Towards a semi-local study of parabolic invariant curves for fibred holomorphic maps

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Abstract

We introduce the study of the local dynamics around a parabolic indifferent invariant curve for fibred holomorphic maps. As in the classical non-fibred case, we show that petals are the main ingredient. Nevertheless, one expects that the properties of the base rotation number should play an important role in the arrangement of the petals. We exhibit examples where the existence and the number of petals depend not just on the complex coordinate of the map, but on the base rotation number. Furthermore, under additional hypotheses on the arithmetic and smoothness of the map, we present a theorem that allows a characterization of the local dynamics around a parabolic invariant curve.

1 Introduction

Among the first results on one-dimensional complex dynamics one finds the understanding of the behavior of points near a fixed point under the iteration of a holomorphic map. Since the works of König and Poincaré, one knows that the derivative of the map at the fixed point determines the local dynamics in the hyperbolic case. When the derivative at the fixed point is not hyperbolic, one says that the fixed point is indifferent. In this case, finer techniques are needed because the very fast exponential convergence due to hyperbolicity is no longer available.

When the derivative is a rotation by a rational angle, one says that the fixed point is rationally indifferent. An elegant result due to Leau and Fatou states that the local dynamics is actually determined by the derivative at the fixed point. Indeed, the local dynamics follows a nice and simple pattern known as a Leau-Fatou flower:

Let $f(z) = z + az^{n+1} + \dots,\ a \neq 0$. A simply connected open set U is called an attracting petal for f if

$$0 \in \partial U\ ,\ f(\overline{U}) \subset U \cup \{0\} \text{ and } \bigcap_{j\geq 0} f^j(\overline{U}) = \{0\}.$$

One defines the notion of a repelling petal in an analogous way.

Theorem 1 (Leau-Fatou flower, see [3, 5, 14, 15]) *Let f be as above. Then there exist n disjoint attracting petals and n disjoint repelling petals. These $2n$ petals, together with the fixed origin itself, form a neighborhood of the origin.*

Recently, F. Le Roux [13] has shown a topological version of this result for surface homeomorphisms with Lefschetz index greater than 1.

In the irrational indifferent case, in most cases the local dynamics is determined by the derivative. In fact, by assuming an extra arithmetical hypothesis on the rotation angle, the local dynamics is conjugate to the corresponding rigid rotation (the Brjuno arithmetical condition has full Lebesgue measure, see for instance [21]).

In the last years, a special interest has been given to skew-product dynamical systems, and many results about complex skew-product dynamics have appeared (see [20], [11], [17], [19], [4], [10]). On the one hand, skew-product dynamics presents a rich source of examples, and on the other hand, it is considered as an intermediate step towards higher dimensional dynamics. This paper is devoted to the study of continuous maps of the form

$$\begin{array}{rcl} F : \mathbb{T}^1 \times \mathbb{D} & \longrightarrow & \mathbb{T}^1 \times \mathbb{C} \\ (\theta, z) & \longmapsto & (\theta + \alpha, f_\theta(z)) \end{array}$$

where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, α is an irrational angle and each $f_\theta : \mathbb{D} \to \mathbb{C}$ is holomorphic. We call these *fibred holomorphic maps* (abbreviated as *fhm*). Since $\theta \mapsto \theta + \alpha$ is minimal, F has no fixed points. However, the natural object that plays the role of a fixed point is an *invariant curve*, that is, a continuous curve $u : \mathbb{T}^1 \to \mathbb{D}$ such that

$$F(\theta, u(\theta)) = (\theta + \alpha, u(\theta + \alpha)).$$

The aim of this paper is to understand orbits of points close to an invariant curve. In [16], the author shows that a Poincaré-König Theorem holds for transversally hyperbolic invariant curves. Also, a version of Siegel's Theorem holds for Diophantine transversal rotation numbers.

In this work we introduce the study of the local dynamics around a parabolic indifferent invariant curve. As in the classical non-fibred case, we find that petals are the main ingredient (see Proposition 6). Nevertheless, one expects that the properties of the base rotation number should play an important role in the arrangement of the petals. We exhibit examples where the existence and the number of petals depend not just on the complex coordinate of the map, but on the base rotation number (see Section 4.1). Furthermore, under additional hypotheses on the arithmetic and smoothness of the map, we present a theorem that allows us to characterize the local dynamics around a parabolic invariant curve (see Theorem 10).

Given a curve $u : \mathbb{T}^1 \to \mathbb{D}$ which is invariant under F, we can perform a fibred translation in order to locate u at the zero section $\mathbb{T}^1 \times \{0\}$. Therefore, in the sequel we will deal only with complex maps of the form $f_\theta(z) = a_1(\theta)z + a_2(\theta)z^2 + \ldots$. The invariant curve will be called *parabolic indifferent* if $a_1(\theta) \equiv \lambda$, where λ is a constant primitive root of the unity.

Remark 2 *Under additional hypotheses on the arithmetic and smoothness of the map F, we can relax the above definition by requiring that the number*

$$\frac{1}{2\pi i} \int \log a_1(\theta) d\theta$$

be a rational number. Indeed, by solving a cohomological equation and performing a fibred homotopy we can recover the above form for the map f_θ. A natural direction of research is to determine whether or not the results contained in this work still hold for this weak definition of a parabolic curve when those additional hypotheses are absent.

