Commutator methods with applications to the spectral analysis of dynamical systems

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1 Commutator methods for self-adjoint operators

Commutator methods are a tool for the spectral theory and the scattering theory of self-adjoint operators in Hilbert spaces. They have been introduced by Éric Mourre in the 80’s for the study of Schrödinger operators in $L^2(\mathbb{R}^d)$ (and further developed by Amrein, Boutet de Monvel, Georgescu, Gérard, Jensen, Perry, Sahbani, …)
1.1 Classical mechanics as a motivation

- $M$, symplectic/Poisson manifold with Poisson bracket $\{ \cdot, \cdot \}$
- $H \in \mathcal{C}^\infty(M)$, Hamiltonian with complete flow $\{ \varphi_t \}_{t \in \mathbb{R}}$
- Hamiltonian evolution equation for an observable $f \in \mathcal{C}^\infty(M)$:

$$\frac{d}{dt} f \circ \varphi_t = \{ f, H \} \circ \varphi_t, \quad t \in \mathbb{R}.$$
For instance, if $H(q, p) := |p|^2 + V(q)$ on $M := T^*\mathbb{R}^d$ with $V \in C^\infty_c(\mathbb{R}^d)$, let’s say that we don’t want orbits bounded in $|q|^2$.

We want something like:

\[ |q|^2 \circ \varphi_t \]
Since, $\frac{\mathrm{d}^2}{\mathrm{d}t^2} |q|^2 \circ \varphi_t = \{\{|q|^2, H\}, H\} \circ \varphi_t$, it is sufficient to check that
\[
\{\{|q|^2, H\}, H\} \geq \delta > 0.
\]

In the example $H(q, p) = |p|^2 + V(q)$, we get
\[
\{\{|q|^2, H\}, H\} = \{\{|q|^2, |p|^2 + V(q)\}, H\} = \{4(q \cdot p), |p|^2 + V(q)\} = 8|p|^2 - 4q \cdot (\nabla V)(q).
\]

Thus, $|p|^2 > \frac{1}{2} \sup_{q \in \mathbb{R}^n} |q \cdot (\nabla V)(q)|$ implies $\lim_{|t| \to \infty} |q|^2 \circ \varphi_t = +\infty$.

(If the kinetic energy $|p|^2$ is large enough, all the trajectories go to infinity . . . )
To some extent, the idea behind commutators methods for self-adjoint operators is to translate the last example into the language of the (quantum) Hilbertian theory with the following heuristic dictionary in mind:

<table>
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<th>Poisson manifold $M$</th>
<th>$\iff$</th>
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<tr>
<td>Poisson bracket ${\cdot, \cdot}$</td>
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<td>Hamiltonian $H \in C^\infty(M)$</td>
<td>$\iff$</td>
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<td>$\frac{d}{dt} f \circ \varphi_t = {f, H} \circ \varphi_t$</td>
<td>$\iff$</td>
<td>$\frac{d}{dt} e^{-itH} F e^{itH} = e^{itH} [iF, H] e^{-itH}$</td>
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<tr>
<td>bounded orbits of $H$</td>
<td>$\iff$</td>
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1.2 Self-adjoint operators

References:

- W. O. Amrein, Hilbert Space Methods In Quantum Mechanics, EPFL Press, 2009
- J. Weidmann, Linear Operators in Hilbert Spaces, Springer Verlag, 1980
An operator $H$ with dense domain $\mathcal{D}(H)$ in a Hilbert space $\mathcal{H}$ is symmetric if

$$\langle H\varphi, \psi \rangle = \langle \varphi, H\psi \rangle \quad \text{for all } \varphi, \psi \in \mathcal{D}(H).$$

A vector $\eta \in \mathcal{H}$ belongs to $\mathcal{D}(H^*)$ if there exists $\eta^* \in \mathcal{H}$ such that

$$\langle \eta^*, \varphi \rangle = \langle \eta, A\varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(H).$$

In this case, one sets $H^*\eta := \eta^*$ and one calls $H^*$ the adjoint of $H$.

A symmetric operator $H$ is self-adjoint if

$$\{H, \mathcal{D}(H)\} = \{H^*, \mathcal{D}(H^*)\},$$

which is verified if and only if the ranges $\text{Ran}(H \pm i) = \mathcal{H}$. 
If $H$ is self-adjoint, then the set $\mathcal{D}(H)$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{D}(H)} := \langle \varphi, \psi \rangle + \langle H\varphi, H\psi \rangle, \quad \varphi, \psi \in \mathcal{D}(H),$$

and the induced norm

$$\|\varphi\|_{\mathcal{D}(H)}^2 := \langle \varphi, \varphi \rangle_{\mathcal{D}(H)}, \quad \varphi \in \mathcal{D}(H),$$

defines a Hilbert space (a complete inner space).

A subspace $\mathcal{D} \subset \mathcal{D}(H)$ is a core for $H$ if the closure of $\mathcal{D}$ in $\mathcal{D}(H)$ is equal to $\mathcal{D}(H)$; that is,

$$\overline{\mathcal{D}} \cap \mathcal{D}(H) = \mathcal{D}(H).$$
Example 1.1. The multiplication operator $Q$ in $\mathcal{H} := L^2(\mathbb{R})$ given by

$$(Q\varphi)(x) := x\varphi(x), \quad \varphi \in \mathcal{H}_1(\mathbb{R}) := \left\{ \varphi \in \mathcal{H} \mid \int_{\mathbb{R}} (1 + |x|^2)|\varphi(x)|^2 < \infty \right\},$$

is self-adjoint.

Example 1.2. The operator $P$ in $\mathcal{H} := L^2(\mathbb{R})$ given by

$$(P\varphi)(x) := -i\varphi'(x), \quad \varphi \in \mathcal{H}^1(\mathbb{R}) := \mathcal{F}\mathcal{H}_1(\mathbb{R}),$$

with $\mathcal{F}$ the 1-dimensional Fourier transform, is self-adjoint.

(the operator $P$ is just the Fourier transform of the operator $Q$; that is, $Q = \mathcal{F}P\mathcal{F}^{-1}$)

The space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$ is a core for $Q$ and $P$, since $\mathcal{S}(\mathbb{R})$ is dense in the (Sobolev) spaces $\mathcal{H}_1(\mathbb{R})$ and $\mathcal{H}^1(\mathbb{R})$. 
Let $\mathcal{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$ and let $H$ be a self-adjoint operator $H$ in $\mathcal{H}$.

The set

$$\rho(H) := \{ z \in \mathbb{C} \mid (H - z)^{-1} \text{ exists and belongs to } \mathcal{B}(\mathcal{H}) \}$$

is the resolvent set of $H$; it is an open subset of $\mathbb{C}$.

The set $\sigma(H) := \mathbb{C} \setminus \rho(H)$ is the spectrum of $H$; it is a closed subset of $\mathbb{R}$. 
A spectral family on a Hilbert space $\mathcal{H}$ is a function $E : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H})$ such that

- $E(\lambda)$ is an orthogonal projection for each $\lambda \in \mathbb{R}$, i.e.,
  $$E(\lambda) = E(\lambda)^* = E(\lambda)^2 \quad \text{for each } \lambda \in \mathbb{R},$$

- $E(\mu) \leq E(\lambda)$ for all $\mu \leq \lambda$, i.e.,
  $$\langle \varphi, E(\mu)\varphi \rangle \leq \langle \varphi, E(\lambda)\varphi \rangle \quad \text{for all } \varphi \in \mathcal{H}, \mu \leq \lambda \quad \text{(monotonicity)},$$

- $\operatorname{s-lim}_{\varepsilon \searrow 0} E(\lambda + \varepsilon) = E(\lambda)$ for each $\lambda \in \mathbb{R}$ (right continuity),

- $\operatorname{s-lim}_{\lambda \to -\infty} E(\lambda) = 0$ and $\operatorname{s-lim}_{\lambda \to \infty} E(\lambda) = 1$.

For intervals, one defines the spectral measure

$$E((a, b]) := E(b) - E(a), \quad E((a, b)) := \operatorname{s-lim}_{\varepsilon \searrow 0} E(b - \varepsilon) - E(a), \quad \text{etc.}$$

and one extends these definitions to $E(\mathcal{V})$ for any Borel set $\mathcal{V} \subset \mathbb{R}$. 
Theorem 1.3 (Spectral theorem). A self-adjoint operator $H$ in a Hilbert space $\mathcal{H}$ admits exactly one spectral family $E^H$ such that

$$H = \int_\mathbb{R} \lambda E^H(d\lambda),$$

with the strong integral $\int_\mathbb{R} \lambda dE^H(d\lambda)$ satisfying

$$\left\langle \varphi, \int_\mathbb{R} \lambda E^H(d\lambda) \psi \right\rangle := \int_\mathbb{R} \lambda \left\langle \varphi, E^H(d\lambda) \psi \right\rangle, \quad \varphi \in \mathcal{H}, \ \psi \in \mathcal{D}(H).$$

Furthermore, one has for $-\infty < a < b < \infty$ that

$$E^H((a, b]) = \frac{1}{\pi} \text{s-lim}_{\delta \downarrow 0} \text{s-lim}_{\varepsilon \downarrow 0} \int_{a+\delta}^{b+\delta} d\lambda \ \text{Im}(H - \lambda - i\varepsilon)^{-1}.$$  

(Stone’s Formula)
Two comments:

- The support of the spectral family $E^H$ is the set of points of non-constancy and coincides with the spectrum of $H$

$$\text{supp}(E^H) = \{ \lambda \in \mathbb{R} \mid E^H(\lambda + \varepsilon) - E^H(\lambda - \varepsilon) \neq 0 \ \forall \varepsilon > 0 \} = \sigma(H).$$

- Formally, one has

$$\|H\psi\|^2 = \langle H\psi, H\psi \rangle = \int_{\mathbb{R}} \lambda \int_{\mathbb{R}} \mu \langle E^H(\mathrm{d}\mu)\psi, E^H(\mathrm{d}\lambda)\psi \rangle$$

$$= \int_{\mathbb{R}} \lambda \int_{\mathbb{R}} \mu \langle \psi, E^H(\mathrm{d}\mu \cap \mathrm{d}\lambda)\psi \rangle$$

$$= \int_{\mathbb{R}} \lambda^2 \langle \psi, E^H(\mathrm{d}\lambda)\psi \rangle,$$

so that $\psi \in \mathcal{D}(H)$ if and only if $\int_{\mathbb{R}} \lambda^2 \langle \psi, E^H(\mathrm{d}\lambda)\psi \rangle < \infty$. 
Example 1.4. The spectral projection $E^Q(\lambda)$ of the operator $Q$ in $\mathcal{H} := L^2(\mathbb{R})$ is the operator of multiplication by the characteristic function $\chi_{(-\infty,\lambda]}$, i.e.,

$$E^Q(\lambda)\varphi := \chi_{(-\infty,\lambda]} \varphi, \quad \varphi \in \mathcal{H}.$$

