Naishul's Theorem for fibered holomorphic maps

Mario Ponce*

May 4, 2011

Abstract

We show that the fibered rotation number associated to an indifferent invariant curve for a fibered holomorphic map is a topological invariant.

The rotation number of circle homeomorphisms was introduced by Poincaré in order to compare a general circle transformation with the simplest nontrivial dynamical models, namely the Euclidean rotations. Poincaré shows that this number characterizes the cyclic order of the orbits, thus controlling the topology or the *shape* of them. As a consequence, the circle rotation number is a topological conjugacy invariant. Moreover, under some arithmetical and smoothness conditions, one may show that the circle rotation number is a characterization of the full conjugacy class. Indeed, these conditions imply that the map is conjugated to the corresponding rotation (Denjoy, Arnold, Herman, Yoccoz; see for instance [4] and references therein).

An analogous rotation number can be associated to a differentiable surface local homeomorphism having an indifferent fixed point, the so-called *nonlinear rotations*. The derivative at the fixed point is a pure rotation, and it is natural to expect that this plane rotation number has some control on the dynamics of points that are close to the fixed point. Under some hypothesis, one can give results in this direction. For example, in the holomorphic case, the dynamics is actually conjugated to the pure rotation map provided that the Brjuno arithmetical condition is satisfied. Even in the absence of this nice behavior, the plane rotation number determines the shape of orbits, as it is shown by the classical

Theorem 1 (Naishul [6]) *Let f and g be two orientation-preserving nonlinear rotations that are topologically conjugated by a conjugacy that preserves orientation and the fixed point. If f is holomorphic (or area-preserving), then the plane rotation numbers of f and g are equal.*

In a subsequent work, Gambaudo and Pécou [2] realized that the smoothness condition (holomorphic or area-preserving) is not the intrinsic ingredient which turns this result true. They define the *linking property* for nearby orbits and show that the topological invariance for the plane rotation number follows from this topologically flavored property. Fortunately, area-preserving and irrationally indifferent holomorphic maps enjoy this property. In [1], Gambaudo-Le Calvez-Pécou show that a nonlinear rotation verifies either the linking property or a Birkhoff–Pérez-Marco property associated to the existence of completely-invariant nontrivial compact sets (the so-called $\mathcal{P}$ condition). Further, using the action on the prime-ends, they show that Naishul's

*Partially supported by FONDECYT 11090003

result holds under this alternative condition. Hence, the panorama for the invariance of the plane rotation number is fairly complete in the plane setting. Coming back to the holomorphic case, it is easy to note that a rationally indifferent map does not verify the linking property. However, the $\mathcal{P}$ condition holds and Naishul's result follows. Alternatively, the Leau-Fatou flower Theorem asserts that the local dynamics is combinatorially finite and quite simple, and the invariance of the plane rotation number directly follows from this. Recently, Le Roux [5] shows that for every homeomorphic nonlinear rotation, this simpler alternative occurs. Indeed, Le Roux' result asserts that in the absence of the linking property, the map is topologically conjugated to a rationally indifferent polynomial, and a topological Leau-Fatou flower appears. Summarizing, in the plane setting, Naishul's result is a consequence of both the linking property and the existence of a Leau-Fatou-Le Roux flower. The prime-ends machinery can, therefore, be bypassed.

In higher dimensions, Gambaudo and Pécou [2] treat the case of a differentiable local homeomorphism admitting an invariant, real codimension-two torus. They also assume that the restricted dynamics on this torus is conjugated to an irrational translation. They show that the complementary tangent direction wraps around this invariant torus with a well-defined asymptotic speed (a rotation number). Whenever the linking property holds, this rotation number is invariant under topological conjugacy. For example, a volume preserving homeomorphism defined in a neighborhood of the invariant torus verifies the linking property.

