Fibred quadratic polynomials can admit two attracting invariant curves

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Abstract

We present an example of a fibred quadratic polynomial admitting two attracting invariant curves. This phenomena can not occur in the non-fibred setting.

A classical result due to Fatou and Julia asserts that quadratic polynomials may have at most one attracting cycle. This is an easy consequence of the fact that every immediate basin of attraction must contain a critical point (see [1]). In this note we deal with a more general version of quadratic dynamics, namely, fibred quadratic polynomials over an irrational rotation (also known as quasi periodically forced quadratic maps).

In this short note we show a new phenomena on fibred quadratic polynomials that do not occur in the non-fibred setting. We show an example of a fibred quadratic polynomial admitting two attracting invariant curves.

A *fibred quadratic polynomial* is a continuous map

$$\begin{array}{rcl} P : \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ (\theta, z) & \mapsto & \big(\theta + \alpha, P_\theta(z)\big). \end{array}$$

We denote $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since α is irrational the map P has no fixed points (nor periodic orbits). A natural invariant object that extends the notion of a fixed point is an *invariant curve*, that is, a continuous curve $\gamma : \mathbb{T}^1 \to \mathbb{C}$ such that $P(\theta, (\gamma(\theta)) = (\theta + \alpha, \gamma(\theta + \alpha))$. The real number

$$\kappa(\gamma) = \frac{1}{2\pi} \int_{\mathbb{T}^1} \log |\partial_z P_\theta\big(\gamma(\theta)\big)| d\theta$$

is called the *multiplier* (or Lyapunov exponent in the complex direction) and measures the expanding or contracting infinitesimal nature of the invariant curve. In [2] the author shows that an expanding or contracting multiplier (that is $\kappa(\gamma) > 0$ or $\kappa(\gamma) < 0$) implies actually that the map is locally conjugated to a linear fibred map which is expanding or

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contracting respectively.

The example is quite simple: given a continuous function $a : \mathbb{T}^1 \to \bar{\mathbb{C}}^*$ construct the fibred quadratic polynomial

$$P(\theta, z) = (\theta + \alpha, z(1 + a(\theta)(z - 1)))\,.$$

The curves $\{z = 0\}_{\mathbb{T}^1}$, $\{z = 1\}_{\mathbb{T}^1}$ are both invariants. These curves are attracting provided that we can choose the function a so that

$$\int_{\mathbb{T}^1} \log|1 + a(\theta)|d\theta < 0 \quad \text{and} \quad \int_{\mathbb{T}^1} \log|1 - a(\theta)|d\theta < 0.$$

Let's show that it is possible: by Jensen Fomula one has $\int_{\mathbb{T}^1} \log|1 + e^{i\theta}|d\theta = 0$. Note that $-e^{i\theta} = e^{i(\theta+\pi)}$ and hence $\int_{\mathbb{T}^1} \log|1 + e^{i\theta}|d\theta = \int_{\mathbb{T}^1} \log|1 - e^{i\theta}|d\theta = 0$. Also note that $|1 + e^{i\theta}|$ is the distance from 1 to $-e^{i\theta}$. We choose $a : \mathbb{T}^1 \to \bar{\mathbb{C}}^*$ as an ellipse having the major axis equal to $[-1, 1]$, and the minor axis equal to $i[1 - \varepsilon, -(1 - \varepsilon)]$ for small $\varepsilon > 0$, parametrized in polar coordinates. In this way the distances $|1 + a(\theta)| < |1 + e^{i\theta}|$ for every $\theta \in \mathbb{T}^1 \setminus \{0, \pi\}$. Hence

$$\int_{\mathbb{T}^1} \log|1 + a(\theta)|d\theta = \int_{\mathbb{T}^1} \log|1 - a(\theta)|d\theta < 0.$$

Concluding remarks. The curve $\{(\theta, \frac{a(\theta)-1}{2a(\theta)})\}_{\mathbb{T}^1}$ contains the critical point of each fibre. In the non-fibred quadratic setting the critical point must be attracted to the (unique) attracting cycle. In fibred quadratic polynomial, having an attracting invariant curve η, the set $\Omega(\eta)$ of $\theta \in \mathbb{T}^1$ such that the critical point $(\theta, \frac{a(\theta-1)}{2a(\theta)})$ is attracted by η is open. Moreover, a result due to Sester says that the set $\Omega(\eta)$ is non-empty (see [3], Proposition 3.2). Since $\gamma_1 \cap \gamma_2 = \emptyset$, the set $\mathbb{T}^1 \setminus (\Omega(\gamma_1) \cup \Omega(\gamma_2))$ is non-empty by connexity. Any fibred quadratic polynomial close enough to P still have two attracting invariant curves and the same remark on the critical points holds. Let's suppose that $\hat{P}$ is close to P and hyperbolic (that is, the post critical set does not accumulate the fibred Julia set, see [3] for definitions). In that case, there should be a critical point going to infinity or accumulating around a non-attracting object in the fibred Fatou set (since the circle cannot be covered by disjoint open sets).

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References

[1] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[2] Mario Ponce. Local dynamics for fibred holomorphic transformations. *Nonlinearity*, 20(12):2939–2955, 2007.

[3] Olivier Sester. Hyperbolicité des polynômes fibrés. *Bull. Soc. Math. France*, 127(3):393–428, 1999.