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2 Fatou coordinates and cohomological equation

In the one-dimensional case the study of the local complex dynamics near a parabolic fixed point is achieved by seeing the map as acting on a neighborhood of infinity. More precisely, one introduces a conjugacy by the map $i(z) = -z^{-1}$, the so called Fatou coordinates. This procedure will also give us good information about the local dynamics of a fibred holomorphic map on a neighborhood of a parabolic invariant curve. Let's consider the fibred Fatou coordinates

$$\begin{array}{rcl} I : \mathbb{T}^1 \times \overline{\mathbb{C}} & \longrightarrow & \mathbb{T}^1 \times \overline{\mathbb{C}} \\ (\theta, z) & \longmapsto & (\theta, Z = -z^{-1}). \end{array}$$

Let F be a *fhm* of the form

$$F(\theta, z) = \big(\theta + \alpha, z + a_2(\theta)z^2 + a_3(\theta)z^3 + \dots\big) \tag{1}$$

defined on $\mathbb{T}^1 \times D(0, r)$ for some $r > 0$. By looking at infinity one gets the fibred map $\tilde{F} = I \circ F \circ I^{-1}$ which is defined on $\mathbb{T}^1 \times I(D(0, r))$ and takes the form

$$\tilde{F}(\theta, Z) = \left(\theta + \alpha, \frac{-1}{\frac{-1}{Z} + \frac{a_2(\theta)}{Z^2} - \frac{a_3(\theta)}{Z^3} + \dots}\right) = \left(\theta + \alpha, Z\left[\frac{1}{1 - \frac{a_2(\theta)}{Z} + \frac{a_3(\theta)}{Z^2} - \dots}\right]\right).$$

Since the radius of convergence of the series $a_2(\theta)z^2 + a_3(\theta)z^3 + \dots$ is at least r for every $\theta \in \mathbb{T}^1$, there exists $C > 0$ such that

$$\left| \frac{a_2(\theta)}{Z} - \frac{a_3(\theta)}{Z^2} + \dots \right| < 1$$

for $|Z| > C$. Thus, for $|Z| > C$ we have

$$\tilde{F}(\theta, Z) = \left(\theta + \alpha, Z + a_2(\theta) + \frac{a_2(\theta)^2 - a_3(\theta)}{Z} + \dots\right).$$

At first glance, for a big enough $|Z|$ this dynamics is much like a fibred translation by $a_2(\theta)$. In fact, we will see that under some mild hypotheses, the map $\tilde{F}$ can actually be conjugated to a fibred translation. Let us explain the general procedure: we look for a conjugacy

$$T_c(\theta, Z) = (\theta, Z + c(\theta))$$

in such a way that $T_c \circ \tilde{F} \circ T_c^{-1}$ takes the form

$$(\theta, Z) \longmapsto \left(\theta + \alpha, Z + k + \frac{b_1(\theta)}{Z} + \frac{b_2(\theta)}{Z^2} + \dots \right), \tag{2}$$

where $k = \int_{\mathbb{T}^1} a_2(\theta) d\theta$. This leads us to consider the cohomological equation

$$c(\theta + \alpha) - c(\theta) = -a_2(\theta) + k. \tag{3}$$

This equation has been widely studied. It is known that a continuous solution for (3) exists if and only if the Birkhoff sums of the function $a_2(\cdot) - k$ are bounded (Gottschalk-Hedlund's Lemma, see [7]). Moreover, such a solution is unique if we require $\int_{\mathbb{T}^1} c(\theta) d\theta = 0$ (which will always be our choice). Our purpose is not to discuss the precise and optimal hypotheses for the existence of continuous solutions for this equation. We only mention the following result:

Lemma 3 (see [12]) *Let α be a $CD(\tau)$ Diophantine number and $a_2(\theta)$ a differentiable $C^{1+\tau}$ function. Then equation (3) has a continuos solution $c : \mathbb{T}^1 \to \mathbb{C}$* ■

For more precise results, the reader can refer to Herman's works [8, 9]. The next lemma is trivial, but useful in the sequel:

Lemma 4 *Let α be irrational and let $a_2(\theta)$ be a trigonometric polynomial. Then equation (3) has a solution which is a trigonometric polynomial* ■

3 Fibred flowers

In this section we show that the classical theory of the Leau-Fatou flower also holds in the fibred setting. However, we need to know that the leading coefficient has non-zero mean. In the sequel, we write $\int_{\mathbb{T}^1} \cdot$ instead of $\int_{\mathbb{T}^1} \cdot \, d\theta$.

3.1 Case $z + Az^2 + \dots$

We continue with the discussion of the previous section. Assume that $k = \int_{\mathbb{T}^1} a_2 \neq 0$. Equation (3) could provide the required fibred translation T_c. Of course, there are two situations we need to deal with:

Equation (3) has a continuous solution. In this case, and by conjugating by an extra homothetic map $A_k(\theta, Z) = (\theta, k^{-1} Z)$, the map $\tilde{F}$ can be thought of as having the form

$$\tilde{F}(\theta, Z) = \left(\theta + \alpha, Z + 1 + \frac{b_1(\theta)}{Z} + \frac{b_2(\theta)}{Z^2} + \dots \right) \tag{4}$$

defined and convergent for $|Z| > C > 0$, where C is a positive constant (possibly greater than the C defined above) and $b_j : \mathbb{T}^1 \to \mathbb{C}$ are continuous functions.

Equation (3) has no continuous solution. Let $l : \mathbb{T}^1 \to \mathbb{C}$ be a trigonometric polynomial such that

i) $\hat{l}(0) = \int_{\mathbb{T}^1} l(\theta) = k$.

ii) $|l(\theta) - a_2(\theta)| < \frac{k}{1000}$ for every $\theta \in \mathbb{T}^1$.