In this work we deal with a similar situation. We consider fibered holomorphic maps over irrational rotations. Given an indifferent invariant curve, a fibered rotation number is computed as the mean of the rotation angles at each fiber. We show the following

Theorem 2 (Naishul's Theorem for fibered holomorphic maps) *Let F and G be two fibered holomorphic maps that admit the zero section as an indifferent, zero-degree invariant curve. Suppose F and G are topologically conjugated by a fibered conjugacy isotopic to the identity. Then the fibered rotation numbers $\varrho_T(F)$, $\varrho_T(G)$ are equal.*

At the time of this writing, we still don't know whether the linking property holds in our setting, or at least in the non-parabolic situation. In addition, no well-understood theory of *parabolic* behavior for fibered maps is available and so, we don't have an alternative *à la Le Roux*.

The methods in this paper allow to prove an analogous Naishul's result for *in mean-volume preserving fibered maps*, that is, fibered maps over an irrational rotation verifying that each fiber map is a positive local diffeomorphism and such that the area is multiplied by a positive number $\beta(\theta)$ satisfying $\int_{\mathbb{T}^1} \log \beta(\theta) d\theta = 0$.

Acknowledgements. I would like to thank Jean-Marc Gambaudo for very useful conversations and comments, and Tobias Jäger for suggesting me an idea of proof for Lemma 8. Also, I must acknowledge the sympathy, patience and guidance of Patrice Le Calvez in face of many wrong comments I made regarding the prime-ends technique applied to this problem. Finally, I want to thank the Referee by many useful indications, corrections and improvements.

Fibered holomorphic maps

Let α be an irrational angle in $\mathbb{T}^1$. We denote by $\mathbb{D}_\delta$ the open ball in $\mathbb{C}$ centered at 0 with radius $\delta > 0$. A *fibered holomorphic map* is a continuous transformation

$$\begin{array}{rcl} F : \mathbb{T}^1 \times \mathbb{D}_\delta & \longrightarrow & \mathbb{T}^1 \times \mathbb{C} \\ (\theta, z) & \longmapsto & \big(\theta + \alpha, f_\theta(z)\big) \end{array}$$

such that the functions $f_\theta : \mathbb{D}_\delta \to \mathbb{C}$ are univalent for all $\theta \in \mathbb{T}^1$. In all what follows, we assume that the zero section $\mathbb{T}^1 \times \{0\}$ is an invariant curve, that is, $f_\theta(0) = 0$ for every $\theta \in \mathbb{T}^1$. Invariant curves play the role of a center around which the dynamics of F is organized, thus generalizing the role of a fixed point for the local dynamics of an holomorphic map (see [7]). We say that the invariant curve is *indifferent* if

$$\int_{\mathbb{T}^1} \log \big|\partial_z f_\theta(0)\big| d\theta = 0.$$

We recall that F is injective and so the differential $\partial_z f_\theta$ is everywhere nonzero. As it is shown in [7], a non-indifferent invariant curve is either attracting or repelling in the sense that there exists a topological tube that is attracted (or repelled) to the curve in the future. In fact, this is an equivalent topological definition for being indifferent. Let us suppose that the topological degree of the application $\theta \mapsto \partial_z f_\theta(0)$ is zero, that is, the application $\theta \mapsto \partial_z f_\theta(0)$ is homotopic in $\mathbb{C} \setminus \{0\}$ to a constant. In this case we say that curve is a *zero-degree invariant curve.* Under this hypothesis, we can define the logarithm of $\partial_z f_\theta(0)$. We define a number that represents the average rotation speed of the dynamics around the invariant curve:

$$\varrho_T(F) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log \partial_z f_\theta(0) d\theta.$$

This is a real number and is called the *fibered rotation number.* Notice that the log above is well-defined $\mod 2\pi i$, and hence the number $\varrho_T(F)$ is well-defined $\mod (\mathbb{Z})$. In [7] the author also shows that for analytic fibered holomorphic maps, and under an extra diophantine condition on the pair $(\alpha, \varrho_T(F))$, the map is conjugated to the pure linear map $(\theta, z) \mapsto (\theta + \alpha, e^{2\pi i \varrho_T(F)} z)$.

