Spectral properties of horocycle flows for compact surfaces of constant negative curvature

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Commutator methods

- $\mathcal{H}$, Hilbert space with norm $\| \cdot \|$ and scalar product $\langle \cdot, \cdot \rangle$
- $\mathcal{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathcal{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- $A, H$, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral measures $E^A(\cdot), E^H(\cdot)$, and spectra $\sigma(A), \sigma(H)$
Commutator methods

Definition

$S \in \mathcal{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if the map

$$\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H})$$

is strongly of class $C^k$.

$S \in C^1(A)$ if and only if

$$\left| \langle \varphi, S A \varphi \rangle - \langle A \varphi, S \varphi \rangle \right| \leq \text{Const.} \| \varphi \|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator associated to the continuous extension of the quadratic form is denoted by $[S, A]$, and

$$[iS, A] = s- \frac{d}{dt} \bigg|_{t=0} e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H}).$$
### Definition

A self-adjoint operator $H$ is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If $H$ is of class $C^1(A)$, then

$$[A, (H - z)^{-1}] = (H - z)^{-1}[H, A](H - z)^{-1},$$

with $[H, A]$ the operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}.$$
Theorem (Mourre 1981, and others in the 1990’s)

Let \( H \) be of class \( C^2(A) \). Assume there exist a bounded Borel set \( I \subset \mathbb{R} \), a number \( a > 0 \) and \( K \in \mathcal{K}(\mathcal{H}) \) such that

\[
E^H(I)[iH, A]E^H(I) \geq aE^H(I) + K. \tag{★}
\]

Then, \( H \) has at most finitely many eigenvalues in \( I \) (multiplicities counted), and \( H \) has no singular continuous spectrum in \( I \).

- The inequality (★) is called a Mourre estimate for \( H \) on \( I \).
- The operator \( A \) is called a conjugate operator for \( H \) on \( I \).
- If \( K = 0 \), \( H \) has purely absolutely continuous spectrum in \( I \cap \sigma(H) \).
Let $M$ be a smooth manifold with probability measure $\mu$, and $\{F_t\}_{t \in \mathbb{R}}$ a $C^1$ measure preserving flow on $M$ with $C^0$ vector field $X_F$.

Ergodicity, weak mixing and strong mixing of $\{F_t\}_{t \in \mathbb{R}}$ with respect to $\mu$ are expressible in terms of the self-adjoint operator $H_F := iX_F$ in $L^2(M, \mu)$.

- $\{F_t\}_{t \in \mathbb{R}}$ is ergodic if and only if $0$ is a simple eigenvalue of $H$,
- $\{F_t\}_{t \in \mathbb{R}}$ is weakly mixing if and only if $H_F$ has purely continuous spectrum in $\mathbb{R} \setminus \{0\}$.
- $\{F_t\}_{t \in \mathbb{R}}$ is strongly mixing if and only if
  \[
  \lim_{t \to \infty} \langle \varphi, e^{-itH_F} \varphi \rangle = 0 \quad \text{for all } \varphi \in \{C \cdot 1\}^\perp.
  \]

\[
\begin{array}{cccc}
\text{a.c. spectrum in } \{C \cdot 1\}^\perp & \Rightarrow & \text{strong mixing} & \Rightarrow \text{weak mixing} & \Rightarrow \text{ergodicity}
\end{array}
\]
Minimal $W^u$ flows

- $M$, compact connected Riemannian manifold with distance $d$,
- $\{f_t\}_{t \in \mathbb{R}}$, $C^{1+\varepsilon}$ Anosov flow on $M$; that is, a $C^{1+\varepsilon}$ flow on $M$ without fixed points, with three submanifolds $W^u(x)$, $W^s(x)$, $\text{Orb}(x)$ passing through each $x \in M$,

$$W^u(x) = \left\{ y \in M \mid \lim_{t \to -\infty} d(f_t(x), f_t(y)) = 0 \right\} \quad \text{unstable manifold},$$