We solve the cohomological equation

$$c(\theta + \alpha) - c(\theta) = -l(\theta) + k$$

and obtain a trigonometric polynomial $c : \mathbb{T}^1 \to \mathbb{C}$ which provides a conjugacy T_c such that

$$T_c \circ \tilde{F} \circ T_c^{-1}(\theta, Z) = \left(\theta + \alpha, Z + k + (l(\theta) - a_2(\theta)) + \frac{b_1(\theta)}{Z} + \dots\right).$$

An extra homothetic conjugacy allows us to consider $\tilde{F}$ as having the form

$$\tilde{F}(\theta, Z) = \left(\theta + \alpha, Z + 1 + \tilde{l}(\theta) + \frac{b_1(\theta)}{Z} + \frac{b_2(\theta)}{Z^2} + \dots\right)$$

where $|\tilde{l}(\theta)| << 1$ and $b_j : \mathbb{T}^1 \to \mathbb{C}$ are continuous functions.

In any case, we get a form for the fibred map that allows us to follow the classical construction which provides the attracting and repelling petals. We perform the fibred version of this procedure in the next section.

Near infinity dynamics and invariant regions. Let $A > 0$ be a constant such that the regions

$$\Omega_A^+ = \{Z = x + iy \big| x > A - |y|\} \quad , \quad \Omega_A^- = \{Z = x + iy \big| x < -A + |y|\}$$

are contained in $\mathbb{C} \setminus D(0, C)$. The region $\Omega_A = \Omega_A^+ \cup \Omega_A^-$ is a neighborhood of ∞. For C_2 large enough and $Re(Z) > C_2$ we have

$$|Re\tilde{F}_\theta^n(Z)| > Re(Z) + \frac{n}{2}.$$

Note that $\tilde{F}(\Omega_A^+) \subset \Omega_A^+$.

Now, let us consider the region $\mathcal{R}_\theta \subset \mathbb{C}$, delimited by the line $\mathcal{L} = \{x = L\}$, with $L > A$, and its image $\mathcal{L}_\theta = \tilde{F}_{\theta-\alpha}(\mathcal{L})$. For each $\theta \in \mathbb{T}^1$ there exists a homeomorphism H_θ from $\mathcal{R}_\theta$ into the region $\{Z \in \mathbb{C} \bigm| 0 \le Re(Z) \le 1\}$. We can pick H_θ verifying $H_\theta(L + iy) = iy$ and $H_\theta\left(\tilde{F}_{\theta-\alpha}(L + iy)\right) = 1 + iy$. Moreover, we can take the map $H(\theta, Z) = (\theta, H_\theta(Z))$ to be continuous. If we restrict the dynamics of $\tilde{F}$ to $\mathbb{T}^1 \times \{Re(Z) \ge L\}$, the subset $\bigcup_{\theta \in \mathbb{T}^1} \{\theta\} \times \mathcal{R}_\theta$ becomes a fundamental domain and we can extend the homeomorphisms H_θ in order to get a homeomorphism

$$\begin{array}{rcl} H : \mathbb{T}^1 \times \{Re(Z) \ge L\} & \longrightarrow & \mathbb{T}^1 \times \{Re(Z) \ge 0\} \\ (\theta, Z) & \longmapsto & (\theta, H_\theta(Z)) \end{array}$$

verifying

$$H \circ \tilde{F} \circ H^{-1}(\theta, Z) = (\theta + \alpha, Z + 1)$$

for every $(\theta, Z) \in \mathbb{T}^1 \times \{Re(Z) \geq 0\}$. Indeed, for (θ, Z) with $Re(Z) \geq 0$, there exists a unique $n \in \mathbb{N}$ such that $0 \leq Re(Z - n) < 1$. Then we define

$$H^{-1}(\theta, Z) = \left(\theta, \tilde{F}^n_{\theta - n\alpha}\left(H^{-1}_{\theta - n\alpha}(Z - n)\right)\right).$$

By using an analogous construction we can conjugate $\tilde{F}^{-1}$ to $(\theta - \alpha, Z - 1)$ into $Re(Z) < -L$.

If we look at the regions Ω^+_A, Ω^-_A in the $z-$plane (the original coordinates in a neighborhood of the origin) we get, for each fibre, two topological discs $\mathcal{P}^+_\theta, \mathcal{P}^-_\theta$ depending continuously on θ, such that

1. $0 \in \partial\mathcal{P}^+_\theta$.
2. $\mathcal{P}^+_\theta \cup \mathcal{P}^-_\theta$ is a neighborhood of the origin.
3. $\tilde{F}_\theta(\mathcal{P}^+_\theta) \subset \mathcal{P}^+_{\theta+\alpha}$ and $\tilde{F}^{-1}_\theta(\mathcal{P}^-_\theta) \subset \mathcal{P}^-_{\theta-\alpha}$.