Let us study the behavior of this rotation number under conjugacies. A continuous map $h : U \subset (\mathbb{C}, 0) \to \mathbb{C}$, defined on a neighborhood of the origin will be called a *local positive homeomorphism* if it is an orientation preserving homeomorphism onto its image and left fixed the origin. We will consider continuous change of coordinates H defined on a tubular neighborhood of the invariant curve in the form

$$(\theta, z) \overset{H}{\longmapsto} \big(\theta, h_\theta(z)\big),$$

where functions h_θ are local positive homeomorphisms. We say that H is a *fibered conjugacy.* Conjugating the map F by H we get a new fibered map $\hat{F} = H^{-1} \circ F \circ H$ having the zero section as an invariant curve. The topological characterization of an indifferent invariant curve implies that this curve is also indifferent. Suppose for a while that h_θ is holomorphic and $\theta \mapsto \partial_z h_\theta(0)$ has zero topological degree (as defined above). An easy computation implies that the zero section is a zero-degree invariant curve and the fibered rotation number does not change, that is,

$\varrho_T(F) = \varrho_T(\hat{F})$.

We are interested in to show that the fibered rotation number is a topological characterization of local dynamics, and not just a differentiable one. Hence, we are interested in the invariance of the fibered rotation number under fibered conjugacies. The zero degree of the derivative at the origin needs a topological counterpart: we require that H is isotopic to the identity.

Finite fibers maps

Circle maps. The results contained in this section follow from the classical theory of circle homeomorphisms and proofs have been omitted (the reader is referred to [8]). Let $g_0, g_1, \ldots, g_{n-1}$ be positive circle homeomorphisms. We define the *finite fibers circle homeomorphism* G by

$$\begin{array}{rcl} G : \mathbb{Z}_n \times \mathbb{T}^1 & \longrightarrow & \mathbb{Z}_n \times \mathbb{T}^1 \\ (j, x) & \longmapsto & \big(j+1, g_i(x)\big). \end{array}$$

Let us consider $\tilde{g}_j : \mathbb{R} \to \mathbb{R}$ a *lift* of g_j to the real line, that is: $\tilde{g}_j(x+1) = \tilde{g}_j(x) + 1$ and $\Pi \circ \tilde{g}_j = g_j \circ \Pi$, where $\Pi : \mathbb{R} \to \mathbb{T}^1$ stands for the natural projection. These lifts are continuous, increasing and define a finite fibers dynamics

$$\begin{array}{rcl} \tilde{G} : \mathbb{Z}_n \times \mathbb{R} & \longrightarrow & \mathbb{Z}_n \times \mathbb{R} \\ (j, x) & \longmapsto & \big(j+1, \tilde{g}_j(x)\big). \end{array}$$

In order to take account of the rotation speed of orbits we define the m^{th}*-step* of a point $(j, x) \in \mathbb{Z}_n \times \mathbb{R}$ by $\Psi^{(m)}(j, x) = \Pi_{\mathbb{R}} \tilde{G}^m(j, x) - x$, where $\Pi_{\mathbb{R}} : \mathbb{Z}_n \times \mathbb{R} \to \mathbb{R}$ stands for the second coordinate projection. Using a sub-additive argument one can prove the following

Proposition 3 *For every $j \in \mathbb{Z}_n$, $x \in \mathbb{R}$ the limit*

$$\rho_{\mathrm{ff}}(\tilde{G}, j, x) = \lim_{m \to \infty} \frac{\Psi^{(m)}(j, x)}{m}$$

exists and belongs to $\mathbb{R}$. Moreover, this limit is independent of the choice of (j, x).

We define the *finite fibers circle rotation number of $\tilde{G}$* by a (any) circle rotation number $\rho_{\mathrm{ff}}(\tilde{G}) = \rho_{\mathrm{ff}}(\tilde{G}, j, x)$. The following properties hold

- i) Let $\hat{G}$ be another lift of G. Then there exists an integer p such that $\rho_{\mathrm{ff}}(\tilde{G}) = \rho_{\mathrm{ff}}(\hat{G}) + \frac{p}{n}$.
- ii) For every $m \in \mathbb{Z}$ one has $\rho_{\mathrm{ff}}(\tilde{G}^m) = m\rho_{\mathrm{ff}}(\tilde{G})$.