One verifies that

$$\sigma(Q) = \text{supp}(E^Q) = \mathbb{R}.$$
Example 1.5. The multiplication operator $Q^2 := \sum_{j=1}^{d} Q_j^2$ in
\[ \mathcal{H} := L^2(\mathbb{R}^d) \]
given by

\[(Q^2 \varphi)(x) := x^2 \varphi(x), \quad \varphi \in \mathcal{H}_2(\mathbb{R}^d), \quad x^2 := \sum_{j=1}^{d} x_j^2,\]

is self-adjoint, and its spectral family is given by

\[E^{Q^2}(\lambda) \varphi := \begin{cases} 
\chi_{[-\lambda^{1/2}, \lambda^{1/2}]} \varphi & \text{if } \lambda > 0 \\
0 & \text{if } \lambda \leq 0, 
\end{cases} \quad \varphi \in \mathcal{H}.\]

One verifies that

\[\sigma(Q^2) = \text{supp}(E^{Q^2}) = [0, \infty).\]
The Laplacian $-\Delta$ in $\mathcal{H} := L^2(\mathbb{R}^d)$ satisfies on $\mathcal{L}(\mathbb{R}^d)$ (and thus on $\mathcal{H}^2(\mathbb{R}^d)$)

$$-\Delta = \sum_{j=1}^{d} P_j^2 \equiv P^2 = \mathcal{F}^{-1} Q^2 \mathcal{F},$$

with $\mathcal{F}$ the $d$-dimensional Fourier transform. So, one has

$$E^{-\Delta} = E^{\mathcal{F}^{-1} Q^2 \mathcal{F}} \quad \text{(Stone)} \quad \mathcal{F}^{-1} E Q^2 \mathcal{F}.$$
Let $\mathcal{A}_B$ be the Borel $\sigma$-algebra of $\mathbb{R}$ and $|\mathcal{V}|$ be the Lebesgue measure of $\mathcal{V} \in \mathcal{A}_B$.

If $H$ is a self-adjoint operator in $\mathcal{H}$, one has the orthogonal decompositions

$$\mathcal{H} = \mathcal{H}_p(H) \oplus \mathcal{H}_{sc}(H) \oplus \mathcal{H}_{ac}(H)$$

$$H = H|_{\mathcal{H}_p(H)} \oplus H|_{\mathcal{H}_{sc}(H)} \oplus H|_{\mathcal{H}_{ac}(H)},$$

with

$$\mathcal{H}_p(H) := \overline{\text{Span}\{\text{eigenvectors of } H\}}$$

$$\mathcal{H}_{sc}(H) := \{\varphi \in \mathcal{H} \mid \lambda \mapsto \|E^H(\lambda)\varphi\|^2 \text{ is continuous}$$

$$\text{ and } \exists \mathcal{V} \in \mathcal{A}_B \text{ with } |\mathcal{V}| = 0 \text{ and } E^H(\mathcal{V})\varphi = \varphi\}$$

$$\mathcal{H}_{ac}(H) := \{\varphi \in \mathcal{H} \mid \lambda \mapsto \|E^H(\lambda)\varphi\|^2 \text{ is absolutely continuous}\}.$$
The subspaces $\mathcal{H}_p(H)$, $\mathcal{H}_{sc}(H)$, $\mathcal{H}_{ac}(H)$ are the pure point subspace of $H$, the singular continuous subspace of $H$ and the absolutely continuous subspace of $H$.

The decomposition of $\mathcal{H}$ induces a decomposition of $\sigma(H)$

$$\sigma(H) = \sigma_p(H) \cup \sigma_{sc}(H) \cup \sigma_{ac}(H),$$

with

$\sigma_p(H) := \sigma(H|_{\mathcal{H}_p(H)})$ the pure point spectrum of $H$,

$\sigma_{sc}(H) := \sigma(H|_{\mathcal{H}_{sc}(H)})$ the singular continuous spectrum of $H$,

$\sigma_{ac}(H) := \sigma(H|_{\mathcal{H}_{ac}(H)})$ the absolutely continuous spectrum of $H$.

The sets $\sigma_p(H)$, $\sigma_{sc}(H)$, $\sigma_{ac}(H)$ are closed and (in general) not mutually disjoint.
Example 1.6. For each $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{H} := L^2(\mathbb{R})$, one has
\[
\left\| E^Q(\lambda) \varphi \right\|^2 = \left\| \chi_{(-\infty, \lambda]} \varphi \right\|^2 = \int_{-\infty}^{\lambda} dx \, |\varphi(x)|^2 = \text{integral of a } L^1\text{-function} = \text{absolutely continuous function.}
\]

So, $\mathcal{H} = \mathcal{H}_{ac}(Q)$ and $Q$ has purely absolutely continuous spectrum $\sigma(Q) = \sigma_{ac}(Q) = \mathbb{R}$.

In fact, $Q$ has Lebesgue spectrum since
\[
e^{itP} e^{isQ} e^{-itP} = e^{ist} e^{isQ}, \quad s, t \in \mathbb{R} \quad \iff \quad e^{itP} Q e^{-itP} = Q + t, \quad t \in \mathbb{R}.
\]
\[\text{\ldots Stone-von Neumann theorem \ldots}\]
Example 1.7. Let \( f : [0, 1] \to [0, 1] \) be the Cantor function, and let

\[
M_f \varphi := f \varphi, \quad \varphi \in \mathcal{H} := L^2([0, 1]),
\]

be the corresponding bounded multiplication operator.
The spectral family of $M_f$ is

$$E^{M_f}(\lambda)\varphi := \begin{cases} \chi_{f^{-1}([0,\lambda])} \varphi & \text{if } \lambda \in [0,1] \\ 0 & \text{if } \lambda \in \mathbb{R} \setminus [0,1], \end{cases} \quad \varphi \in \mathcal{H}.$$ 

One verifies that

$$\sigma(M_f) = \text{supp}(E^{Q^2}) = \text{Cantor ternary set}$$

and that the function

$$[0,1] \ni \lambda \mapsto \left\| E^{M_f}(\lambda)\varphi \right\|^2 = \left\| \chi_{f^{-1}([0,\lambda])} \varphi \right\|^2 = \int_0^1 dx \chi_{f^{-1}([0,\lambda])}(x) |\varphi(x)|^2$$

is continuous but not absolutely continuous.

So, $\mathcal{H} = \mathcal{H}_{sc}(M_f)$ and $M_f$ has purely singular continuous spectrum $\sigma(M_f) = \sigma_{sc}(M_f) = \text{Cantor ternary set}$. 
1.2 Self-adjoint operators

An interesting link between spectral theory and dynamics is provided by the following:

**Theorem 1.8 (RAGE theorem).** Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let $C \in \mathcal{B}(\mathcal{H})$ be such that $C(H+i)^{-1}$ is compact. Then,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \left\| C e^{-itH} \varphi \right\|^2 = 0 \quad \text{for all } \varphi \in \mathcal{H}_{sc}(H) \oplus \mathcal{H}_{ac}(H).$$

RAGE theorem says that, as time evolves, the state $\varphi$ in the continuous subspace of $H$ escapes (in Cesàro mean) from the range of the operator $C$.

(the typical example is when $H$ is a Schrödinger operator in $\mathbb{R}^d$ and $C$ the orthogonal projection onto a compact subset of $\mathbb{R}^d$)
1.3 Commutator methods for self-adjoint operators

References:

- W. O. Amrein, A. Boutet de Monvel and V. Georgescu, $C_0$-groups, commutator methods and spectral theory of $N$-body Hamiltonians, Birhäuser, 1996


1.3 Commutator methods for self-adjoint operators

- $\mathcal{H}$, Hilbert space with norm $\| \cdot \|$ and scalar product $\langle \cdot , \cdot \rangle$
- $\mathcal{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathcal{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- $A, H$, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral families $E^A(\cdot), E^H(\cdot)$ and spectra $\sigma(A), \sigma(H)$
- The adjoint space of a Banach space $\mathcal{B}$ is defined by

$$\mathcal{B}^* := \{ \text{anti-linear continuous functions } \phi : \mathcal{B} \rightarrow \mathbb{C} \}$$

$$\| \phi \|_{\mathcal{B}^*} := \sup \{ |\phi(\varphi)| \mid \varphi \in \mathcal{B}, \| \varphi \|_{\mathcal{B}} \leq 1 \}$$
Definition 1.9. An operator \( S \in \mathcal{B}(\mathcal{H}) \) satisfies \( S \in C^k(A) \) if the map

\[
\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H})
\]

is strongly of class \( C^k \).

In other terms, \( S \in C^k(A) \) if there exist

\[
B_0(t) \equiv e^{-itA} S e^{itA}, B_1(t), B_2(t), \ldots, B_k(t) \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R},
\]

such that

\[
\lim_{h \to 0} \left\| \frac{B_j(t + h) - B_j(t)}{h} \phi - B_{j+1}(t) \phi \right\| = 0 \quad \text{for all } t \in \mathbb{R}, \ \phi \in \mathcal{H},
\]

for \( j = 0, 1, \ldots, k - 1 \).
$S \in \mathcal{C}^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle A \varphi, S \varphi \rangle - \langle \varphi, SA \varphi \rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$; that is, if

$$\left| \langle A \varphi, S \varphi \rangle - \langle \varphi, SA \varphi \rangle \right| \leq \text{Const.} \| \varphi \|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator corresponding to the continuous extension of the quadratic form is denoted by $[A, S]$, and one has

$$-\left[ iA, S \right] = s \left. \frac{d}{dt} e^{-itA} S e^{itA} \right|_{t=0} \in \mathcal{B}(\mathcal{H}).$$
Example 1.10. Let $f \in L^\infty(\mathbb{R})$ be an absolutely continuous function with $f' \in L^\infty(\mathbb{R})$, and let

$$M_f \varphi := f \varphi, \quad \varphi \in \mathcal{H} := L^2(\mathbb{R}),$$

be the corresponding bounded multiplication operator.

Then, one has for each $\varphi \in \mathcal{H}$

$$\frac{d}{dt} e^{-itP} M_f e^{itP} \varphi = \frac{d}{dt} M_{f(-t)} \varphi = -M_{f'(-t)} \varphi,$$

and thus $M_f \in C^1(P)$ with $[iP, M_f] = M_{f'}$. 