$$W^s(x) = \left\{ y \in M \mid \lim_{t \to +\infty} d(f_t(x), f_t(y)) = 0 \right\} \quad \text{stable manifold},$$

$$\text{Orb}(x) = \left\{ f_t(x) \mid t \in \mathbb{R} \right\} \quad \text{orbit},$$

with respective tangent spaces $E^u_x$, $E^s_x$, $E_x$ continuous in $x$ and satisfying

$$T_x M = E^u_x \oplus E^s_x \oplus E_x.$$ 

The flow $\{f_t\}_{t \in \mathbb{R}}$ has a $C^\varepsilon$ vector field $X_f$. 
Assume that $\{f_t\}_{t \in \mathbb{R}}$ is a codimension 1 Anosov flow. More specifically: 

$\{W^u(x)\}_{x \in M}$ is a 1-dimensional orientable $C^0$ foliation of $M$ (in particular each $W^u(x)$ is a curve), which supports a $C^0$ minimal flow $\{\phi_s\}_{s \in \mathbb{R}}$ whose orbits are the unstable manifolds.\(^1\)

$\{\phi_s\}_{s \in \mathbb{R}}$ is called minimal $W^u$ flow or minimal $W^u$ parametrisation.

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\(^1\)A flow on a compact metric manifold is minimal if each of its orbit is dense.
Example

The iconic example of Anosov flow $\{f_t\}_{t \in \mathbb{R}}$ and $W^u$ flow $\{\phi_s\}_{s \in \mathbb{R}}$ are the geodesic flow and the horocycle flow on the unit tangent bundle of a compact connected orientable surface of (possibly variable) negative curvature.
Geodesic flow in the Poincaré half plane
Positive horocycle flow in the Poincaré half plane
(from Bekka/Mayer’s book)
Geodesics and horocycles in the Poincaré half plane
(from Hasselblatt/Katok’s book)
Anosov stable and unstable foliations for the geodesic flow on the unit tangent bundle of a surface of constant negative curvature

(from http://kyokan.ms.u-tokyo.ac.jp/~showroom/)
Some facts from [Marcus 75], [Marcus 77], [Bowen-Marcus 77]:

(i) $\{\phi_s\}_{s \in \mathbb{R}}$ is uniquely ergodic w.r.t. a probability measure $\mu$ on $M$.

This means that for any $h \in C(M)$ we have

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t ds \ (h \circ \phi_s)(x) = \int_M d\mu(y) \ h(y)
$$

uniformly in $x \in M$.

(ii) There exists $s^* : \mathbb{R} \times \mathbb{R} \times M \to \mathbb{R}$ such that

$$(f_t \circ \phi_s \circ f_{-t})(x) = \phi_{s^*(t,s,x)}(x), \quad s, t \in \mathbb{R}, \ x \in M,$$

(the Anosov flow $\{f_t\}_{t \in \mathbb{R}}$ expands the $W^u$ orbits).
(iii) \( \{ W^u(x) \}_{x \in M} \) admits a \( C^0 \) parametrisation \( \{ \tilde{\phi}_s \}_{s \in \mathbb{R}} \) such that

\[
    f_t \circ \tilde{\phi}_s \circ f_{-t} = \tilde{\phi}_{\lambda^t s}, \quad s, t \in \mathbb{R}, \quad \lambda > 1 \quad \text{(that is, } s^*(t, s, x) = \lambda^t s) \]

(uniformly expanding parametrisation).

(iv) \( \{ \tilde{\phi}_s \}_{s \in \mathbb{R}} \) is uniquely ergodic w.r.t. a probability measure \( \tilde{\mu} \) given in terms of \( \mu \).

(v) \( \tilde{\mu} \) is invariant under the Anosov flow \( \{ f_t \}_{t \in \mathbb{R}} \).
Assumption 1

\( \{ \phi_s \}_{s \in \mathbb{R}} \) is \( C^1 \), and \( \{ \tilde{\phi}_s \}_{s \in \mathbb{R}} \) is a \( C^1 \) reparametrisation of \( \{ \phi_s \}_{s \in \mathbb{R}} \).

