Phantom Categories.

X = smooth proj.

\[ D(X) = \text{Proj}(\text{coh}(X)) \]

If \( A \subseteq D(X) \) is a subcategory, then it is full and \( A \to D(X) \) has left and right adjoint.

If \( X \) is as above, and \( D(X) = \langle A, \beta \rangle \) is a seminormal monad.

\[ \beta_A = \bigwedge \beta \] is a quotient.

\( \Rightarrow \) \( A \to A \) is flat admissible.

By a result of Bondal-Kapranov, \( A, \beta \) are admissible.

For an admissible subcategory \( A \subseteq D(X) \) we have that

\[ K_0(A) = K(X) \]

are direct summands.

\[ \oplus \]

Hence:

\[ K(A) = \text{Hom}_{A}(A) \]

By:

\[ \Rightarrow A \text{ is admissible } \Rightarrow K(A) = \text{Hom}_{A}(A) \]

Thus:

Surface:

- "Phantom" exists
- Bohm
- Grayson-Zabrocki
- Some
- A.E.\'s
- Indevor
- Some
- Indevor

Phantom:

\[ K(A) = 0 \]

Quasi-Phantom:

\[ K_0(A) = \text{finite} \]

\[ \text{H}(A) = 0 \]

Let \( A \subseteq D(X) \) be admissible.

\[ \Rightarrow A \subseteq \text{Proj}(\text{coh}(X)) \]

\[ A \to X \to A \]

Ex.

\[ K(A) \to K(X) \to K(A) \]
\( \Rightarrow \mathcal{K}(A) \) in a right module of \( \mathcal{K}(X) \).

\[ \text{II. } \text{ Hochschild cohomology} \]

\[ \text{HH}_n(X) = \mathcal{K}^n(X, \mathcal{X}, \Delta, \mathcal{X}) \quad (\text{natural}) \]

\[ \text{where } X \hookrightarrow X \times X \text{ in degree zero.} \]

\[ \text{Claim: } A = (\text{comm. alg. alg. then } \text{HH}^n(X) = \text{Ext}^n_{\text{alg}}(A, A) \]

\[ \text{and } \text{HH}_n(X) = \text{Tor}_{n-1}(A, A). \]

\[ T = \Delta \text{-act. } (\text{which splits alg. acts.}) \text{ in } A \hookrightarrow A \text{ in } \text{Mod}(T) \]

\[ \text{such that } \Phi_i \hookrightarrow \exists B, C \text{ s.t. } A \cong B \otimes C \text{ with } i \text{ in } k_{B, C}. \]

\[ E \in T \text{ is a split generator of the smallest } \Delta \text{-ad. subobj. of } T \text{ and } \exists \text{ closed under taking direct summands is itself.} \]

\[ \text{Suppose } T \text{ has a split generator } E. \text{ Set } A = \text{End}_\mathcal{A}(E, E). \]

\[ \text{There is a } \text{a } \Delta \text{-equivalence } \mathcal{D}(A \text{-mod}) \rightarrow T \]

\[ \exists \mathcal{M} \rightarrow \mathcal{M} \rightarrow E \]

\[ \text{Ex. } X = \mathbb{P}^1 \text{, } T = D(X) = \langle \mathcal{O}(1), 0 \rangle \text{, } E = \mathcal{O}(1) \otimes 0 \]

\[ \text{RHom}(E, E) \cong \text{C} \text{ as a vector space.} \]

\[ \text{If } \text{the alg. of the group } \Phi_i \text{ is } \otimes \text{ then } \text{c.} \]

\[ D(X) \cong \mathcal{D}(A \text{-mod}) \]
Then $X = \text{smooth proj} \Rightarrow X$ is split generable.

For $L = $ very ample line bundle. Then, $\mathcal{O}_X$ has a rank:

$\mathcal{O}_X \to V_L(L^{\otimes n}) \to V_L(L^{\otimes n}) \to 0 \Rightarrow \mathcal{O}_X \to 0$

known. If $k > 2 \dim X$, then $\text{Ext}(\mathcal{M}, \mathcal{N}) = 0$

for $\mathcal{M}, \mathcal{N} \in \text{coh}(X \times X)$

$\Rightarrow \mathcal{O}_X \to (\ldots \to \mathcal{O}_X \to 0) \Rightarrow \mathcal{O}_X \to 0$

where $k > 2 \dim X$, or set $C = C \otimes \mathcal{O}_X \otimes \mathcal{P} \in \mathcal{D}(X \times X)$

Regarding $\mathcal{D}(X)$ as the derived category of coherent sheaves, we see that $L^n, L^{2n}$ split generable.

Then $X = \text{smooth proj}, E = \text{split even}, A = R\text{Hom}(E, E)$

$\Rightarrow \mathcal{H}^0_A(X) = \mathcal{H}^0_A(A), \text{ etc.}$

Finally, $\mathcal{D}(X) = \langle A, B \rangle$ then derive $H^n(A) = H^n(A')$

where $A' = R\text{Hom}(E_A, E_A)$ where $E_A = \text{proj of } E$ to $A'$.

Finally, we have the HKR lemma $\Delta^* \Delta_0 X \Rightarrow \bigoplus_{p=0}^{n^2} \mathcal{H}^p(X, \mathcal{O}_X)$

$\Rightarrow \mathcal{H}^n(X) \cong \bigoplus_{p=0}^{n^2} \mathcal{H}^p(X, \mathcal{O}_X)$