In the case of (unbounded) self-adjoint operators, we have a similar definition:

**Definition 1.11.** A self-adjoint operator $H$ is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \rho(H)$.

If $H$ is of class $C^1(A)$, then

$$[A, (H - z)^{-1}] = (H - z)^{-1}[H - z, A](H - z)^{-1}$$

$$= (H - z)^{-1}[H, A](H - z)^{-1},$$

with $[H, A]$ the bounded operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ associated with the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}.$$
Theorem 1.12 (Virial Theorem). Let $A, H$ be self-adjoint operators with $H$ of class $C^1(A)$. Then,

$$E^H(\{\lambda\})[A, H]E^H(\{\lambda\}) = 0 \quad \text{for each } \lambda \in \mathbb{R}.$$ 

Thus, one has $\langle \varphi, [A, H]\varphi \rangle = 0$ if $\varphi$ is an eigenvector of $H$. 
Proof. We must show that if \( \varphi_1, \varphi_2 \in \mathcal{D}(H) \) satisfy \( H \varphi_j = \lambda \varphi_j \) for some \( \lambda \in \mathbb{R} \), then \( \langle \varphi_1, [A, H] \varphi_2 \rangle = 0 \).

But,

\[
\langle \varphi_1, [A, H] \varphi_2 \rangle \\
= \langle (\lambda - i)(H - i)^{-1} \varphi_1, [A, H](\lambda + i)(H + i)^{-1} \varphi_2 \rangle \\
= -(\lambda + i)^2 \langle \varphi_1, [A, (H + i)^{-1}] \varphi_2 \rangle \\
= -(\lambda + i)^2 \lim_{\tau \to 0} \langle \varphi_1, \left[ \frac{1}{i\tau}(e^{i\tau A} - 1), (H + i)^{-1} \right] \varphi_2 \rangle \\
= -(\lambda + i)^2 \lim_{\tau \to 0} \frac{1}{i\tau} \left\{ \langle \varphi_1, e^{i\tau A}(H + i)^{-1} \varphi_2 \rangle - \langle (H - i)^{-1} \varphi_1, e^{i\tau A} \varphi_2 \rangle \right\} \\
= -(\lambda + i)^2 \lim_{\tau \to 0} \frac{1}{i\tau} \{0\}.
\]

\[\square\]
Corollary 1.13 (Point spectrum of $H$). Let $A, H$ be self-adjoint operators with $H$ of class $C^1(A)$. Assume there exist a Borel set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathcal{K}(\mathcal{H})$ such that

$$E^H(I)[iH, A]E^H(I) \geq a E^H(I) + K. \quad (1.1)$$

Then, $H$ has at most finitely many eigenvalues in $I$ (multiplicities counted).

Some comments:

- If $I$ is bounded, one has

$$\underbrace{E^H(I)}_{\in \mathcal{B}(\mathcal{D}(H)^*, \mathcal{H})} \underbrace{[iH, A]}_{\in \mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)} \underbrace{E^H(I)}_{\in \mathcal{B}(\mathcal{H}, \mathcal{D}(H))} \in \mathcal{B}(\mathcal{H}).$$

- If $I$ is not bounded, the inequality (1.1) holds in the sense of quadratic forms on $\mathcal{D}(H)$.

- The inequality (1.1) is called a Mourre estimate.
Proof. If \( \varphi \in \mathcal{H} \) is an eigenvector of \( H \) with \( \|\varphi\| = 1 \) and with eigenvalue in \( I \), the Mourre inequality (1.1) implies that

\[
0 \geq a \langle \varphi, E^H(I)\varphi \rangle + \langle \varphi, K\varphi \rangle \implies \langle \varphi, K\varphi \rangle \leq -a.
\]

Now, if the claim were false, there would exist an infinite orthonormal sequence \( \{\varphi_j\} \) of eigenvectors of \( H \) in \( E^H(I)\mathcal{H} \). In particular, one would have \( \text{w-}\lim_{j \to \infty} \varphi_j = 0 \). Since \( K \in \mathcal{K}(\mathcal{H}) \), this would imply that \( \lim_{j \to \infty} \langle \varphi_j, K\varphi_j \rangle = 0 \), which contradicts the inequality \( \langle \varphi_j, K\varphi_j \rangle \leq -a < 0 \).

Note that the proof shows that if \( K = 0 \), then \( H \) is purely continuous in \( I \cap \sigma(H) \).
Example 1.14 (Finite dimension). If $\dim(\mathcal{H}) < \infty$, then $A, H$ are hermitian matrices and $H \in C^\infty(A)$.

Furthermore, one has

$$[iH, A] = 1 + ([iH, A] - 1) = 1 + \text{compact operator},$$

and the corollary implies (without surprise) that $H$ has at most finitely many eigenvalues in $\sigma(H)$. 
Definition 1.15. $S \in C^{1+0}(A)$ if $S \in C^{1}(A)$ and

$$\int_{0}^{1} \frac{dt}{t} \left\| e^{-itA}[A, S] e^{itA} - [A, S] \right\|_{\mathcal{B}(\mathcal{H})} < \infty.$$ 

Similarly, a self-adjoint operator $H$ is of class $C^{1+0}(A)$ if $(H - z)^{-1} \in C^{1+0}(A)$ for some $z \in \rho(H)$.

If we regard $C^{1}(A)$, $C^{1+0}(A)$ and $C^{2}(A)$ as subspaces of $\mathcal{B}(\mathcal{H})$, we have the inclusions

$$C^{2}(A) \subset C^{1+0}(A) \subset C^{1}(A) \subset \mathcal{B}(\mathcal{H}).$$
Example 1.16. Let $f \in L^\infty(\mathbb{R})$ be an absolutely continuous function with $f' \in L^\infty(\mathbb{R})$ Dini-continuous, and let $M_f$ be the corresponding multiplication operator in $\mathcal{H} := L^2(\mathbb{R})$.

Then, we know that $M_f \in C^1(P)$ with $[iP, M_f] = M_{f'}$, and

$$
\int_0^1 \frac{dt}{t} \| e^{-itP} [P, M_f] e^{itP} - [P, M_f] \|_{\mathcal{B}(\mathcal{H})} = \int_0^1 \frac{dt}{t} \| M_{f'(\cdot - t) - f'} \|_{\mathcal{B}(\mathcal{H})} \\
= \int_0^1 \frac{dt}{t} \| f'(\cdot - t) - f' \|_{L^\infty(\mathbb{R})} < \infty
$$

due to the Dini-continuity of $f'$. So, one has $M_f \in C^{1+0}(P)$. 
Spectral result of Mourre
(and Amrein, Boutet de Monvel, Georgescu, Sahbani, . . . )

Theorem 1.17 (Spectral properties of $H$). Let $H$ be of class $C^{1+0}(A)$. Assume there exist an open set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathcal{K}(\mathcal{H})$ such that

$$E^H(I)[iH, A]E^H(I) \geq a E^H(I) + K.$$

Then, $H$ has at most finitely many eigenvalues in $I$ (multiplicities counted), and $H$ has no singular continuous spectrum in $I$. 
Some comments:

- the operator $A$ is called a conjugate operator for $H$ on $I$

- if $K = 0$, then $H$ is purely absolutely continuous in $I \cap \sigma(H)$

- if $H$ has a spectral gap or satisfies an additional invariance assumption, then one can replace the condition $C^{1+0}(A)$ by a weaker condition $C^{1,1}(A)$
Sketch of the proof of Mourre (i)

One has for $\mu \in \sigma(H)$ and $\varepsilon \in \mathbb{R}$ that

$$\|(H - \mu - i\varepsilon)^{-1}\| = \left\|x \mapsto (x - \mu - i\varepsilon)^{-1}\right\|_{\mathcal{L}^\infty(\mathbb{R})} = |\varepsilon|^{-1}.$$ 

Thus, $(H - \mu - i\varepsilon)^{-1}$ cannot have a limit in $\mathcal{B}(\mathcal{H})$ as $\varepsilon \to \pm 0$.

However, for some $\varphi \in \mathcal{H} \setminus \{0\}$, the holomorphic function

$$F: \rho(H) \to \mathbb{C}, \quad z \mapsto \langle \varphi, (H - z)^{-1}\varphi \rangle,$$

may have a limit

$$F(\mu) := \lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon)$$

uniformly on each interval $[a, b] \subset I$. 

In such a case, Stone’s Formula and Lebesgue’s dominated convergence theorem imply for $\lambda \in (a, b]$ that

$$\|E^H((a, \lambda]) \varphi\|^2 = \langle \varphi, E^H((a, \lambda]) \varphi \rangle = \frac{1}{\pi} \int_a^\lambda \, d\mu \, \text{Im} \, F(\mu).$$

But, $F$ is continuous on $[a, b]$ due to the uniform convergence of the sequence $F_\varepsilon(\cdot) := \langle \varphi, (H - (\cdot) - i\varepsilon)^{-1} \varphi \rangle$. Thus,

$$\text{Im} \, F(\mu) \in L^1([a, b]) \quad \text{and} \quad E^H(I) \varphi \in \mathcal{H}_{ac}(H).$$

Therefore, if there is a dense set of vectors $\varphi \in \mathcal{H}$ satisfying what precedes, then $E^H(I)\mathcal{H} \subset \mathcal{H}_{ac}(H)$ and $H$ is purely absolutely continuous in $I \cap \sigma(H)$. 
Sketch of the proof of Mourre (ii)

Let’s show the existence of the limit \( \lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon) \) in the homogeneous case \([iH, A] = H\).

(in such case, one has \( e^{-itA} H e^{itA} = e^t H \), and thus we already know that \( H \) has homogeneous spectrum on \( \mathbb{R} \setminus \{0\} \))

One has for \( z \in \rho(H) \)

\[
z \frac{d}{dz} (H - z)^{-1} = z (H - z)^{-2} = (H - z)^{-1} H (H - z)^{-1} - (H - z)^{-1}
= [iA, (H - z)^{-1}] - (H - z)^{-1}
\]

which gives for \( \varphi \in \mathcal{D}(A) \)

\[
z \frac{d}{dz} F(z) = -F(z) - \langle iA\varphi, (H - z)^{-1} \rangle - \langle (H - \bar{z})^{-1}\varphi, iA\varphi \rangle.
\]
But, if $z = \mu + i\varepsilon$ with $\varepsilon > 0$, then

$$\|(H - \mu - i\varepsilon)^{-1}\varphi\|^2 = \|(H - \mu + i\varepsilon)^{-1}\varphi\|^2$$

$$= \langle\varphi, |H - \mu - i\varepsilon|^{-2}\varphi\rangle$$

$$= |\langle\varphi, \varepsilon^{-1}\text{Im}(H - \mu - i\varepsilon)^{-1}\varphi\rangle|$$

$$= \varepsilon^{-1} |\text{Im} F(\mu + i\varepsilon)|.$$
Thus, we get for $z = \mu + i\epsilon$ with $\mu \neq 0$ fixed and $\epsilon > 0$ that

$$\left| z \frac{d}{dz} F(z) \right| = \left| -F(z) - \langle iA\varphi, (H - z)^{-1} \rangle - \langle (H - \bar{z})^{-1}\varphi, iA\varphi \rangle \right|$$

$$\implies |\mu + i\epsilon| \left| \frac{d}{d\epsilon} F(\mu + i\epsilon) \right| \leq |F(\mu + i\epsilon)| + 2\|A\varphi\| \| (H - \mu - i\epsilon)^{-1}\varphi \|$$

$$\implies \left| \frac{d}{d\epsilon} F(\mu + i\epsilon) \right| \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \| (H - \mu - i\epsilon)^{-1}\varphi \|$$

$$\implies \left| \frac{d}{d\epsilon} F(\mu + i\epsilon) \right| \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \epsilon^{-1/2} |F(\mu + i\epsilon)|^{1/2}.$$
Now,
\[
|F(\mu + i\varepsilon)| \geq |\text{Im} F(\mu + i\varepsilon)| = \varepsilon \|(H - \mu - i\varepsilon)^{-1}\varphi\|^2 > 0
\]
if \(\varepsilon > 0\) and \(\varphi \neq 0\).