Under this assumption, we have:

- \( \tilde{\mu} = \mu / \tilde{\rho} \) with \( \tilde{\rho} = \rho \int_M d\mu \rho^{-1} \) and \( \rho \in C(M; (0, \infty)) \).

- The group in \( \mathcal{H} := L^2(M, \mu) \) given by
  \[
  U_s^\phi \varphi := \varphi \circ \phi_s, \quad s \in \mathbb{R}, \ \varphi \in \mathcal{H},
  \]
  is strongly continuous, unitary, with essentially self-adjoint generator
  \[
  H_\phi \varphi = iX_\phi \varphi, \quad \varphi \in C^1(M),
  \]

- The group in \( \mathcal{H} \) given by
  \[
  U_t^f \varphi := \varphi \circ f_t, \quad t \in \mathbb{R}, \ \varphi \in \mathcal{H},
  \]
  is strongly continuous, but not unitary if \( \rho \neq 1 \).
Assumption 2

The derivative

\[ u_{t,s}(x) := (\partial_1 \partial_2 s^*)(t, s, x) \]

exists and is continuous in \( s, t \in \mathbb{R} \) and \( x \in M \).

Under this assumption, using the unique ergodicity of \( \{\phi_s\}_{s \in \mathbb{R}} \), Marcus has proved that \( \{\phi_s\}_{s \in \mathbb{R}} \) is strongly mixing w.r.t. \( \mu \). Therefore,

\[ H_\phi \text{ has purely continuous spectrum in } \mathbb{R} \setminus \{0\}. \]

So, let’s prove that \( H_\phi \) has purely absolutely continuous spectrum in \( \mathbb{R} \setminus \{0\} \) under some additional regularity assumption.
Mourre estimate

Assumption 3

\[ X_f \text{ and } X_\phi \text{ are } C^1, \ X_f(\rho) \in C(M) \text{ and } \rho^{-1}X_f(\rho) \in C^1(M). \]

Intuitively, the conjugate operator is constructed as follows:

1) Sum \( 2iX_f \) and its “divergence” \( i\rho^{-1}X_f(\rho) \) to get a symmetric operator \( 2iX_f + i\rho^{-1}X_f(\rho) \) on \( C^1(M) \).

2) Take the Birkhoff average of \( 2iX_f + i\rho^{-1}X_f(\rho) \) along the flow \( \{\phi_s\}_{s \in \mathbb{R}} \) to take into account the unique ergodicity of \( \{\phi_s\}_{s \in \mathbb{R}} \).
Proposition (Conjugate operator)

Suppose that Assumptions 1, 2, 3 are satisfied. Then, the operator

\[
A_t \varphi := \frac{1}{t} \int_0^t ds \, U_s^\phi (2iX_f + i\rho^{-1}X_f(\rho)) U_{-s}^\phi \varphi, \quad t > 0, \, \varphi \in C^1(M),
\]

is essentially self-adjoint in \( \mathcal{H} \).

Idea of the proof.

The operator \( 2iX_f + i\rho^{-1}X_f(\rho) \) is symmetric on \( C^1(M) \), and the operations \( U_s^\phi(\cdots)U_{-s}^\phi \) and \( \frac{1}{t} \int_0^t ds \, (\cdots) \) preserve this property. So, \( A_t \) is symmetric on \( C^1(M) \).

Furthermore, \( A_t \) can be written as \( i(X_t + g_t) \) on \( C^1(M) \), with \( X_t \) a \( C^1 \) vector field and \( g_t \in C^1(M; \mathbb{R}) \).

