $\{C_s\}_{s\geq 0}$ be a family of compact sets. We say that $\{C_s\}$ *converges in scale to the unit circle* if

$$\tilde{C}_s := \frac{2}{diam(C_s)} C_s$$

converges in the Hausdorff topology to the unit circle $C_0(1)$. For example, the family $\{C_0(s)\}_{s\geq 0}$ converges in scale to the unit circle. The meaning of this definition is that the shape of the sets C_s *converges to the shape of the unit circle.*

Theorem 8 *Let $K \subset \mathbb{R}^2$ be a compact set. The family $\{\mathcal{C}_K(r)\}_{r>|K|}$ converges in scale to the unit circle.*

Proof. Using Proposition 6, we can deduce that for $r > |K|$

$$\tilde{\mathcal{C}}_K(r) = \left\{ \frac{2}{diam(\mathcal{C}_K(r))} \eta_{K,r}(\theta) e^{i\theta} \ : \ \theta \in \mathcal{S}^1 \right\}.$$

Thus the Hausdorff distance $dist_H$ between $\tilde{\mathcal{C}}_K(r)$ and $C_0(1)$ is bounded by

$$dist_H(\tilde{\mathcal{C}}_K(r), C_0(1)) \leq \max_{\theta \in \mathcal{S}^1} \left| 1 - \frac{2\eta_{K,r}(\theta)}{diam(\mathcal{C}_K(r))} \right|. \tag{1}$$

In the proof of the above proposition we saw that $\eta_{K,r}(\theta) \in [r - |K|, r + |K|]$ and then $diam(\mathcal{C}_K(r)) \in [2(r - |K|), 2(r + |K|)]$. From these we see that

$$\frac{r - |K|}{r + |K|} \leq \frac{2\eta_{K,r}(\theta)}{diam(\mathcal{C}_K(r))} \leq \frac{r + |K|}{r - |K|},$$

and then we conclude that the right hand side in (1) goes to 0 as $r \to \infty$. □

Large generalized circles are almost circular curves. The remaining part of this note is devoted to study the plane curve $\gamma_{K,r}(\theta) := \eta_{K,r}(\theta)e^{i\theta}$ from the differentiable point of view and to compare it with the parametrized circle $C_0(r)$. A classical exercise in differential geometry of curves is to show that a differentiable curve verifyong that at each point the tangent vector is orthogonal to the the position must be an arc of circle. The estimate (2) bellow serves to measure how the curve $\gamma_{K,r}$ approaches a circle from this point of view.

Theorem 9 *For every $\tilde{\theta} \in \mathcal{S}^1$, the curve $\gamma_{K,r}$ has well defined right and left tangent vectors, that is, the following limits exist:*

$$t_{r,+}(\tilde{\theta}) := \lim_{\theta \to \tilde{\theta}^+} \frac{\gamma_{K,r}(\theta) - \gamma_{K,r}(\tilde{\theta})}{\theta - \tilde{\theta}} \ , \ t_{r,-}(\tilde{\theta}) := \lim_{\theta \to \tilde{\theta}^-} \frac{\gamma_{K,r}(\theta) - \gamma_{K,r}(\tilde{\theta})}{\theta - \tilde{\theta}}.$$

Moreover, the angle $\alpha_{r,+}(\tilde{\theta})$ between $e^{i\tilde{\theta}}$ and $t_{r,+}(\tilde{\theta})$ verifies

$$\cos \alpha_{r,+}(\tilde{\theta}) < \frac{|K|}{r}, \tag{2}$$

and analogously for $\alpha_{r,-}(\tilde{\theta})$. Finally, the curve $\gamma_{K,r}$ is differentiable at $\tilde{\theta}$ if and only if the intersection $K \cap C_{\gamma_{K,r}(\tilde{\theta})}(r)$ has exactly one point.

Proof. We will use the Figure 1. bellow, where the following elements have been marked:

$a := \gamma_{K,r}(\tilde{\theta})$.

$\Gamma := \overline{B}(0, |K|) \cap C_a(r)$.

$b :=$ is the point over $\Gamma \cap K$ and such that there is no other point of $\Gamma \cap K$ between b and the upper point of $C_a(r) \cap C_0(|K|)$.