So, one can divide the last inequality by \(|F(\mu + i\varepsilon)|^{1/2}\) to get
\[
\frac{|\frac{d}{d\varepsilon} F(\mu + i\varepsilon)|}{|F(\mu + i\varepsilon)|^{1/2}} \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \varepsilon^{-1/2}
\]
\[
\iff \left| \frac{d}{d\varepsilon} F(\mu + i\varepsilon)^{1/2} \right| \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \frac{1}{2\varepsilon^{1/2}}
\]
\[
\int_{\varepsilon}^{1} d\varepsilon \implies |F(\mu + i)^{1/2} - F(\mu + i\varepsilon)^{1/2}| \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) (1 - \varepsilon^{1/2})
\]
\[
\varepsilon \in (0,1) \implies |F(\mu + i\varepsilon)|^{1/2} \leq |F(\mu + i)|^{1/2} + \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|).
\]
Putting the last estimate in the inequality

\[
\left| \frac{d}{d\varepsilon} F(\mu + i\varepsilon) \right| \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \varepsilon^{-1/2} \left| F(\mu + i\varepsilon) \right|^{1/2},
\]

one gets for each $|\mu| \geq \delta > 0$ and $\varepsilon \in (0, 1)$

\[
\left| \frac{d}{d\varepsilon} F(\mu + i\varepsilon) \right| \leq \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \varepsilon^{-1/2} \left\{ \left| F(\mu + i) \right|^{1/2} + \frac{1}{|\mu|} (\|\varphi\| + 2\|A\varphi\|) \right\}
\]

\[
\leq \frac{1}{\delta} (\|\varphi\| + 2\|A\varphi\|) \varepsilon^{-1/2} \left\{ \|\varphi\| + \frac{1}{\delta} (\|\varphi\| + 2\|A\varphi\|) \right\}
\]

\[
\leq c(\delta, \varphi) \varepsilon^{-1/2} (\|\varphi\|^2 + \|A\varphi\|^2).
\]
It follows that \( \{ F(\mu + i/m) \}_{m \in \mathbb{N}^*} \) is a Cauchy sequence since

\[
|F(\mu + i/m) - F(\mu + i/n)| = \left| \int_{1/n}^{1/m} d\epsilon \frac{d}{d\epsilon} F(\mu + i\epsilon) \right|
\]

\[
\leq c(\delta, \varphi)(\|\varphi\|^2 + \|A\varphi\|^2) \left| \int_{1/n}^{1/m} d\epsilon \epsilon^{-1/2} \right|
\]

\[
= 2c(\delta, \varphi)(\|\varphi\|^2 + \|A\varphi\|^2) |m^{-1/2} - n^{-1/2}|
\]

\[
\to 0 \quad \text{as} \quad m, n \to \infty.
\]

Thus, the limit \( \lim_{\epsilon \searrow 0} F(\mu + i\epsilon) \) exists uniformly on \( |\mu| \geq \delta \).
1.4 Schrödinger operators

Let $M_V$ be the self-adjoint multiplication operator in $\mathcal{H} := L^2(\mathbb{R})$ given by $V \in L^\infty(\mathbb{R}; \mathbb{R})$. Then, the 1-dimensional Schrödinger operator

$$H\varphi := -\Delta \varphi + M_V \varphi, \quad \varphi \in \mathcal{D}(H) := \mathcal{H}^2(\mathbb{R}),$$

is self-adjoint due to the Kato-Rellich theorem.

(self-adjointness is preserved under the perturbation by a bounded self-adjoint operator)

In quantum mechanics, the operator $H$ describes a non-relativistic particle in $\mathbb{R}$ in presence of a scalar (electric) potential $V$. 
Can we (under some assumptions on $V$) determine the spectral nature of $H$?

Can we do it using commutator methods?
The family of operators \( \{U_t\}_{t \in \mathbb{R}} \) in \( \mathcal{H} \) given by

\[
(U_t \varphi)(x) := e^{t/2} \varphi(e^t x), \quad \varphi \in \mathcal{S}(\mathbb{R}), \ x, t \in \mathbb{R},
\]
defines a strongly continuous unitary group (the dilation group).

The self-adjoint generator \( A := i \left( s - \frac{d}{dt} U_t \big|_{t=0} \right) \) of \( \{U_t\}_{t \in \mathbb{R}} \) acts as

\[
A \varphi := \frac{1}{2} (QP + PQ) \varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}).
\]
The operator $A$ is the quantum analogue of the classical observable $q \cdot p$ on $M := T^*\mathbb{R}$ which appeared at the beginning:

$$\{\{q^2, p^2 + V(q)\}, p^2 + V(q)\} = \{4(q \cdot p), p^2 + V(q)\}$$

$$= 8p^2 - 4q \cdot (\nabla V)(q).$$

... just replace the observables $q$ and $p$ on $M := T^*\mathbb{R}$ by the self-adjoint operators $Q$ and $P$ in $\mathcal{H}$, and be cautious with the domains and the self-adjointness of the unbounded operators...
One has

\[
e^{-itA} (-\triangle + i)^{-1} e^{itA}
\]

\[
= \mathcal{F}^{-1}(\mathcal{F} e^{-itA} \mathcal{F}^{-1}) \mathcal{F} (-\triangle + i)^{-1} \mathcal{F}^{-1}(\mathcal{F} e^{itA} \mathcal{F}^{-1}) \mathcal{F}
\]

\[
= \mathcal{F}^{-1} U_t (Q^2 + i)^{-1} U_t \mathcal{F}
\]

\[
= \mathcal{F}^{-1} ((e^{-t}Q)^2 + i)^{-1} \mathcal{F}
\]

\[
= (e^{-2t} (-\triangle) + i)^{-1}
\]

Thus,

\[
s-\frac{d}{dt} e^{-itA} (-\triangle + i)^{-1} e^{itA} \bigg|_{t=0} = (-\triangle + i)^{-1} 2 (-\triangle) (-\triangle + i)^{-1},
\]

and $-\triangle$ is of class $C^\infty(A)$ with $[iA, -\triangle] = -2(-\triangle)$. 
Similarly, one has
\[ e^{-itA} M_V e^{itA} = M_{V(e^t \cdot)}. \]

Thus, if \( V \) is absolutely continuous with \( \text{id}_\mathbb{R} \cdot V' \in L^\infty(\mathbb{R}) \), one has
\[ s - \frac{d}{dt} e^{-itA} M_V e^{itA} \bigg|_{t=0} = M_{\text{id}_\mathbb{R} \cdot V}, \]
and \( M_V \in C^1(A) \) with \([iA, M_V] = M_{\text{id}_\mathbb{R} \cdot V'}\).

Furthermore, if \( V' \) is Dini-continuous, one has \( M_V \in C^{1+0}(A) \) since
\[
\int_0^1 \frac{dt}{t} \left\| e^{-itA} [A, M_V] e^{itA} - [A, M_V] \right\|_{\mathcal{B}(\mathcal{H})} = \int_0^1 \frac{dt}{t} \left\| (\text{id}_\mathbb{R} \cdot V') (e^t \cdot) - \text{id}_\mathbb{R} \cdot V' \right\|_{L^\infty(\mathbb{R})} < \infty.
\]
We infer that $H$ is of class $C^{1+0}(A)$, with

$$[iH, A] = 2(-\Delta) - M_{\text{id}_\mathbb{R}} \cdot V' = 2H - M(2V - \text{id}_\mathbb{R} \cdot V').$$

Now, assume that

$$\lim_{|x| \to \infty} (2V - \text{id}_\mathbb{R} \cdot V')(x) = 0.$$

Then, a standard result tells us that

$$M(2V - \text{id}_\mathbb{R} \cdot V') (\Delta + i)^{-1} \in \mathcal{K}(\mathcal{H}).$$

(the products $f(Q)g(P)$ with $f, g \in C(\mathbb{R})$ vanishing at infinity are compact operators)
Given an open bounded set $I \subset \mathbb{R}$, it follows that

$$E^H(I)[\dot{i}H, A]E^H(I)$$

$$= 2E^H(I)HE^H(I) - E^H(I)M(2V - \text{id}_\mathbb{R} \cdot V')E^H(I)$$

$$\geq 2 \inf(I)E^H(I) - E^H(I)M(2V - \text{id}_\mathbb{R} \cdot V')(-\Delta + i)^{-1}(-\Delta + i)E^H(I)$$

$$= 2 \inf(I)E^H(I) + \text{compact operator}.$$ 

Thus, Theorem 1.17 implies that $H$ has at most finitely many eigenvalues in each open bounded set $I \subset (0, \infty)$ (multiplicities counted), and that $H$ has no singular continuous spectrum in $(0, \infty)$.

(in fact, since $M_V(-\Delta + i)^{-1}$ is compact, one has $\sigma_{\text{ess}}(H) = [0, \infty)$, so that $\sigma_{\text{sc}}(H) = \emptyset$ and $\sigma_{\text{ac}}(H) = [0, \infty)$)
Countless variations/generalisations of this example can be found in the literature:

- the potential $V$ may have singularities (for instance of Coulomb-type)
- the potential $V$ may have anisotropies at infinity
- the Schrödinger operator $H$ may contain a magnetic field
- the Schrödinger operator $H$ can be replaced by an $N$-body Schrödinger operator
- the Schrödinger operator $H$ can be replaced by a quantum field Hamiltonian
- the Schrödinger operator $H$ can be replaced by a Dirac operator
• the operator $-\Delta$ can be replaced by the Laplace-Beltrami operator (on functions or differential forms) on various types of non-compact manifolds

$$M_c \quad M_\infty$$

• the operator $-\Delta$ can be replaced by the combinatorial Laplacian (adjacency matrix) on various types of infinite graphs

• etc...
1.5 Time changes of horocycles flows

References:


- R. Tiedra, Commutator methods for the spectral analysis of uniquely ergodic dynamical systems, preprint on arXiv
Horocycle flow

- $\Sigma$, compact Riemann surface of genus $\geq 2$
- $M := T^1\Sigma$, unit tangent bundle of $\Sigma$
- $\mu_\Omega$, probability measure on $M$ induced by a volume form $\Omega$

The horocycle flow $\{F_{1,t}\}_{t \in \mathbb{R}}$ and the geodesic flow $\{F_{2,t}\}_{t \in \mathbb{R}}$ are one-parameter groups of diffeomorphisms on $M$.