Operators of this type are essentially self-adjoint on \( C^1(M) \).
With some calculations on $C^1(M)$ using properties of the flows $\{f_t\}_{t \in \mathbb{R}}$, $\{\phi_s\}_{s \in \mathbb{R}}$, $\{\bar{\phi}_s\}_{s \in \mathbb{R}}$ and the function $u_{t,s}(x) = (\partial_1 \partial_2 s^*)(t, s, x)$, we obtain the following:

**Lemma (Regularity of $H_\phi$)**

Suppose that Assumptions 1, 2, 3 are satisfied. Then, for $t > 0$ we have $(H_\phi - i)^{-1} \in C^2(A_t)$, and

$$[i(H_\phi - i)^{-1}, A_t] = 2(H_\phi - i)^{-1} c_t H_\phi (H_\phi - i)^{-1} - [(H_\phi - i)^{-1}, c_t]$$

with

$$c_t := \frac{1}{t} \int_0^t ds \left( u_{0,0} \circ \phi_s \right).$$
Because of the general formula
\[
[i(H - z)^{-1}, A] = -(H - z)^{-1}[iH, A](H - z)^{-1},
\]
we infer from the lemma that
\[
E^{H\phi}(I)[iH\phi, -A_t] E^{H\phi}(I) = 2E^{H\phi}(I)c_t H\phi E^{H\phi}(I) - (H\phi - i)E^{H\phi}(I)[(H\phi - i)^{-1}, c_t] (H\phi - i) E^{H\phi}(I)
\]
for each bounded Borel set \( I \subset \mathbb{R} \).
Can we get some positivity out of the last equation?
Proposition (Mourre estimate)

Suppose that Assumptions 1, 2, 3 are satisfied, and take $I \subset (0, \infty)$ compact with $I \cap \sigma(H_\phi) \neq \emptyset$. Then, there exist $t > 0$ and $a > 0$ such that

$$E^{H_\phi}(I)[iH_\phi, -A_t]E^{H_\phi}(I) \geq aE^{H_\phi}(I).$$

A similar result holds for $I \subset (-\infty, 0)$.

Idea of the proof.

The unique ergodicity of $\{\phi_s\}_{s \in \mathbb{R}}$ w.r.t. $\mu$ implies that

$$\lim_{t \to \infty} c_t = \lim_{t \to \infty} \frac{1}{t} \int_0^t ds \left(u_{0,0} \circ \phi_s\right) = \int_M d\mu u_{0,0}$$

uniformly on $M$. Moreover, some calculations show that

$$\int_M d\mu u_{0,0} = \ln(\lambda) > 0.$$
Idea of the proof (continued).

So, one has for $t > 0$ large enough

$$E^{H_\phi}(I)[iH_\phi, -A_t] E^{H_\phi}(I)$$

$$= 2 E^{H_\phi}(I) c_t H_\phi E^{H_\phi}(I)$$

$$- (H_\phi - i) E^{H_\phi}(I) [(H_\phi - i)^{-1}, c_t - \ln(\lambda)] (H_\phi - i) E^{H_\phi}(I)$$

$$\approx 2 E^{H_\phi}(I) \ln(\lambda) H_\phi E^{H_\phi}(I)$$

$$\geq 2 \ln(\lambda) \inf(I) E^{H_\phi}(I)$$

which gives

$$E^{H_\phi}(I)[iH_\phi, -A_t] E^{H_\phi}(I) \geq a E^{H_\phi}(I) \quad \text{with} \quad a \in (0, 2 \ln(\lambda) \inf(I)).$$
Using Mourre’s theorem, we conclude that:

**Theorem (Absolutely continuous spectrum)**

Suppose that Assumptions 1, 2, 3 are satisfied. Then, $H_\phi$ has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbb{C} \cdot 1$.

- The theorem applies in particular to generators of reparametrisations of the horocycle flow on the unit tangent bundle of a compact connected orientable surface of **constant** negative curvature.

- For reparametrisations of the horocycle flow on the unit tangent bundle of a compact connected orientable surface of **variable** negative curvature the question is open.
Gracias !


• R. Tiedra de Aldecoa. Spectral analysis of time changes of horocycle flows. J. Mod. Dyn., 2012


• R. Tiedra de Aldecoa. Spectral properties of horocycle flows for compact surfaces of constant negative curvature. Proyecciones, 2017