Goal: Give further details about the structure and construction of $D(K)$.

Recall: $A$ is an abelian category $\mathbf{C}$, and $K(A)$ is complex of objects in $A$.

We define the $K(A)$: i.e. up to homotopy.

Homotopy between $f, g \in \text{Hom}_A(A, B)$ is a collection of maps $h^t: A \to B^{-t}$ s.t.

$f - g = \sum h^t \circ d^t_A + d^{-t}_B \circ h^t.

Exercise: Show that $K(A)$ is a well-defined additive category using the pull, push.

Recall: $\text{Hom}(g, h) \equiv h \circ g$.

$\text{Hom}_A(A, B) \equiv \{ f \in \text{Hom}_A(A, B) \mid f \circ g = h \circ g \}$

Exercise: Show $\text{Hom}_A(A, B)$ is a well-defined additive category using the pull, push.

Diagram:

Two functors $\varphi, \psi: A \to B$ are equivalent if there exists a natural isomorphism $\mu$ such that the following (commute) in $K(A)$:

$\mu$ is an isomorphism.

$A \xrightarrow{\varphi} B \xrightarrow{\psi} B$

$C \xrightarrow{\psi} C \xrightarrow{\psi}$

$\varphi \circ C \xrightarrow{\mu}$

Remark: $\text{K}(\mathbf{C})$ is introduced as an intermediate step because 2 loops cannot be composed to get a 3rd diagram in $\text{K}(\mathbf{C})$. 
Not: If \( f : A \rightarrow B \) is a morphism, we denote it as \( A \rightarrow f \rightarrow B \).

Composition:

\[
\begin{array}{c}
\text{let } \phi \in \text{Mor}(\text{Kom}(A)) \\
\phi : A \rightarrow B \\
\end{array}
\]

(i) We want \( \phi \circ \psi \) and (ii) we need to show it is unique.

Need the mapping cone for this:

Recall:\( F : A \rightarrow B \in \text{Mor}(\text{Kom}(A)) \)

\[
\Rightarrow C(F) \in \text{Kom}(A) \text{ is obtained by } C(F) = A \oplus B, \quad \delta_C = \begin{pmatrix} \pi_{A} \\ \pi_{B} \end{pmatrix}
\]

Let \( t \) be a non-inclusion \( t : B \rightarrow C(F) \)

projection \( \pi : C(F) \rightarrow A[1] \).

Exercise:

(a) \( B \rightarrow C(F) \rightarrow A[1] \) is a short exact seq. of complexes

(b) \( A \rightarrow B \rightarrow C(F) \) has no

Answer:

(1) \( \Rightarrow \) is as in coh \( \Rightarrow \text{H}^i \text{(B)} \rightarrow \text{H}^i \text{(C)} \rightarrow \text{H}^{i+1}(A) \rightarrow \ldots \)

Note that any commutative diag:

\[
\begin{array}{c}
\text{A}_1 & \xrightarrow{f} & B_1 & \xrightarrow{g} & \text{A}[1] \\
\text{A}_2 & \xrightarrow{h} & B_2 & \xrightarrow{a} & \text{A}[1] \\
\end{array}
\]

(Recall TR 3)

Prop:

Let \( f : A \rightarrow B \in \text{Mor}(\text{Kom}(A)) \) and \( A \rightarrow B \rightarrow C(F) \rightarrow A[1] \)

be the \( \Delta \) above. Then, \( F : A[1] \rightarrow C(F) \in \text{Mor}(\text{Kom}(A)) \)

which is on each in \( K(A) \) s.t. comm. diag in \( K(F) \):

\[
\begin{array}{c}
\text{C(F)E} \rightarrow \text{B} \rightarrow \text{C(F)} \rightarrow \text{C(F)} \\
\text{A} \rightarrow \text{B} \rightarrow \text{C(F)} \rightarrow \text{C(F)} \\
\end{array}
\]

(Recall TR 2)
Sketch of Proof:

Define $g_1 : A_1 \to C(t)$ as

$$A_1^g = A_{t^g}$$

$$C(t) = \beta^{\gamma} \otimes C_f$$

Define $h : C(t) \to A_1$ as $h = (g)$.

Exercise: Check that $h \in \text{Hom}(A_1, C(t))$.