Both flows correspond to right translations on $M$ when $M \cong \Gamma \backslash \text{PSL}(2; \mathbb{R})$, for some cocompact lattice $\Gamma$ in $\text{PSL}(2; \mathbb{R})$. 
Geodesic in the Poincaré half plane
Horocycle flow in the Poincaré half plane
The operators

\[ U_j(t)\varphi := \varphi \circ F_{j,t}, \quad t \in \mathbb{R}, \quad \varphi \in C(M), \]

define strongly continuous unitary groups in \( \mathcal{H} := L^2(M, \mu_\Omega) \) with essentially self-adjoint generators

\[ H_j\varphi := -i \mathcal{L}_{X_j}\varphi, \quad \varphi \in C^\infty(M), \]

where \( X_j \) is the divergence-free vector field associated with \( \{F_{j,t}\}_{t \in \mathbb{R}} \) and \( \mathcal{L}_{X_j} \) the corresponding Lie derivative.

The horocycle flow \( \{F_{1,t}\}_{t \in \mathbb{R}} \) is uniquely ergodic [Furstenberg73], mixing of all orders [Marcus78], and \( U_1(t) \) has countable Lebesgue spectrum for each \( t \neq 0 \) [Parasyuk53].
The horocycle flow and the geodesic flow satisfy the commutation relation (see for instance [Bachir/Mayer 00])

\[ U_2(s)U_1(t)U_2(-s) = U_1(e^s t), \quad s, t \in \mathbb{R}, \quad (1.2) \]

which is a consequence of the matrix identity in \( \text{SL}(2, \mathbb{R}) \):

\[
\begin{pmatrix}
e^{s/2} & 0 \\
0 & e^{-s/2}
\end{pmatrix}
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
e^{-s/2} & 0 \\
0 & e^{s/2}
\end{pmatrix} =
\begin{pmatrix}
1 & e^s t \\
0 & 1
\end{pmatrix}.
\]

Therefore, by applying the strong derivatives \( \frac{d}{dt} \big|_{t=0} \) and \( \frac{d}{ds} \big|_{s=0} \) to (1.2), one obtains that \( H_1 \) is of class \( C^\infty(H_2) \) with

\[
[iH_1, H_2] = H_1.
\]
Time changes of horocycle flows

Consider a $C^1$ vector field with the same orientation and proportional to $X_1$; that is, $fX_1$ with $f \in C^1(M;(0,\infty))$.

The reparametrised time coordinate $h(p,t)$ given by

$$ t = \int_0^{h(p,t)} \frac{ds}{f(F_{1,s}(p))}, \quad t \in \mathbb{R}, \ p \in M, $$

is such that $h(p,0) = 0$, $\lim_{t \to \pm \infty} h(p,t) = \pm \infty$ and

$$ \frac{d}{dt} h(p,t) = f(F_{1,h(p,t)}(p)). $$

The function $\mathbb{R} \ni t \mapsto \tilde{F}_{1,t}(p) \in M$ given by $\tilde{F}_{1,t}(p) := F_{1,h(p,t)}(p)$ satisfies

$$ \frac{d}{dt} \tilde{F}_1(p,t) = (fX_1)_{\tilde{F}_1(p,t)}, \ \tilde{F}_1(p,0) = p, $$

and thus $\{\tilde{F}_{1,t}\}_{t \in \mathbb{R}}$ is the flow of $fX_1$. 
The operators

\[ \tilde{U}_1(t) \varphi := \varphi \circ \tilde{F}_{1,t}, \quad t \in \mathbb{R}, \ \varphi \in C(M), \]

define a strongly continuous unitary group in \( \tilde{\mathcal{H}} := L^2(M, \mu_{\Omega}/f) \).

The generator \( \tilde{H} := -i \mathcal{L}_{f X_1} \) of \( \{ \tilde{U}_1(t) \}_{t \in \mathbb{R}} \) is essentially self-adjoint on \( C^1(M) \) and unitarily equivalent to the operator in \( \mathcal{H} \) given by

\[ H := f^{1/2} H_1 f^{1/2}. \]

(\ldots the unitary operator \( \mathcal{U} : \mathcal{H} \to \tilde{\mathcal{H}}, \ \varphi \mapsto f^{1/2} \varphi \) realises the unitary equivalence \ldots )
What is the spectral nature of $\tilde{H}$ (or equivalently of $H$)?

- Spectral properties are in general not preserved under time changes even though basic ergodic properties are preserved under time changes.

- In 1974, Kushnirenko shows that the flow $\{\tilde{F}_{1,t}\}_{t \in \mathbb{R}}$ is strongly mixing if $f$ is of class $C^\infty$ and $f - L_{X_2}(f) > 0$. So, $\tilde{H}$ has purely continuous spectrum in $\mathbb{R} \setminus \{0\}$ in this case.

- In 2006, Katok and Thouvenot conjecture that $\tilde{H}$ has absolute continuous spectrum (and even countable Lebesgue spectrum) if $f$ is sufficiently smooth.
Mourre estimate

Let \( z \in \mathbb{C} \setminus \mathbb{R} \) and assume for a moment that \( f \equiv 1 \), so that \( H \equiv H_1 \). Then, one has \( (H + z)^{-1} \in C^1(H_2) \) with

\[
[i(H + z)^{-1}, H_2] = -(H + z)^{-1}[iH, H_2](H + z)^{-1} \\
= -(H + z)^{-1}H(H + z)^{-1}.
\]

It follows that

\[
[i(H^2 + 1)^{-1}, H_2] \\
= (H + i)^{-1}[i(H - i)^{-1}, H_2] + [i(H + i)^{-1}, H_2](H - i)^{-1} \\
= -(H^2 + 1)^{-1}H(H - i)^{-1} - (H + i)^{-1}H(H^2 + 1)^{-1} \\
= -(H^2 + 1)^{-1}H(H + i)(H^2 + 1)^{-1} - (H^2 + 1)^{-1}(H - i)H(H^2 + 1)^{-1} \\
= -(H^2 + 1)^{-1}2H^2(H^2 + 1)^{-1}.
\]
Thus $H^2$ is of class $C^\infty(H_2)$ with $[iH^2, H_2] = 2H^2$, and
\[
E^{H^2}(I)[iH^2, H_2] E^{H^2}(I) = E^{H^2}(I) 2H^2 E^{H^2}(I) \geq 2 \inf(I) E^{H^2}(I)
\]
for each open bounded set $I \subset (0, \infty)$.

Therefore, in the case $f \equiv 1$, Mourre’s theorem applies to the operator $H^2$ on the interval $(0, \infty)$.

So, let’s try the same approach in the case $f \not\equiv 1$...
If \( f \neq 1 \), one has \((H + z)^{-1} \in C^1(H_2)\) with
\[
[i(H + z)^{-1}, H_2] = -(H + z)^{-1}(Hg + gH)(H + z)^{-1}
\]
and
\[
g := \frac{1}{2} - \frac{1}{2} \mathcal{L}_{x_2}(\ln(f)).
\]
(note that \( g \equiv \frac{f - \mathcal{L}_{x_2}(f)}{2f} > 0 \) under Kushnirenko’s condition)

A calculation as in the case \( f \equiv 1 \) shows that
\[
[i(H^2 + 1)^{-1}, H_2] = -(H^2 + 1)^{-1}(H^2g + 2HgH + gH^2)(H^2 + 1)^{-1},
\]
which means that \((H^2 + 1)^{-1} \in C^1(H_2)\) with
\[
[iH^2, H_2] = H^2g + 2HgH + gH^2.
\]
If $g > 0$ and $f$ is of class $C^2$, one has

\[
H^2 g + g H^2
\]

\[
= \left[ H^2, g^{1/2} \right] g^{1/2} + 2 g^{1/2} H^2 g^{1/2} + g^{1/2} \left[ g^{1/2}, H^2 \right]
\]

\[
\geq \left[ H^2, g^{1/2} \right] g^{1/2} + g^{1/2} \left[ g^{1/2}, H^2 \right]
\]

\[
= H \left[ H, g^{1/2} \right] g^{1/2} + \left[ H, g^{1/2} \right] H g^{1/2} + g^{1/2} \left[ g^{1/2}, H \right] H + g^{1/2} H \left[ g^{1/2}, H \right]
\]

\[
= H \left[ H, g^{1/2} \right] g^{1/2} + \left[ H, g^{1/2} \right] g^{1/2} H + \left[ H, g^{1/2} \right] \left[ H, g^{1/2} \right]
\]

\[
+ g^{1/2} \left[ g^{1/2}, H \right] H + H g^{1/2} \left[ g^{1/2}, H \right] + \left[ g^{1/2}, H \right] \left[ g^{1/2}, H \right]
\]

\[
= H \left[ H, g^{1/2} \right] g^{1/2} + \left[ H, g^{1/2} \right] g^{1/2} H + 2 \left[ H, g^{1/2} \right]^2 + g^{1/2} \left[ g^{1/2}, H \right] H
\]

\[
+ H g^{1/2} \left[ g^{1/2}, H \right]
\]

\[
= 2 \left[ H, g^{1/2} \right]^2
\]

\[
\geq 0.
\]
Thus, making everything rigorous, one obtains that

\[ E^{H^2} (I) \left[ iH^2, H_2 \right] E^{H^2} (I) \]

\[ = E^{H^2} (I) \left( H^2 g + 2H g H + gH^2 \right) E^{H^2} (I) \]

\[ \geq a E^{H^2} (I) \quad \text{with} \quad a := 2 \inf(I) \cdot \inf_{p \in M} g(p) > 0 \]

for each bounded open set \( I \subset (0, \infty) \).

Since we also have \( (H^2 + 1)^{-1} \in C^2 (H_2) \), we conclude by Mourre’s theorem that \( H^2 \) is purely absolutely continuous outside \( \{0\} \), where it has a simple eigenvalue corresponding to the constant functions.

Standard arguments then imply that \( H \) has the same spectral properties as \( H^2 \).
Summing up:

**Theorem 1.18.** Under Kushnirenko’s condition and for time changes \( f \) of class \( C^2 \), the self-adjoint operator \( \tilde{H} \) associated with the vector field \( fX_1 \) has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.

*Proof.* \( H \) and \( \tilde{H} \) are unitarily equivalent. \( \square \)

In fact, this also holds for noncompact surfaces \( \Sigma \) of finite volume.