Remark 4 *The map $\tilde{G}^n\big|_{\{0\}\times\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is the lift of a positive circle homeomorphism. The circle rotation number $\rho(\tilde{G}^n\big|_{\{0\}\times\mathbb{R}})$ (the real representative corresponding to the lift $\tilde{G}$) coincides with the finite fibers circle rotation number $\rho_{\mathrm{ff}}(\tilde{G}^n)$ and hence we get $\rho(\tilde{G}^n\big|_{\{0\}\times\mathbb{T}^1}) = n\rho_{\mathrm{ff}}(\tilde{G})$.*

Local plane homeomorphisms. In the same manner as in the circle case, we will consider finite fibers maps of positive local homeomorphisms. Let us recall the generalized version of the Naishul's Theorem by Gambaudo-Le Calvez-Pécou: define $\mathcal{H}_+(0)$ as the set of positive local homeomorphisms. An element $f \in \mathcal{H}_+(0)$ is called *attractive* (resp. *repulsive*) if, in every neighborhood V of zero, there exists a simple closed curve C surrounding zero, that does not intersect its image, and such that its interior D verifies $\cap_{n\geq 0} f^n(D) = \{0\}$ (resp. $\cap_{n\geq 0} f^{-n}(D) = \{0\}$). An element $f \in \mathcal{H}_+(0)$ is *indifferent* if it is nor attractive nor repulsive.

A polar coordinates systems in $\mathbb{C}$ is a homeomorphism $h : \mathbb{C} \setminus \{0\} \to \mathbb{T}^1 \times (0, \infty)$, such that $h(z) = (\theta(z), r(z))$ verifies $\lim_{z\to 0} r(z) = 0$. We say that $f \in \mathcal{H}_+(0)$ is $\mathbb{T}^1$*-extendible* if there exists a polar coordinates system h such that $h \circ f \circ h^{-1}$ extends continuously to the boundary $\mathbb{T}^1 \times \{0\}$ as a homeomorphism. The circle rotation number of the positive circle homeomorphism obtained in this way is called an *angle of* f and the set of angles of f is denoted by $\mathcal{A}(f)$. The set $\mathcal{A}(f)$ is invariant by conjugacies in $\mathcal{H}_+(0)$. More precisely, if $\tilde{f}$ is a lift of f to (a neighborhood of $\mathbb{R} \times \{0\}$ in) $\mathbb{R} \times (0, \infty)$, each $\mathbb{T}^1$-extension defines a positive homeomorphism of $\mathbb{R}$, that is the lift of a positive circle homeomorphism. We denote by $\mathcal{A}(\tilde{f})$ the set of (real) circle rotation numbers obtained in this way.

Theorem 5 (Gambaudo-Le Calvez-Pécou, see [2]) *If $f \in \mathcal{H}_+(0)$ is $\mathbb{T}^1$-extendible and indifferent then the set $\mathcal{A}(f)$ has exactly one element. More precisely, for each lift $\tilde{f}$ of f, the set $\mathcal{A}(\tilde{f})$ reduces exactly to one element.*

This result generalizes the Naishul's Theorem since both area-preserving differentiable maps in $\mathcal{H}_+(0)$ and holomorphic maps with an indifferent derivative are indifferent in the sense above.

Let $f_0, f_1, \ldots, f_{n-1}$ be elements in $\mathcal{H}_+(0)$, defined in a neighborhood V of zero. We define the *finite fibers local positive homeomorphism* F by

$$\begin{array}{rcl} F : \mathbb{Z}_n \times V \subset & \longrightarrow & \mathbb{Z}_n \times \mathbb{C} \\ (j, z) & \longmapsto & \big(j+1, f_j(z)\big). \end{array}$$