The disjoint union of the sets $\{\theta\} \times \mathcal{P}^+_\theta$ forms an open tube having the invariant zero section on the boundary. Moreover, this tube is forward invariant and converges to the invariant curve by forward iteration. We call such a set a *fibred attracting petal* (we define the notion of a *fibred repelling petal* analogously). Summarizing, we can state the following:

Proposition 5 *Let F be a fibred holomorphic map as in (1). If $\int_{\mathbb{T}^1} a_2 \neq 0$ then there exists one fibred attracting petal and one fibred repelling petal. The union of these two sets form a tubular neighborhood of the invariant curve* ■

Exterior direction of fibred petals. We have shown that the original map F is conjugate to the map $(\theta, z) \mapsto \left(\theta + \alpha, z + z^2 + \dots\right)$ via the fibred homeomorphism $L = A_k \circ T_c \circ I$. Explicitly, one has

$$L(\theta, z) = \left(\theta, k^{-1}\left(\frac{-1}{z} + c(\theta)\right)\right).$$

In Section 3.2 we will need to cut the plane $\mathbb{C}$ along the repulsive direction of the fibred petal, hence we compute it here: In the coordinates of Z this direction corresponds to the direction of the curve $\gamma_t : t \in \mathbb{R} \mapsto t \in \mathbb{C}$ with $t \to +\infty$. Thus, the direction we look for is the direction of $L^{-1}(\theta, \gamma_t)$ as $t \to +\infty$

$$\begin{aligned} \frac{d}{dt}\frac{-1}{kt - c(\theta)} &= \frac{k}{(kt - c(\theta))^2} \\ &= \frac{1}{kt^2}\frac{1}{\left(1 + \frac{c(\theta)}{kt}\right)^2} \end{aligned}$$

that is parallel to k^{-1}. Thus, the exterior direction of the fibred petal does not depend on θ and corresponds to the direction of k^{-1}.

3.2 Case $z + Az^{n+1} + \dots$

In this section we consider the following fibred map

$$F(\theta, z) = \big(\theta + \alpha, z + a_{n+1}(\theta)z^{n+1} + a_{n+2}(\theta)z^{n+2} + \dots\big) \qquad (5)$$

with $n > 1$. We assume that $\int_{\mathbb{T}^1} a_{n+1} \neq 0$. Write $\int_{\mathbb{T}^1} a_{n+1} = re^{i\Theta}$ with $r > 0$ and $\Theta \in [0, 2\pi)$. We consider the *repulsive directions* as the n unitary vectors $e_j = e^{i(\frac{2\pi j}{n} - \Theta)}$ for $0 \leq j \leq n-1$. We call $\mathcal{R}_j$ the open region limited by two consecutive repulsive directions e_j, e_{j+1}. Pick $\mathcal{R} = \mathcal{R}_j$ to be one of these regions. We define the homeomorphism

$$\begin{array}{rcl} \Phi_n : \mathbb{T}^1 \times \mathcal{R} & \longrightarrow & \mathbb{T}^1 \times \{\mathbb{C} \setminus \mathcal{L}_{-\Theta}\} \\ (\theta, z) & \longmapsto & (\theta, w = z^n) \end{array}$$

where $\mathcal{L}_{-\Theta} = \mathbb{R}_+ e^{-i\Theta}$. We conjugate F by Φ_n, and thus obtain a fibred holomorphic map on $\mathbb{T}^1 \times \{\mathbb{C} \setminus \mathcal{L}_{-\Theta}\}$

$$(\theta, w) \longmapsto \big(\theta + \alpha, w + na_{n+1}(\theta)w^2 + \dots\big).$$

Since $\int_{\mathbb{T}^1} na_{n+1} \neq 0$, Proposition 5 implies that this map presents one fibred attracting petal and one fibred repelling petal (cut by the line $\mathcal{L}_{-\Theta}$). Pasting together the regions $\mathcal{R}_j$ we get

Proposition 6 *Let F be a fibred holomorphic map as in (5). If $\int_{\mathbb{T}^1} a_{n+1} \neq 0$ then there exist n fibred attracting petals and n fibred repelling petals. The union of these $2n$ fibred petals and the invariant curve form a tubular neighborhood of the invariant curve* ■

3.3 Case $\lambda z + Az^2 + \dots,\ \lambda^n = 1$

Let

$$F(\theta, z) = \big(\theta + \alpha, \lambda z + a_2(\theta)z^2 + a_3(\theta)z^3 + \dots\big)$$

where λ is a n^{th} primitive root of the unity. The iterate F^n of F has the form

$$F^n(\theta, z) = \big(\theta + n\alpha, z + b_2(\theta)z^2 + b_3(\theta)z^3 + \dots\big),$$

hence previous results can be applied. In the very same way as in the classical non fibred case, if F^n presents a p petals parabolic behavior and $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_p$ are the fibred attracting petals, then $F(\mathcal{P}_j)$ equals $\mathcal{P}_i$ for some i (in the sense of germs). Thus, F permutes the petals in cycles of length n. We conclude that n divides p.

4 Case $\int_{\mathbb{T}^1} a_2 = 0$

Let's come back to the situation of the map (2) and assume that $\int_{\mathbb{T}^1} a_2 = 0$. Suppose that a continuous solution $c_2 : \mathbb{T}^1 \to \mathbb{C}$ exists for the cohomological equation (3). By performing the fibred translation and then coming back to the original coordinates, it is not hard to see that the map F is conjugate to

$$F_{(1)}(\theta, z) \longmapsto \big(\theta + \alpha, z - d_1 z^3 + d_2 z^4 + (d_1^2 - d_3)z^5 \dots\big)$$

where

$$
\begin{aligned}
b_1 &= a_2^2 - a_3 \\
b_2 &= a_4 - 2a_2a_3 + a_2^2 \\
b_3 &= -a_5 + a_3^2 + 2a_2a_4 - 3a_2^2a_3 + a_2^4 \\
d_1 &= b_1 \\
d_2 &= b_1c_2 + b_2 \\
d_3 &= b_1c_2^2 + 2b_2c_2 + b_3.
\end{aligned}
$$