Fine, but... Forni and Ulcigrai have obtained the same result (and also Lebesgue maximal spectral type) without assuming Kushnirenko’s condition (for compact surfaces and for time changes in a Sobolev space of order \( > 11/2 \)).
So, can we get rid off Kushnirenko’s condition?
Mourre estimate (one more time)

Lemma 1.19 (Conjugate operator). Let \( f \in C^3(M;(0,\infty)) \) and \( L > 0 \). Then, the operator

\[
A_L \varphi := \frac{1}{L} \int_0^L dt \ e^{itH} \ H_2 \ e^{-itH} \varphi, \quad \varphi \in C^1(M),
\]

is essentially self-adjoint in \( \mathcal{H} \).

Idea of the proof. A calculation on \( C^1(M) \) shows that

\[
\frac{1}{L} \int_0^L dt \ e^{itH} \ H_2 \ e^{-itH} = -i(\mathcal{L}_X + \frac{1}{2} \text{div}_\Omega X),
\]

for a certain vector field \( X \) on \( M \). Furthermore, if \( f \) is of class \( C^3 \), then the r.h.s. is the self-adjoint generator of a strongly continuous unitary group (see [Abraham/Marsden 78]).

\( \square \)

(\( \ldots \) if someone knows how to do it for \( f \) of class \( C^2 \ldots \))
Replacing $H_2$ by $A_L$ in the previous calculations and noting that

$$\frac{1}{L} \int_0^L dt \, e^{itH} \, g \, e^{-itH} = \frac{1}{L} \int_0^L dt \, e^{it \mathcal{U}^* \tilde{H} \mathcal{U}} \, g \, e^{-it \mathcal{U}^* \tilde{H} \mathcal{U}}$$

$$= \frac{1}{L} \int_0^L dt \, \mathcal{U}^* \, e^{it \tilde{H}} \, g \, e^{-it \tilde{H}} \, \mathcal{U}$$

$$= \frac{1}{L} \int_0^L dt \, (g \circ \tilde{F}_{1,-t}),$$

we obtain that $(H^2 + 1)^{-1} \in C^2(A_L)$ with

$$[i(H^2+1)^{-1}, A_L] = -(H^2+1)^{-1} (H^2 g_L + 2H g_L H + g_L H^2)(H^2 + 1)^{-1},$$

where

$$g_L := \frac{1}{L} \int_0^L dt \, (g \circ \tilde{F}_{1,-t}).$$
The flow \( \{\tilde{F}_{1,t}\}_{t \in \mathbb{R}} \) is uniquely ergodic, since it is a reparametrised version of the uniquely ergodic flow \( \{F_{1,t}\}_{t \in \mathbb{R}} \) [Humphries74].

So, the Cesàro mean \( g_L = \frac{1}{L} \int_0^L dt \left( g \circ \tilde{F}_{1,-t} \right) \) converges uniformly on \( M \) to \( \int_M d\tilde{\mu_\Omega} \, g_L \); that is,

\[
\lim_{L \to \infty} g_L = \int_M d\tilde{\mu_\Omega} \, g_L = \frac{1}{2} - \frac{1}{2} \int_M d\tilde{\mu_\Omega} \, \mathcal{L}_{X_2}(\ln(f))
\]

\[
= \frac{1}{2} + \frac{1}{2} \int_M f^{-1} d\mu_\Omega \int_M d\mu_\Omega \, \mathcal{L}_{X_2}(f^{-1})
\]

\[
= \frac{1}{2} + \frac{i}{2} \int_M f^{-1} d\mu_\Omega \langle 1, H_2 f^{-1} \rangle
\]

\[
= \frac{1}{2}.
\]

\[\implies g_L > 0 \text{ if } L > 0 \text{ is big enough.}\]
So, we got rid off Kushnirenko’s condition, and thus have proved the following:

**Theorem 1.20.** For time changes $f$ of class $C^3$, the self-adjoint operator $\tilde{\mathcal{H}}$ associated with the vector field $fX_1$ has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.

(...) if someone knows how to prove countable Lebesgue spectrum ...
2 Commutator methods for unitary operators

Commutator methods for unitary operators is the unitary analogue of commutator methods for self-adjoint operators.

The theory applies to general unitary operators $U$ (not necessarily of the type $e^{iH}$), up to the regularity class $C^{1+0}(A)$. 
2.1 Unitary operators

A unitary operator $U$ in a Hilbert space $\mathcal{H}$ is a surjective isometry; that is,

$$U^* U = UU^* = 1.$$ 

Since $U^* U = UU^*$, the spectral theorem for normal operators implies that $U$ admits exactly one complex spectral family $E^U$ with support

$$\text{supp}(E^U) = \sigma(U) \subset \mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$$

such that

$$U = \int_{\mathbb{C}} z \, E^U(dz),$$

where $E^U(\lambda + i\mu) := E^\text{Re}(U)(\lambda) E^\text{Im}(U)(\mu)$ for each $\lambda, \mu \in \mathbb{R}$, and

$$\text{Re}(U) := \frac{1}{2} (U + U^*) \quad \text{and} \quad \text{Im}(U) := \frac{1}{2i} (U - U^*).$$
One has $U = \int_{\mathbb{R}} e^{is} \tilde{E}^U (ds)$ with

$$\tilde{E}^U (s) := \begin{cases} 
0 & \text{if } s < 0 \\
E^U (\{e^{i\tau} \mid \tau \in \left[0, s\right]\}) & \text{if } s \in [0, 2\pi) \\
1 & \text{if } s \geq 2\pi.
\end{cases}$$

So, one can use the real spectral family $\tilde{E}^U$ to obtain orthogonal decompositions

$$\mathcal{H} = \mathcal{H}_p(U) \oplus \mathcal{H}_{sc}(U) \oplus \mathcal{H}_{ac}(U)$$

$$U = U|_{\mathcal{H}_p(U)} \oplus U|_{\mathcal{H}_{sc}(U)} \oplus U|_{\mathcal{H}_{ac}(U)}$$

as in the self-adjoint case.
Example 2.1 (1-parameter groups of unitary operators). If $H$ is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then

$$U_t := e^{-itH}$$

is a unitary operator for each $t \in \mathbb{R}$, and the family $\{U_t\}_{t \in \mathbb{R}}$ defines a strongly continuous 1-parameter group of unitary operators.

Example 2.2 (Koopman operator). Let $T : X \to X$ be an automorphism of a probability space $X$ with probability measure $\mu$. Then, the Koopman operator $U_T$ in $\mathcal{H} := L^2(X, \mu)$ given by

$$U_T : \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.
Ergodicity, weak mixing and strong mixing of an automorphism $T : X \to X$ are expressible in terms of spectral properties of the Koopman operator $U_T$:

- $T$ is ergodic if and only if 1 is a simple eigenvalue of $U_T$.
- $T$ is weakly mixing if and only if $U_T$ has purely continuous spectrum in $\{\mathbb{C} \cdot 1\}^\perp$.
- $T$ is strongly mixing if and only if

$$
\lim_{n \to \infty} \langle \varphi, U_T^n \varphi \rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C} \cdot 1\}^\perp.
$$

\[
\text{strong mixing} \implies \text{weak mixing} \implies \text{ergodicity}
\]
2.2 Commutator methods for unitary operators

References:


In [Astaburuaga/Bourget/Cortés/Fernández06], the authors show an analogue of Mourre’s theorem for a unitary operator $U$ in a Hilbert space $\mathcal{H}$.

However...

- the regularity assumption is $U \in C^2(A)$,
- the proofs rely once more on differential inequalities for “resolvents” of $U$.

We want to obtain this result with the weaker assumption $U \in C^{1+0}(A)$ and with a simpler proof!
At the end of the day, we obtain:

**Theorem 2.3** (Spectral properties of $U$). Let $U \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset S^1$, a number $a > 0$ and $K \in \mathcal{H}(\mathcal{H})$ such that

$$E^U(\Theta)U^*[A, U]E^U(\Theta) \geq aE^U(\Theta) + K.$$  

Then, $U$ has at most finitely many eigenvalues in $\Theta$ (multiplicities counted), and $U$ has no singular continuous spectrum in $\Theta$. 
Sketch of the proof (i)

Why the “commutator” $U^*[A, U]$ is the right expression to consider?

Imagine that $U = e^{-iH}$ with $H \in C^1(A)$, then one has

$U^*[A, U]$

\[
= i \left( \frac{d}{dt} e^{iH} e^{-itA} e^{-iH} e^{itA} \right)_{t=0}
\]

\[
= i \left( \frac{d}{dt} \int_0^1 d\mu \frac{d}{d\mu} e^{i\mu H} e^{-itA} e^{-i\mu H} e^{itA} \right)_{t=0}
\]

\[
= - \int_0^1 d\mu \frac{d}{dt} \left( e^{i\mu H} H e^{-itA} e^{-i\mu H} e^{itA} - e^{i\mu H} e^{-itA} H e^{-i\mu H} e^{itA} \right)_{t=0}
\]

\[
= - \int_0^1 d\mu \left( e^{i\mu H} H [i e^{-i\mu H}, A] - e^{i\mu H} \left[ iH e^{-i\mu H}, A \right] \right)
\]

\[
= \int_0^1 d\mu e^{i\mu H} [iH, A] e^{-i\mu H}.
\]
Thus,

\[ U^*[A, U] = \int_{0}^{1} d\mu \ e^{i\mu H} [iH, A] e^{-i\mu H}, \]

and positivity of \([iH, A]\) leads to positivity of \(U^*[A, U]\) and vice versa.

(the idea of using \(U^*[A, U]\) dates back to Putnam in the 60's)
Sketch of the proof (ii)

As in the self-adjoint case, one can show a Virial theorem which implies the following:

**Corollary 2.4 (Point spectrum of \( U \)).** Let \( U \) and \( A \) be respectively a unitary and a self-adjoint operator in \( \mathcal{H} \), with \( U \in C^1(A) \). Assume there exist a Borel set \( \Theta \subset S^1 \), a number \( a > 0 \) and \( K \in \mathcal{K}(\mathcal{H}) \) such that

\[
E^U(\Theta)U^*[A,U]E^U(\Theta) \geq aE^U(\Theta) + K.
\]

Then, \( U \) has at most finitely many eigenvalues in \( \Theta \) (multiplicities counted).
If $U \in C^1(A)$ and

$$E^U(\Theta) U^*[A, U] E^U(\Theta) \geq a E^U(\Theta) + K,$$

then the corollary implies that $U$ has at most finitely many eigenvalues in $\Theta$ (multiplicities counted).

So, there exists $\theta \in \Theta$ which is not an eigenvalue of $U$, and the range $\text{Ran}(1 - \tilde{\theta} U)$ of $1 - \tilde{\theta} U$ is dense in $\mathcal{H}$.