We say that F is *indifferent* if $F^n(0, \cdot)$ is an indifferent map in $\mathcal{H}_+(0)$. A *finite fibers polar coordinate system* is a finite fibers homeomorphism $H(j, z) = (j, \theta_j(z), r_j(z))$ such that each (θ_j, r_j) is a polar coordinates system. We say that F is $\mathbb{Z}_n \times \mathbb{T}^1$-extendible if there exists a finite fibers polar coordinates system H such that $H \circ F \circ H^{-1}$ extends continuously to the boundary $\mathbb{Z}_n \times \mathbb{T}^1 \times \{0\}$ (as a finite fibers positive circle homeomorphism). Given a lift $\tilde{F} = \tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{n-1}$ of F, each $\mathbb{Z}_n \times \mathbb{T}^1$-extension defines a homeomorphism of $\mathbb{Z}_n \times \mathbb{R}$, which is the lift of a finite fibers positive circle homeomorphism. We denote by $\mathcal{A}(\tilde{F})$ the set of finite fibers circle rotation numbers obtained in this way. Using the Remark 4 and Theorem 5 one gets

Proposition 6 *Let F be a finite fibers positive local homeomorphism that is indifferent and $\mathbb{T}^1$-extendible. For every lift $\tilde{F}$ of F the set $\mathcal{A}(\tilde{F})$ reduces to exactly one element* ∎

Fibered circle homeomorphisms

In this section we review some features of the also called quasi-periodically forced circle homeomorphisms introduced by M.Herman in [3]: let $\alpha \in \mathbb{T}^1$ be an irrational angle and the continuous map

$$\begin{array}{rcl} W:\mathbb{T}^1\times\mathbb{T}^1 & \longrightarrow & \mathbb{T}^1\times\mathbb{T}^1 \\ (\theta,\omega) & \longmapsto & (\theta+\alpha, g_\theta(\omega)) \end{array}$$

where each map g_θ is a positive circle homeomorphism. Assume that W is isotopic to the identity. In [3] Herman defines the *fibered circle rotation number* $\rho_T(W)$ that can be computed as follows: given a lift $\tilde{W}$ of W to $\mathbb{R}\times\mathbb{R}$, and any point (θ,ω), the limit

$$\rho_T(\tilde{W}) = \lim_{n\to\infty} \frac{\Pi_{\mathbb{R}}\tilde{W}^n(\theta,\omega)-\omega}{n}, \tag{1}$$

always exists and is independent of (θ,ω). Moreover, the convergence is uniform on θ and ω. Another lift $\hat{W}$ of W produces a fibered circle rotation number $\rho_T(\hat{W})$ differing from $\rho_T(\tilde{W})$ by an integer number, and hence, the fibered circle rotation number $\rho_T(W)$ can be defined as $\rho_T(\tilde{W}) \mod \mathbb{Z}$. These torus homeomorphisms are *pseudo rotations*, in the sense that its rotation vectors set is composed exactly by one element, namely $(\alpha,\rho_T(W))$. The fibered circle rotation number is invariant under fibered conjugacies, that is, under a conjugacy by a isotopic to the identity map in the form

$$(\theta,\omega)\longmapsto(\theta, h_\theta(\omega)).$$

Example 7 *Let $\tau:\mathbb{T}^1\to\mathbb{T}^1$ be a continuous function with zero topological degree. Define $W(\theta,\omega)=(\theta+\alpha,\omega+\tau(\theta))$. Let $\tilde{\tau}:\mathbb{T}^1\to\mathbb{R}$ be a lift of τ. The Lebesgue measure is invariant and the Birkhoff ergodic theorem implies*

$$\rho_T(W) = \int_{\mathbb{T}^1}\tilde{\tau}(\theta)d\theta+\mathbb{Z}. \tag{2}$$

In terms of lifts, the above equality says that for every lift $\tilde{W}$ of W there exists an integer p so that $\rho_T(\tilde{W}) = p+\int_{\mathbb{T}^1}\tilde{\tau}(\theta)d\theta$.

The following is a useful way for computing the fibered circle rotation number. Let us consider a sequence of rational approximations $\frac{p_n}{q_n}$ of α. We define the *finite approximations of W at θ* as the following finite fibers circle homeomorphisms:

$$\begin{array}{rcl} W_{n,\theta}:\mathbb{Z}_{q_n}\times\mathbb{T}^1 & \longrightarrow & \mathbb{Z}_{q_n}\times\mathbb{T}^1 \\ (j,\omega) & \longmapsto & \left(j+1, g_{\theta+\frac{jp_n}{q_n}}(\omega)\right). \end{array}$$

For any lift $\tilde{W}$ of W the lifts $\tilde{W}_{n,\theta}$ are lifts of the corresponding finite fibers circle homeomorphisms.