We call $F_{(1)}$ the *order 2-reduction* of $F = F_{(0)}$. In general, if the leading coefficient (let's say a_{n+1}) verifies $\int_{\mathbb{T}^1} a_{n+1} = 0$ and the corresponding cohomological equation has a continuous solution, we can perform a reduction in order to get a fibred map with leading coefficient of order $\tilde{n} > n+1$. We call it the order $(n+1)$-reduction (see Section 4.2 for details). We can easily obtain formulas for the new coefficients $\{\tilde{a}_j\}_{j \geq n+2}$ in terms of the old ones $\{a_j\}_{j \geq n+1}$ and the solution h_{n+1} to the corresponding cohomological equation. For example:

$n+2)$ Provided that $n \geq 2$ one has

$$\tilde{a}_{n+2} = a_{n+2}.$$

$n+3)$ Provided that $n \geq 3$ one has

$$\tilde{a}_{n+3} = a_{n+3}.$$

In the case $n = 2$ one has:

$2+3)$

$$\tilde{a}_5 = 3a_3h_3 + a_5.$$

4.1 Some pathological examples

In this section we exhibit some examples that show new phenomena on parabolic fibred maps. First of all, three examples show that counting petals is not easy *a priori*, in contrast with the well known parabolic theory of one-dimensional complex dynamics. The last two examples are toy models for more complicated behavior.

1. Let us take
$$F(\theta, z) = \left(\theta + \alpha, z + \sin(\theta)z^2\right).$$
In this case $\int_{\mathbb{T}^1} a_2 = 0$ and the cohomological equation has a continuous solution for every irrational α. By performing the order 2-reduction, we find out that F is topologically equivalent to
$$(\theta, z) \longmapsto \left(\theta + \alpha, z - \sin^2(\theta)z^3 + \dots\right).$$
Since $\int_{\mathbb{T}^1} \sin^2(\theta)d\theta \neq 0$, F has a fibred flower with 2 attracting petals and 2 repelling petals . Also note that every coefficient a_j of F verifies $\int_{\mathbb{T}^1} a_j = 0$.

2. Let us take

$$F(\theta, z) = \left(\theta + \alpha, z + \sin(\theta) z^2 + \sin(\theta)^2 z^3 + \cos(\theta)^2 z^4\right).$$

Note that $\int_{\mathbb{T}^1} a_2 = 0$ and $\int_{\mathbb{T}^1} a_3 \neq 0$. We could guess a 2 petals parabolic behavior; nevertheless, the order 2-reduction says that F is topologically equivalent to

$$(\theta, z) \longmapsto \left(\theta + \alpha, z + \left(1 - 2\sin^3(\theta)\right) z^4 + \dots\right),$$

hence a 3 petals parabolic behavior appears!

3. In the next example, we will construct a parabolic fibred map presenting petals, but in such a way that the number of petals depends on the base rotation number. This suggests that the number of petals can not be computed by means of an integral just depending on the complex coordinate of the map (as in the one-dimensional case). We use the notations from the beginning of Section 4. We will construct a fibred polynomial of degree 5 with trigonometric polynomial coefficients:

$$F_\alpha(\theta, z) = \left(\theta + \alpha, z + a_2(\theta) z^2 + a_3(\theta) z^3 + a_4(\theta) z^4 + a_5(\theta) z^5\right).$$

We use the sub-script in order to make the dependence on the base rotation number explicit. Take a non identically zero a_2 such that $\int_{\mathbb{T}^1} a_2 = 0$. Let h_2^α be the solution to the corresponding cohomological equation. Note that h_2^α depends on α (we put α as a super-script in order to stress this). Take a_3 as being constant and equal to $\int_{\mathbb{T}^1} a_2^2$. The order 2-reduction gives

$$\tilde{F}_\alpha(\theta, z) = \left(\theta + \alpha, z - d_1 z^3 + d_2^\alpha z^4 + (d_1^2 - d_3^\alpha) z^5 + \dots\right).$$

The above choice of a_3 yields $\int_{\mathbb{T}^1} d_1 = 0$. We can make the order 3-reduction and construct the solution h_3^α to the corresponding cohomological equation. We conjugate the original map to

$$(\theta, z) \longmapsto \left(\theta + \alpha, z + d_2^\alpha z^4 + \left[-3 d_1 h_3^\alpha + d_1^2 - d_3\right] z^5 + \dots\right). \tag{6}$$

Recall that

$$d_2^\alpha = b_1 h_2^\alpha + b_2 = (a_2^2 - a_3) h_2^\alpha + a_4 - 2 a_2 a_3 + a_2^2.$$

Fix $\alpha = \alpha^*$ and pick a_4 to be a constant such that $\int_{\mathbb{T}^1} d_2^{\alpha^*} = 0$. By performing the order 4-reduction we can conjugate F_{α^*} to

$$(\theta, z) \longmapsto \left(\theta + \alpha^*, z + \left[-3 d_1 h_3^{\alpha^*} + d_1^2 - d_3\right] z^5 + \dots\right).$$

Finally, choose a_5 (and thus d_3) such that $\int_{\mathbb{T}^1} [-3 d_1 h_3^{\alpha^*} + d_1^2 - d_3] \neq 0$. Hence F_{α^*} presents a 4 petals parabolic behavior.