Exercise: Check that $h = g_1$ in $\text{K}(A)$.

The commutativity in (1) is clear in $\text{K}(A)$, but it does not commute

in $\text{K}(A)$, only in $\text{K}(A)$.

Exercise: Check that $g$ commutes in $\text{K}(A)$ using $h \circ \pi = \pi$ and $g^{-1} = h$ in $\text{K}(A)$.

Proposition: Suppose $f : A \to B$ is a g'fim and $g : C \to B$ is an arbitrary

morphism in $\text{K}(A)$. Then $f$ a commutator in $\text{K}(A)$.

\[ \begin{array}{ccc}
\text{C} & \xrightarrow{g} & \text{A} \\
\downarrow & & \downarrow \\
\text{B} & \xrightarrow{f} & \text{C} \\
\end{array} \]

Take $C_0 = C(T_{fg})[-1]$ and then (1) is the required diagram. Only

need to show that $h$ is a g'fim take $f$, e.g., in column $i$.

$$H^i(C(f)) = 0 \quad \forall i$$

Since $H(f)$ is an isomorphism $\forall i$.

$$\Rightarrow H^i(h)$$ is an isomorphism $\forall i$. Thus $h$ is a g'fim.
Com: composition in $\mathcal{D}(A)$, as proposed above, is well-defined.

Exercise: Check that the given class $A \to X$ is unique.

Exercise: Check that $\mathcal{D}(A)$ is an additive category.

Exercise: Check that $A \to X$ & $A \to X$ are fully faithful, i.e., $\text{Hom}_{\mathcal{D}(A)}(A,B) \cong \text{Hom}_{\mathcal{D}}(X,A) \otimes \text{Hom}_{\mathcal{D}}(X,B)$ etc.

Exercise: Let $0 \to A \to B \to C \to 0$ be a s.e.s. in $\mathcal{A}$.

\[\begin{array}{ccc}
\delta & \delta & \delta \\
\sigma & \sigma & \sigma \\
\tau & \tau & \tau
\end{array}\]

Exercise: (a) Let $A \in \text{Ob}(\mathcal{D}(A))$. Then, $A = 0$ if $H^n(A) = 0$ $\forall n$.

(b) $f \in \text{Hom}_{\mathcal{D}(A)}(A,B)$. Show that $f = 0 \iff \exists$ a morphism $g: C \to A$ s.t. $f \circ g = 0$.

Remark: It may happen that $f: A \to B$ satisfies $H^n(f) = 0$ $\forall n$ but $f \neq 0$ in $\text{Mor}(\mathcal{D}(A))$.

Exercise: Let $A \to B \to C$ s.e.s. in $\mathcal{A} \Rightarrow A \to B \to C \to A[1]$ as comup. A with $C = C(f)$. Note $H^n(A[1]) \cong H^{n+1}(A)$

& $H^n(h): H^n(C) \to H^n(A[1]) = A \Rightarrow H^n(h) = 0 \forall n$.

Remark: $h \in \text{Hom}(C, A[1]) = \text{Ext}^1(C, A)$

But in general $h \neq 0$ in $\mathcal{D}(A)$.

Remark: $h \in \text{Hom}(C, A[1]) = \text{Ext}^1(C, A)$

We will show that the extension class of the seq. $(h)$ is non-zero.
\[ a \Delta : A_{1} \to A_{2} \to A_{3} \to A_{4}[1] \text{ in } K(A), \text{ resp. } D(A) \]

is called distinguished if it is isomorphic to a \( \Delta \) of the form
\[ A \to B \to C(\mathbb{Z}) \to A[1] \text{ in } K(A), \text{ resp. } D(A). \]

**Prop.** with the distinguished \( \Delta \) as in question above and the natural shift of complexes, \( A \mapsto A[1], K(A), \text{ resp. } D(A) \) become \( \Delta \)-ind categories.

**Exerc.** Let \( A = \text{Vect}_{\mathbb{F}}(k) \); \( k \)-dim vector spaces over \( k \).

Show that \( D(A) = T_{1} A \), i.e., \( T_{1} A \in D(A) \), where \( T_{1} \) is the extension \( A \cong \bigoplus_{i \in \mathbb{Z}} H^{i}(A)[i] \) (complex with zero differentials).