Lemma 8 *For any lift $\tilde{W}$ of W, the finite fibers circle rotation numbers $\rho_{\mathrm{ff}}(\tilde{W}_{n,\theta})$ converge uniformly (on θ) to the fibered circle rotation number $\rho_T(\tilde{W})$.*

Proof. Remember that the convergence in (1) is uniform. That implies that, for $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that for every (θ,ω) one has

$$\left|\frac{\Pi_{\mathbb{R}}\tilde{W}^k(\theta,\omega)-\omega}{k}-\rho_T(\tilde{W})\right|<\varepsilon. \tag{3}$$

For large enough n and every (θ,ω) one has

$$\left|\Pi_{\mathbb{R}}\tilde{W}_n^k(\theta,\omega)-\Pi_{\mathbb{R}}\tilde{W}^k(\theta,\omega)\right|<k\varepsilon. \tag{4}$$

Let us define inductively

$$\begin{array}{rclcrcl}\theta_0 & = & \theta & , & \omega_0 & = & \omega \\ \theta_{j+1} & = & \theta_j+k\frac{p_n}{q_n} & , & \omega_{j+1} & = & \Pi_{\mathbb{R}}\tilde{W}_n^k(\theta_j,\omega_j).\end{array}$$

For every $r\in\mathbb{N}$ one has

$$\frac{\Pi_{\mathbb{R}}\tilde{W}_n^{rk}(\theta,\omega)-\omega}{rk}=\frac{1}{r}\sum_{j=0}^{r-1}\frac{\Pi_{\mathbb{R}}\tilde{W}_n^k(\theta_j,\omega_j)-\omega_j}{k}.$$

Using (3) and (4) one can write

$$\frac{\Pi_{\mathbb{R}}\tilde{W}_n^k(\theta_j,\omega_j)-\omega_j}{k}=\rho_T(\tilde{W})+\varepsilon_j$$

with $|\varepsilon_j|<2\varepsilon$. Hence

$$\frac{\Pi_{\mathbb{R}}\tilde{W}_n^{rk}(\theta,\omega)-\omega}{rk}=\rho_T(\tilde{W})+\frac{1}{r}\sum_{j=0}^{r-1}\varepsilon_j,$$

which implies that

$$\left|\frac{\Pi_{\mathbb{R}}\tilde{W}_n^{rk}(\theta,\omega)-\omega}{rk}-\rho_T(\tilde{W})\right|<2\varepsilon.$$

Letting $r\to\infty$ in the above inequality one gets

$$|\rho_{\mathrm{ff}}(\tilde{W}_{n,\theta})-\rho_T(\tilde{W})|<2\varepsilon \quad \blacksquare$$

Torus extensions of fibered local homeomorphisms

We say that a continuous map $F:\mathbb{T}^1\times\mathbb{D}_\delta\to\mathbb{T}^1\times\mathbb{C}$ is a *fibered local positive homeomorphism* if

$$F(\theta,z)=(\theta+\alpha,f_\theta(z)) \tag{5}$$

is such that $\alpha\in\mathbb{T}^1$ is an irrational angle and each fiber map f_θ is a positive local homeomorphism. We say that a homeomorphism $\mathcal{L}:\mathbb{T}^1\times(\mathbb{C}\setminus\{0\})\to\mathbb{T}^1\times\big(\mathbb{T}^1\times(0,+\infty)\big)$ is a *fibered polar coordinates system* if it has the form

$$\mathcal{L}(\theta,z)=(\theta,(\omega(\theta,z),r(\theta,z))),$$

and such that $\lim_{z\to 0}r(\theta,z)=0$. We say that F as in (5) is $\mathbb{T}^1\times\mathbb{T}^1$-*extendible* if there exists a fibered polar coordinates system $\mathcal{L}$ such that $\mathcal{L}\circ F\circ\mathcal{L}^{-1}$ extends continuously to the boundary $\mathbb{T}^1\times\big(\mathbb{T}^1\times\{0\}\big)$ as a homeomorphism that is isotopic to the identity. In such case the extension results into a fibered circle homeomorphism.