On the other hand, pick $\alpha = \alpha^{**}$ such that in (6) we have $\int_{\mathbb{T}^1} d_2^{\alpha^{**}} \neq 0$. Then, the reduction procedure stops there and $F_{\alpha^{**}}$ presents a 3 petals parabolic behavior.

4. Let α be an irrational number not belonging to the Brjuno class (see [21] for definitions). The following theorem is due to Yoccoz:

Theorem 7 (see [21]) *The quadratic polynomial $P(z) = e^{2\pi i\alpha}z + z^2$ is not linearizable. Furthermore, there exist periodic orbits approximating the fixed point $z = 0$* ■

Consider the following fibred holomorphic map:

$$Q(\theta, z) = (\theta + \alpha, P(z)) .$$

Of course, this fibred map does not present a parabolic behavior since there are periodic curves converging to the invariant curve. By performing the (not isotopic to the identity) change of coordinates $(\theta, z) \mapsto (\theta, e^{-2\pi i\theta}z)$ we get the map

$$(\theta, z) \mapsto \left(\theta + \alpha, z + e^{2\pi i(\theta-\alpha)}z^2\right)$$

which seems to be a fibred parabolic dynamics. Note that $\int_{\mathbb{T}^1} a_j = 0$ for every $j \geq 2$. We will come back to this example in Section 4.2.

5. This next example is interesting since, to some extent, it should model the non-reducible case which is the source of new and rich phenomena in fibred holomorphic maps. Let $a : \mathbb{T}^1 \to \mathbb{C}$ be a continuous function such that $\int_{\mathbb{T}^1} a = 0$. We consider the fibred map defined in $\mathbb{T}^1 \times \overline{\mathbb{C}}$

$$\begin{aligned} F_a(\theta, z) &= \left(\theta + \alpha, \frac{z}{1 - a(\theta)z}\right) \\ &= \left(\theta + \alpha, z + a(\theta)z^2 + a(\theta)^2 z^3 + \dots\right) . \end{aligned}$$

By the change of coordinates at infinity we get the map

$$\tilde{F}_a(\theta, Z) = (\theta + \alpha, Z + a(\theta)) . \tag{7}$$

In the literature, (7) is known as a *cylindrical cascade* and has been widely studied (see for instance [7], [2], [1], and the fairly complete introduction [6]).

If the corresponding cohomological equation

$$c(\theta) - c(\theta + \alpha) = a(\theta) \tag{8}$$

has a continuous solution, then the map $\tilde{F}_a$, and a posteriori F_a, is topologically equivalent to the fibred identity map $(\theta, Z) \mapsto (\theta + \alpha, Z)$. On the other hand, let's concentrate on the situation where we cannot solve the cohomological equation. In this case one says that the dynamical system (7) is *non-integrable*. The most outstanding result concerning topological dynamics of cylindrical cascades is

Theorem 8 (Atkinson '78, [1]) *Let $a : \mathbb{T}^1 \to \mathbb{C}$ be a continuous function with $\int_{\mathbb{T}^1} a = 0$. If $\tilde{F}_a$ is non-integrable then there exists a non-zero complex number τ such that the real cylindrical cascade*

$$(\theta, t) \longmapsto (\theta + \alpha, t + < \tau, a(\theta) >)$$

is topologically transitive. ■

Note that the above real cylindrical cascade is a topological factor of $\tilde{F}_a$, and hence $\tilde{F}_a$ is far from exhibiting any parabolic behavior. On the other hand, Besikovitch showed that cylindrical cascades are never minimal.

4.2 Infinitely reducible maps

In this section we look for the possibility of conjugating a parabolic fibred map to the fibred identity map

$$Id_\alpha(\theta, z) = (\theta + \alpha, z)$$

via a fibred holomorphic change of coordinates. From a formal point of view, there should exist $H(\theta, z) = \big(\theta, z + h_2(\theta)z^2 + h_3(\theta)z^3 + \dots\big)$ such that

$$F \circ H = H \circ Id_\alpha. \tag{9}$$

By writing out the formal power series in the above equality we get a recursive definition for the coefficients h_k:

$$\begin{array}{lrcl} (1) & h_1(\theta) & = & 1 \\ (2) & h_2(\theta+\alpha) - h_2(\theta) & = & a_2(\theta) \\ \vdots & \vdots & = & \vdots \\ (ec_k) & h_k(\theta+\alpha) - h_k(\theta) & = & \sum_{j=2}^k a_j \left\{ \sum_{r_1+\dots+r_j=k} h_{r_1} \cdots h_{r_j} \right\}(\theta). \end{array}$$

However, each cohomological equation (ec_k) makes sense only if equations $(2), \dots, (ec_{k-1})$ have continuous solutions $h_2, \dots, h_{k-1}$. A necessary condition for the existence of the coefficients for H is the vanishing of every mean

$$(Hyp_k) \quad \int_{\mathbb{T}^1} \sum_{j=2}^k a_j \left\{ \sum_{r_1+\dots+r_j=k} h_{r_1} \cdots h_{r_j} \right\}(\theta) d\theta \quad = \quad 0.$$

Of course, the hypothesis (Hyp_k) makes sense only when $(Hyp_2), \dots, (Hyp_{k-1})$ hold and equations $(2), \dots, (ec_{k-1})$ have continuous solutions $h_2, \dots, h_{k-1}$. This formal computation suggests an algorithm for studying the dynamics of F near the invariant curve:

1. If (Hyp_k) does not hold, then we can use the continuous change of coordinates $H^{k-1}(\theta, z) = \big(\theta, z + h_2(\theta)z^2 + \dots + h_{k-1}(\theta)z^{k-1}\big)$ in order to conjugate the original map F to
$$F_k(\theta, z) = \Big(\theta + \alpha, z + \tilde{a}_k z^k + \dots\Big)$$
with $\int_{\mathbb{T}^1} \tilde{a}_k \neq 0$. Hence F_k, and a posteriori F, presents a $k-1$ petals parabolic dynamics.