$\tau := C_b(r)$ where we only draw a portion of the arc passing trough a.

l_θ is the infinite ray starting at the origin with direction θ.

$a_\theta := \tau \cap l_\theta$.

Let $\theta > \tilde{\theta}$ and such that $\theta - \tilde{\theta}$ is small. In order to find $\eta_{K,r}(\theta)$ we can take a closed disc radius r whose center moves from infinity approaching K along l_θ. In fact, $\eta_{K,r}(\theta)$ corresponds to the first (an unique) time this disc meets K (in the set K_θ). There are two possibilities for this first intersection:

i. $b \in K_\theta$ that implies $\eta_{K,r}(\theta)$ occurs in a_θ.

ii. $b \notin K_\theta$, which implies that $\eta_{K,r}(\theta)$ occurs *before* a_θ. This implies that there is a point $b_\theta \in K_\theta$ inside the triangular shaded region delimited by Γ, the circle $C_0(|K|)$ and the circle $C_{a_\theta}(r)$. Using these definitions we can check that $\eta_{K,r}(\theta)$ occurs in $C_{b_\theta}(r) \cap l_\theta$.

Since $b_\theta \notin \Gamma$, one has $b_\theta \to b$ in the direction of Γ as $\theta \searrow \tilde{\theta}$. This yields that $\gamma_{K,r}(\theta) \to a$ in the direction of τ as $\theta \searrow \tilde{\theta}$. Hence $t_{+,r}(\tilde{\theta})$ exists and is parallel to τ. The angle $\alpha_{r,+}(\tilde{\theta})$ is then the angle between $l_{\tilde{\theta}}$ and τ and (2) follows. For the left tangent vector we proceed analogously, this time taking b as the lowest point in $K \cap \Gamma$. In the case $K \cap \Gamma$ consists in a single point then both tangent vectors coincide and $\gamma_{K,r}$ is differentiable at $\tilde{\theta}$. □

Looking closer at the above proof we can state the next

Proposition 10 *If for every $\theta \in \mathcal{S}^1$ the intersection $K \cap C_{\gamma_{K,r}(\theta)}(r)$ consists in a single point then the direction of the tangent vector $t_{K,r}(\theta)$ increases. In particular $\mathcal{C}_K(r)$ is convex.*

Proof. We refer to the Figure 1. bellow. Since b_θ is very close to b, the difference of direction between $t_r(\theta)$ and $t_r(\tilde{\theta})$ is almost the same than the difference between the tangent vectors of $C_b(r)$ at a_θ and a respectively. This angle equals $\angle aba_\theta$, that is positive since a_θ is very close to a. □

Corollary 11 (compare with Theorem 1) *If K is convex then $\mathcal{C}_K(r)$ is convex. In particular $\gamma_{K,r}$ is of class C^1.*

Proof. Since K is convex we can apply the above proposition and to deduce that the direction of the tangent vector is increasing. The Darboux' intermediate value theorem allows to conclude. □

Remark 12 *It is an interesting exercise to show that the set of θ's so that $K \cap C_{\gamma_{K,r}(\theta)}(r)$ has more than one point is countable (cf. Theorem 4.1 in [4]).*

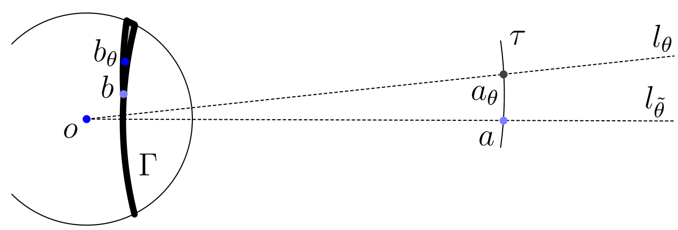

Figure 1.

Acknowledgments. Mario Ponce was partially supported by the Fondecyt Grant 11090003.

References

[1] M. Brown. Sets of constant distance from a planar set. *Michigan Math. J.* **47**, (1972), 321-323.

[2] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.* **93**, (1959), 418-491.

[3] S. Ferry. When ε-boundaries are manifolds. *Fundamenta Mathematicae* **XC**, (1976), 199-210.

[4] P. Pikuta. On sets of constant distance from a planar set. *Topological Methods in Nonlinear Analysis.* **21**, (2003), 369-374.

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