Example 9 *Let F be a fibered holomorphic map*

$$\begin{aligned} F : \mathbb{T}^1 \times \mathbb{D}_\delta &\longrightarrow \mathbb{T}^1 \times \mathbb{C} \\ (\theta, z) &\longmapsto \big(\theta + \alpha, \rho_1(\theta) z + \rho_2(\theta) z^2 + \dots\big). \end{aligned} \tag{6}$$

Then F is a fibered positive local homeomorphism. Moreover, F is $\mathbb{T}^1 \times \mathbb{T}^1$-extendible if we consider the fibered action of the complex derivative over the directions at the invariant curve. Indeed, the natural $\mathbb{T}^1 \times \mathbb{T}^1$-extension is

$$W_F(\theta, \omega) = \left(\theta + \alpha, \omega + \mathcal{R}\frac{1}{2\pi i} \log \rho_1(\theta)\right)$$

where $\mathcal{R}\cdot$ stands for the real part. As expected, the polar coordinates system is the usual polar coordinates.

Let us define $\mathcal{A}(F)$ as the set of all fibered circle rotation numbers corresponding to fibered circle homeomorphisms extending F. This set is clearly invariant under fibered conjugacies that are isotopic to the identity. More precisely, for every lift $\tilde{F}$ of F to (a neighborhood of $\mathbb{R}\times\mathbb{R}\times\{0\}$ in) $\mathbb{R} \times \mathbb{R} \times (0, \infty)$, each $\mathbb{T}^1 \times \mathbb{T}^1$-extension defines a homeomorphism of $\mathbb{R} \times \mathbb{R}$, that is a lift of a fibered circle homeomorphism. We denote by $\mathcal{A}(\tilde{F})$ the set of (real) fibered circle rotation numbers obtained in this way.

Lemma 10 *Let F be a fibered holomorphic map as in (6). Then F is $\mathbb{T}^1 \times \mathbb{T}^1$-extendible. Moreover, for every lift $\tilde{F}$ of F there exists a real representative $\varrho_T(\tilde{F})$ of $\varrho_T(F)$ that belongs to $\mathcal{A}(\tilde{F})$.*

Proof. The discussion above yields $\rho_T(W_F) \in \mathcal{A}(F)$. Example 7 implies $\varrho_T(F) = \rho_T(W_F)$ ∎

Proposition 11 *Let F be a fibered holomorphic map as in (6). Then $\mathcal{A}(F) = \{\varrho_T(F)\}$. More precisely, for every lift $\tilde{F}$ of F there exists a real representative $\varrho_T(\tilde{F})$ of $\varrho_T(F)$ such that $\mathcal{A}(\tilde{F}) = \{\varrho_T(\tilde{F})\}$.*

Proof. Let $\mathcal{L} = (\theta, \omega, r)$ be a fibered polar coordinates system and $W : \mathbb{T}^1 \times \mathbb{T}^1 \to \mathbb{T}^1 \times \mathbb{T}^1$ be the fibered circle homeomorphism extending F in these coordinates. We are going to consider the *finite approximations of F at θ*

$$\begin{aligned} F_{n,\theta} : \mathbb{Z}_{q_n} \times \mathbb{D}_\delta &\longrightarrow \mathbb{Z}_{q_n} \times \mathbb{C} \\ (j, z) &\longmapsto \left(j + 1, \rho_1\left(\theta + \frac{jp_n}{q_n}\right) z + \rho_2\left(\theta + \frac{jp_n}{q_n}\right) z^2 + \dots\right). \end{aligned}$$

Since $\int_{\mathbb{T}^1} \log|\rho_1(\theta)| d\theta = 0$, the intermediate value theorem gives $\theta_n \in \mathbb{T}^1$ such that

$$\sum_{j=0}^{q_n-1} \log\left|\rho_1\left(\theta_n + \frac{jp_n}{q_n}\right)\right| = 0. \tag{7}$$