2. If (Hyp_k) holds and there exists a continuous solution h_k for (ec_k), we iterate this algorithm for (Hyp_{k+1}) and (ec_{k+1}).

3. If (Hyp_k) holds but (ec_k) does not admit a continuos solution, then the dynamics of F can be as *strange* as a cylindrical cascade (see Example 5 in Section 4.1) and we cannot say much more. We call F a *non-reducible* map.

If this algorithm stops for some k, then we have a sufficient understanding of the local dynamics of F around the invariant curve. On the other hand, it may occur that at each step we fall on the point *(2)* of the above algorithm. In this case we say that F is *infinitely reducible.*

A natural question in the infinitely reducible case is whether or not F is conjugate to the fibred identity map Id_α. At least formally, this is true due to (9). The next example shows that in order to get a topological conjugacy (that is, the uniform convergence of the series $z + h_2(\theta)z^2 + \dots$), we need to require additional hypotheses:

Proposition 9 *Let α be an irrational number not belonging to the Brjuno class. Then the fibred map*

$$Q(\theta, z) = \left(\theta + \alpha, z + e^{i(\theta - \alpha)} z^2\right)$$

is infinitely reducible but not topologically conjugate to the fibred identity.

Proof. By Yoccoz's Theorem, Q is not topologically conjugate to the fibred identity. We need to show that Q is infinitely reducible. Indeed, every (Hyp_k) holds, otherwise, Proposition 6 implies parabolic behavior. Moreover, each equation (ec_k) has a continuous solution since every $\{h_j, a_j\}_{j<k}$ are trigonometric polynomials. ■

Under good hypotheses on F and α, the situation is rather simple and we obtain the following dichotomy, which represents the main result of this work:

Theorem 10 *Let*

$$\begin{aligned} F : B_\delta \times \mathbb{D} &\longrightarrow B_\delta \times \mathbb{C} \\ (\theta, z) &\longmapsto \left(\theta + \alpha, z + a_2(\theta) z^2 + \dots\right) \end{aligned}$$

be a fibred holomorphic map, analytic on $B_\delta \times \mathbb{D}$, where $B_\delta = \{\theta \in \mathbb{C}/\mathbb{Z} \mid Im(\theta) < \delta\}$ for some $\delta > 0$ and $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. If α verifies a Diophantine arithmetic condition $CD(c, \tau)$ for some $c > 0, \tau \geq 0$ then one of the following statements holds:

1. *F is not infinitely reducible and there exists $k \geq 1$ such that $F : B_\delta \times \mathbb{D} \to B_\delta \times \mathbb{C}$ presents a k petals parabolic behavior around the invariant curve.*
2. *F is infinitely reducible and there exists*

$$\begin{aligned} H : B_{\delta/2} \times D(0, r) &\longrightarrow B_{\delta/2} \times \mathbb{C} \\ (\theta, z) &\longmapsto \left(\theta, z + h_2(\theta) z^2 + \dots\right) \end{aligned}$$

analytic on $B_{\delta/2} \times D(0, r)$ for some $0 < r \leq 1$, such that

$$H^{-1} \circ F \circ H = Id_\alpha.$$

Proof. The first part follows easily from Lemma 3 and Proposition 6. For the proof of the second part we will need to estimate the growth of the coefficients h_k. For that, we will follow closely the original proof by Siegel of the linearization theorem for holomorphic germs with Diophantine rotation number. We start by recalling some technical lemmas from Siegel's paper [18].

Siegel's Lemmas. Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a real sequence verifying

$$\varepsilon_n < (2n)^\nu \tag{10}$$

for some $\nu > 0$. Let's define the sequence $\vartheta_1, \vartheta_2, \dots$ recursively by first defining $\vartheta_1 = 1$. For $k > 1$, denote by μ_k the largest value among the products $\vartheta_{r_1} \vartheta_{r_2} \cdots \vartheta_{r_j}$ with

$$r_1 + r_2 + \dots + r_j = k > r_1 \geq r_2 \geq \dots \geq r_j \geq 1$$

and $2 \leq j \leq k$. Define $\vartheta_k = \varepsilon_{k-1} \mu_k$.

Lemma 11 *The following holds*

$$\vartheta_k \leq k^{-2\nu} 2^{(5\nu+1)(k-1)}$$

for every $k \geq 1$ ∎

Let us define the sequence $\tau_1, \tau_2, \ldots$ recursively by

$$\begin{aligned}\tau_1 &= 1\\ \tau_k &= \sum \tau_{r_1}\tau_{r_2}\cdots\tau_{r_j}\end{aligned}$$

where the above sum is taken over every integer solution of $r_1 + r_2 + \cdots + r_j = k$ with $2 \leq j \leq k$.