Indeed, if $\psi \in \mathcal{H}$ is such that $\psi \perp \text{Ran}(1 - \tilde{\theta} U)$, then

$$\langle \psi, (1 - \tilde{\theta} U) \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{H} \quad \Rightarrow \quad (1 - \theta U^*) \psi = 0$$

$$\Rightarrow \quad U \psi = \theta \psi$$

$$\Rightarrow \quad \psi = 0.$$
Furthermore, the Cayley transform of $U$ at the point $\theta$; that is, the operator

$$H_\theta := -i(1 + \bar{\theta}U)(1 - \bar{\theta}U)^{-1}, \quad \mathcal{D}(H_\theta) := \text{Ran}(1 - \bar{\theta}U),$$

is self-adjoint.

Indeed, $H_\theta$ is self-adjoint if and only if

$$\text{Ran}(H_\theta + i) = \text{Ran}(H_\theta - i) = \mathcal{H}$$

$$\iff \text{Ran} \left( -2i\bar{\theta}U(1 - \bar{\theta}U)^{-1}|_{\text{Ran}(1 - \bar{\theta}U)} \right) = \mathcal{H}$$

$$= \text{Ran} \left( -2i(1 - \bar{\theta}U)^{-1}|_{\text{Ran}(1 - \bar{\theta}U)} \right) = \mathcal{H}$$

$$\iff -2i\bar{\theta}U\mathcal{H} = -2i\mathcal{H} = \mathcal{H}$$

$$\iff \mathcal{H} = \mathcal{H} = \mathcal{H}.$$
For any Borel set $\Theta \subset S^1$, the spectral measure $E^{H_\theta}$ of $H_\theta$ satisfies

$$E^{H_\theta}(I) = E^{U}(\Theta) \quad \text{with} \quad I := \left\{ -i \frac{1 + \bar{\theta}z}{1 - \theta z} \mid z \in \Theta \right\}.$$

Cayley transform of $\mathbb{R}$ (for $\theta = -i$)
Sketch of the proof (iii)

One has
\[
(H_\theta - i)^{-1} = \left\{ ( -i(1 + \bar{\theta}U) - i(1 - \bar{\theta}U) ) (1 - \bar{\theta}U)^{-1} \right\}^{-1} \\
= \left\{ -2i(1 - \bar{\theta}U)^{-1} \right\}^{-1} \\
= -\frac{1}{2i} (1 - \bar{\theta}U).
\]

Thus,
\[
[A, (H_\theta - i)^{-1}] = \left[ A - \frac{1}{2i} (1 - \bar{\theta}U) \right] = \frac{\bar{\theta}}{2i} [A, U],
\]

and the regularity condition \( U \in C^{1+0}(A) \) implies the regularity condition \( (H_\theta - i)^{-1} \in C^{1+0}(A) \).
Sketch of the proof (iv)

A calculation in $\mathcal{B}(\mathcal{D}(H_\theta), \mathcal{D}(H_\theta)^*)$ shows that

$$[iH_\theta, A] = [(1 + \bar{\theta}U)(1 - \bar{\theta}U)^{-1}, A]$$

$$= (1 + \bar{\theta}U)[(1 - \bar{\theta}U)^{-1}, A] + [(1 + \bar{\theta}U), A](1 - \bar{\theta}U)^{-1}$$

$$= 2 \{(1 - \bar{\theta}U)^{-1}\}^* U^* [A, U](1 - \bar{\theta}U)^{-1}$$

So, the positivity of $U^*[A, U]$ on a Borel set $\Theta \subset S^1$ implies the positivity of $[iH_\theta, A]$ on the corresponding set $I \subset \mathbb{R}$.

Since $H_\theta$ is of class $C^{1+0}(A)$, the usual (self-adjoint) Mourre’s theorem implies that $H_\theta$ has no singular continuous spectrum in $I$. 
Now, suppose by absurd that $U$ has some singular continuous spectrum in $\Theta \setminus \{\theta\}$. Then, there exist $\varphi \in \mathcal{H} \setminus \{0\}$ and $\mathcal{V} \subset [0, 2\pi)$ such that

$$\operatorname{closure}(e^{i\mathcal{V}}) \subset \Theta \setminus \{\theta\}, \quad |\mathcal{V}| = 0 \quad \text{and} \quad \tilde{E}^U(\mathcal{V})\varphi = \varphi.$$ 

This implies that

$$\tilde{E}^U(\mathcal{V})\varphi = \varphi \iff E^U(e^{i\mathcal{V}})\varphi = \varphi \iff E^{H_\theta}(J)\varphi = \varphi,$$

with

$$J := \left\{ -i \frac{1 + \theta e^{iv}}{1 - \theta e^{iv}} \mid v \in \mathcal{V} \right\} \subset I.$$
But, the function

\[ \mathcal{V} \ni v \mapsto -i \frac{1 + \bar{\theta} e^{iv}}{1 - \bar{\theta} e^{iv}} \in J \]

has the Luzin N property. So \(|J| = 0\), and thus \(\varphi = 0\) since \(H_\theta\) has no singular continuous spectrum in \(J \subset I\).

Since \(\varphi \in \mathcal{H} \setminus \{0\}\), this is a contradiction. So, \(U\) has no singular continuous spectrum in \(\Theta \setminus \{\theta\}\), and thus no singular continuous spectrum in \(\Theta\).

No need to re-do any proof with differential inequalities. We just used the Cayley transform and the pre-existing self-adjoint theory.
We also have the following perturbation result:

**Corollary 2.5** (Perturbations of $U$). Let $U, V$ be unitary, with $U, V \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset S^1$, a number $a > 0$ and $K \in \mathcal{K}(\mathcal{H})$ such that

$$E^U(\Theta)U^*[A, U]E^U(\Theta) \geq a E^U(\Theta) + K. \quad (2.1)$$

Suppose also that $(V - 1) \in \mathcal{K}(\mathcal{H})$ is compact. Then, $VU$ has at most finitely many eigenvalues in each closed subset of $\Theta$ (multiplicities counted), and $VU$ has no singular continuous spectrum in $\Theta$.

- the Mourre estimate (2.1) depends on $U$ only ($V$ is the perturbation)

- $UV$ and $VU$ are unitarily equivalent since $UV = U(VU)U^*$
2.3 Perturbations of bilateral shifts

Let $U$ be a bilateral shift on a Hilbert space $\mathcal{H}$ with wandering subspace $\mathcal{M} \subset \mathcal{H}$, i.e.,

$$\mathcal{M} \perp U^n(\mathcal{M}) \text{ for each } n \in \mathbb{Z} \setminus \{0\} \text{ and } \mathcal{H} = \bigoplus_{n \in \mathbb{Z}} U^n(\mathcal{M}).$$

Using the notation $\varphi \equiv \{\varphi_n\} \in \mathcal{H}$, define the (number) operator

$$A\varphi := \{n\varphi_n\}, \quad \varphi \in \mathcal{D}(A) := \{\psi \in \mathcal{H} \mid \sum_{n \in \mathbb{Z}} n^2 \|\psi_n\|^2 < \infty\},$$

which is self-adjoint since it is a maximal multiplication operator in a $\ell^2$-space.
One has for each \( \varphi \in D(A) \)
\[
\langle A\varphi, U\varphi \rangle - \langle \varphi, UA\varphi \rangle = \langle \{n\varphi_n\}, \{\varphi_{n+1}\} \rangle - \langle \{\varphi_n\}, U\{n\varphi_n\} \rangle
\]
\[
= \langle \{\varphi_n\}, \{(n + 1)\varphi_{n+1}\} \rangle - \langle \{\varphi_n\}, \{n\varphi_{n+1}\} \rangle
\]
\[
= \langle \varphi, U\varphi \rangle,
\]
meaning that \( U \in C^\infty(A) \subset C^{1+0}(A) \) with \( U^*[A, U] = U^*U = 1 \).

Thus, Theorem 2.3 implies that \( U \) has purely absolutely continuous spectrum, as it is well known.

In fact, the conditions \( U \in C^1(A) \) and \([A, U] = U\) imply that
\[
\frac{d}{dt} \ e^{-itA} \ U \ e^{i\tau A} = -i \ e^{-itA} \ U \ e^{i\tau A} \iff \ e^{-itA} \ U \ e^{i\tau A} = e^{-i\tau} \ U.
\]

So, \( U \) is unitarily equivalent to \( e^{-it} \ U \) for each \( t \in \mathbb{R} \), and thus has purely Lebesgue spectrum covering the whole circle \( S^1 \).
Let $V$ be another unitary operator with $V \in C^{1+0}(A)$ and $(V - 1) \in \mathcal{K}(\mathcal{H})$.

We deduce from Corollary 2.5 that $VU$ has purely absolutely continuous spectrum except, possibly, at a finite number of points of $S^1$, where $VU$ may have eigenvalues of finite multiplicity.
2.4 Perturbations of the Schrödinger free evolution

The Schrödinger free evolution \( \{U_t\}_{t \in \mathbb{R}} \) in \( \mathcal{H} := L^2(\mathbb{R}^d) \) given by

\[
U_t := e^{-itP^2}, \quad t \in \mathbb{R},
\]

satisfies

\[
\sigma(U_t) = \sigma_{ac}(U_t) = \mathbb{S}^1 \quad \text{for each } t \neq 0.
\]

Indeed, one has for each \( s \in [0, 2\pi) \) and \( t \neq 0 \) that

\[
E^{e^{-itP^2}}(e^{is}) = E^{\cos(-tP^2)}(\cos(s)) E^{\sin(-tP^2)}(\sin(s))
\]

\[
= E^{-tP^2}([0, s] + 2\pi \mathbb{Z}) E^{-tP^2}([0, s] + 2\pi \mathbb{Z})
\]

\[
= E^{P^2} \left( [0, -s/t] + \frac{2\pi}{t} \mathbb{Z} \right).
\]
What can we say about perturbations of the type $V U_t$?

The operator

$$A := \frac{1}{2} \left\{ (P^2 + 1)^{-1} P \cdot Q + Q \cdot P (P^2 + 1)^{-1} \right\}$$

is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ (because the vector field $X_x := x(x^2 + 1)^{-1} \in \mathbb{R}^d$ is complete), and calculations on $C_c^\infty(\mathbb{R}^d)$ show that $U_t \in C^1(A)$ with

$$(U_t)^* [A, U_t]$$

$$= \frac{1}{2} \sum_j \left\{ (P^2 + 1)^{-1} P_j \left[ Q_j, e^{-itP^2} \right] + \left[ Q_j, e^{-itP^2} \right] P_j (P^2 + 1)^{-1} \right\}$$

$$= t e^{itP^2} \sum_j \left\{ (P^2 + 1)^{-1} P_j^2 e^{-itP^2} + e^{-itP^2} P_j^2 (P^2 + 1)^{-1} \right\}$$

$$= 2tP^2(P^2 + 1)^{-1}.$$
Further commutations on $C_c^\infty(\mathbb{R}^d)$ show that $U_t \in C^2(A)$.