The map F_{n,θ_n} is a finite fibers local positive homeomorphism. Furthermore, it is $\mathbb{Z}_{q_n} \times \mathbb{T}^1$-extendible. The equation (7) says that the derivative of $F^{q_n}_{n,\theta_n}(0,\cdot)$ at the origin is a rotation, and, since the fibers are holomorphic maps, we conclude that F_{n,θ_n} is indifferent in the sense

of Proposition 6. Indeed, a holomorphic map with a fixed point having a neighborhood that is contracted (resp. expanded) is holomorphically conjugated to a linear contraction (resp. linear expansion), hence, its derivative at the fixed point is a linear contraction (resp. linear expansion).
Let $\tilde{F}$ be a lift of F and $\tilde{W}, \tilde{F}_{n,\theta_n}, \tilde{W}_{n,\theta_n}$ be the corresponding lifts. On the one hand, Proposition 6 implies that $\rho_{\mathrm{ff}}(\tilde{W}_{n,\theta_n})$ does not depend on the $\mathbb{T}^1 \times \mathbb{T}^1$-extension of F. On the other hand, Lemma 8 says that $\rho_{\mathrm{ff}}(\tilde{W}_{n,\theta_n})$ converges to $\rho_T(\tilde{W})$. Hence the fibered circle rotation number $\rho_T(\tilde{W})$ can assume exactly one value and Lemma 10 above allows to conclude ■

Proof of the Theorem 2. Since F, G are topologically conjugated by a fibered map that is isotopic to the identity, $\mathcal{A}(F)$ and $\mathcal{A}(G)$ must coincide. The above Proposition implies $\varrho_T(F) = \varrho_T(G)$ □

Remark. Let us stress that the assumption that the fiber maps are holomorphic is only used in the proof of the proposition 11. There, we conclude that $F^{q_n}_{n,\theta_n}$ is indifferent since its derivative at the fixed point is a rotation. All the other results hold under the weaker assumption of that the fiber maps are C^1 diffeomorphisms, and such that the logarithm of the Jacobian determinant at the invariant curve has zero mean. An additional hypothesis allowing to conclude that the finite fibers maps obtained in the proof of the proposition 11 are indifferent is hence sufficient to obtain a related Naishul's result. For example, assume that each fiber map is a positive local diffeomorphism and such that the area is multiplied by a positive number $\beta(\theta)$ satisfying $\int_{\mathbb{T}^1} \log \beta(\theta) d\theta = 0$.

References

[1] Jean-Marc Gambaudo, Patrice Le Calvez, and Élisabeth Pécou. Une généralisation d'un théorème de Naishul. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(4):397–402, 1996.

[2] Jean-Marc Gambaudo and Elisabeth Pécou. A topological invariant for volume preserving diffeomorphisms. *Ergodic Theory Dynam. Systems*, 15(3):535–541, 1995.

[3] Michael-R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol′d et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58(3):453–502, 1983.

[4] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[5] Frédéric Le Roux. A topological characterization of holomorphic parabolic germs in the plane. *Fund. Math.*, 198(1):77–94, 2008.

[6] V. A. Naĭshul′. Topological invariants of analytic and area-preserving mappings and their application to analytic differential equations in $\mathbf{C}^2$ and $\mathbf{C}P^2$. *Trudy Moskov. Mat. Obshch.*, 44:235–245, 1982.

[7] Mario Ponce. Local dynamics for fibred holomorphic transformations. *Nonlinearity*, 20(12):2939–2955, 2007.

[8] J. Stark, U. Feudel, P. A. Glendinning, and A. Pikovsky. Rotation numbers for quasi-periodically forced monotone circle maps. *Dyn. Syst.*, 17(1):1–28, 2002.

Mario Ponce
Facultad de Matemáticas, Pontificia Universidad Católica de Chile.
Casilla 306, Santiago 22, Chile
E-mail: mponcea@mat.puc.cl