Lemma 12 *The power series*

$$\sum_{k=1}^{\infty} \tau_k z^k$$

converges on the disc $|z| < 3 - 2\sqrt{2}$ ∎

Let us define the sequence $\gamma_1, \gamma_2, \ldots$ recursively by

$$\begin{aligned}\gamma_1 &= 1\\ \gamma_k &= \varepsilon_{k-1}\sum \gamma_{r_1}\gamma_{r_2}\cdots\gamma_{r_j}\end{aligned}$$

where the above sum is taken over every integer solution of $r_1 + r_2 + \cdots + r_j = k$ with $2 \leq j \leq k$.

Lemma 13 *The following inequality holds*

$$\gamma_k \leq \vartheta_k \tau_k.$$

Consequently, the power series

$$\sum_{k=1}^{\infty} \gamma_k z^k$$

converges on the disc $|z| < (3 - 2\sqrt{2})2^{-5\nu-1}$ ∎

Estimates for the analytic cohomological equation. Consider the cohomological equation

$$h(\theta + \alpha) - h(\theta) = g(\theta) \tag{11}$$

where $g : B_\delta \to \mathbb{C}$ is analytic on the strip B_δ and α verifies the Diophantine condition $CD(c, \tau)$. Also assume that $\int_{\mathbb{T}^1} g(\theta)d\theta = 0$. The classical method of Fourier series gives the formal solution

$$h(\theta) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\hat{g}(n)}{e^{in\alpha} - 1} e^{in\theta} \tag{12}$$

where $\hat{g}(n)$ stands for the Fourier coefficient. We pick $\hat{h}(0) = \int_{\mathbb{T}^1} h(\theta)d\theta = 0$.

Lemma 14 *Denote by $\|g\|_\delta = \sup_{\theta \in B_\delta} |g(\theta)|$. Suppose $\|g\|_\delta \leq E$. With the above hypotheses and notation one has that the solution h is analytic on the strip B_δ. Furthermore, there exists a constant $C = C(c)$ such that for every $d < \delta$ one has*

$$\|h\|_{\delta - d} \leq \frac{EC}{d^{3+\tau}}.$$

Proof. The Fourier coefficient can be estimated by

$$|\hat{g}(n)| \leq E e^{-|n|\delta}$$

for $n \in \mathbb{Z} \setminus \{0\}$. The Diophantine condition yields

$$|e^{in\alpha} - 1| \leq \frac{C}{|n|^{2+\tau}}$$

for some constant $C = C(c)$. Using (12) and the above estimates one gets

$$\|h\|_{\delta - d} \leq EC \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-|n|d} |n|^{2+\tau}.$$

Lemma 15 bellow completes the proof ■

Lemma 15 *For every $s > 0$ there exists $C = C(s)$ such that for every $x \in (0,1)$ one has*

$$\sum_{n \geq 0} x^n n^s \leq \frac{C}{(1-x)^{s+1}}$$

Proof. Note that

$$\begin{aligned} \sum_{n \geq 0} x^n n^s &< \sum_{n \geq 0} (n+s)(n+s-1) \cdots (n+1) x^n \\ &= \frac{\partial^s}{\partial x^s} \left(\sum_{n \geq 0} x^{n+s} \right) \\ &= \frac{\partial^s}{\partial x^s} \left(\frac{x^s}{1-x} \right) \quad ■ \end{aligned}$$

Now, we can give the proof of Theorem 10. Let $\nu = \nu(\delta, c, \tau)$ be such that

$$\sum_{k \geq 2} \left(\frac{C(k-1)}{(2(k-1))^\nu} \right)^{\frac{1}{3+\tau}} < \frac{\delta}{2}. \tag{13}$$

For $k \geq 2$ define

$$d_k = \left(\frac{C(k-1)}{(2(k-1))^\nu} \right)^{\frac{1}{3+\tau}} \quad \text{and} \quad \varepsilon_{k-1} = \frac{C(k-1)}{d_k^{3+\tau}}.$$

By construction, condition (10) holds. Put $\delta_1 = \delta$ and recursively $\delta_k = \delta_{k-1} - d_k$. Note that by (13) we have $\delta_k > \frac{\delta}{2}$ for every $k \geq 1$. As $F(\theta, \cdot)$ has a uniform convergence radius, there exists $a > 0$ such that $|a_j(\theta)| < a^{j-1}$ for every $j \geq 2$ and $\theta \in B_\delta$. By considering the change of coordinates $z \mapsto \frac{z}{a}$ we can assume that $|a_j(\theta)| \leq 1$ for every $j \geq 2, \theta \in B_\delta$.

Lemma 16 *For every $k \geq 1$ the following holds*

$$\|h_k\|_{\delta_k} \leq \gamma_k.$$

Proof. The desired inequality holds for $k = 1$. Assume the result for every $j < k$. Lemma 14 asserts that the solution h_k for the equation (ec_k) verifies

$$\begin{aligned}
\|h_k\|_{\delta_{k-1}-d_k} &\leq \frac{C}{d_k^{3+\tau}} \left\| \sum_{j=2}^{k} a_j \left\{ \sum_{r_1+\cdots+r_j=k} h_{r_1} \cdots h_{r_j} \right\} \right\|_{\delta_{k-1}} \\
\|h_k\|_{\delta_k} &\leq \frac{C(k-1)}{d_k^{3+\tau}} \sum_{r_1+\cdots+r_j=k} \gamma_{r_1} \cdots \gamma_{r_j} \\
&= \varepsilon_{k-1} \sum_{r_1+\cdots+r_j=k} \gamma_{r_1} \cdots \gamma_{r_j} = \gamma_k \quad \blacksquare
\end{aligned}$$

Finally, putting together Lemmas 13 and 16, the proof of the Theorem 10 is complete ■

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