Moreover, if $t > 0$ and closure$(\Theta) \cap \{1\} = \emptyset$, there exists $\delta > 0$ such that

$$E^{U_t}(\Theta)(U_t)^*[A, U_t] E^{U_t}(\Theta) \geq 2t \delta (\delta + 1)^{-1} E^{U_t}(\Theta).$$

So, all the assumptions for $U_t$ are satisfied, and we have:

**Lemma 2.6.** If $V \in C^{1+0}(A)$ and $(V - 1) \in \mathcal{K}(\mathcal{H})$, then the eigenvalues of $VU_t$ outside $\{1\}$ are of finite multiplicity and can accumulate only at $\{1\}$. Furthermore, $VU$ has no singular continuous spectrum.

This extends previous results on the Schrödinger free evolution perturbed by “periodic kicks” ($V = e^{iB}$ with $B = B^*$ of finite rank).
2.5 Skew products over translations

Let \( \{y_t\}_{t \in \mathbb{R}} \) be a \( C^1 \) one-parameter subgroup of a compact metric abelian Banach Lie group \( X \) with normalised Haar measure \( \mu \) (such group \( X \) is isomorphic to a subgroup of \( \mathbb{T}^{\aleph_0} \equiv (\mathbb{R}/\mathbb{Z})^{\aleph_0} \)).

Let \( \{F_t\}_{t \in \mathbb{R}} \) be the corresponding translation flow,

\[
F_t(x) := y_t x, \quad t \in \mathbb{R}, \ x \in X,
\]

and let \( \{V_t\}_{t \in \mathbb{R}} \) the corresponding strongly continuous unitary group in \( \mathcal{H} := L^2(X, \mu) \),

\[
V_t \varphi := \varphi \circ F_t, \quad t \in \mathbb{R}, \ \varphi \in C(X).
\]
The generator $H$ of $\{V_t\}_{t \in \mathbb{R}}$ given by

$$H\varphi := -i \mathcal{L}_Y \varphi, \quad \varphi \in C^\infty(X),$$

with $Y$ the vector field associated with $\{F_t\}_{t \in \mathbb{R}}$ and $\mathcal{L}_Y$ the corresponding Lie derivative, is essentially self-adjoint on $C^\infty(X)$. 
Let $G$ be a compact metric abelian group with Haar measure $\nu$ and character group $\hat{G}$, and let $\phi : X \to G$ be a measurable function (cocycle).

We want to apply commutator methods to the Koopman operator

$$W\psi := \psi \circ T, \quad \psi \in L^2(X \times G, \mu \times \nu),$$

with $T$ the (measure-preserving invertible) skew product

$$T : X \times G \to X \times G, \quad (x, z) \mapsto (y_1 x, \phi(x) z).$$
The operator $W$ is reduced by the orthogonal decomposition (given by the Peter-Weyl theorem)

$$L^2(X \times G, \mu \times \nu) = \bigoplus_{\chi \in \hat{G}} L_\chi, \quad L_\chi := \{ \varphi \otimes \chi \mid \varphi \in \mathcal{H} \},$$

and $W|_{L_\chi}$ is unitarily equivalent to the unitary operator

$$U_\chi \varphi := (\chi \circ \phi)V_1 \varphi, \quad \varphi \in \mathcal{H}.$$

Furthermore, the operator $U_\chi$ satisfies the following purity law:

*If $F_1$ is ergodic, the spectrum of $U_\chi$ has uniform multiplicity and is either purely punctual, purely singular continuous or purely Lebesgue (see [Helson86] in the case $X = G = \mathbb{T}$).*
We assume the following:

**Assumption 2.7.** The translation $F_1$ is ergodic and $\phi : X \to G$ satisfies $\phi = \xi \eta$, where

(i) $\xi : X \to G$ is a continuous group homomorphism,

(ii) $\eta \in C(X; G)$ has a Lie derivative $\mathcal{L}_Y(\chi \circ \eta)$ which satisfies

$$
\int_0^1 \frac{dt}{t} \left\| \mathcal{L}_Y(\chi \circ \eta) \circ F_t - \mathcal{L}_Y(\chi \circ \eta) \right\|_{L^\infty(X)} < \infty.
$$

Two comments:

- $\chi \circ \xi$ encodes the “topological degree” of the cocycle $\chi \circ \phi$.

- (ii) means that $\mathcal{L}_Y(\chi \circ \eta)$ is of Dini-type along the translation flow $\{F_t\}_{t \in \mathbb{R}}$. 
Define
\[
\xi_0 := \left. \frac{d}{dt} (\chi \circ \xi)(y_t) \right|_{t=0}, \quad g := |\xi_0|^2 - \xi_0 \frac{\mathcal{L}_Y(\chi \circ \eta)}{\chi \circ \eta} \quad \text{and} \quad A := -i \xi_0 H,
\]
and observe that \( g : X \to \mathbb{R} \) is of Dini-type along \( \{F_t\}_{t \in \mathbb{R}} \) and that \( A \) is self-adjoint with \( \mathcal{D}(A) \supset \mathcal{D}(H) \).

Since \( A \) and \( V_1 \) commute, we have for each \( \varphi \in C^\infty(X) \) that
\[
\langle A \varphi, U_\chi \varphi \rangle - \langle \varphi, U_\chi A \varphi \rangle = \langle \varphi, [A, \chi \circ \phi] V_1 \varphi \rangle = \langle \varphi, -\xi_0 \mathcal{L}_Y(\chi \circ \phi) V_1 \varphi \rangle,
\]
with
\[
\mathcal{L}_Y(\chi \circ \phi) = \mathcal{L}_Y(\chi \circ \xi)(\chi \circ \eta) + (\chi \circ \xi) \mathcal{L}_Y(\chi \circ \eta)
= \left( \xi_0 + \frac{\mathcal{L}_Y(\chi \circ \eta)}{\chi \circ \eta} \right) (\chi \circ \phi).
\]
It follows that
\[
\langle A \varphi, U_\chi \varphi \rangle - \langle \varphi, U_\chi A \varphi \rangle = \langle \varphi, g U_\chi \varphi \rangle,
\]
with \( g \in L^\infty(X) \). So, one has \( U_\chi \in C^1(A) \) with \( [A, U_\chi] = g U_\chi \) due to the density of \( C^\infty(X) \) in \( D(A) \).

Since \( g \) is of Dini-type along \( \{F_t\}_{t \in \mathbb{R}} \), the equalities
\[
\int_0^1 \frac{dt}{t} \left\| \frac{e^{-itA} [A, U_\chi] e^{itA} - [A, U_\chi]}{t} \right\| \mathcal{B}(\mathcal{H})
\]
\[
= \int_0^1 \frac{dt}{t} \left\| e^{-itA} g U_\chi e^{itA} - g U_\chi \right\| \mathcal{B}(\mathcal{H})
\]
\[
= \int_0^1 \frac{dt}{t} \left\| (e^{-itA} g e^{itA} - g) e^{-itA} U_\chi e^{itA} + g(e^{-itA} U_\chi e^{itA} - U_\chi) \right\| \mathcal{B}(\mathcal{H})
\]
imply that \( U_\chi \in C^{1+0}(A) \).
If the function $g$ were strictly positive, we would be able to apply Theorem 2.3 since

$$(U_{\chi})^*[A, U_{\chi}] = (U_{\chi})^* g U_{\chi} \geq \inf_{x \in X} g(x) > 0.$$ 

But, this is a priori not the case since

$$g = |\xi_0|^2 - \xi_0 \frac{\mathcal{L}_Y (\chi \circ \eta)}{\chi \circ \eta} \equiv \text{positive constant + total derivative}.$$ 

Nonetheless, the same averaging of the conjugate operator $A$ as the one used for horocycle flows may work and lead to a strictly positive function $g$. 
2.5 Skew products over translations

Since $U_\chi \in C^1(A)$, we have $U_\chi^\ell \in C^1(A)$ and $U_\chi^\ell D(A) = D(A)$ for each $\ell \in \mathbb{Z}$, and thus the operator

$$A_n \varphi := \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} A U_\chi^\ell \varphi = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} [A, U_\chi^\ell] \varphi + A \varphi,$$

is self-adjoint since $\frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} [A, U_\chi^\ell]$ is bounded.

Doing the same calculations as before with $A_n$ instead of $A$, one obtains that $U_\chi \in C^{1+0}(A_n)$ with

$$[A_n, U_\chi] = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} [A, U_\chi^\ell] U_\chi^\ell = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} (g U_\chi) U_\chi^\ell = g_n U_\chi$$

and

$$g_n := \left( \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} g U_\chi^\ell \right) = \frac{1}{n} \sum_{\ell=0}^{n-1} g \circ F_{-\ell}.$$
Since $F_1$ is ergodic, we know (see [Cornfeld/Fomin/Sinaï82]) that the flow $\{F_\ell\}_{\ell \in \mathbb{Z}}$ is uniquely ergodic and that

$$\xi_0 = \frac{d}{dt} (\chi \circ \xi)(y_t) \bigg|_{t=0} \neq 0 \quad \text{if} \quad \chi \circ \xi \neq 1.$$ 

Using the notation $\chi \circ \eta = e^{i f_x,\eta}$, we infer that

$$\lim_{n \to \infty} g_n = \int_X d\mu \, g = |\xi_0|^2 - \xi_0 \int_X d\mu \frac{\mathcal{L}_Y(\chi \circ \eta)}{\chi \circ \eta}$$

$$= |\xi_0|^2 + \xi_0 \langle 1, H f_{\chi,\eta} \rangle$$

$$= |\xi_0|^2$$

uniformly on $X$. 
Thus, \( g_n > 0 \) if \( n \) is large enough, and

\[
(U_\chi)^* [A_n, U_\chi] = (U_\chi)^* g_n U_\chi \geq \inf_{x \in X} g_n(x) > 0
\]

as desired.

Putting everything together, we obtain the following:

**Theorem 2.8** (Spectral properties of \( W \)). Let \( F_1 \) be ergodic and let \( \phi \) satisfy Assumption 2.7 with \( \chi \circ \xi \neq 1 \). Then, \( U_\chi \) has purely Lebesgue spectrum. In particular, the restriction of \( W \) to the subspace \( \bigoplus_{\chi \in \hat{G}, \chi \circ \xi \neq 1} L_\chi \subset L^2(X \times G, \mu \times \nu) \) has countable Lebesgue spectrum.
Two remarks:

- In the case $X = \mathbb{T}^d$, $G = \mathbb{T}^{d'}$ with $d, d' \geq 1$, this complements previous results of [Iwanik/Lemańczyk/Rudolph 93-99], where $\mathcal{L}_Y(\chi \circ \eta)$ is of bounded variation instead of Dini-type. (bounded variation and Dini-continuity are mutually independent)

- If we do not assume that $\mathcal{L}_Y(\chi \circ \eta)$ is of Dini-type, we can already infer that $W$ has purely continuous spectrum in $\bigoplus_{\chi \in \hat{G}, \chi \circ \xi \neq 1} L_\chi$ due to the corollary on the point spectrum (Corollary 2.4) and the purity law.